1
Introduction to and History of the Fibonacci Sequence

A brief look at mathematical proportion calculations and some interesting facts about this ratio.

The origins of the Fibonacci sequence are well known to architects, artists and technical analysts, but knowledge of the importance of the Golden Section was known further back in ancient history, definitely as far as the Greeks and, depending on which source is read, as far back as ancient Egyptians and Sumerians. However, evidence for understanding and usage in ancient Sumer is tenuous at best.

Taking a line of any length, the ancients discovered that there was a point on the line where the proportions of the whole to the larger section was the same proportion of the smaller section to the larger section. This point on the line is called the Golden Section.

Knowledge of irrational numbers was known in antiquity, and for the Greeks, especially the Pythagorean school, came as a shock. In ancient times, rational numbers (1, 2, 3, etc.) were believed to have the secret of all knowledge and that any length could be measured using whole number units only; e.g. 9.65 was actually 965 units of some smaller measure. The discovery of pi (π) came as a surprise to the Greeks looking at the relationship between the diameter of a circle and its circumference, as the multiplication factor to find the circumference was not a whole number. Imagine the additional shock of discovering that in a square of side one unit the diagonal was not a whole number that could be counted? That is to say, within the line section that gives the Golden Mean, there is no measure, no matter how small, that will give the result that one part of the line section is a whole number of measuring units and the smaller is also a whole number. The inability to find common measures that will give whole numbers for the two sections means that the proportion is incommensurable.

This meant that there was no number representing the hypotenuse of the triangle of sides equal to one, or within the line section, that could be seen as the product of two others, no matter how they searched. That was just the start as more and more of what we now call irrational numbers were discovered. It is into this group that the Golden Section belongs. The Golden Section is an incommensurable number, i.e. it cannot be represented as a fraction, and was represented by the Greek letter τ (tau), being the first letter of the word for 'the cut', τομη (to-mi) in Greek. Contemporary symbolism for the Golden Section is 'ϕ', which was suggested in the early 20th century by Mark Barr, an American mathematician, as a homage to Phidias, the classical Greek sculptor and builder of the Athenian Parthenon and of the Temple of Zeus at Olympus. What greater honour could there be?

Much later, in the 15th century in Pisa, Italy, Leonardo de Fibonacci constructed a simple series after observing the population expansion of a pair of rabbits. He noted that it took one generation before each new pair reached sexual maturity and the population exploded. The total number of pairs (breeding and immature) was noted down. In Figure 1.1, taken from data in Table 1.1, the normal notation from biological science is used, where F_n is the filial generation and n is the number of

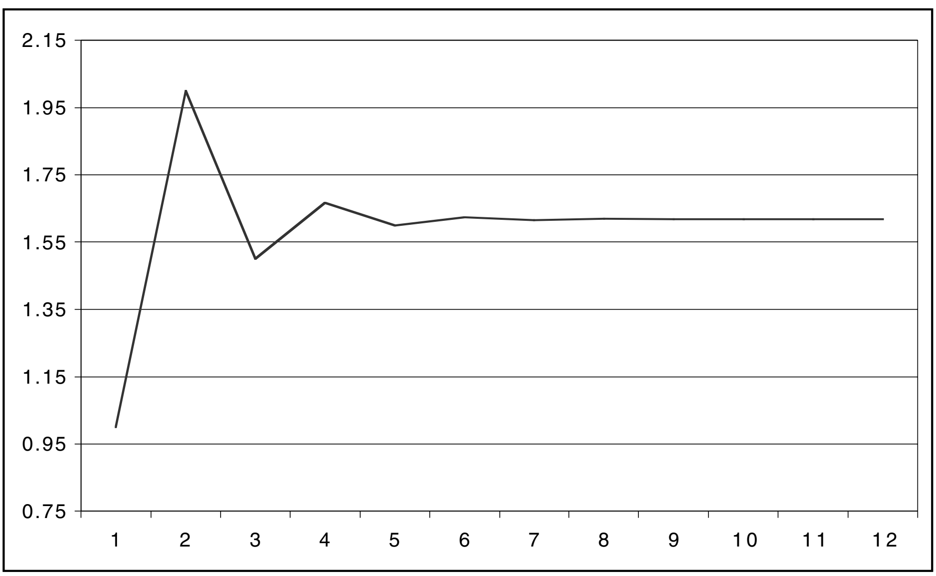

Figure 1.1

Table 1.1

n	1	2	3	4	5	6	7	8	9	10	11	12	13
F	1	1	2	3	5	8	13	21	34	55	89	144	233
		1	2.000	1.500	1.667	1.600	1.625	1.615	1.619	1.618	1.618	1.618	1.618

that generation. Taking this series (1, 1, 2, 3, 5, 8, 13 and so on), each subsequent filial generation is seen as the sum of the previous two generations as follows:

$$F_n = F_{n-2} + F_{n-1}$$

This is an infinite series without limit.

