Introduction

I.1 OUTLINE OF THE BOOK

Recent trends in the development of vacuum microwave electronics and the physics of electron beams have been shaped in part by competition with solid-state high-frequency electronics. So practically all information technology and microwave devices of small power and limited frequency are based on solid-state electronics. Contemporary vacuum microwave electronics and the physics of charged-particle beams include the formation and transport of intense and relativistic electron beams, electron optics, powerful microwave devices together with millimeter- and submillimeter-wave devices, charged-particle accelerators, material procession, and free electron lasers. This narrowing of focus has led to considerable progress in the following aspects of theory and engineering:

- Theory of electromagnetic fields
- Dynamics of charged-particle beams
- Interaction of electron beams with high-frequency fields
- Electron emission and optics
- Development and application of vacuum electron devices
- Methods of computer simulation
- Technology and powerful and high-voltage experimental techniques and equipment

It would thus be very difficult to embrace within a single volume a complete description of the state of the art in this field. There are many books on the subject of vacuum electronics (many of which we cite in the book). They do not, however, provide a thorough treatment of the theory of both electron beams and microwave

Electron Beams and Microwave Vacuum Electronics. By Shulim E. Tsimring

electronics. Two books, *Modern Microwave and Millimeter-Wave Power Electronics* (Barker et al., 2005) and *High Power Microwave Sources and Technologies* (Barker and Schamiloglu, 2001), are the exception. These volumes are characterized by an exceptionally wide scope of information. They do not, however, include a systematic exposition of the theory of basic processes assuming the reader's familiarity with the theory. In this book I strike a compromise, providing the reader with a foundation in the physics and theory of electron beams and vacuum microwave electronics. The material is presented in historical sequence, and classical results and concepts are treated alongside contemporary issues.

The book is divided into two parts: Part I, Electron Beams (Chapters 1 to 5), and Part II, Vacuum Microwave Electronics (Chapters 6 to 10). Auxiliary information (e.g., equations of motion, Maxwell's equations, Hamiltonian formalism, the Liouville theorem) is presented in the Introduction. This material cannot, however, replace corresponding background fundamental guides. It serves a reference function and provides notation and definitions.

Part I begins in Chapter 1 with a discussion of the motion of charged particles in static electric and magnetic fields. Special attention is devoted to an analysis of relativistic beams and the motion of charged particles in weakly inhomogeneous fields (e.g., adiabatic invariants, drift equations).

In addition to classical paraxial electron optics, in Chapter 2 we describe the theory and applications of quadrupole lenses, which are important elements of accelerators and effective correctors of aberration in paraxial electron-optical systems. Principles of electronic image construction are an important element in electron beam formation in microwave devices and accelerators.

In Chapters 3 and 4, an extensive area of the physics of intense electron beams that are used in most high-frequency electron devices is considered. The self-consistent equation of steady-state space-charge beams is derived. Self-consistent solutions for certain space-charge curvilinear flows as well as gun synthesis methods are described. A number of electron guns with compressed electron beams are discussed. The theory of noncongruent space-charge beams and its application to the design of magnetron-injected guns are considered.

Electron guns that use explosive electron emission, making it possible to obtain electron beams with energy on the order of MeV and currents of hundreds of kiloamperes, have acquired great significance in powerful high-frequency electronics and electron beam technology. Guns using planar explosive emission and magnetically insulated diodes are considered.

Transport problems of lengthy intense electron beams that are key problems for microwave devices are discussed in Chapter 5. A group of relevant problems is connected with the transport of nonrelativistic and relativistic Brillouin beams of various configurations. The transport of intense beams in an infinite magnetic field approximation and centrifugal focusing is also discussed. A theory of intense axially symmetric paraxial electron beams with arbitrary shielding of the cathode magnetic field is described. A criterion for stiffness beam formation is formulated.

Finally, the transport of intense electron beams in spatially periodic fields is considered. A theory of periodic magnetic focusing, which has the most practical value for beam-type tubes, is expounded.

Part II opens (Chapter 6) with an analysis of quasistationary microwave devices in which the electric field is potential but the energy integral is not conserved. Analysis of the simplest element of these systems, a planar electron gap, demonstrates two principal effects: bunching of electrons and phasing of bunches. The latter, and also the effects of velocity and energy modulation, are crucial for all vacuum microwave devices. All these effects in the electron gap are not optimal, however. Their implementation led to the first truly microwave amplifiers and oscillators: klystrons based on electron-stimulated transition radiation. In Chapter 7 a number of klystron systems, including reflex and relativistic klystrons, are considered.

Linear and nonlinear theories of traveling-wave tubes of O type (TWTOs) based on the synchronous radiation of rectilinearly moved electrons in the field of a slow electromagnetic wave are discussed in Chapter 8. These tubes and backward-wave oscillators (BWOs), in which an electron beam interacts with an electromagnetic wave whose phase and group velocities are opposite, possess unique properties as wideband oscillators. Relativistic TWTOs are considered there as well. These tubes have output power on the order of gigawatts and should provide very high gain, because only in this case can conventional low-power input sources be used. Powerful relativistic TWTs have spatially extended electromagnetic structures. Therefore, mode selection is an important problem in these tubes.

The energy of the electromagnetic field in TWTs and BWOs of O type is fed by electron kinetic energy. Decrease in electron velocities in the process of interaction violates the synchronism. So the efficiency of these tubes, especially of BWOs, is comparatively low. An essentially different mechanism is implemented when electron beams interact with electromagnetic fields in crossed static electric and magnetic fields (M-type systems). In this case the energy of the electromagnetic field is extracted from the potential energy of particles. As a result, synchronism is maintained along a deep conversion of the electron energy. M-type systems can have an efficiency close to 100%. In Chapter 9, typical devices of M type are considered: magnetrons, injected-beam traveling-wave and backward-wave amplifiers and oscillators, and amplifiers of magnetron type. The very high efficiency and high pulse power of the latter allow them to be used as basic high-frequency sources in radar systems and electronic countermeasure devices. Also, the high efficiency, compactness, and low cost of low-power magnetrons explain their exceptional use in domestic microwave ovens. Relativistic magnetrons that use explosive emission cathodes are also considered in Chapter 9. These oscillators are very promising high-frequency sources in radar systems and countermeasure means.

A very interesting power oscillator that utilizes crossed fields is the magnetically insulated line oscillator (MILO). In this tube, the magnetic field of the electron beam replaces the external magnetic field of a conventional magnetron. This requires a very high beam current that can be provided only by explosive electron emission. The constructive simplicity of such systems provides potential advantages with respect to other pulse sources of electromagnetic oscillation with power on the order of gigawatts in the L and S frequency bands.

The microwave amplifiers and oscillators mentioned above exploit radiation of electrons executing rectilinear or close to rectilinear particle motion: transition and Cerenkov radiation. In the latter, synchronous radiation of particles is possible, due to their interaction with slow and therefore surface electromagnetic fields. The output power and efficiency of corresponding devices inevitably drop with the frequency. So the shortest nonrelativistic BWOs have a maximum output power on the order of milliwatts in the submillimeter-wavelength range. Relativistic devices of O type are an exception, but the possibilities for their practical application, especially in continuous-wave (CW) regimes, are limited.

New ideas were put forward at the end of the 1950s and the beginning of the 1960s. The natural attenuation was turned to electron beams with curvilinear periodical trajectories of particles in which electrons radiate at an arbitrary ratio of their velocity to the phase velocity of a wave in a given medium. This concept is the idea underlying classical electron masers (CEMs), where stimulated radiation of oscillating electrons takes place.

Chapter 10 is devoted to the mechanism, theory, and sources of stimulated radiation of classical electron oscillators. This area of vacuum electronics reflects perhaps the most significant tendencies in modern high-frequency electronic developments. The analysis of an ensemble of classical electron oscillators in electromagnetic fields displays two important mechanisms: linear and quadratic bunching. The latter is the result of the nonisochronism of oscillators. Among the examples of subrelativistic classical electron masers considered in the book, the gyrotron and the ubitron are notable, in which takes places the stimulated bremsstralung of electrons in uniform and spatially periodic magnetic fields, respectively. The surprising property of the gyrotron is the existence of a strong essentially relativistic quadratic bunching for subrelativistic energies of electrons (on the order of tens of keV). Another important property of a gyrotron is the possibility of using spatially developed electrodynamic and electron-optical systems, due to the existence of effective mode selection methods for gyrotrons. That allows one to obtain record average output power in the millimeter- and submillimeter-wave ranges. Unique gyromonotrons have been developed that deliver CW output power up to 1 MW in a 2-mm wavelength. Similar gyromonotrons find wide application in controlled fusion experiments (e.g., electron–cyclotron resonance heating and electron–cyclotron current drive in tokamak–stellarator plasmas). A substantial part of Chapter 10 covers an analysis of the gyrotron mechanism, gyrotron electron-optical systems, methods of mode selection in gyrotrons, and various gyrotron applications.

The efficiency of gyrotrons drops, however, as the electron energy approaches the relativistic energy, because the decrease in the relativistic electron mass in the process of radiation violates the synchronism between oscillating electrons and the electromagnetic field. In this case, cyclotron autoresonance masers (CARMs) and free-electron masers (FELs) are alternatives. In a CARM, synchronism is supported when the phase velocity is closed to the light velocity c, due to compensation of the electron relativistic gyrofrequency shift and the Doppler shift stipulated by a change in the electron drift velocity. In a FEL, an ultrarelativistic version of the ubitron, the stimulated radiation of electrons in wiggler (undulator) devices with a spatially periodic magnetic field, is used. A very important property of a FEL is the bremsstrahlung Doppler frequency up-conversion, according to which the radiation frequency in the laboratory frame of reference increases approximately proportionally to the square of the relativistic mass of the moving radiating particle. This property, together with those specific for the FEL pondermotive bunching effect, allows one to obtain powerful coherent radiation in the infrared, optical, ultraviolet, and potentially, even hard x-ray ranges.

CARMs and FELs are considered in the book comparatively briefly. FELs were invented in 1971, and at present the number of published papers dedicated to FELs is on the order of 10^4. Due to its wavelengths, coherent properties, frequency tunability, and high output power, FELs open an unprecedented range of applications. A list that is very far from complete includes such topics as biology, biomedicine, surgery, solid-state physics, chemistry and the chemical industries, defense, micromachining, photophysics of polyatomic molecules, and military and domestic applications.

Unfortunately, a reader will find very little material on electron emission, stochastic oscillations in electron beams and microwave tubes, and charged-particle beam problems in accelerators. That restriction certainly narrows the scope of the book but opens additional possibilities for a more detailed discussion of theory and methods in the main topics mentioned above. Also, there was no room for numerous constructive implementations of devices. This is a very complicated problem taking into account the modern dynamics of high-power vacuum electronics development.

Finally, computation algorithms and softwares that have reached a very high level of sophistication are not considered in this book. Surely, they should now be treated as an independent area of vacuum electronics. Certainly, numerical methods are a necessary component of the design of all electron devices. However, the application of the numerical simulation can turn out to be useless without a clear understanding of theoretical foundations.

I.2 LIST OF SYMBOLS

Vector values are denoted in bold face. MKS units are used.

$\mathbf{A}$	magnetic vector potential
$\mathbf{B}, B$	magnetic induction
$c = 2.997925 \times 10^8$ m/s	light velocity
$\mathbf{E}, E$	electric field
$e_0 = 1.602177 \times 10^{-19}$C	absolute value of an electron charge
$h = 6.626076 \times 10^{-34}$ J $\cdot$ s	Planck's constant
$I_0 = \dfrac{4\pi\varepsilon_0 c^3}{\eta} \approx 17$ kA	relativistic current
i	imaginary unity
$\mathbf{j}, j$	current density
$k = 1.38066 \times 10^{-23} J/K$	Boltzmann's constant
m	electron relativistic mass
$m_0 = 9.109390 \times 10^{-31}$ kg	electron rest mass
n	particle density
$\mathbf{p}, p$	momentum

$\mathbf{v}$, v	velocity of a particle
v_g	group velocity
v_{ph}	phase velocity
$w_0 = m_0c^2 = 8.187111 \times 10^{-14}$ J	electron rest energy
$\beta = \dfrac{v}{c}$	dimensionless velocity
β	propagation constant
$\gamma = \dfrac{m}{m_0} = 1/\sqrt{1-\beta^2}$	relativistic factor
$\varepsilon_0 = (\mu_0 c^2)^{-1} = 8.854188 \times 10^{-12}\mathrm{F/m}$	permittivity of free space
$\eta = \dfrac{e_0}{m_0} = 1.758820 \times 10^{11}\ \mathrm{C/kg}$	electron specific charge
$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H/m}$	permeability of free space
ρ	space-charge density
φ	electric potential
$\varphi_0 = \dfrac{m_0c^2}{e_0} \approx 511 \times 10^3\ \mathrm{V}$	reduced electron rest energy
$\omega_g = \dfrac{e_0B}{m} = \dfrac{e_0B}{m_0\gamma}$	gyrofrequency (electron–cyclotron frequency)
$\omega_p = \left(\dfrac{n_e e_0^2}{m_0\varepsilon_0}\right)^{\frac{1}{2}}$	electron plasma frequency
ω_q	reduced plasma frequency

I.3 ELECTROMAGNETIC FIELDS AND POTENTIALS

Considered below are electromagnetic fields acting on a *moving particle* with location $\mathbf{r}(t)$; therefore, the fields and potentials are expressed as

$$\mathbf{E} = \mathbf{E}[\mathbf{r}(t),t], \qquad \mathbf{B} = \mathbf{B}[\mathbf{r}(t),t], \qquad \varphi[\mathbf{r}(t),t], \qquad \mathbf{A} = \mathbf{A}[\mathbf{r}(t),t] \tag{I.1}$$

