Chapter 1

BASIC PRINCIPLES FOR ELECTRIC MACHINE ANALYSIS

1.1 INTRODUCTION

There are several basic concepts that must be established before the analysis of electric machines can begin. The principle of electromechanical energy conversion is perhaps the cornerstone of machine analysis. This theory allows us to establish an expression of electromagnetic torque in terms of machine variables, generally the currents and the displacement of the mechanical system. Other principles that must be established are (1) the derivation of equivalent circuit representations of magnetically coupled circuits, (2) the concept of a sinusoidally distributed winding, (3) the concept of a rotating air-gap magnetomotive force (MMF), and (4) the derivation of winding inductances. The above-mentioned basic principles are presented in this chapter, concluding with the voltage equations of a 3-phase synchronous machine and a 3-phase induction machine. It is shown that the equations, which describe the behavior of alternating-current (ac) machines, contain time-varying coefficients due to the fact that some of the machine inductances are functions of the rotor displacement. This establishes an awareness of the complexity of these voltage equations and sets the stage for the change of variables (Chapter 3), which reduces the complexity of the voltage equations by eliminating the time-dependent inductances.

1.2 MAGNETICALLY COUPLED CIRCUITS

Magnetically coupled electric circuits are central to the operation of transformers and electric machines. In the case of transformers, stationary circuits are

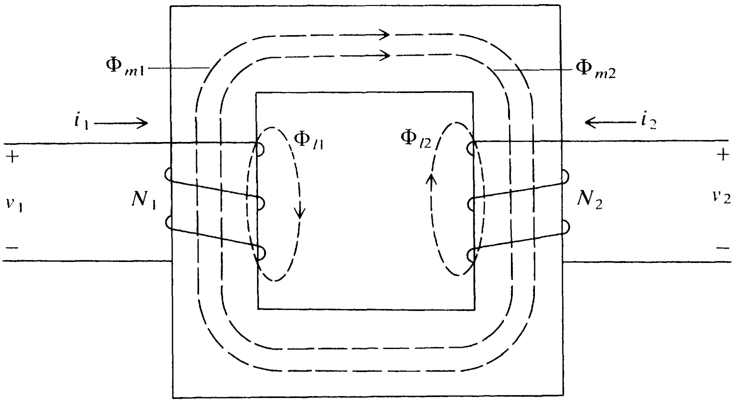

Figure 1.2-1 Magnetically coupled circuits.

magnetically coupled for the purpose of changing the voltage and current levels. In the case of electric machines, circuits in relative motion are magnetically coupled for the purpose of transferring energy between mechanical and electrical systems. Because magnetically coupled circuits play such an important role in power transmission and conversion, it is important to establish the equations that describe their behavior and to express these equations in a form convenient for analysis. These goals may be achieved by starting with two stationary electric circuits that are magnetically coupled as shown in Fig. 1.2-1. The two coils consist of turns N_1 and N_2, respectively, and they are wound on a common core that is generally a ferromagnetic material with permeability large relative to that of air. The permeability of free space, μ_0, is $4\pi \times 10^{-7}$ H/m. The permeability of other materials is expressed as $\mu = \mu_r\mu_0$ where μ_r is the relative permeability. In the case of transformer steel the relative permeability may be as high 2000 to 4000.

In general, the flux produced by each coil can be separated into two components: a leakage component denoted with an l subscript and a magnetizing component denoted by an m subscript. Each of these components is depicted by a single streamline with the positive direction determined by applying the right-hand rule to the direction of current flow in the coil. Often, in transformer analysis, i_2 is selected positive out of the top of coil 2, and a dot is placed at that terminal.

The flux linking each coil may be expressed as

$$\Phi_1 = \Phi_{l1} + \Phi_{m1} + \Phi_{m2} \tag{1.2-1}$$

$$\Phi_2 = \Phi_{l2} + \Phi_{m2} + \Phi_{m1} \tag{1.2-2}$$

The leakage flux Φ_{l1} is produced by current flowing in coil 1, and it links only the turns of coil 1. Likewise, the leakage flux Φ_{l2} is produced by current flowing in coil 2, and it links only the turns of coil 2. The magnetizing flux Φ_{m1} is produced by current flowing in coil 1, and it links all turns of coils 1 and 2. Similarly, the magnetizing flux Φ_{m2} is produced by current flowing in coil 2, and it also links all turns of coils 1 and 2. With the selected positive direction of current flow and the manner in which the coils are wound (Fig. 1.2-1), magnetizing flux produced by positive current in one coil adds to the magnetizing flux produced by positive current in the other coil. In other words, if both currents are actually flowing in the same direction, the magnetizing fluxes produced by each coil are in the same direction, making the total magnetizing flux or the total core flux the sum of the instantaneous magnitudes of the individual magnetizing fluxes. If the actual currents are in opposite directions, the magnetizing fluxes are in opposite directions. In this case, one coil is said to be magnetizing the core, the other demagnetizing.