An interesting corollary of this series is that there is a relationship between each filial total. Taking

$$\frac{F_n}{F_{n-1}}$$

the series

4.236, 2.618, 1.618, 0.618, 0.382, 0.236, 0.146

very quickly tends to 1.618, as represented graphically by Figure 1.1. Further relationships are found by taking F_n with F_{n-2}, F_n with F_{n-3}, etc., resulting in the limits given in Figure 1.2, taken from the data in Table 1.2. These are important values for the technical analyst, for from these our 'common or garden' Fibonacci ratio of 61.8% is derived.

Reversing the ratio will give similar limits, with 0.618, 0.382, 0.236 as key here. These are the main ratios used in technical analysis and a discussion and application chapter follows later in the book. The table of Fibonacci ratios is

1.618, 2.618, 4.236, 0.618, 0.382, 0.236 and 0.146

Normally in technical analysis, these are expressed as percentages:

161.8%, 261.8%, 243.6%, 61.8%, 38.2%, 23.6% and 14.6%

The interesting number of 1.618 is also derived from the following infinite fraction:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

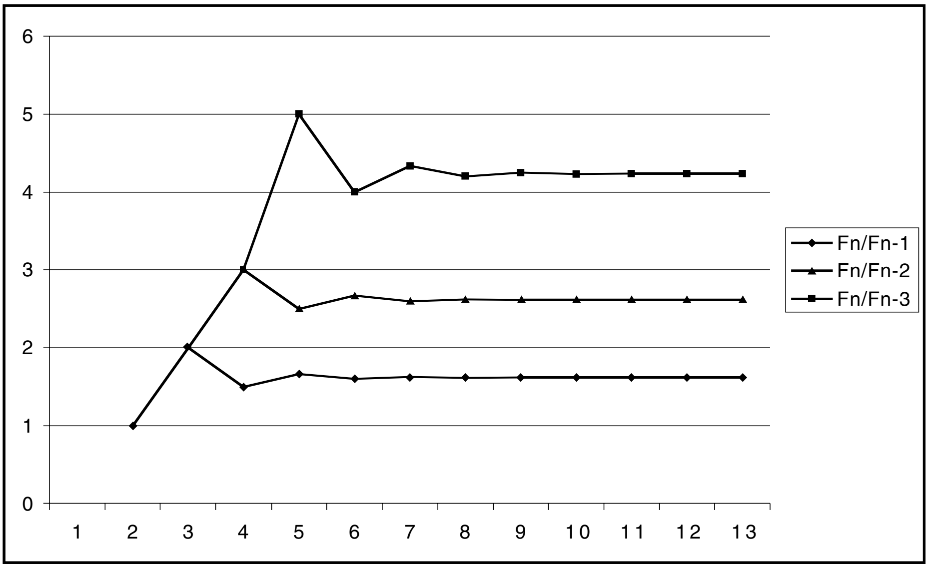

Figure 1.2

Table 1.2

n	1	2	3	4	5	6	7	8	9	10	11	12	13
F	1	1	2	3	5	8	13	21	34	55	89	144	233
		1	2.000	1.500	1.667	1.600	1.625	1.615	1.619	1.618	1.618	1.618	1.618
			2.000	3.000	2.500	2.667	2.600	2.625	2.615	2.619	2.618	2.618	2.618
				3.000	5.000	4.000	4.333	4.200	4.250	4.231	4.238	4.235	4.236

Although this looks complicated, making the above equal to x, it breaks down to

$$x = 1 + \frac{1}{x}$$

resulting in $x^2 = x + 1$ once both sides are multiplied by x. Therefore, using the quadratic solution of

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{with } a = 1, b = -1 \text{ and } c = -1 \text{ (from } x^2 - x - 1 = 0)$$

gives

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1(-1)}}{2.1} = x = \frac{1 \pm \sqrt{1 + 4}}{2} = x = \frac{1 \pm \sqrt{5}}{2}$$

which results in $x = 1.618\,033\,9$ (ϕ) and $-0.618\,033\,9$, which is $-1/\phi$.

There are many volumes that look at the interesting properties and occurrences of this ratio in nature. Some key examples of this are the famous nautilus shell chambers, the sunflower head seed pattern and the spiral in a galaxy, and in architecture the ratio of the length to the width of the Parthenon of Phidias, which is seen as the epitome of classical proportion. In other art forms such as fresco and oil painting, the proportions of the setting are often seen in the above ratio, especially in the work of Leonardo da Vinci and in the 20th century in the religious art of Salvador Dali. Closer to home, the human ear length needs to be 1.618 greater than the width to be said to be 'in proportion', as are the relationships between limbs and the ratio of the navel to the feet and total height, as in the work of Le Corbusier (Charles Edouard Jeanneret), in *The Modulor: A Harmonious Measure to the Human Scale Universally Applicable to Architecture and Mechanics* and *Modulor 2* (*Let the User Speak Next*) with the Red and Blue scales of proportion. However, Le Corbusier had to force his proportion system to appear as the Golden Ratio, given that his original premise was that the male figure in his drawings had to be British and not French in order to get the height of the figure with arm outstretched above equal to 220 cm.