For static fields, $\partial/\partial t = 0$ and

$$\mathbf{E} = \mathbf{E}[\mathbf{r}(t)], \qquad \mathbf{B} = \mathbf{B}[\mathbf{r}(t)], \qquad \varphi[\mathbf{r}(t)], \qquad \mathbf{A} = \mathbf{A}[\mathbf{r}(t)] \tag{I.2}$$

Field–potential relations according to Maxwell's equations are

$$\mathbf{E} = -\mathrm{grad}\,\phi - \frac{\partial A}{\partial t}, \qquad \mathbf{B} = \mathrm{curl}\,\mathbf{A} \tag{I.3}$$

For static fields:

$$\mathbf{E} = -\mathrm{grad}\,\varphi, \qquad \mathbf{B} = \mathrm{curl}\,\mathbf{A} \tag{I.4}$$

Maxwell's equations (in free space):

$$\mathrm{curl}\,\mathbf{B} = \mu_0\,\mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{I.5}$$

$$\mathrm{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{I.6}$$

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{I.7}$$

$$\operatorname{div} \mathbf{B} = 0 \tag{I.8}$$

For static fields according to Eqs. (I.4) and (I.7), Poisson's equation is valid:

$$\text{div grad } \varphi = -\frac{\rho}{\varepsilon_0} \tag{I.9}$$

This equation is reduced to the following forms in Cartesian and cylindrical coordinates, respectively:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = -\frac{1}{\varepsilon_0} \rho(x,y,z) \tag{I.10}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} = -\frac{1}{\varepsilon_0} \rho(r,\theta,z) \tag{I.11}$$

I.4 PRINCIPLE OF LEAST ACTION. LAGRANGIAN. GENERALIZED MOMENTUM. LAGRANGIAN EQUATIONS

The *principle of least action* (*Hamilton's principle*), is stated: For each mechanical system, the functional (the action integral of specific function L) exists as

$$S = \int_{t_1}^{t_2} L(q_1, q_2, \ldots, q_N, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N, t)\, dt \tag{I.12}$$

For real trajectories, $q_i = [q_i(t)]_{\text{real}}$ S has the least (in general, extreme) value. Here L is the Lagrangian, q_i and $\dot{q}_i (i = 1, 2, \ldots, N)$ are generalized coordinates and velocities, and N is the number of degrees of freedom. A system of n particles has $N = 3n$ degrees of freedom. All possible (comparable) trajectories belong to the class

$$q_i(t_1) = q_i^{(1)}, \qquad q_i(t_2) = q_i^{(2)}, \qquad i = 1, 2, \ldots, N \tag{I.13}$$

It can be shown that a necessary condition for realization of the extreme of the functional S is the system of equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, N \tag{I.14}$$

which determines the real trajectories. These are the famous *Lagrangian equations*. *Generalized momenta* are defined as

$$P_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad i = 1, 2, \ldots, N \tag{I.15}$$

It is convenient for the particle ($N = 3$) to use the vector notation $\mathbf{r}(q_1, q_2, q_3)$, $\mathbf{v}(\dot{q}_1, \dot{q}_2, \dot{q}_3)$. Then the generalized momentum is

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{v}} \tag{I.16}$$

An alternative *Lagrangian equation* to (I.14) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = \frac{d\mathbf{P}}{dt} - \frac{\partial L}{\partial \mathbf{r}} = 0 \tag{I.17}$$

The *Lagrangian* for the electron in the electromagnetic field is given by (see, e.g., Landau and Lifshitz, 1987)

$$L = -m_0 c^2 \sqrt{1 - \beta^2} - e_0\, \mathbf{A}\mathbf{v} + e_0 \varphi \tag{I.18}$$

The *electron generalized momentum* according to Eqs. (I.16) and (I.18) is

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{v}} = \frac{m_0 \mathbf{v}}{\sqrt{1 - \beta^2}} - e_0 \mathbf{A} = \mathbf{p} - e_0 \mathbf{A} \tag{I.19}$$

where the mechanical momentum $\mathbf{p} = m_0 \gamma \mathbf{v}$. Then the *electron Lagrangian* may be written as

$$L = -\frac{m_0 c^2}{\sqrt{1 - \beta^2}}(1 - \beta^2) - e_0 \mathbf{A}\mathbf{v} + e_0 \phi = \mathbf{P}\mathbf{v} - w \tag{I.20}$$

where the quantity

$$w = mc^2 - e_0 \varphi \tag{I.21}$$

is the electron energy [see Eq. (1.3)] and $m = m_0/\sqrt{1 - \beta^2}$ is the relativistic electron mass. The equation of motion according to Eqs. (I.17) and (I.19) is

$$\frac{d\mathbf{p}}{dt} - e_0 \frac{d\mathbf{A}}{dt} + e_0 \frac{\partial(\mathbf{A}\mathbf{v})}{\partial \mathbf{r}} - e_0 \frac{\partial \varphi}{\partial \mathbf{r}} = 0 \tag{I.22}$$

Using a known relation of vector analysis,

$$\frac{\partial(\mathbf{A}\mathbf{v})}{\partial \mathbf{r}} = (\mathbf{v}\nabla)\mathbf{A} + \mathbf{v} \times \operatorname{curl} \mathbf{A} \tag{I.23}$$

we obtain

$$\frac{d\mathbf{p}}{dt} - e_0 \frac{\partial \mathbf{A}}{\partial t} - e_0(\mathbf{v}\nabla)\mathbf{A} + e_0(\mathbf{v}\nabla)\mathbf{A} + e_0 \mathbf{v} \times \mathbf{B} - e_0 \text{ grad } \varphi = 0 \tag{I.24}$$

So, corresponding to Eqs. (I.3), Eq. (I.24) becomes

$$\frac{d\mathbf{p}}{dt} = -e_0\mathbf{E} - e_0\mathbf{v} \times \mathbf{B} \tag{I.25}$$

This equation can also be expressed in terms of the relativistic factor $\gamma = m/m_0$:

$$\frac{d(\gamma \mathrm{v})}{dt} = -\eta(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{I.26}$$

The equation of motion in the nonrelativistic approximation ($\gamma = 1$) is

$$\frac{d\mathbf{v}}{dt} = -\eta(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{I.27}$$

I.5 HAMILTONIAN. HAMILTONIAN EQUATIONS (e.g., Landau and Lifshitz, 1987)

According to Eq. (I.20), the energy in terms of the momentum and the Lagrangian is

$$w = \mathbf{P}\mathbf{v} - L \tag{I.28}$$

Here L and w are functions of **r, v,** and t. Using the formula for generalized momentum (I.19), we can express **v** in terms of **P**, **r**, and **t**. The energy as a function of these variables is called the *Hamiltonian*:

$$H = w(\mathbf{P}, \mathbf{r}, t) = \mathbf{P}\mathbf{v} - L \tag{I.29}$$

We can express the equation of motion via the Hamiltonian. The total differential of H is

$$dH = \mathbf{P}\, dv + \mathbf{v}\, d\mathbf{P} - \frac{\partial L}{d\mathbf{r}} d\mathbf{r} - \frac{\partial L}{\partial \mathbf{v}} d\mathbf{v} - \frac{\partial L}{\partial t} dt \tag{I.30}$$

Taking the Lagrangian equation [Eq. (I.17)] and the Lagrangian momentum definition [Eq. (I.16)], we find that

$$dH = \frac{\partial H}{\partial \mathbf{r}} d\mathbf{r} + \frac{\partial H}{\partial \mathbf{P}} d\mathbf{P} + \frac{\partial H}{\partial t} dt = \mathbf{v}\, d\mathbf{P} - \frac{d\mathbf{P}}{dt} d\mathbf{r} - \frac{\partial L}{\partial t} dt \tag{I.31}$$

Equating corresponding terms in Eq. (I.31), we obtain the *Hamiltonian equations*

$$\frac{d\mathbf{P}}{dt} = -\frac{\partial H}{\partial \mathbf{r}}, \qquad \frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{P}} \tag{I.32}$$

Also, we find that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{I.33}$$

The Hamiltonian equations are the equations of motion in the variables **P** and **r**. Note that for static fields according to Eq. (I.33), $dH/dt = 0$. This relation is equivalent to the conservation of energy.

The explicit expression of the Hamiltonian for an electron in the electromagnetic field as a function of **P, r** and t according to Eq. (I.19) is

$$H = mc^2 - e_0\varphi = c\sqrt{m_0^2c^2 + (\mathbf{P} + e_0\mathbf{A})^2} - e_0\varphi \tag{I.34}$$

Here **A** and φ are functions of **r** and t. We also used the obvious relationship for the relativistic mass:

$$m^2c^2 = m_0^2c^2 + p^2$$

If one differentiates the first of Eqs. (I.32) with respect to **P** and the second with respect to **r** and add them, we obtain

$$\frac{\partial \dot{\mathbf{P}}}{\partial \mathbf{P}} + \frac{\partial \dot{\mathbf{r}}}{\partial \mathbf{r}} = 0 \tag{I.35}$$

I.6 LIOUVILLE THEOREM

I.6.1 Liouville Theorem for Interaction Particles

When $F(P, q, t)$ is the distribution function in the phase space (i.e., a number of particles in the unity phase volume),

$$F = \frac{dn}{dP\,dq}$$

where dn is the number of particles in the phase volume $dP\,dq$. The value F denotes density in the phase space. In general, if there are N interacting particles, the Lagrangian for each particle depends on the coordinates and velocities of all particles, and the phase space has $6N$ dimensions.

Each element of $dP_i\,dq_i$ is really a six-dimensional element in Euclidean space. The $6N$ dimension's current density of these particles is

$$\mathbf{j} = \mathbf{v}\frac{dn}{dP\,dq} = \mathbf{v}\mathrm{F}$$

where the $6N$ velocity $\mathbf{v}$ has the components $\dot{P}$ and $\dot{q}$. According to a continuity theorem (conservation of the particle number),

$$\operatorname{div}\mathbf{j} + \frac{\partial F}{\partial t} = 0 \quad \text{or} \quad \frac{\partial j_P}{\partial P} + \frac{\partial j_q}{\partial q} + \frac{\partial F}{\partial t} = 0$$

Then

$$\frac{\partial(F\dot{P})}{\partial P} + \frac{\partial(F\dot{q})}{\partial q} + \frac{\partial F}{\partial t} = 0 \qquad \text{or}$$
$$\dot{P}\frac{\partial F}{\partial P} + F\frac{\partial \dot{P}}{\partial P} + \dot{q}\frac{\partial F}{\partial q} + F\frac{\partial \dot{q}}{\partial q} + \frac{\partial F}{\partial t} = 0 \tag{I.36}$$

By letting the particles move along trajectories, Eq. (I.35) becomes valid, and the second and fourth terms in Eq. (I.36) cancel. We obtain

$$\frac{\partial F}{\partial P}\dot{P} + \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t} = 0 \quad \text{or} \quad \frac{DF}{Dt} = 0 \tag{I.37}$$

This is a version of the *Liouville theorem*: The density F of particles moving along trajectories is constant.

Consider a group of N particles that move along their trajectories. The number of the particles in the group is constant:

$$N = \int_{\Omega} F\, dP\, dq = \text{const.} \tag{I.38}$$

Because $F = \text{const.}$, we obtain

$$\int_{\Omega} dP\, dq = \text{const.} \tag{I.39}$$

The volume of the $6N$ phase space that encloses a chosen group of the particles is constant. This is the second version of the Liouville theorem.

Note that the Hamiltonian equations (I.32) are valid if the forces acting on the particles have a potential. These forces are called *Hamiltonian forces*. For *non-Hamiltonian forces* (i.e., radiation losses, frictional forces, etc.) the first of Eqs. (I.32) must be changed (Landau and Lifshitz, 1987). In this case, the Liouville theorem is not valid, and the phase density as well as the phase volume of the group of particles chosen has no more invariants.

I.6.2 Liouville Theorem for Noninteraction Identical Particles

In this case, the Lagrangian of each particle will depend on the coordinate and momentum of the particle. Then we can write the Hamiltonian equations for each particle in the same six-dimensional phase space and use a distributive function in this space for identical particles:

$$f = \frac{dn}{dP\,dq}$$

Furthermore, formulas (I.37)–(I.39) are repeated and we obtain Liouville's theorem for six-dimensional phase space. In particular,

$$f = \text{const.} \quad \text{or} \quad \frac{\partial f}{\partial \mathbf{P}}\dot{\mathbf{P}} + \frac{\partial f}{\partial \mathbf{r}}\dot{\mathbf{r}} + \frac{\partial f}{\partial t} = 0, \qquad \int_{V_6} dP\,dq = \text{const.} \tag{I.40}$$

where V_6 is the six-dimensional phase volume.