Before proceeding, it is appropriate to point out that this is an idealization of the actual magnetic system. Clearly, all of the leakage flux may not link all the turns of the coil producing it. Likewise, all of the magnetizing flux of one coil may not link all of the turns of the other coil. To acknowledge this practical aspect of the magnetic system, the number of turns is considered to be an equivalent number rather than the actual number. This fact should cause us little concern because the inductances of the electric circuit resulting from the magnetic coupling are generally determined from tests.

The voltage equations may be expressed in matrix form as

$$\mathbf{v} = \mathbf{ri} + \frac{d\boldsymbol{\lambda}}{dt} \tag{1.2-3}$$

where $\mathbf{r} = \mathrm{diag}\,[r_1 \ \ r_2]$, a diagonal matrix, and

$$(\mathbf{f})^T = [f_1 \ \ f_2] \tag{1.2-4}$$

where f represents voltage, current, or flux linkage. The resistances r_1 and r_2 and the flux linkages λ_1 and λ_2 are related to coils 1 and 2, respectively. Because it is assumed that Φ_1 links the equivalent turns of coil 1 and Φ_2 links the equivalent turns of coil 2, the flux linkages may be written as

$$\lambda_1 = N_1\Phi_1 \tag{1.2-5}$$

$$\lambda_2 = N_2\Phi_2 \tag{1.2-6}$$

where Φ_1 and Φ_2 are given by (1.2-1) and (1.2-2), respectively.

Linear Magnetic System

If saturation is neglected, the system is linear and the fluxes may be expressed as

$$\Phi_{l1} = \frac{N_1 i_1}{\mathcal{R}_{l1}} \tag{1.2-7}$$

$$\Phi_{m1} = \frac{N_1 i_1}{\mathcal{R}_m} \tag{1.2-8}$$

$$\Phi_{l2} = \frac{N_2 i_2}{\mathcal{R}_{l2}} \tag{1.2-9}$$

$$\Phi_{m2} = \frac{N_2 i_2}{\mathcal{R}_m} \tag{1.2-10}$$

where $\mathscr{R}_{l1}$ and $\mathscr{R}_{l2}$ are the reluctances of the leakage paths and $\mathscr{R}_m$ is the reluctance of the path of the magnetizing fluxes. The product of N times i (ampere-turns) is the MMF, which is determined by application of Ampere's law. The reluctance of the leakage paths is difficult to express and impossible to measure. In fact, a unique determination of the inductances associated with the leakage flux cannot be made by tests; instead, it is either calculated or approximated from design considerations. The reluctance of the magnetizing path of the core shown in Fig. 1.2-1 may be computed with sufficient accuracy from the well-known relationship

$$\mathscr{R} = \frac{l}{\mu A} \tag{1.2-11}$$

where l is the mean or equivalent length of the magnetic path, A is the cross-sectional area, and μ is the permeability.

Substituting (1.2-7)–(1.2-10) into (1.2-1) and (1.2-2) yields

$$\Phi_1 = \frac{N_1 i_1}{\mathscr{R}_{l1}} + \frac{N_1 i_1}{\mathscr{R}_m} + \frac{N_2 i_2}{\mathscr{R}_m} \tag{1.2-12}$$

$$\Phi_2 = \frac{N_2 i_2}{\mathscr{R}_{l2}} + \frac{N_2 i_2}{\mathscr{R}_m} + \frac{N_1 i_1}{\mathscr{R}_m} \tag{1.2-13}$$

Substituting (1.2-12) and (1.2-13) into (1.2-5) and (1.2-6) yields

$$\lambda_1 = \frac{N_1^2}{\mathscr{R}_{l1}} i_1 + \frac{N_1^2}{\mathscr{R}_m} i_1 + \frac{N_1 N_2}{\mathscr{R}_m} i_2 \tag{1.2-14}$$

$$\lambda_2 = \frac{N_2^2}{\mathscr{R}_{l2}} i_2 + \frac{N_2^2}{\mathscr{R}_m} i_2 + \frac{N_2 N_1}{\mathscr{R}_m} i_1 \tag{1.2-15}$$