The human eye sees proportion in interesting ways: what is pleasing to the eye generally is seen as beauty. It does not take long to see that something is 'out of proportion' in nature, and no more so than the frequent occurrences of the ratio on and within the human body. Artists and architects have used this relationship, often called the 'Golden Mean', for centuries to produce work that is pleasing to the eye.

The following derivation of the Fibonacci spiral contains some very basic algebra which I hope will not confuse the reader so early on in this general work.

Beginning with a square of side unit equal to 1, one of the sides is extended so that the ratio of the new line to the old side of the square is in the Golden Mean, i.e. the new total length is ϕ, being the original size of the square edge + the new line ($\phi - 1$). Now, completing a new square adjacent to the original, this will have a side of length $\phi - 1$. Again extending the side of this square so that the new length equals that of the original square, i.e. size = 1, the length of this addition is calculated from

$$1 = x + (1 + \phi), \text{ where } x \text{ is the length of the line extension}$$

Solving gives $x = 2 - \phi$. Repeating this process, the next line extension will join back to one of the corners of the original square.

This unknown (y) can be calculated from some of the previous lengths as follows. The initial extension line was of size $\phi - 1$ and part of that is the x found above. Therefore,

$$\phi - 1 = x + y = 2 - \phi + y$$

and so

$$y = \phi - 1 - 2 + \phi = 2\phi - 3$$

Again repeating this move, a square is formed with sides equal to y and an extension line z is drawn. This can be calculated as $y + z + \phi - 1 = 1$ (the side of the original square):

$$z = 1 + 1 - \phi - y = 2 - \phi - (2\phi - 3) = 5 - 3\phi$$

Again a square is completed, now with side z, and an extension line is also drawn. The length of this extension line is calculated from the knowledge gained before: $\phi - 1$, the initial extension line length, is equal to $x + z + q$, where q is the new length. Thus

$$\begin{aligned} q = \phi - 1 - x - z &= \phi - 1 - (2 - \phi) - (5 - 3\phi) \\ &= \phi - 1 - 2 + \phi - 5 + 3\phi = 5\phi - 8 \end{aligned}$$

Continuing this process, the next line length is r. As we know that $r + z = y$, then

$$r = y - z = 5 - 3\phi - 5\phi + 8 = 13 - 8\phi$$

The natural occurrence of the Fibonacci ratio is most famously seen in the developing chambers of the nautilus shell. Here each new chamber is 1.618 greater than the previous one as the crustacean grows in size.

In addition, the pattern of seeds in a sunflower head also show this relationship. Here two concentric spirals compete and grow as the flower head develops. The spirals increase in size as the flower grows and the spiral increases at 1.618 as well. On a grander scale, the spiral galaxy also grows at this rate.

These proportions also exist in human anatomy. Taking the unit as that distance from the navel to the feet, the distance from the navel to the top of the head is 0.618. Similar relationships are seen within this position within the body itself, e.g. from the total arm length and the shoulder to the elbow.

It is a simple step to link all these occurrences together and from there to suggest that 'natural' systems of growth should show this relationship in some form or other. Mathematically, the Golden Ratio displays interesting characteristics. Taking $\phi = 0.618$, the following becomes clear:

$\phi^2 = 0.618 \times 0.618 = 0.382$, a Fibonacci retracement level (see Chapters 3 and 5)
$\phi^3 = 0.618 \times 0.618 \times 0.618 = 0.236$, a Fibonacci retracement level (see Chapters 3 and 5)

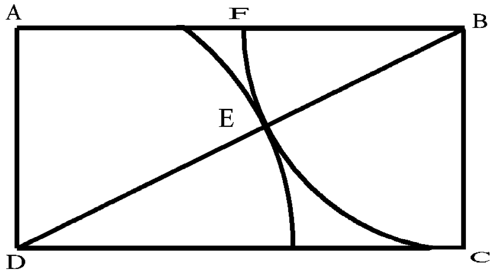

Figure 1.3

$1/\phi = 1/0.618 = 1.618$, a Fibonacci extension level (see Chapters 3 and 5)
$1/\phi^2 = 1/(0.618 \times 0.618) = 2.618$, a Fibonacci extension level (see Chapters 3 and 5)

It can therefore be seen that adding a unit to ϕ is the same as multiplying by ϕ.

A further method of constructing the Fibonacci ratio comes from simple geometry (see Figure 1.3). Take a rectangle with two sides of one unit and with two others of two units. The diagonal of this shape has the value $\sqrt{5}$. Taking an arc from one corner of radius 1, the diagonal is cut as shown. Then using an arc from the opposite corner with a radius measured along the diagonal to the previous cut, curving this to the long side gives the following:

Side AB:AF = AF:FB

These are exactly the proportions necessary to complete the Golden Section mentioned above. It can be seen from the above various derivations that the Golden Section, ϕ, is very important. It is the 'naturalness' and frequency of occurrence that gives the proportion to financial market analysis, as technical analysts believe that 'price' is the physical outcome of a natural system at any point in time, the natural system being the result of the action between buyers and sellers. This is covered in more depth in the next chapter.