I.6.3 Liouville Theorem for a Phase Space of Lesser Dimensions

The invariant phase volume in the Liouville theorem may be shortened even more if some degree of freedom of the noninteraction particles is independent of the other(s). In this case, the corresponding Hamiltonian may be represented as the sum of independent components, and for each component we obtain invariants corresponding to Eq. (I.40) (Moss, 1968; Reiser, 1994). For example, if the motion in the transverse plane ($p_\perp$, $q_\perp$) and the longitudinal motion (p_z, q_z) are independent, the invariants are

$$f(P_\perp, q_\perp) = \text{const.}, \qquad \int_{V_4} d\dot{P}_\perp\, dq_\perp = \text{const.} \tag{I.41}$$

$$f(P_z, q_z) = \text{const.}, \qquad \int_{V_2} dP_z\, dq_z = \text{const.} \tag{I.42}$$

Finally, if motions in the x and y directions are independent, the invariant of Eq. (1.41) is split to

$$\begin{aligned} f(P_x, x) &= \text{const.}, & f(P_y, y) &= \text{const.} \\ \int_{V_2} dP_x\, dx &= \text{const.}, & \int_{V_2} dP_y\, dy &= \text{const.} \end{aligned} \tag{I.43}$$

I.7 EMITTANCE. BRIGHTNESS (Humphries, 1990; Lawson, 1988; Lejuene and Aubert, 1980)

I.7.1 Emittance in a Zero Magnetic Field

In this case, $\mathbf{P} = \mathbf{p} + e_0\mathbf{A} = \mathbf{p}$. Consider the first of the integrals in Eq. (I.43):

$$\int dp_x\, dx = \int \frac{dp_x}{p_z} p_z\, dx = \int p_z\, dx\, dx' = \text{const.} \tag{I.44}$$

Suppose that p_z is constant in each transverse section of the beam. Then

$$p_z \int dx\, dx' = p_z V_2 = \text{const.} \tag{I.45}$$

If the particles move with constant energy in the z-direction ($p_z = $ const.),

$$\Xi_x = \int dx\, dx' = V_2 = \text{const.} \tag{I.46}$$

The (x, x') phase space is called a *trace space*. The integral Ξ_x is called an *emittance*. In the literature, the emittance is often defined as $(1/\pi)\Xi_x$. According to Eq. (I.40), the area of the trace space for some groups of the particles with constant axial momentum along a trajectory is *invariant*. Note that the configuration of the contour enclosing a chosen group of particles in the trace space can vary, but the area into the contour is conserved. A reader can find many interesting pictures displaying contour transformations in the literature (see, e.g., Humphries, 1990; Lawson, 1988; Lejuene and Aubert, 1980).

If the beam is axially symmetric, the single emittance of Eq. (I.46) may be utilized. For the nonsymmetrical paraxial beams with constant p_z, only the invariance of a *hyperemittance*

$$\Xi_h = \int dx\, dx' dy\, dy' = V_4 \tag{I.47}$$

is correct. If the beam has two planes of symmetry x and y, the hyperemittance is

$$\Xi_h = \int_{V_2} dx\, dx' \int_{V_2} dy\, dy' = \Xi_x \Xi_y$$

For a beam with variable p_z momentum, the invariant is

$$p_z^2 \Xi_h \sim \beta^2 \gamma^2 \Xi_h = \beta^2 \gamma V_4 \tag{I.48}$$

In nonrelativistic approximation, that is equivalent to

$$\varphi(z)\Xi_h = \text{const.} \tag{I.49}$$

I.7.2 Brightness

Microscopic brightness is, by definition,

$$B_m = \frac{dI}{dV_4} = \frac{dI}{dx\, dx' dy\, dy'} \qquad \text{A/rad}^2 \cdot \text{m}^2 \tag{I.50}$$

where I is the beam current. Usually, the average brightness value is utilized:

$$B = \frac{I}{V_4} = \frac{I}{\Xi_h} \tag{I.51}$$

For the small beam angles, this definition is equivalent to

$$B \approx \frac{j}{\Omega} = \frac{j}{\pi\alpha^2(z)} \qquad \mathrm{A/m^2 \cdot st} \tag{I.52}$$

where j is a current density, Ω is a solid angle enclosing all the rays emerging from a point on the z-axis, and $\alpha(z)$ is the angle between the z-axis and the beam edging rays. According to Eq. (I.47), the brightness is invariant if $\beta\gamma$ or $\varphi(z)$ is a constant. Otherwise, invariants are

$$B_{\mathrm{inv}} = \frac{B\overline{\beta}^2\overline{\gamma}^2}{\beta^2\gamma^2} \quad \text{(relativistic)}, \qquad B_{\mathrm{inv}} = \frac{B_{\overline{\varphi}}}{\varphi(z)} \quad \text{(nonrelativistic)} \tag{I.53}$$

where $\overline{\beta}$, $\overline{\gamma}$, and $\overline{\varphi}$ are average values.

I.7.3 Maximum Langmuir Brightness for Thermionic Emitters

Assuming a Maxwellian distribution of initial velocities and neglecting all other factors that can limit the beam current density, Langmuir (1937) obtained the following formula for a theoretical maximum of current density:

$$j(z) = j_c \frac{e_0\varphi(z)}{kT} \sin^2\alpha(z) \tag{I.54}$$

where j_c is the cathode current density, T is the cathode temperature, and k is Boltzmann's constant. This formula is valid in a nonrelativistic approximation and for $\varphi(z)/kT \gg 1$.

Substituting Eq. (I.54) for small $\alpha(z)[\sin\alpha(z) \approx \alpha(z)]$ into Eq. (I.52), we obtain the *maximum brightness*:

$$B_{\max} = j_c \frac{e_0\varphi(z)}{\pi kT} \tag{I.55}$$

Correspondingly, the *invariance maximum brightness* [Eq. (I.53)] is

$$B_{\mathrm{inv,\,max}} = j_c \frac{e_0}{\pi kT}\overline{\varphi} \tag{I.56}$$

According to Eq. (I.54), for small α the maximum current density is

$$j_{\max} = j_c \frac{e_0\overline{\varphi}}{kT}\alpha(z)^2 \tag{I.57}$$

PART I

ELECTRON BEAMS

Electron beams are flows of free electrons moving in the direction chosen. This is called the *axis* of the beam. The trajectories of the beam particles often come close to being rectilinear, although recently, specific devices with periodic curvilinear beams, in particular classical electron masers (CEMs) and free electron lasers (FELs), are attracting more and more attention. A beam's axis may be either straight or curved. Finally, beams may have different symmetry: for example, a cylindrical, a sheet, or a strip.

Let us list some properties of electrons as elementary particles. These make electron beams very important for creating and controlling images, material processing, transferring energy with high density, generating other particles, and generating and transforming high-frequency signals.

- Electrons are charged particles that provide effective control of a particle's movement by means of electric and magnetic fields. At the same time, interaction between the particles through self fields (space-charge and self magnetic fields) raises many specific effects and problems connected with the control and stability of intense electron beams.
- Electrons are stable (long-life) particles.
- Electrons are very light particles. They are the lightest among all long-living elementary charged particles and the longest living of the light charged particles. The small mass and correspondingly, inertia of electrons forms the functioning base of high-frequency systems. The transfer of an electron's beams is not accompanied by a large mass transport. So a 100-kA beam transfers only 2 g of electrons during 1 hour. Self fields are relatively weak, due to their high level of electron mobility.
- Electrons are chemically neutral particles. Their transfer does not change the chemical content of electrode surfaces.

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- Simple and effective methods are available to extract electrons from solid, liquid, or gaseous media. It is essential that each type of electron emission provide many important applications of corresponding electronic devices.

Usually, beams are thought of as thin flows with transverse dimensions (at least, a single transverse dimension for sheet beams) and are quite small compared to beam length.

The physics of electron beams certainly does not exhaust the topic of vacuum electronics. Nevertheless, the sphere of electron beam applications is great. A list of some important applications follows:

- Electron optics (e.g., lenses, electron guns, beam deflection systems)
- Microwave vacuum electronics, including relativistic microwave electronics and free-electron lasers
- Electron microscopy, including electron-probe microanalysis
- Electron beam tubes in television, radar, and radio-meter systems
- Electron beam technology (e.g., precise cutting, drilling, welding, melting, high-resolution lithography, manufacture of integrated contours)
- Electron accelerators
- Formation and transport of intense electron beams

To stay within the scope of the book, we will not give a detailed exposition of the huge family of microwave devices and in general, devices using electron beams, or catalogue their numerous specific properties. Only a description of key devices and clarification of their design schemes are offered.

CHAPTER ONE

Motion of Electrons in External Electric and Magnetic Static Fields

1.1 INTRODUCTION

Taking external fields into account—the sources of fields that are independent of beam electrons—is the simplest approximation. In general, the motion of charged particles is also determined by interaction with other particles in the beams. For pure electron beams, this interaction is realized through macroscopic (collective) forces created by electrons themselves (self fields, i.e., space-charged electric and magnetic fields) and by microscopic fields (in practice, through collisions). Considering motion and ignoring space-charge fields is sufficient for rarefied beams (e.g., for beams in electron lenses). But to study dense beams, knowledge of motion principles in external fields is also necessary. Beams with the essential effect of self fields are considered in Chapters 3 to 5.

In this chapter we limit ourselves to a study of static fields. Aspects of electron motion in nonstationary fields are considered in Chapters 6 to 10.

1.2 ENERGY OF A CHARGED PARTICLE

The important principle of charged-particle motion in static fields is energy conservation. Let us multiply Eq. (I.25) by $\mathbf{v}$:

$$\mathbf{v}\frac{d\mathbf{p}}{dt} = -e_0\,\mathbf{v}\mathbf{E} \tag{1.1}$$

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The transformation of Eq. (1.1) for the static fields ($\partial\varphi/\partial t = 0$) gives

$$\frac{d\varphi[\mathbf{r}(t)]}{dt} = \frac{\partial\varphi}{\partial\mathbf{r}}\frac{d\mathbf{r}}{dt} = -\mathbf{vE}$$

$$\mathbf{v}\frac{d\mathbf{p}}{dt} = \frac{1}{m}\,\mathbf{p}\frac{d\mathbf{p}}{dt} = \frac{1}{2m}\frac{dp^2}{dt} = \frac{1}{2m}\frac{d}{dt}(m^2c^2 - m_0^2c^2) = \frac{d}{dt}(mc^2)$$

Here we used the relation

$$m^2c^2 = p^2 + m_0^2c^2 \tag{1.2}$$

following the expression for the relativistic mass $m = m_0/\sqrt{1 - v^2/c^2}$. Substituting these formulas in Eq. (1.1), we obtain the *conservation of energy*:

$$\frac{d}{dt}(mc^2 - e_0\varphi) = 0, \qquad w = mc^2 - e_0\varphi = \text{const.} \tag{1.3}$$

The energy w can be called a *full energy*, one that is equal to the sum of the electron potential $e_0\varphi$ and kinetic mc^2 energies. Often, the kinetic energy is assumed as the difference $w_k = w - w_0 = (m - m_0)c^2 = m_0c^2(\gamma - 1)$. Note that in the important *quasistatic approximation*, where the equation $\mathbf{E} = -\text{grad}\,\varphi$ is still correct but the electric field depends on the time explicitly, the following relation is true:

$$\frac{dw}{dt} = \frac{\partial\varphi}{\partial t} \tag{1.4}$$

In *nonrelativistic approximation*, the energy is $mc^2 = (m_0v^2/2) + m_0c^2$. Omitting m_0c^2, we obtain the conservation of nonrelativistic energy:

$$w_n = \frac{m_0v^2}{2} - e_0\varphi = \text{const.} \tag{1.5}$$

1.3 POTENTIAL–VELOCITY RELATION (STATIC FIELDS)

Let us assume that an electron leaves a cathode with velocity v_c and mass m_c correspondingly. We believe that the cathode potential $\varphi_{\text{cath}} = 0$. Below m and φ are the electron mass at an arbitrary point $\mathbf{r}$; $e_0\varphi_c$ is the initial energy of the electron at the cathode. Then, according to energy conservation,

$$mc^2 - e_0\varphi = m_0c^2 + e_0\varphi_c \tag{1.6}$$

Let us divide Eq. (1.6) by m_0c^2:

$$\gamma = \frac{m}{m_0} = 1 + \frac{e_0(\varphi + \varphi_c)}{m_0c^2} = 1 + \frac{e_0\varphi^*}{m_0c^2} = 1 + \frac{\varphi^*}{\varphi_0} = 1 + \frac{\eta\varphi^*}{c^2} \tag{1.7}$$

where $\varphi_0 = m_0c^2/e_0 = c^2/\eta$ is the reduced electron rest energy in volts; $\varphi^* \equiv \varphi + \varphi_c$. Taking 1 MV as a unit of potential, we obtain $\varphi_0 \approx 0.51100$ MV, and

$$\gamma \approx 1 + 1.9569\phi^*_{\mathrm{MV}} \approx 1 + 2\phi^*_{\mathrm{MV}} \tag{1.8}$$

For zero initial velocity, $\varphi^* = \varphi$ and

$$\gamma = 1 + \frac{\varphi}{\varphi_0} \approx 1 + 1.9569\varphi_{\mathrm{MV}} \approx 1 + 2\varphi_{\mathrm{MV}} \tag{1.9}$$

where φ_{MV} is the potential in megavolts. In nonrelativistic approximation and for $\varphi_{\mathrm{cath}} = 0$, we obtain from Eq. (1.5) the nonrelativistic dimensionless velocity

$$\beta_n = \frac{v}{c} = \sqrt{\frac{2\varphi^*}{\varphi_0}} \approx \frac{1}{16}\sqrt{\phi^*_{\mathrm{kV}}} \tag{1.10}$$

where φ^*_{kV} is the reduced potential in kilovolts. Using the dependence $\gamma = 1/\sqrt{1-\beta^2}$ and Eq. (1.7), it is easy to find that

$$\beta = \frac{\sqrt{\varphi^*(\varphi^* + 2\varphi_0)}}{\varphi^* + \varphi_0}, \qquad \beta_{\varphi_c = 0} = \frac{\sqrt{\varphi(\varphi + 2\varphi_0)}}{\varphi + \varphi_0} \tag{1.11}$$

In the extreme relativistic limit $\varphi_0/\varphi \ll 1$, according to Eq. (1.11),

$$\beta_{\mathrm{ext}} \approx 1 - \frac{1}{2}\left(\frac{\varphi_0}{\varphi}\right)^2 \tag{1.12}$$

The values of $\beta_n, \beta,$ and β_{ext} are given in Table 1.1 for different values of φ. According to the table, nonrelativistic approximation "works" until $\varphi \sim 10$ kV. Extreme relativistic approximation is acceptable after $\varphi = 5$ MV.