When the magnetic system is linear, the flux linkages are generally expressed in terms of inductances and currents. We see that the coefficients of the first two terms on the right-hand side of (1.2-14) depend upon the turns of coil 1 and the reluctance of the magnetic system, independent of the existence of coil 2. An analogous statement may be made regarding (1.2-15). Hence the self-inductances are defined as

$$L_{11} = \frac{N_1^2}{\mathscr{R}_{l1}} + \frac{N_1^2}{\mathscr{R}_m} = L_{l1} + L_{m1} \tag{1.2-16}$$

$$L_{22} = \frac{N_2^2}{\mathscr{R}_{l2}} + \frac{N_2^2}{\mathscr{R}_m} = L_{l2} + L_{m2} \tag{1.2-17}$$

where L_{l1} and L_{l2} are the leakage inductances and L_{m1} and L_{m2} are the magnetizing inductances of coils 1 and 2, respectively. From (1.2-16) and (1.2-17) it follows that the magnetizing inductances may be related as

$$\frac{L_{m2}}{N_2^2} = \frac{L_{m1}}{N_1^2} \tag{1.2-18}$$

The mutual inductances are defined as the coefficient of the third term of (1.2-14) and (1.2-15).

$$L_{12} = \frac{N_1 N_2}{\mathscr{R}_m} \tag{1.2-19}$$

$$L_{21} = \frac{N_2 N_1}{\mathscr{R}_m} \tag{1.2-20}$$

Obviously, $L_{12} = L_{21}$. The mutual inductances may be related to the magnetizing inductances. In particular,

$$L_{12} = \frac{N_2}{N_1} L_{m1} = \frac{N_1}{N_2} L_{m2} \tag{1.2-21}$$

The flux linkages may now be written as

$$\boldsymbol{\lambda} = \mathbf{Li} \tag{1.2-22}$$

where

$$\mathbf{L} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} L_{l1} + L_{m1} & \frac{N_2}{N_1} L_{m1} \\ \frac{N_1}{N_2} L_{m2} & L_{l2} + L_{m2} \end{bmatrix} \tag{1.2-23}$$

Although the voltage equations with the inductance matrix $\mathbf{L}$ incorporated may be used for purposes of analysis, it is customary to perform a change of variables that yields the well-known equivalent T circuit of two magnetically coupled coils. To set the stage for this derivation, let us express the flux linkages from (1.2-22) as

$$\lambda_1 = L_{l1} i_1 + L_{m1}\left(i_1 + \frac{N_2}{N_1} i_2\right) \tag{1.2-24}$$

$$\lambda_2 = L_{l2} i_2 + L_{m2}\left(\frac{N_1}{N_2} i_1 + i_2\right) \tag{1.2-25}$$

Now we have two choices. We can use a substitute variable for $(N_2/N_1)i_2$ or for $(N_1/N_2)i_1$. Let us consider the first of these choices

$$N_1 i_2' = N_2 i_2 \tag{1.2-26}$$

whereupon we are using the substitute variable i_2' that, when flowing through coil 1, produces the same MMF as the actual i_2 flowing through coil 2. This is said to be referring the current in coil 2 to coil 1 whereupon coil 1 becomes the reference coil. On the other hand, if we use the second choice, then

$$N_2 i_1' = N_1 i_1 \tag{1.2-27}$$

Here, i_1' is the substitute variable that produces the same MMF when flowing through coil 2 as i_1 does when flowing in coil 1. This change of variables is said to refer the current of coil 1 to coil 2.

We will demonstrate the derivation of the equivalent T circuit by referring the current of coil 2 to coil 1; thus from (1.2-26) we obtain

$$i_2' = \frac{N_2}{N_1} i_2 \tag{1.2-28}$$

Power is to be unchanged by this substitution of variables. Therefore,

$$v_2' = \frac{N_1}{N_2} v_2 \tag{1.2-29}$$

whereupon $v_2 i_2 = v_2' i_2'$. Flux linkages, which have the units of volt-second, are related to the substitute flux linkages in the same way as voltages. In particular,

$$\lambda_2' = \frac{N_1}{N_2} \lambda_2 \tag{1.2-30}$$

If we substitute (1.2-28) into (1.2-24) and (1.2-25) and then multiply (1.2-25) by N_1/N_2 to obtain λ_2' and we further substitute $(N_2^2/N_1^2)L_{m1}$ for L_{m2} into (1.2-24), then

$$\lambda_1 = L_{l1} i_1 + L_{m1}(i_1 + i_2') \tag{1.2-31}$$

$$\lambda_2' = L_{l2}' i_2' + L_{m1}(i_1 + i_2') \tag{1.2-32}$$

where

$$L_{l2}' = \left(\frac{N_1}{N_2}\right)^2 L_{l2} \tag{1.2-33}$$

The voltage equations become

$$v_1 = r_1 i_1 + \frac{d\lambda_1}{dt} \tag{1.2-34}$$

$$v_2' = r_2' i_2' + \frac{d\lambda_2'}{dt} \tag{1.2-35}$$

where

$$r_2' = \left(\frac{N_1}{N_2}\right)^2 r_2 \tag{1.2-36}$$

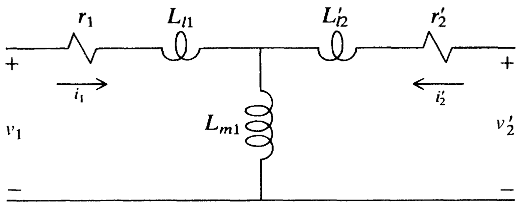

Figure 1.2-2 Equivalent circuit with coil 1 selected as the reference coil.

The above voltage equations suggest the T equivalent circuit shown in Fig. 1.2-2. It is apparent that this method may be extended to include any number of coils wound on the same core.

Example 1A It is instructive to illustrate the method of deriving an equivalent T circuit from open- and short-circuit measurements. For this purpose let us assume that when coil 2 of the two-winding transformer shown in Fig. 1.2-1 is open-circuited, the power input to coil 2 is 12 W with an applied voltage is 100 V (rms) at 60 Hz and the current is 1 A (rms). When coil 2 is short-circuited, the current flowing in coil 1 is 1 A when the applied voltage is 30 V at 60 Hz. The power during this test is 22 W. If we assume $L_{l1} = L'_{l2}$, an approximate equivalent T circuit can be determined from these measurements with coil 1 selected as the reference coil.

The power may be expressed as

$$P_1 = |\tilde{V}_1||\tilde{I}_1|\cos\phi \tag{1A-1}$$

where $\tilde{V}$ and $\tilde{I}$ are phasors and ϕ is the phase angle between $\tilde{V}_1$ and $\tilde{I}_1$ (power-factor angle). Solving for ϕ during the open-circuit test, we have

$$\phi = \cos^{-1}\frac{P_1}{|\tilde{V}_1||\tilde{I}_1|} = \cos^{-1}\frac{12}{110 \times 1} = 83.7^\circ \tag{1A-2}$$

With $\tilde{V}_1$ as the reference phasor and assuming an inductive circuit where $\tilde{I}_1$ lags $\tilde{V}_1$, we obtain

$$Z = \frac{\tilde{V}_1}{\tilde{I}_1} = \frac{110/\underline{0^\circ}}{1/\underline{-83.7^\circ}} = 12 + j109.3\,\Omega \tag{1A-3}$$

If we neglect hysteresis (core) losses, which in effect assumes a linear magnetic system, then $r_1 = 12\,\Omega$. We also know from the above calculation that $X_{l1} + X_{m1} = 109.3\,\Omega$.

For the short-circuit test we will assume that $i_1 = -i'_2$ because transformers are designed so that $X_{m1} \gg |r'_2 + jX'_{l2}|$. Hence using (1A-1) again we obtain

$$\phi = \cos^{-1}\frac{22}{30 \times 1} = 42.8^\circ \tag{1A-4}$$

In this case the input impedance is $(r_1 + r_2') + j(X_{l1} + X_{l2}')$. This may be determined as follows:

$$Z = \frac{30/\underline{0^\circ}}{1/\underline{-42.8^\circ}} = 22 + j20.4\,\Omega \tag{1A-5}$$

Hence, $r_2' = 10\,\Omega$ and, because it is assumed that $X_{l1} = X_{l2}'$, both are $10.2\,\Omega$. Therefore $X_{m1} = 109.3 - 10.2 = 99.1\,\Omega$. In summary,

$$r_1 = 12\,\Omega \qquad L_{m1} = 262.9\,\text{mH} \qquad r_2' = 10\,\Omega$$
$$L_{l1} = 27.1\,\text{mH} \qquad L_{l2}' = 27.1\,\text{mH}$$

Nonlinear Magnetic System

Although the analysis of transformers and electric machines is generally performed assuming a linear magnetic system, economics dictate that in the practical design of these devices some saturation occurs and that heating of the magnetic material exists due to hysteresis losses. The magnetization characteristics of transformer or machine materials are given in the form of the magnitude of flux density versus magnitude of field strength (B–H curve) as shown in Fig. 1.2-3. If it is assumed that the magnetic flux is uniform through most of the core, then B is proportional to Φ and H is proportional to MMF. Hence a plot of flux versus current is of the same shape as