TABLE 1.1 Reduced Nonrelativistic, Relativistic, and Extreme Relativistic Velocities as Functions of the Potential

φ	1.0 V	100 V	1 kV	10 kV	100 kV	300 kV	1 MV	5 MV	25 MV
β_n	0.001978	0.01978	0.06256	0.1976	0.625	1.083	1.976	4.42	9.9
β	0.001978	0.01978	0.06247	0.1950	0.5482	0.7253	0.9411	0.9957	0.9998
β_{ext}	-13×10^{10}	-13×10^{6}	-13×10^{4}	-1300	-12	-0.451	0.8694	0.9948	0.9998

1.4 ELECTRONS IN A LINEAR ELECTRIC FIELD $e_0E = kx$

Equations of motion:

$$\frac{dp}{dt} = -e_0E_x = -kx \tag{1.13}$$

$$\frac{dx}{dt} = \frac{p}{m} = \frac{pc}{\sqrt{p^2 + m_0^2c^2}} \tag{1.14}$$

1.4.1 Nonrelativistic Approximation

Equation (1.14) becomes

$$\frac{dx}{dt} = \frac{p}{m_0} \tag{1.14a}$$

Equations (1.13) and (1.14*a*) are the equations of a harmonic oscillator. Their solution is

$$\begin{aligned} x(t) &= x_0 \cos(\omega t + \alpha) \\ p(t) &= -x_0 m_0 \omega \sin(\omega t + \alpha) \end{aligned} \tag{1.15}$$

where $\omega = \sqrt{k/m_0}$ is the oscillation frequency. The trajectory in a phase space for this oscillator is the ellipse:

$$\frac{x^2}{x_0^2} + \frac{p^2}{x_0^2\,km_0} = 1 \tag{1.16}$$

1.4.2 Relativistic Oscillator

Return to the relativistic system of equations. Dividing Eq. (1.14) by Eq. (1.13), we obtain

$$-kx\,dx = \frac{cp\,dp}{\sqrt{p^2 + m_0^2c^2}} \tag{1.17}$$

Integration of this equation yields the trajectory in phase space:

$$a^2 - \frac{kx^2}{2} = c\sqrt{p^2 + m_0^2c^2} \tag{1.18}$$

where a^2 is an arbitrary constant. It is readily verified that this trajectory describes a finite (periodic) motion. Actually, the variables x and p have maximum values (amplitudes):

$$\begin{aligned} x_{\max} &= x_{p=0} = c\sqrt{\frac{2\,m_0}{k}(u-1)} \\ p_{\max} &= p_{x=0} = m_0c\sqrt{u^2-1} \end{aligned} \tag{1.19}$$

where $u = a^2/m_0c^2 > 0$ is a reduced arbitrary constant. The results of a numerical integration of Eqs. (1.13) and (1.14) have been given by Humphries (1990). It was shown that phase trajectories have the form of distorted ellipses. According to Eq. (1.19), they are transformed to correct nonrelativistic ellipses when $u - 1 \approx kx^2/2\,m_0c^2 \ll 1$. This inequality means that the potential energy of an oscillator is negligible compared with the particle rest energy.

1.5 MOTION OF ELECTRONS IN HOMOGENEOUS STATIC FIELDS

Homogeneous fields certainly are idealizations. However, the principles and results of this theory are a basis for the solution of much more complicated problems.

1.5.1 Electric Field

We assume that the electric field is opposite the y-axis (Fig. 1.1), so that $E_y = -E$. Consider two cases:

1. *Initial momentum p_0 of the electron turns toward the x-axis* (Fig. 1.1). The equations of motion are

$$\frac{d\mathbf{p}}{dt} = -e_0\,\mathbf{E} \tag{1.20}$$

$$\frac{dp_x}{dt} = 0, \qquad \frac{dp_y}{dt} = e_0 E \tag{1.21}$$

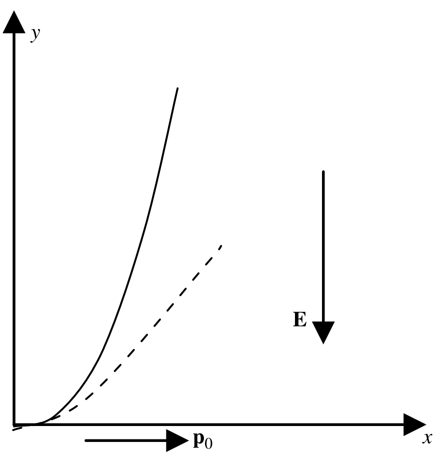

FIGURE 1.1 Motion of an electron in a homogeneous electric field. Solid curve, relativistic trajectory; dashed curve, nonrelativistic trajectory.

Integrating the first equation in Eq. (1.21), we obtain

$$p_x = mv_x = p_0 \tag{1.22}$$

As follows from the second equation in Eq. (1.21),

$$\frac{dp_y}{dt} = \frac{dp_y}{dx}\frac{dx}{dt} = \frac{p_0}{m}\frac{dp_y}{dx} = e_0 E \tag{1.23}$$

According to Eq. (1.2), the mass may be written as

$$m = \frac{1}{c}\sqrt{p^2 + m_0^2 c^2} = \frac{1}{c}\sqrt{p_0^2 + m_0^2 c^2 + p_y^2} = \frac{1}{c}\sqrt{\frac{W_0^2}{c^2} + p_y^2} \tag{1.24}$$

where $W_0 = c\sqrt{m_0^2 c^2 + p_0^2}$ is the initial kinetic energy of the particle. Substituting m into Eq. (1.16), we obtain

$$\frac{dp_y}{\sqrt{(W_0^2/c^2) + p_y^2}} = \frac{e_0 E}{cp_0}dx$$

Integration of this equation gives

$$\sinh^{-1}\frac{p_y c}{W_0} = \frac{e_0 E}{cp_0}x \tag{1.25}$$

Note that

$$p_y = m\frac{dy}{dt} = m\frac{dy}{dx}\frac{dx}{dt} = p_x\frac{dy}{dx} = p_0\frac{dy}{dx}$$

After a corresponding transformation (1.25) and integration, we obtain

$$y = \frac{W_0}{e_0 E}\left(\cosh\frac{e_0 Ex}{cp_0} - 1\right) \tag{1.26}$$

This is the equation of a catenary curve. In the nonrelativistic approximation ($e_0 Ex/cp_0 \ll 1$), the energy $W_0 \approx m_0 c^2$. The first term of the Taylor expansion of cosh in Eq. (1.19) gives the well-known parabolic trajectory

$$y = \frac{e_0 E}{2\, m_0 v_0^2}x^2$$

In Fig. 1.1 the catenary curve is situated above the parabola because of the more rapid (almost exponential) change of the relativistic y-coordinate as a function of x [Eq. (1.26)].

2. *Initial momentum p of the particle is parallel to the y-axis* (Fig. 1.1). According to Eqs. (1.21),

$$\begin{aligned} p_x &= 0 \\ p_y &= m\frac{dy}{dt} = e_0Et + p \end{aligned} \tag{1.27}$$

Then using relation (1.2) for m, we find that

$$dy = \frac{c(e_0Et + p)}{\sqrt{m_0^2c^2 + (e_0Et + p)^2}}dt$$

Integrating this equating with the initial condition $y_0 = 0$, we obtain

$$y = \frac{c}{e_0E}\left(\sqrt{m_0^2c^2 + (e_0Et + p)^2} - \sqrt{m_0^2c^2 + p^2}\right)$$

1.5.2 Magnetic Field

The equation of motion

$$\frac{dm\,\mathbf{v}}{dt} = -e_0\,\mathbf{v} \times \mathbf{B} \tag{1.28}$$

Since the force is perpendicular to the velocity, the modulus of $\mathbf{v}$ and the electron mass m are constants:

$$v = |\mathbf{v}| = \text{const.}, \qquad m = \text{const.} \tag{1.29}$$

The same result follows from the energy integral (1.3) because the electric field is zero. Then the potential $\varphi = \text{const.}$ and $m = \text{const.}$ Equation (1.28) becomes

$$\frac{d\mathbf{v}}{dt} = -\frac{e_0}{m}\,\mathbf{v} \times \mathbf{B} \tag{1.30}$$

Let us assume that v_B and $\mathbf{v}_\perp$ are velocity components parallel and perpendicular to $\mathbf{B}$, respectively. We obtain from Eq. (1.30)

$$v_B = \text{const.} \tag{1.31}$$

$$\frac{d\mathbf{v}_\perp}{dt} = -\frac{e_0}{m}\,\mathbf{v}_\perp \times \mathbf{B} \tag{1.32}$$

From Eqs. (1.29) and (1.31) follow

$$v_\perp \equiv |\,\mathbf{v}_\perp| = \sqrt{v^2 - v_B^2} = \text{const.} \tag{1.33}$$

So in a static magnetic field, the values $m, v, v_\perp$, and v_B are constants. The quantity $\mathbf{a}_\perp = d\mathbf{v}_\perp/dt$ is the acceleration of a plane motion. In this case, it may be represented as a sum of the tangential and the centripetal components:

$$a_n = \frac{v_\perp^2}{R}, \qquad a_\tau = \frac{dv_\perp}{dt} \tag{1.34}$$

where R is the radius of curvature in the given point of the trajectory. According to Eq. (1.33), $a_\tau = 0$. Then

$$a_n = \frac{v_\perp^2}{R} = \frac{e_0}{m}|\,\mathbf{v}_\perp \times \mathbf{B}| = \frac{e_0}{m} v_\perp B \tag{1.35}$$

The radius of the curvature

$$R = r_\perp = \frac{v_\perp}{(e_0/m)B} = \text{const.} \tag{1.36}$$

Hence, the electron trajectory in the plane perpendicular to $\mathbf{B}$ is a circle. The quantity

$$\omega_g \equiv \frac{e_0}{m} B = \frac{\eta B}{\gamma} \tag{1.37}$$

is the angular velocity of gyration.

We find that the motion of the electron (in general, of a charged particle) is the superposition of two motions: uniform drift along the magnetic field with velocity v_B and uniform gyration on a circle of radius $r_\perp$ [Eq. (1.36)] with frequency ω_g [Eq. (1.37)]. Therefore, the spatial trajectory of the particle is a helical line (Fig. 1.2) with pitch $h = (2\pi/\omega_g)v_B$. It is evident that equations of the trajectory for $x_0 = y_0 = z_0 = 0$ are

$$\begin{aligned} x &= r_\perp \sin(\omega_g t) \\ y &= r_\perp[\cos(\omega_g t) - 1] \\ z &= v_B t \end{aligned} \tag{1.38}$$

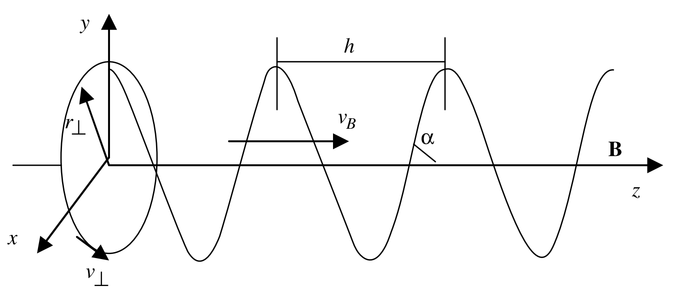

FIGURE 1.2 Motion of an electron in a homogeneous magnetic field.

The circle is called a *Larmor circle*, the radius is a *Larmor radius*, the center of the Larmor orbit is a *guiding center*, and ω_g is the *gyrofrequency* or *cyclotron frequency*. It is essential that the relativistic gyrofrequency depends on the particle's kinetic energy:

$$\omega_g = \frac{e_0 c^2}{w} B = \frac{\eta B}{\gamma} \tag{1.39}$$

Therefore, electrons in magnetic fields behave as nonisochronous oscillators. This property of electrons has very important applications in electron masers (Chapter 10).

In the nonrelativistic approximation $\gamma = 1$, the gyrofrequency is $\omega_g = \eta B$. In this case, the electrons are isochronous oscillators.

Example 1.1 Find the trajectory parameters of an electron that is injected into the uniform magnetic field $B = 1$ T at an angle $\alpha = 30^\circ$ (Fig. 1.2). The electron energy is 1 MeV.

For $w = 1$ MeV, $\gamma = 1.957$ [Eq. (1.12)], and $\beta = 0.941$ (Table 1.1). The gyrofrequency is $\omega_g = \eta B/\gamma = 8.99 \times 10^{10}$ rad/s. The velocity components (Fig. 2.1) are $v = \beta c = 2.82 \times 10^8$ m/s, $v_B = v\cos\alpha = 2.43 \times 10^8$ m/s, and $v_\perp = v \sin\alpha = 1.41 \times 10^8$ m/s. The pitch of the helical trajectory is $h = (2\pi/w_g)v_B = 1.7 \times 10^{-2}$ m. The radius of the helix is $r_\perp = v_\perp/\omega_g = 1.6 \times 10^{-3}$ m.

1.5.3 Parallel Electric and Magnetic Fields

Now assume that the electric and magnetic fields are oriented along the z-axis. Let $E_z = -E$ and $B_z = B$ (Fig. 1.3). The equations of motion are

$$\frac{d\mathbf{p}_\perp}{dt} = -e_0 \mathbf{v}_\perp \times \mathbf{B} \tag{1.40}$$

$$\frac{dp_z}{dt} = e_0 E \tag{1.41}$$

Nonrelativistic Approximation *($m = m_0$)* Equations (1.40) and (1.41) become

$$\frac{d\mathbf{v}_\perp}{dt} = -\eta \mathbf{v}_\perp \times \mathbf{B} \tag{1.42}$$

$$\frac{dv_z}{dt} = \eta E \tag{1.43}$$

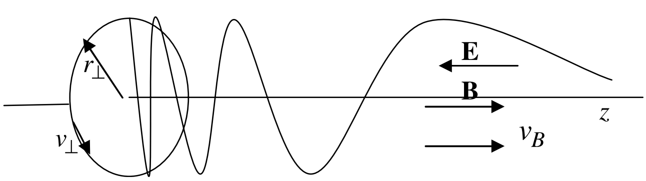

FIGURE 1.3 Motion of an electron in parallel fields **E** and **B**.

Comparison of Eq. (1.42) with Eq. (1.32) shows that motion of the electron perpendicular to the z-axis plane is the same as motion without an electric field; the electron gyrates on the Larmor circle with gyrofrequency $\omega_g = \eta B$. However, the velocity v_B (the velocity of the guiding center along the magnetic field) is now proportional to t. As a result, the trajectory is a helix with variable pitch h. Note that for large t the velocity v_B reaches arbitrarily large values that contradict relativity theory. In fact, for a velocity of order c, the mass becomes variable and the motion in the transversal plane is not described by Eq. (1.42).

Relativistic Motion Let us turn to Eqs. (1.40) and (1.41). According to Eq. (1.41),

$$p_z = e_0 E t \tag{1.44}$$

(for simplicity we take $p_{z0} = 0$). Since $\mathbf{p}_\perp$ is perpendicular to $\mathbf{v}_\perp \times \mathbf{B}$ and according to Eq. (1.40), the magnitude of the perpendicular momentum

$$p_\perp \equiv |\, \mathbf{p}_\perp| = \text{const.} \tag{1.45}$$

Then the full momentum of the particle is

$$p = \sqrt{p_\perp^2 + p_z^2} = \sqrt{p_\perp^2 + (e_0 E t)^2} \tag{1.46}$$

The mass is [Eq. (1.2)]

$$m = \frac{1}{c}\sqrt{p^2 + m_0^2 c^2} = \frac{1}{c}\sqrt{p_\perp^2 + m_0^2 c^2 + (e_0 E t)^2} = \frac{1}{c}\sqrt{\frac{w_0^2}{c^2} + (e_0 E t)^2} \tag{1.47}$$

where w_0 is the initial kinetic energy of the particle.

The perpendicular velocity now decreases with time:

$$v_\perp = \frac{p_\perp}{m} = \frac{p_\perp c}{\sqrt{(w_0^2/c^2) + (e_0 E t)^2}} \tag{1.48}$$

The magnitude of the full acceleration of the particle in the plane perpendicular to $\mathbf{B}$ is

$$a = \left|\frac{d\mathbf{v}_\perp}{dt}\right| \tag{1.49}$$

This value may be found from Eq. (1.4):

$$\frac{d}{dt}(m\mathbf{v}_\perp) = m\frac{d\,\mathbf{v}_\perp}{dt} + \mathbf{v}_\perp \frac{dm}{dt} = e_0 \mathbf{v}_\perp \times \mathbf{B} \tag{1.50}$$

and

$$a = \frac{1}{m}\left| e_0 \mathbf{v}_\perp \times \mathbf{B} - \mathbf{v}_\perp \frac{dm}{dt}\right| = \sqrt{\frac{(e_0 v_\perp B)^2}{m^2} + \frac{v_\perp^2}{m^2}\left(\frac{dm}{dt}\right)^2} \tag{1.51}$$

It is readily verified that the second term under the square root in Eq. (1.51) is a_τ^2, where a_τ is the tangential acceleration in the transverse plane:

$$a_\tau = \frac{dv_\perp}{dt} = \frac{d}{dt}\left(\frac{p_\perp}{m}\right) = -\frac{p_\perp}{m^2}\frac{dm}{dt} = -\frac{v_\perp}{m}\frac{dm}{dt} \tag{1.52}$$

Therefore, the first term equals a_n^2, where a_n is the centripetal acceleration. Thus,

$$a_n = \frac{v_\perp^2}{R} = \frac{e_0 v_\perp B}{m} \tag{1.53}$$

where R is the curvature radius of the trajectory in the perpendicular plane. We obtain

$$R = \frac{mv_\perp}{e_0 B} = \frac{p_\perp}{e_0 B} = \text{const.} \tag{1.54}$$

The angular velocity is

$$\omega = \frac{v_\perp}{R} = \frac{e_0}{m}B \tag{1.55}$$

This value is analogous to the gyrofrequency ω_g [Eq. (1.33)], but m increases with t [Eq. (1.47)]. We find that the trajectory of the particle is a helical line of constant radius, variable pitch, and variable angular velocity (Fig. 1.3). It may be seen from Eqs. (1.44), (1.47), and (1.55) that $\omega \sim 0, v_z \sim c$ for very large t and the trajectory becomes a straight line parallel to the z-axis at a distance from it of $R = p_\perp/e_0 B$.

1.5.4 Perpendicular Fields E and B

Nonrelativistic Approximation The coordinate system and directions of fields are shown in Fig. 1.4. For this configuration

$$E_y = -E = \text{const.}, \qquad B_x = B = \text{const.} \tag{1.56}$$

The nonrelativistic equations of motion are

$$\frac{dv_x}{dt} = 0 \tag{1.57}$$

$$\frac{d\mathbf{v}_\perp}{dt} + \eta \mathbf{v}_\perp \times \mathbf{B} = -\eta \mathbf{E} \tag{1.58}$$

Equation (1.57) corresponds to uniform drift in the x direction with velocity

$$v_x \equiv v_B = \text{const.} \tag{1.59}$$

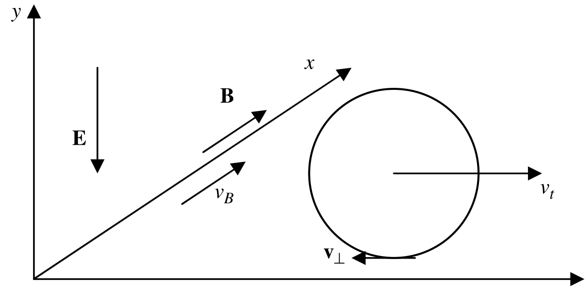

FIGURE 1.4 Motion of an electron in perpendicular fields **E** and **B**.

The drift velocity v_B is determined by the initial condition. Equation (1.58) describes motion in the plane (Y, Z) perpendicular to **B**. This is an ordinary linear inhomogeneous differential equation of first order in $\mathbf{v}_\perp$. The general solution of this equation is the sum of a general solution of the homogeneous equation and a partial solution of the inhomogeneous equation.

The *homogeneous equation is*

$$\frac{d\mathbf{v}_\perp}{dt} + \eta \mathbf{v}_\perp \times \mathbf{B} = 0 \tag{1.60}$$

which we considered in Section 1.4.2. As has been shown, its general solution describes the electron motion as gyration along the Larmor circle in the (Y, Z) plane with gyrofrequency $\omega_g = \eta B$. The magnitude of the gyratic velocity $v_\perp$ is determined by the initial conditions.

The *nonhomogeneous equation* is Eq. (1.58). The right side of this equation is a constant. Because we need a partial solution, we can use any solution of the type $\mathbf{v}_\perp = \mathbf{v}_t = \text{const}$. Then Eq. (1.58) is reduced to

$$\mathbf{v}_t \times \mathbf{B} = -\mathbf{E} \tag{1.61}$$

Vector multiplication of Eq. (1.61) by **B** yields

$$\mathbf{B} \times (\mathbf{v}_t \times \mathbf{B}) = -\mathbf{B} \times \mathbf{E} = \mathbf{E} \times \mathbf{B} \tag{1.62}$$

Using the rule of vector algebra for a double cross product, we have

$$\mathbf{v}_t B^2 - \mathbf{B}(\mathbf{B}\mathbf{v}_t) = \mathbf{E} \times \mathbf{B} \tag{1.63}$$

We can further specialize the partial solution setting $\mathbf{v}_t$ perpendicular to **B**. Then

$$\mathbf{v}_t = \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \qquad v_t = |\mathbf{v}_t| = \frac{E}{B} \tag{1.64}$$

Velocity $\mathbf{v}_t$ is perpendicular to both $\mathbf{E}$ and $\mathbf{B}$. It is the *transversal* (*perpendicular*) *drift*. Its value does not depend on initial conditions. The full velocity can be represented as a sum:

$$\mathbf{v} = \mathbf{v}_\omega + \mathbf{v}_d \tag{1.65}$$

where $\mathbf{v}_\omega$ is the velocity of gyration and $\mathbf{v}_d = \mathbf{v}_t + \mathbf{v}_B$ is the velocity of the guiding center. Its trajectory is evidently a straight line in the (X, Z) plane with slope v_B/v_t. The trajectory in the plane perpendicular to $\mathbf{B}$ is the superposition of the drift $\mathbf{v}_t$ and gyration $\mathbf{v}_\omega$. The character of the trajectory depends on the velocities v_t and v_ω. In the lower part of the trajectory, the resulting velocity is (see Fig. 1.4).

$$v_r = v_t - v_\omega$$

The following are possible types of trajetories:

- *Extended trochoid*: $v_r > 0$, $v_t > v_\omega$
- *Cycloid*: $v_r = 0$, $v_t = v_\omega$
- *Contracted trochoid*: $v_r < 0$, $v_t < v_\omega$

These trajectories are shown in Fig. 1.5 and can be described in kinematic terms as trajectories of the point on the rim of rolling wheel.

Note that for the electric field direction chosen, the magnitude of the electron velocity on the lower point of the trajectory is less than that on the upper point, where the potential is higher.

Perpendicular Fields* E *and* B*: Relativistic Version One method of solving this problem is based on the transformation of electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ by transition to another inertial frame of reference. Below we limit ourselves to a

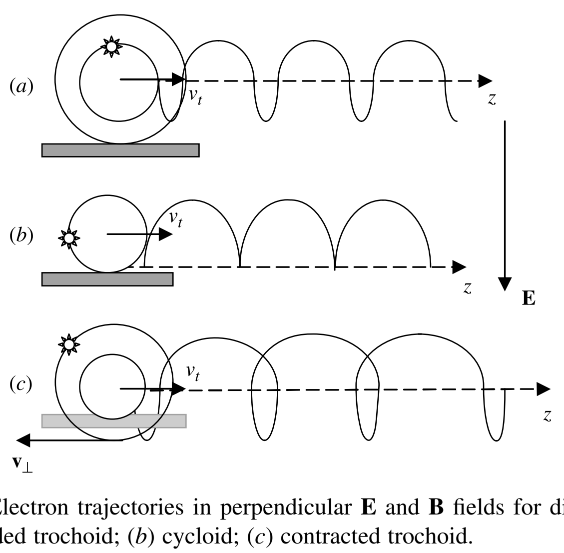

FIGURE 1.5 Electron trajectories in perpendicular $\mathbf{E}$ and $\mathbf{B}$ fields for different values of $v_t/v_\perp$: (*a*) extended trochoid; (*b*) cycloid; (*c*) contracted trochoid.

qualitative discussion. According to the properties of the electromagnetic field tensor (see, e.g., Landau and Lifshitz, 1987; Lehnert, 1964), there are two invariants that are conserved with the transition:

$$E^2 - c^2B^2 = \text{invariant}, \qquad \mathbf{EB} = \text{invariant} \tag{1.66}$$

In the case of perpendicular **E** and **B**, the second invariant is $\mathbf{EB} = 0$. Therefore, it is possible to find an inertial frame of reference in which either the E or the B field is zero.

A field that is not zero depends on the first invariant. Let's assume that $E/B > c$, so the first invariant is positive $E^2 - c^2B^2 > 0$. Hence, it is possible to find an inertial frame of reference with a purely electric field where the first invariant is equal to E'^2 and is also positive. According to Eq. (1.62), $E' = \sqrt{E^2 - c^2B^2}$. As can be shown, the velocity of the reference frame must be $V/c = cB/E$. In the new frame the particle velocity as a function of time would not be periodical. It is clear that after returning to the old frame of reference, the motion will not be periodical either.

In the alternative case, $E^2 - c^2B^2 < 0$, a transition is possible to an inertial frame with a purely magnetic field. In the new inertial frame of reference, the motion evidently would be a superposition of the gyration and the uniform drift along the new magnetic field (it is equal to $B' = \sqrt{B^2 - E^2/c^2}$ and has the same direction as the initial magnetic field). However, the coordinates are functions of time within a moving inertial frame of reference. It is readily verified that this time is

$$dt' = \frac{1 - Ev/Bc^2}{\sqrt{1 - E^2/B^2c^2}}\,dt \tag{1.67}$$

It is not the proper time (the time in the frame of reference linked with the moving particle). We can return to the old system by recalculating coordinates and time simultaneously using the Lorentz transformation. Numerically, this operation is very simple. However, the result computed for the analysis does not have a simple physical interpretation such as that for superposition of the gyration and uniform drift as in nonrelativistic approximation. Therefore, we do not discuss this topic in greater detail. Another possible solution of this problem is given by Landau and Lifshitz (1987), but their result is also not very obvious.