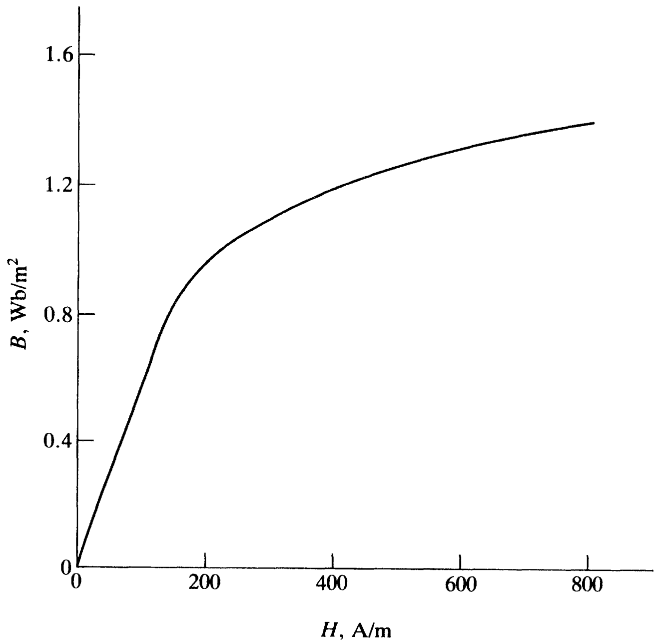

Figure 1.2-3 B–H curve for typical silicon steel used in transformers.

the B–H curve. A transformer is generally designed so that some saturation occurs during normal operation. Electric machines are also designed similarly in that a machine generally operates slightly in the saturated region during normal, rated operating conditions. Because saturation causes coefficients of the differential equations describing the behavior of an electromagnetic device to be functions of the coil currents, a transient analysis is difficult without the aid of a computer. Our purpose here is not to set forth methods of analyzing nonlinear magnetic systems. This procedure is quite straightforward for steady-state operation, but it cannot be used when analyzing the dynamics of electromechanical devices [1]. A method of incorporating the effects of saturation into a computer representation is of interest.

Computer Simulation of Coupled Circuits

Formulating the voltage equations of stationary coupled windings appropriate for computer simulation is straightforward and yet this technique is fundamental to the computer simulation of ac machines. Therefore it is to our advantage to consider this method here. For this purpose let us first write (1.2-31) and (1.2-32) as

$$\lambda_1 = L_{l1} i_1 + \lambda_m \tag{1.2-37}$$

$$\lambda_2' = L_{l2}' i_2' + \lambda_m \tag{1.2-38}$$

where

$$\lambda_m = L_{m1}(i_1 + i_2') \tag{1.2-39}$$

Solving (1.2-37) and (1.2-38) for the currents yields

$$i_1 = \frac{1}{L_{l1}}(\lambda_1 - \lambda_m) \tag{1.2-40}$$

$$i_2' = \frac{1}{L_{l2}'}(\lambda_2' - \lambda_m) \tag{1.2-41}$$

If (1.2-40) and (1.2-41) are substituted into the voltage equations (1.2-34) and (1.2-35) and we solve the resulting equations for flux linkages, the following equations are obtained:

$$\lambda_1 = \int \left[v_1 + \frac{r_1}{L_{l1}}(\lambda_m - \lambda_1) \right] dt \tag{1.2-42}$$

$$\lambda_2' = \int \left[v_2' + \frac{r_2'}{L_{l2}'}(\lambda_m - \lambda_2') \right] dt \tag{1.2-43}$$

Substituting (1.2-40) and (1.2-41) into (1.2-39) yields

$$\lambda_m = L_a \left(\frac{\lambda_1}{L_{l1}} + \frac{\lambda_2'}{L_{l2}'} \right) \tag{1.2-44}$$

where

$$L_a = \left(\frac{1}{L_{m1}} + \frac{1}{L_{l1}} + \frac{1}{L_{l2}} \right)^{-1} \tag{1.2-45}$$

We now have the equations expressed with λ_1 and λ_2' as state variables. In the computer simulation, (1.2-42) and (1.2-43) are used to solve for λ_1 and λ_2' and (1.2-44) is used to solve for λ_m. The currents can then be obtained from (1.2-40) and (1.2-41). It is clear that (1.2-44) could be substituted into (1.2-40)–(1.2-43) and that λ_m could be eliminated from the equations, whereupon it would not appear in the computer simulation. However, we will find λ_m (the mutual flux linkages) an important variable when we include the effects of saturation.