1.5.5 Arbitrary Orientation of Fields E and B. Nonrelativistic Approximation

Let us draw an (X, Y) plane through fields **E** and **B** and decompose vector **E** on the components $\mathbf{E}_B$ and $\mathbf{E}_\perp$ parallel and perpendicular to the magnetic field (Fig. 1.6). The nonrelativistic equations of motion are

$$\frac{dv_B}{dt} = \eta E_B \tag{1.68}$$

$$\frac{d\mathbf{v}_\perp}{dt} + \eta\mathbf{v}_\perp \times \mathbf{B} = -\eta\mathbf{E}_\perp \tag{1.69}$$

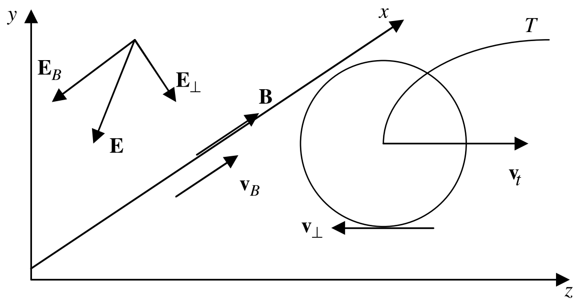

FIGURE 1.6 Electron velocities for arbitrary orientation of fields **E** and **B**. Trajectory T of the guiding center in the (X, Z) plane is parabolic.

Equation (1.68) describes uniformly accelerated drift in the direction of the magnetic field:

$$v_B = \eta E_B t + v_{B0} \tag{1.70}$$

The only difference between Eqs. (1.69) and (1.58) is the replacement of field **E** by $\mathbf{E}_\perp$; therefore, all characteristics of motion perpendicular to the **B** plane coincide in both cases except for the substitution of transversal drift velocity [Eq. (1.64)]:

$$\mathbf{v}_t = \frac{1}{B^2}\,\mathbf{E}_\perp \times \mathbf{B} \tag{1.71}$$

The full velocity of the guiding center is

$$\mathbf{v}_d = \mathbf{v}_B + \mathbf{v}_t = v_B\frac{\mathbf{B}}{B} + \frac{1}{B^2}\,\mathbf{E}_\perp \times \mathbf{B} \tag{1.72}$$

The electron gyrates on the Larmor circle in the plane perpendicular to **B** with gyrofrequency $\omega_g = \eta B$, and the guiding center moves with uniform acceleration along **B**. Evidently, the trajectory of the guiding center is a parabola in the (X, Z) plane (Fig. 1.6).

1.6 MOTION OF ELECTRONS IN WEAKLY INHOMOGENEOUS STATIC FIELDS (Lehnert, 1964; Northrop, 1963; Vandervoort, 1960)

The motion of electrons in weakly inhomogeneous fields is the next approximation in the dynamics of charged particles. A physical system in general may be considered as slowly varying when change in its properties is small on a characteristically finite scale (temporal or spatial) for the system. Small varying systems often display important specific properties. Typical are dynamic oscillatory systems with slowly varying parameters. Here the finite scale is a period T of oscillations. The condition of slow changes is

$$T\left|\frac{dX}{dt}\right| \ll |X| \tag{1.73}$$

or

$$T\left|\frac{dX}{dt}\right| = \varepsilon|X| \tag{1.74}$$

where X is some parameter of the system and $\varepsilon \ll 1$ is the smallness parameter.

1.6.1 Small Variations in Electromagnetic Fields Acting on Moving Charged Particles

If one passes from a laboratory frame where a particle moves in inhomogeneous static fields to a frame moving with a guiding center, the particle will experience action of the variables in time fields. The conditions of slow change in fields (adiabatic approximation) according to Eq. (1.74) are

$$T_g\left|\frac{d\mathbf{B}}{dt}\right| = \varepsilon_B|\,\mathbf{B}\,|, \qquad T_g\left|\frac{d\mathbf{E}}{dt}\right| = \varepsilon_E|\,\mathbf{E}| \tag{1.75}$$

where $T_g = 2\pi/\omega_g$ is the cyclotron period (gyroperiod), and ε_B and ε_E are the smallness parameters. Below we assume for simplicity that $\varepsilon_{B,E} = \varepsilon$ and $\mathbf{B},\mathbf{E} = \mathbf{F}$.

Let us return to the laboratory frame. The velocity magnitude is $|\mathbf{v}| = |d\mathbf{r}/dt|$ and $dt = |d\mathbf{r}|/|\mathbf{v}|$. Substituting in Eq. (1.75) gives conditions for a small inhomogeneity of

$$T_g|\,\mathbf{v}\,|\left|\frac{d\mathbf{F}}{d\mathbf{r}}\right| = \varepsilon|\,\mathbf{F}| \tag{1.76}$$

The full velocity of a particle in homogeneous fields [Eq. (1.65)] is

$$\mathbf{v} = \mathbf{v}_\omega + \mathbf{v}_B + \mathbf{v}_t \tag{1.77}$$

Assume that the gyratic velocity magnitude is

$$|\,\mathbf{v}_\omega| \gg |\,\mathbf{v}_B| + |\,\mathbf{v}_t| \tag{1.78}$$

This means that $|\mathbf{v}| \approx |\mathbf{v}_\omega| = v_\perp$. From Eq. (1.76) we obtain

$$2\pi r_\perp\left|\frac{d\mathbf{F}}{d\mathbf{r}}\right| = \varepsilon|\,\mathbf{F}| \tag{1.79}$$

where $r_\perp = (1/2\pi)T_g v_\perp$ is the Larmor radius. Usually, these conditions are written without a factor 2π. This means that changes in field magnitudes on the Larmor radius scale must be much smaller than full field magnitudes. The alternative condition takes place when

$$|\mathbf{v}_B| \gg |\mathbf{v}_\omega| + |\mathbf{v}_t| \quad \text{and hence} \quad |\mathbf{v}| \approx v_B \tag{1.80}$$

In this case, we obtain from Eq. (1.72)

$$h\left|\frac{d\mathbf{F}}{d\mathbf{r}}\right| = \varepsilon|\mathbf{F}| \tag{1.81}$$

where $h = v_B T_g$ is the pitch of the helical trajectory. Hence, Eq. (1.79) must be fulfilled as a condition of small field changes on the helical trajectory pitch. Note that this is possible only if the component of the electric field E_B parallel to the magnetic field is notzero in the limited time interval, and $E_B \ll E_\perp$. Thus, in the presence of magnetic fields, two natural spatial scales of the weak inhomogeneous electromagnetic field appear: the Larmor radius and the pitch of the helical trajectory.

Example 1.2 It is instructive to calculate the parameter ε_B for electrons in a near-Earth space magnetic field $B_e \sim 2 \times 10^{-5}$ T. The characteristic scale of change of the magnetic field is Earth's radius, $\sim 10^4$ km. Therefore, the value $|\nabla B_e|/B_e$ is on the order of $\sim 10^{-7}\mathrm{m}^{-1}$. Let us assume that the electron moves toward the Earth perpendicular to the magnetic field. Find the value of ε_B for various electron energies.

1. *Nonrelativistic Electron Velocity*. $\beta = 0.2$, $\gamma \approx 1$. The gyrofrequency is $\omega_g = \eta B_e/\gamma \approx 3.5 \times 10^6$ rad/s. The Larmor radius is $R = \beta c/\omega_g \approx 17$ m. $\varepsilon_B = R(|\nabla B_e|/B_e) \approx 1.7 \times 10^{-6}$. As shown in this case, Earth's magnetic field is essentially adiabatic.
2. *Relativistic Velocity*. $\beta = 0.995$, $\gamma = 1/\sqrt{1-\beta^2} \approx 10$, the electron energy is ~ 5 MeV, $\omega_g \approx 3.5 \times 10^5$ rad/s, $R \sim 800$ m, and $\varepsilon_B \sim 8 \times 10^{-5}$. The magnetic field is again adiabatic.
3. *Supreme Relativistic Energy*. ~ 50 GeV ($\gamma = 1 \times 10^5$), $\omega_g \sim 35$ rad/s, $R \sim 10^7$ m, and $\varepsilon_B \sim 1$. The magnetic field is nonadiabatic.

1.6.2 Adiabatic Invariants (Landau and Lifshitz, 1987; Northrop, 1963)

Consider an oscillatory system with a slowly varying parameter X so that condition (1.74) is being fulfilled. Let us assume that only one of the system's degrees of freedom is oscillatory with the coordinate q and generalized momentum P. The Lagrangian equation of the system is

$$\frac{dP}{dt} - \frac{\partial L}{\partial q} = 0 \tag{1.82}$$

Because the Lagrangian L depends on X as a parameter,

$$\frac{\partial L}{\partial q} = \left(\frac{\partial L}{\partial q}\right)_{X=\text{const.}} + \frac{\partial L}{\partial X}\frac{dX}{dq} \tag{1.83}$$

The first term in Eq. (1.83) can be found by integrating Eq. (1.82) with $X = \text{const.}$ Then we obtain a nonperturbed trajectory, and this term must be a strictly periodic function of q. Evidently, the second term is nonperiodic if the function $X = X(t)$ does not have periodicities commensurable with the system oscillation. We can represent this term as

$$\frac{\partial L}{\partial X}\frac{dX}{dq} = \frac{\partial L}{\partial X}\frac{dX}{dt}\frac{dt}{dq} \tag{1.84}$$

Now let us integrate Eq. (1.82) along the closed contour corresponding to the nonperturbed particle trajectory:

$$\frac{d}{dt}\oint P\,dq = \oint\left(\frac{\partial L}{\partial q}\right)_{X=\text{const}} dq + \oint\frac{\partial L}{\partial X}\frac{dX}{dt}\frac{dt}{dq}dq \tag{1.85}$$

The first term on the right-hand side equals zero as an integral over the period of a periodic function. The second term is

$$\int_0^T \frac{\partial L}{\partial X}\frac{dX}{dt}dt = \overline{\frac{\partial L}{\partial X}}\,\overline{\frac{dX}{dt}}\int_0^T dt = \overline{\frac{\partial L}{\partial X}}T\overline{\frac{dX}{dt}}$$

So taking Eq. (1.74) into account, we obtain

$$\frac{d}{dt}\oint P\,dq = \varepsilon\,\overline{\frac{\partial L}{\partial X}}\overline{X} \tag{1.86}$$

This equality means that at least in the first order of the smallness parameter ε, the integral

$$I = \oint P\,dq = \text{const.} \tag{1.87}$$

Remember that the integration in Eq. (1.87) is performed along a nondisturbed oscillating trajectory. This integral is called the *adiabatic invariant*. The number of adiabatic invariants for any system is equal to the number of its oscillatory degrees of freedom. For one particle the maximum possible number of degrees is 3. Particle motion described by three adiabatic invariants is an uncommon event. For example, oscillations of charged particles captured in Earth's Van Allen radiation belt have three invariants, corresponding to three types of oscillating motions (see, e.g., Northrop, 1963): gyration on the Larmor orbits, north–south oscillations, and precession about the Earth.

Transversal Adiabatic Invariant of an Electron in an Electromagnetic Field Let us consider the integral from Eq. (1.87) within the frame of reference of the guiding center. In this frame the contour in Eq. (1.87) is the Larmor circle of the electron gyrating in the static electromagnetic field if $E/cB < 1$. Then, taking into account the fact that the momentum on the circle is $P = P_\perp = \text{const.}$, we obtain

$$\oint P\, dl = \oint P_\perp dl = 2\pi p_\perp r_\perp - e_0 2\pi r_\perp A_\perp \tag{1.88}$$

Application of Stokes' theorem to Eq. (I.3) yields

$$\int_S B_n dS = \int_S \text{curl}_n \mathbf{A}\, dS = \oint A_\perp dl = 2\pi r_\perp A_\perp \tag{1.89}$$

The integral (1.89) is taken over by the Larmor circle, where normal components of a weakly inhomogeneous magnetic field $B_n = B = \text{const.}$ Therefore, $\int_S B_n\, dS = \pi r_\perp^2 B$. By substituting this in Eqs. (1.88) and (1.89), we obtain

$$\oint P\, dl = \pi(2p_\perp r_\perp - e_0 r_\perp^2 B) \tag{1.90}$$

The Larmor radius is equal to

$$r_\perp = \frac{v_\perp}{\omega_g} = \frac{v_\perp m}{e_0 B} = \frac{p_\perp}{e_0 B} \tag{1.91}$$

Then we obtain from Eqs. (1.87) and (1.90),

$$I = \frac{\pi}{e_0} \frac{p_\perp^2}{B} \tag{1.92}$$

This invariant refers to the gyrating degree of freedom and is called the *transversal* (*perpendicular*) *adiabatic invariant*. Usually, I is written

$$I_\perp = \frac{p_\perp^2}{B} = \text{const.} \tag{1.93}$$

In nonrelativistic approximation we can replace $p_\perp$ by $v_\perp$, and

$$I_{\perp n} = \frac{v_\perp^2}{B} \tag{1.94}$$

It is instructive to calculate the invariant $I_\perp$ directly, considering the gyration of an electron in weakly inhomogeneous magnetic fields. Assume that the guiding center drifts along an increasingly magnetic field (Fig. 1.7). The magnetic field lines are shown in Fig. 1.7. The magnetic field in the moving frame would increase in time. In this frame of reference, the electron experiences the action of the inducing

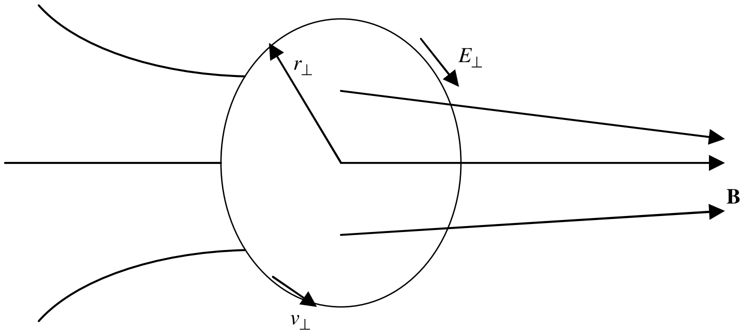

FIGURE 1.7 Perpendicular adiabatic invariant.

electric field $E_\perp$ directed along the Larmor circle clockwise (looking from the magnetic field side; Fig. 1.7). The field $E_\perp$ can be found by integrating Maxwell's equation (I.6) over the Larmor circle and using Stokes' theorem:

$$\int_S \frac{\partial B_n}{\partial t} dS = -\int_S \mathrm{curl}_n\, \mathbf{E}\, ds = -\oint E_\perp\, dl \tag{1.95}$$

Obviously, $B_n = B$, and as a result of Eq. (1.79), fields B and $E_\perp$ are almost uniform on the Larmor circle. We now obtain

$$\frac{dB}{dt}\pi r_\perp^2 = -2\pi r_\perp E_\perp \tag{1.96}$$

$$E_\perp = -\frac{r_\perp}{2}\frac{dB}{dt} = -\frac{p_\perp}{2e_0 B}\frac{dB}{dt} \tag{1.97}$$

This electric field (according to Fig. 1.7) accelerates the electron along the Larmor orbit (the particle has a negative charge). The corresponding change in momentum is

$$\frac{dp_\perp}{dt} = -e_0 E = \frac{p_\perp}{2B}\frac{dB}{dt} \tag{1.98}$$

By integrating this equation, we obtain conservation of the perpendicular adiabatic invariant:

$$I_\perp = \frac{p_\perp^2}{B} = \text{const.} \tag{1.99}$$

The electron motion on the Larmor circle forms the elementary current

$$I = \frac{e_0}{T} = \frac{e_0 \omega_g}{2\pi} \tag{1.100}$$

The magnetic moment of this circular current is $M = (\mu_0/4\pi)IS$ (Northrop, 1963), where S is the area of the circle. Thus, the magnetic moment of the electron is

$$M = \frac{\mu_0}{4\pi}\frac{e_0\omega_g}{2\pi}\pi r_\perp^2 = \frac{\mu_0}{8\pi m}I_\perp \tag{1.101}$$

We see that for $E = 0$ ($m = \text{const.}$) in the nonrelativistic approximation, the magnetic moment of the electron is the adiabatic invariant. The transversal adiabatic invariant has the meaning of a magnetic flux through the Larmor circle:

$$\Psi = \pi r_\perp^2 B = \frac{\pi}{e_0^2}I_\perp \tag{1.102}$$

Accuracy of Conservation of an Adiabatic Invariant According to numerical calculations in many specific problems, the accuracy of the conservation of the transversal adiabatic invariant generally is higher than the first power of ε [see Eq. (1.86)]. But a rigorous consideration of this problem is complicated (Arnold et al., 1988; Kruskal, 1960; Littlejohn, 1980). Let us suppose, following Arnold et al. (1988), that the state of some system depends on a slow parameter λ, which quickly pushes the adiabatic invariant I to its limits: $I(t = -\infty)$ and $I(t = +\infty)$. Then one can introduce an increment of I: $\Delta I = I(+\infty) - I(-\infty)$. Although for finite t, oscillations of I are of order ε, the increment ΔI is much smaller than ε. If λ depends analytically on εt, then $\Delta I \sim O(\exp(-c/\varepsilon))(c > 0)$. *So the increment of an adiabatic invariant decays faster than any power of* ε *as* $\varepsilon \to 0$.

1.6.3 Motion of the Guiding Center

Drift Equations The conditions (1.74) and (1.81) of the small inhomogeneity allow us to set the following scaling orders for the particle trajectory parameters:

$$r_\perp \sim \varepsilon, \qquad T_g \sim \varepsilon, \qquad \omega_g = \frac{2\pi}{T_g} \sim \frac{1}{\varepsilon} \tag{1.103}$$

In the inhomogeneous fields considered, the particles [as seen from Eq. (1.103)] perform oscillations of very high frequency but small amplitude. As a result, the vibrations do not perturb slow movement of the guiding center (evolution of the system). This effect is typical for oscillatory systems with slowly varying parameters. The key method in the theory is the averaging procedure, which separates small oscillations from the drift. This procedure, applying the motion of particles in inhomogeneous static electromagnetic fields, is described briefly below. We use nonrelativistic approximation as well as the conditions

$$\frac{E}{B} < v \qquad E_B \ll E \tag{1.104}$$

Let us write the instantaneous position of the particle (see, e.g., Morozov and Solov'ev, 1960; Northrop, 1963) as

$$\mathbf{r}(t) = \mathbf{R}_d(t) + \mathbf{r}_\perp(t) \tag{1.105}$$

where $\mathbf{R}_d$ is the position of the guiding center and $\mathbf{r}_\perp$ is a radius vector of the particle relative to the guiding center (Fig. 1.8). Thus, the particle velocity is

$$\mathbf{v} = \frac{d\mathbf{R}_d}{dt} + \mathbf{v}_\perp \tag{1.106}$$

Substituting Eq. (1.104) in the nonrelativistic equation of motion, we obtain

$$\ddot{\mathbf{R}} + \ddot{\mathbf{r}}_\perp = -\eta[\mathbf{E}(\mathbf{R} + \mathbf{r}_\perp) + (\dot{\mathbf{R}} + \dot{\mathbf{r}}_\perp) \times \mathbf{B}(\mathbf{R} + \mathbf{r}_\perp)]$$

Expand the fields in a Taylor series about $\mathbf{R}$ up to the first degree of $\mathbf{r}_\perp$. In this case the error is of order $O(\varepsilon)$. We obtain

$$\ddot{\mathbf{R}} + \ddot{\mathbf{r}}_\perp = -\eta\{\mathbf{E}(\mathbf{R}) + \mathbf{r}_\perp \nabla \mathbf{E}(\mathbf{R}) + (\dot{\mathbf{R}} + \dot{\mathbf{r}}_\perp) \times [\mathbf{B}(\mathbf{R}) + \mathbf{r}_\perp \nabla \mathbf{B}(\mathbf{R})]\} + O(\varepsilon) \tag{1.107}$$

Now we must express the gyration of vector $\mathbf{r}_\perp$ explicitly. Let us introduce three orthogonal unit vectors $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$, and $\boldsymbol{\tau}_3$, where $\boldsymbol{\tau}_1 = \mathbf{B}/B$ is parallel to the magnetic field and $\boldsymbol{\tau}_2$ and $\boldsymbol{\tau}_3$ are perpendicular to the $\mathbf{B}$ plane (Fig. 1.8). The particles gyrate in the plane perpendicular to $\mathbf{B}$. Therefore, we can write

$$\mathbf{r}_\perp = r_\perp(\tau_2 \sin\theta + \tau_3 \cos\theta), \qquad \theta = \int \omega_g \, dt \tag{1.108}$$

Repeated differentiation of Eq. (1.108) gives

$$\dot{\mathbf{r}}_\perp = \omega r_\perp(\tau_2 \cos\theta - \tau_3 \sin\theta) + (\mathbf{r}_\perp \tau_2) \sin\theta + (\mathbf{r}_\perp \tau_3) \cos\theta \tag{1.109}$$

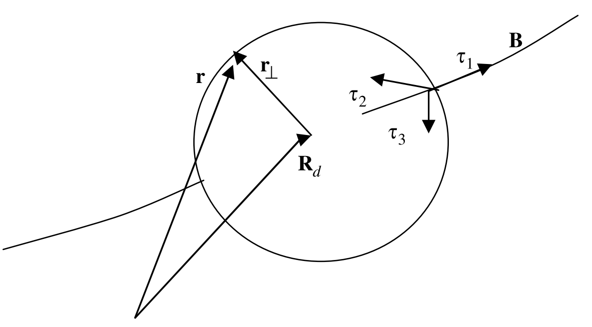

FIGURE 1.8 Motion of a particle as a superposition of the guiding center on a gyrating cyclotron.

We have omitted the corresponding expression of $\ddot{\mathbf{r}}_\perp$ (Northrop, 1963). The next step is the substitution of Eqs. (1.108) and (1.109) and a similar expression for $\ddot{\mathbf{r}}_\perp$ into Eq. (1.107) and integration of the resulting expression:

$$\ddot{\mathbf{R}} = F[\mathbf{R}, \dot{\mathbf{R}}, \mathbf{E}(\mathbf{R}), \mathbf{B}(\mathbf{R}), r_\perp, \theta]$$

over θ with a 2π period $\int_0^{2\pi} (\cdots) d\theta$. After a complicated iterative procedure which takes into account the estimations of Eq. (1.103) and the conditions of Eq. (1.104), the following basic equation of the guiding center's motion is given by Morozov and Solov'ev (1960) and Northrop (1963):

$$\frac{d\mathbf{R}_d}{dt} = v_B \frac{\mathbf{B}}{B} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{v_\perp^2 + 2v_B^2}{2\eta B^3} \mathbf{B} \times \nabla B \tag{1.110}$$

Let us augment the energy relation to this equation:

$$W = \frac{m_0}{2} v^2 - e_0 U = \frac{m_0}{2} (v_B^2 + v_\perp^2) - e_0 \varphi \tag{1.111}$$

where v_B is the velocity component parallel to $\mathbf{B}$. Recall that the *drift equation* (1.110) is correct in nonrelativistic approximation. Relativistic drift equations may be found in an article by Vandervoort (1960).

We see that the guiding center velocity consists of three components: (1) drift along the curvilinear line of the magnetic field, (2) transversal (perpendicular) drift, and (3) gradient drift. The first two components are similar to the drifts in uniform fields. The third component exists only in an inhomogeneous magnetic field.

Equation (1.10) is a first-order differential equation, a significant simplification of the original second-order equation of motion because Eq. (1.110) gives the velocity of the particle directly. It is interesting to compare the accuracy of the numerical calculation of the velocity using the original equation of motion in second order and Eqs. (1.110) and (1.111). The latter gives acceptable accuracy for the times $t \sim \varepsilon^{-1}$, whereas the accuracy of the numerical methods is reduced with time.

Integration of the Drift Equation The right-hand side of Eq. (1.110) can be expressed explicitly in terms of the potential $U = U(\mathbf{r})$ and the magnetic field $B = B(\mathbf{r})$. Let us suppose that the cathode potential and the initial velocity are zeros and the oscillatory velocity $v_{\perp 0}$ is given at the point where $B = B_0$. The full velocity is

$$v = \sqrt{2\eta U(\mathbf{r})} \tag{1.112}$$

The oscillatory velocity can be found from the nonrelativistic adiabatic invariant

$$I = I_n = \frac{v_\perp^2}{B(\mathbf{r})} = \text{const.} \tag{1.113}$$

The value of I is determined by the initial condition

$$I = I_0 = \frac{v_{\perp 0}{}^2}{B_0}$$

The oscillatory velocity then is $v_\perp = v_{\perp 0}\sqrt{B(\mathbf{r})/B_0}$. The longitudinal velocity according to the relation of the energy [Eq. (1.11)] is

$$v_B = \sqrt{v^2 - v_\perp^2} = \sqrt{v^2 - IB(\mathbf{r})} \tag{1.114}$$

The electric field $\mathbf{E}(\mathbf{r}) = -\nabla U(\mathbf{r})$. Thus, all values on the right-hand side of Eq. (1.110) have been expressed as functions of $\mathbf{r}$.

Example 1.3: Drift Motion of Electrons in an Adiabatic Magnetic Trap The scheme of a *magnetic trap* (elementary mirror machine or magnetic bottle) is shown in Fig. 1.9. The particle moves in the magnetic field, which is assumed to be weakly inhomogeneous, so that Eq. (1.113) is valid. As can be seen, the magnetic field in the bottle ends $B = B_M$ is maximum. According to Eq. (1.14), the velocity component v_B is reduced when the particle moves in the direction of increasing magnetic field. This effect can be interpreted as an action of the radial component B_r existing in the region of an inhomogeneous magnetic field. The corresponding force $\mathbf{F}_B = -e_0\mathbf{v}_\perp \times \mathbf{B}_r$ is directed along magnetic lines. In increasing magnetic field, one is opposite to the velocity component v_B. When the oscillatory velocity $v_\perp$ achieves the full velocity v, the value $v_B = 0$ and the particle is reflected. The corresponding planes are named ***magnetic mirrors***. Regions where $v_\perp > v$, called *magnetic corks*, are unattainable for the particle.

Let us suppose that the particle is injected into the magnetic field B at the angle α with the magnetic line (Fig. 1.9). The adiabatic invariant

$$I = \frac{v_\perp^2}{B} = \frac{v^2 \sin^2\alpha}{B} \tag{1.115}$$

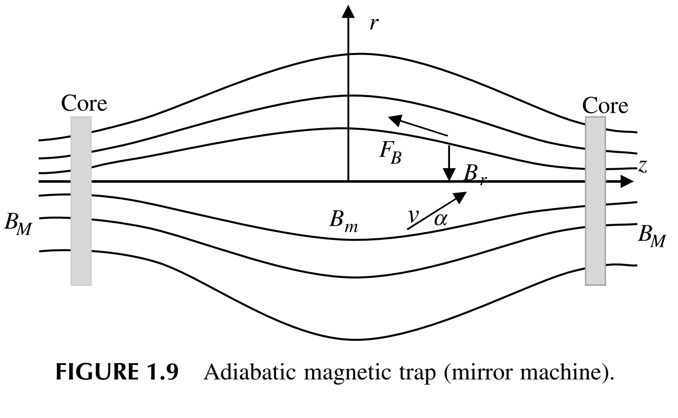

FIGURE 1.9 Adiabatic magnetic trap (mirror machine).