If the magnetization characteristics (magnetization curve) of the coupled winding is known, the effects of saturation of the mutual flux path may be readily incorporated into the computer simulation. Generally, the magnetization curve can be adequately determined from a test wherein one of the windings is open-circuited (winding 2, for example), and the input impedance of the other winding (winding 1) is determined from measurements as the applied voltage is increased in magnitude from zero to say 150% of the rated value. With information obtained from this type of test, we can plot λ_m versus $(i_1 + i_2')$ as shown in Fig. 1.2-4 wherein the slope of the linear portion of the curve is L_{m1}. From Fig. 1.2-4, it is clear that in the region of saturation we have

$$\lambda_m = L_{m1}(i_1 + i_2') - f(\lambda_m) \tag{1.2-46}$$

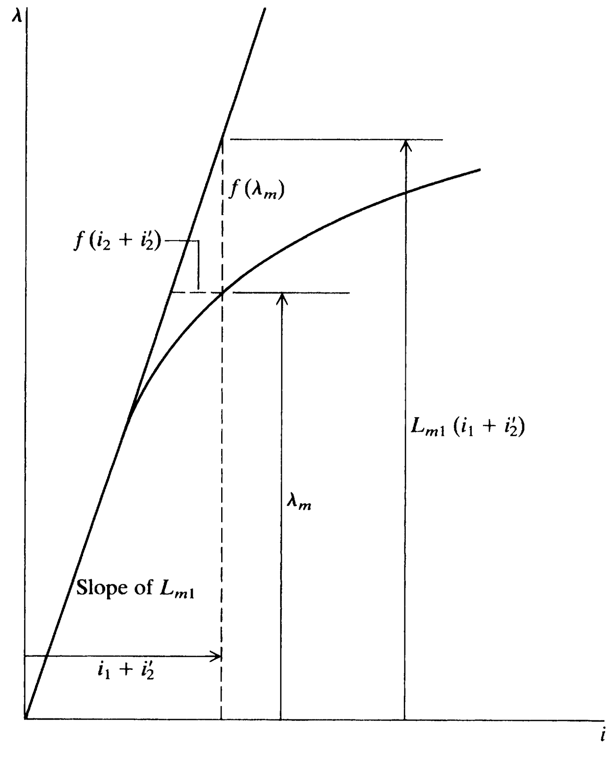

Figure 1.2-4 Magnetization curve.

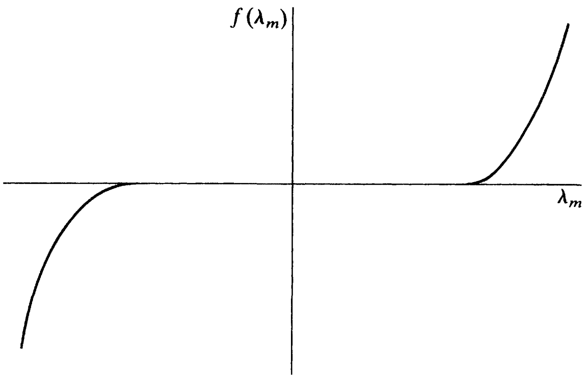

Figure 1.2-5 $f(\lambda_m)$ versus λ_m from Fig. 1.2-4.

where $f(\lambda_m)$ may be determined from the magnetization curve for each value of λ_m. In particular, $f(\lambda_m)$ is a function of λ_m as shown in Fig. 1.2-5. Therefore, the effects of saturation of the mutual flux path may be taken into account by replacing (1.2-39) with (1.2-46) for λ_m. Substituting (1.2-40) and (1.2-41) for i_1 and i'_2, respectively, into (1.2-46) yields the following computer equation for λ_m:

$$\lambda_m = L_a\left(\frac{\lambda_1}{L_{l1}} + \frac{\lambda'_2}{L'_{l2}}\right) - \frac{L_a}{L_{m1}} f(\lambda_m) \qquad (1.2\text{-}47)$$

Hence the computer simulation for including saturation involves replacing λ_m given by (1.2-44) with (1.2-47) where $f(\lambda_m)$ is a generated function of λ_m determined from the plot shown in Fig. 1.2-5.

1.3 ELECTROMECHANICAL ENERGY CONVERSION

Although electromechanical devices are used in some manner in a wide variety of systems, electric machines are by far the most common. It is desirable, however, to establish methods of analysis that may be applied to all electromechanical devices.