The velocity v_B of the particle in the maximum of the magnetic field B_M according to Eq. (1.114) is

$$v_B = \sqrt{v^2 - IB_M} = \sqrt{v^2\left(1 - \frac{B_M}{B}\sin^2\alpha\right)} \tag{1.116}$$

Particles injected with $\sin\alpha > \sqrt{B/B_M}$ are reflected and captured in the adiabatic trap. When the particles emerge in the medium plane of the trap, where the magnetic field B_n is minimal, the critical angle is also minimal.

$$\sin\alpha_{\min} = \sqrt{\frac{B_n}{B_M}} \tag{1.117}$$

1.7 MOTION OF ELECTRONS IN FIELDS WITH AXIAL AND PLANE SYMMETRY. BUSCH'S THEOREM

Similar to the way that field independence (Lagrangian independence) of time leads to a conservation of energy, any symmetry of fields (i.e., independence of the Lagrangian of some other variables) leads to conservation of the corresponding momenta. These laws of conservation are the foundation of mechanics.

Assume that the Lagrangian L does not depend on the generalized coordinate q (i.e., $\partial L/\partial q = 0$), which is a cyclic variable. The Lagrangian equation (I.17) for this case is

$$\frac{dP_q}{dt} = \frac{\partial L}{dq} = 0 \tag{1.118}$$

$$P_q = \text{const.} \tag{1.119}$$

1.7.1 Systems with Axial Symmetry. Busch's Theorem

Consider a cylindrical coordinate system (Fig. 1.10). For axially symmetric fields, the cyclic variable is θ. This tells us that a turning does not change an electromagnetic field in the system. We can find corresponding generalized momentum by writing the Lagrangian as a function of θ and $\dot{\theta}$. According to Eq. (I.18), the Lagrangian is

$$\begin{aligned} L &= -m_0c^2\sqrt{1 - \frac{v^2}{c^2}} - e_0\,\mathbf{Av} + e_0U \\ &= -m_0c^2\sqrt{1 - \frac{1}{c^2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)} - e_0(\dot{r}A_r + r\dot{\theta}A_\theta + \dot{z}A_z) + e_0\varphi \end{aligned} \tag{1.120}$$

where $\partial\mathbf{A}/\partial\theta = \partial\varphi/\partial\theta = 0$. Then the angular generalized momentum is

$$P_\theta = \frac{\partial L}{\partial\dot{\theta}} = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}}\frac{r^2\dot{\theta}}{c^2} - e_0rA_\theta = mr^2\dot{\theta} - e_0rA_\theta = \text{const.} \tag{1.121}$$

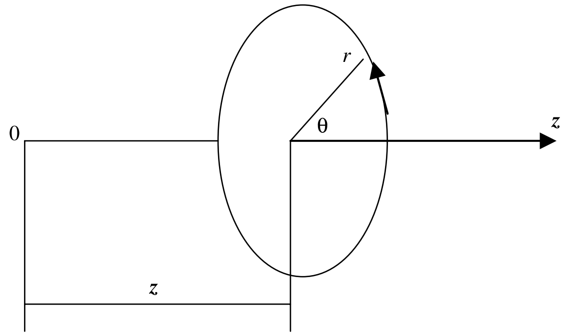

FIGURE 1.10 Cylindrical coordinate system.

To obtain A_θ, let us apply Stokes' theorem to Maxwell's equation (I.3):

$$\int B_n\, dS = \int \mathrm{curl}_n\, \mathbf{A}\, dS = \oint A_\theta\, r d\theta \tag{1.122}$$

Integration is performed along the circle of radius r shown in Fig. 1.1. Here $B_n = B_z$. The value rA_θ is constant on the contour; therefore,

$$rA_\theta = \frac{1}{2\pi}\Psi = \frac{1}{2\pi}\int B_z\, dS \tag{1.123}$$

So the equation of angular momentum conservation is

$$P_\theta = mr^2\dot{\theta} - \frac{e_0}{2\pi}\Psi = \text{const.} \tag{1.124}$$

where $\Psi = 2\pi \int_0^r B_z(r, z) r\, dr$ is the magnetic flux through a circle of radius r. Suppose that the electron leaves the cathode at a distance r_c from the axis of symmetry with angular velocity $\dot{\theta}_c$. Then, according to Eq. (1.124),

$$mr^2\,\dot{\theta} - \frac{e_0}{2\pi}\Psi = m_c r_c^2\,\dot{\theta}_c - \frac{e_0}{2\pi}\Psi_c \tag{1.125}$$

We obtain the formula for angular velocity:

$$\dot{\theta} = \frac{m_c r_c^2}{mr^2}\,\dot{\theta}_c + \frac{e_0}{2\pi m r^2}(\Psi - \Psi_c) = \frac{\gamma_c}{\gamma}\left(\frac{r_c}{r}\right)^2\dot{\theta}_c + \frac{\eta}{2\pi\gamma\, r^2}(\Psi - \Psi_c) \tag{1.126}$$

This relation is known as *Busch's theorem*. In nonrelativistic approximation,

$$\dot{\theta} = \left(\frac{r_c}{r}\right)^2\dot{\theta}_c + \frac{\eta}{2\pi\, r^2}(\Psi - \Psi_c) \tag{1.127}$$

Busch's theorem for zero cathode velocity is

$$\dot{\theta} = \frac{\eta}{2\pi r^2}(\Psi - \Psi_c) \tag{1.128}$$

1.7.2 Formation of Helical Trajectories at a Jump in a Magnetic Field

The electron in an axially symmetric system is injected into a jump in the magnetic field so that in the plane $z = 0$ the magnetic field changes from $B = B_1$ to $B = B_2$ (Fig. 1.11). Before the jump ($z < 0$) the electron moves parallel to the z-axis. In the plane of the jump, the magnetic field is radial. One rotates the particle, and the latter continues to gyrate after the jump in the magnetic field B_2 along the helical trajectory. The magnetic fields on both sides of the jump are assumed uniform.

Let us find the parameters of the electron trajectory. For $z < 0$, $\dot{\theta} = 0$ and we can write Busch's theorem similar to Eq. (1.128):

$$\dot{\theta} = \frac{\eta}{2\pi r^2}(\Psi_2 - \Psi_1) \tag{1.129}$$

Ψ_1 and Ψ_2 are the magnetic fluxes through circles of radii r_1 and r_2, which are the distances of the electron from the z-axis before and just after the jump. We assume that the radial position of the particle does not change across the jump. Therefore, $r_1 = r_2 = r$ and

$$\Psi_1 = \pi r^2 B_1, \qquad \Psi_2 = \pi r^2 B_2 \tag{1.130}$$

The angular velocity according to Eqs. (1.129) and (1.130) is

$$\dot{\theta} = \frac{\eta}{2}(B_2 - B_1) \tag{1.131}$$

The azimuthal velocity is

$$v_\theta = r\dot{\theta} = \frac{\eta r}{2}(B_2 - B_1) = v_\perp \tag{1.132}$$

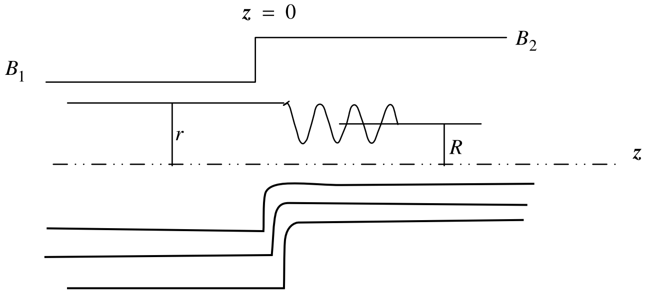

FIGURE 1.11 Jump of a magnetic field as the formation system of a helical trajectory.

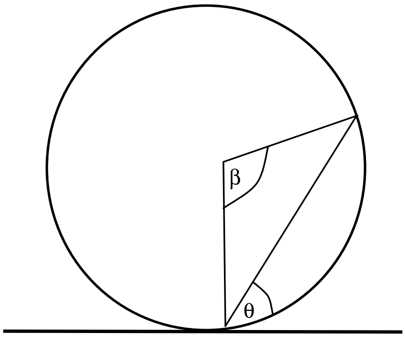

FIGURE 1.12 Motion of an electron injected from a region with zero magnetic field.

where $v_\perp$ is the velocity of the cyclotron gyration. The Larmor radius is

$$r_\perp = \frac{v_\perp}{\omega_{g2}} = \frac{v_\perp}{\eta B_2} = \frac{r}{2}\left(1 - \frac{B_1}{B_2}\right) \tag{1.133}$$

The guiding center radius (see Fig. 1.11) is

$$R = r - r_\perp = \frac{r}{2}\left(1 + \frac{B_1}{B_2}\right) \tag{1.134}$$

Thus, Busch's theorem allows us to solve the problem.

Let us consider two particular cases:

1. *Injection from a Region with Zero Magnetic Field.* $B_1 = 0$. We find from Eqs. (1.131)–(1.134) that

$$\dot{\theta} = \frac{\eta B_2}{2}, \qquad r_\perp = \frac{r}{2}, \qquad R = \frac{r}{2} \tag{1.135}$$

We see that the Larmor circle touches the z-axis. According to Fig. 1.12, angle θ of the turn of the radius vector about the axis equals the half-angle β of the Larmor radius turn. Therefore, $\dot{\theta} = \frac{1}{2}\dot{\beta} = \frac{1}{2}\omega_g$.

2. *Reverse (cusp) of the Magnetic Field.* $B_1 = -B_2$. We find from Eqs. (1.131), (1.133), and (1.134) that $\dot{\theta} = \eta B_2$, $r_\perp = r$, and $R = 0$. So in this case, after the jump the electron gyrates at cyclotron frequency around the guiding center located on the z-axis. The Larmor radius after the cusp is equal to the initial distance of the electron from the axis (Fig. 1.13). Axis-encircling helical electron beams that are formatted in the magnetic cusp are used last time in high-harmonic gyrotrons (Appendix 9, Idehara et al., 2004).

1.7.3 Systems with Plane Symmetry

Consider the Cartesian coordinate system in Fig. 1.14. By definition the plane symmetric system is the electromagnetic system that is uniform along one

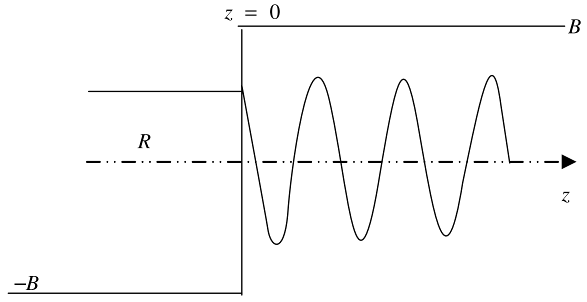

FIGURE 1.13 Motion of an electron through a magnetic cusp.

coordinate (e.g., in the y direction) and symmetrical about a plane parallel to this direction [e.g., the (Y, Z) plane]. It means that the Lagrangian is independent of y; that is,

$$\frac{\partial L}{\partial y} = 0 \tag{1.136}$$

and the potential and z-components of the fields are symmetrical functions of x:

$$\begin{aligned}
&\varphi(x) = \varphi(-x), \qquad &&E_z(x) = E_z(-x), \qquad &&B_z(x) = B_z(-x)\\
&A_y(x) = -A_y(-x), \qquad &&E_x(x) = -E_x(-x), \qquad &&B_x(x) = -B_x(-x)\\
&A_x = A_z = E_y = B_y = 0
\end{aligned} \tag{1.137}$$

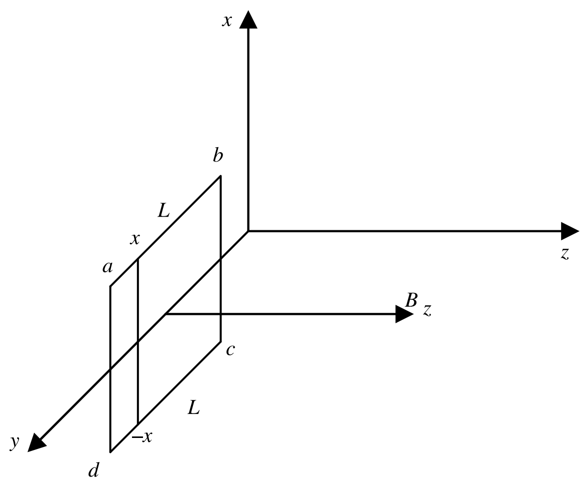

FIGURE 1.14 Derivation of Busch's theorem for a plane symmetrical system.

The generalized momentum for a plane symmetrical system according to Eq. (1.136) is

$$P_y = p_y - e_0 A_y = m\dot{y} - e_0 A_y = \text{const.} \tag{1.138}$$

Applying Eq. (1.3) and Stokes' theorem to the contour *abcd* (Fig. 1.14), we find that

$$\int B_z \;\; ds = \oint_{abcd} A_l \;\; dl \tag{1.139}$$

Taking into account the relations (1.137), we obtain $2L\int_0^x B_z \;\; dx = 2LA_y$; that is,

$$A_y = \Psi = \int_0^x B_z \;\; dx \tag{1.140}$$

where Ψ is the magnetic flux through a strip of width x and unit length in the y direction. So

$$m\dot{y} - e_0\Psi = \text{const.} \tag{1.141}$$

Let us assume that the electron leaves the cathode at distance x_c from the plane of symmetry with the velocity $\dot{y}_c$. Then the velocity for the particle in the question is

$$\dot{y} = \dot{y}_c + \frac{\eta}{\gamma}(\Psi - \Psi_c) = \dot{y}_c + \frac{\eta}{\gamma}\left[\int_0^x B_z(x,z)\; dx - \int_0^{x_c} B_z(x,z_c)\; dx\right] \tag{1.142}$$

Equation (1.142) is a version of Busch's theorem for a plane symmetric system.