Energy Relationships

Electromechanical systems are comprised of an electrical system, a mechanical system, and a means whereby the electrical and mechanical systems can interact. Interaction can take place through any and all electromagnetic and electrostatic fields that are common to both systems, and energy is transferred from one system to the other as a result of this interaction. Both electrostatic and electromagnetic coupling fields may exist simultaneously, and the electromechanical system may have any number of electrical and mechanical systems. However, before considering an involved system it is helpful to analyze the electromechanical system in a simplified form. An electromechanical system with one electrical system, one mechanical system, and

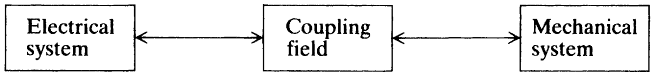

Figure 1.3-1 Block diagram of an elementary electromechanical system.

one coupling field is depicted in Fig. 1.3-1. Electromagnetic radiation is neglected, and it is assumed that the electrical system operates at a frequency sufficiently low so that the electrical system may be considered as a lumped parameter system.

Losses occur in all components of the electromechanical system. Heat loss will occur in the mechanical system due to friction, and the electrical system will dissipate heat due to the resistance of the current-carrying conductors. Eddy current and hysteresis losses occur in the ferromagnetic material of all magnetic fields, whereas dielectric losses occur in all electric fields. If W_E is the total energy supplied by the electrical source and W_M is the total energy supplied by the mechanical source, then the energy distribution could be expressed as

$$W_E = W_e + W_{eL} + W_{eS} \tag{1.3-1}$$

$$W_M = W_m + W_{mL} + W_{mS} \tag{1.3-2}$$

In (1.3-1), W_{eS} is the energy stored in the electric or magnetic fields that are not coupled with the mechanical system. The energy W_{eL} is the heat losses associated with the electrical system. These losses occur due to the resistance of the current-carrying conductors as well as the energy dissipated from these fields in the form of heat due to hysteresis, eddy currents, and dielectric losses. The energy W_e is the energy transferred to the coupling field by the electrical system. The energies common to the mechanical system may be defined in a similar manner. In (1.3-2), W_{mS} is the energy stored in the moving member and compliances of the mechanical system, W_{mL} is the energy losses of the mechanical system in the form of heat, and W_m is the energy transferred to the coupling field. It is important to note that with the convention adopted, the energy supplied by either source is considered positive. Therefore, $W_E(W_M)$ is negative when energy is supplied to the electrical source (mechanical source).

If W_F is defined as the total energy transferred to the coupling field, then

$$W_F = W_f + W_{fL} \tag{1.3-3}$$

where W_f is energy stored in the coupling field and W_{fL} is the energy dissipated in the form of heat due to losses within the coupling field (eddy current, hysteresis, or dielectric losses). The electromechanical system must obey the law of conservation of energy; thus

$$W_f + W_{fL} = (W_E - W_{eL} - W_{eS}) + (W_M - W_{mL} - W_{mS}) \tag{1.3-4}$$

which may be written

$$W_f + W_{fL} = W_e + W_m \tag{1.3-5}$$

This energy relationship is shown schematically in Fig. 1.3-2.

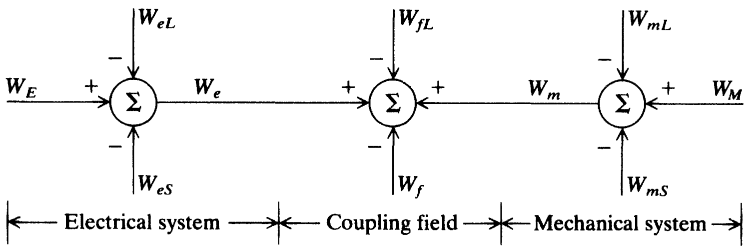

Figure 1.3-2 Energy balance.

The actual process of converting electrical energy to mechanical energy (or vice versa) is independent of (1) the loss of energy in either the electrical or the mechanical systems (W_{eL} and W_{mL}), (2) the energies stored in the electric or magnetic fields that are not common to both systems (W_{eS}), or (3) the energies stored in the mechanical system (W_{mS}). If the losses of the coupling field are neglected, then the field is conservative and (1.3-5) becomes

$$W_f = W_e + W_m \tag{1.3-6}$$

Examples of elementary electromechanical systems are shown in Figs. 1.3-3 and 1.3-4. The system shown in Fig. 1.3-3 has a magnetic coupling field while the electromechanical system shown in Fig. 1.3-4 employs an electric field as a means of transferring energy between the electrical and mechanical systems. In these systems, v is the voltage of the electric source and f is the external mechanical force applied to the mechanical system. The electromagnetic or electrostatic force is denoted by f_e. The resistance of the current-carrying conductors is denoted by r, and l is the inductance of a linear (conservative) electromagnetic system that does not couple the mechanical system. In the mechanical system, M is the mass of the movable member while the linear compliance and damper are represented by a spring constant K and a

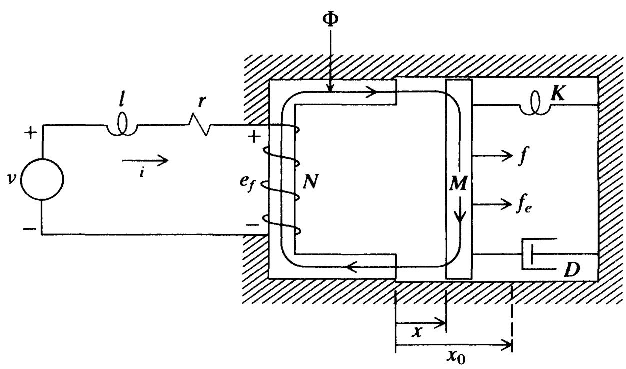

Figure 1.3-3 Electromechanical system with a magnetic field.

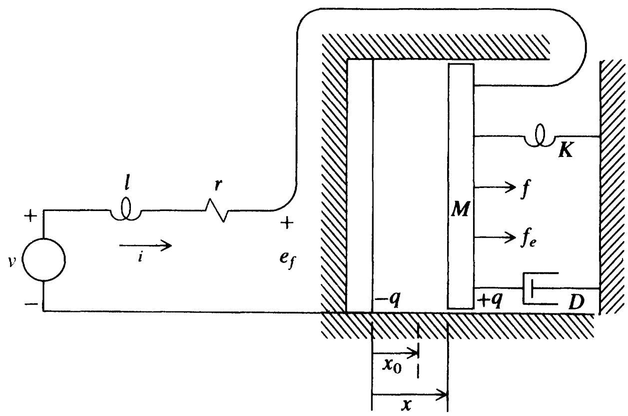

Figure 1.3-4 Electromechanical system with an electric field.

damping coefficient D, respectively. The displacement x_0 is the zero force or equilibrium position of the mechanical system which is the steady-state position of the mass with f_e and f equal to zero. A series or shunt capacitance may be included in the electrical system wherein energy would also be stored in an electric field external to the electromechanical process.

The voltage equation that describes both electrical systems may be written as

$$v = ri + l\frac{di}{dt} + e_f \tag{1.3-7}$$

where e_f is the voltage drop across the coupling field. The dynamic behavior of the translational mechanical system may be expressed by employing Newton's law of motion. Thus,

$$f = M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + K(x - x_0) - f_e \tag{1.3-8}$$

The total energy supplied by the electric source is

$$W_E = \int vi\, dt \tag{1.3-9}$$

The total energy supplied by the mechanical source is

$$W_M = \int f\, dx \tag{1.3-10}$$

which may also be expressed as

$$W_M = \int f\frac{dx}{dt}\, dt \tag{1.3-11}$$

Substituting (1.3-7) into (1.3-9) yields

$$W_E = r\int i^2 dt + l\int i\,di + \int e_f i\,dt \tag{1.3-12}$$

The first term on the right-hand side of (1.3-12) represents the energy loss due to the resistance of the conductors (W_{eL}). The second term represents the energy stored in the linear electromagnetic field external to the coupling field (W_{eS}). Therefore, the total energy transferred to the coupling field from the electrical system is

$$W_e = \int e_f i\,dt \tag{1.3-13}$$

Similarly, for the mechanical system we have

$$W_M = M\int \frac{d^2x}{dt^2}dx + D\int \left(\frac{dx}{dt}\right)^2 dt + K\int (x - x_0)\,dx - \int f_e\,dx \tag{1.3-14}$$

Here, the first and third terms on the right-hand side of (1.3-14) represent the energy stored in the mass and spring, respectively (W_{mS}). The second term is the heat loss due to friction (W_{mL}). Thus, the total energy transferred to the coupling field from the mechanical system is

$$W_m = -\int f_e\,dx \tag{1.3-15}$$

It is important to note that a positive force, f_e, is assumed to be in the same direction as a positive displacement, dx. Substituting (1.3-13) and (1.3-15) into the energy balance relation, (1.3-6), yields

$$W_f = \int e_f i\,dt - \int f_e\,dx \tag{1.3-16}$$

The equations set forth may be readily extended to include an electromechanical system with any number of electrical and mechanical inputs to any number of coupling fields. Considering the system shown in Fig. 1.3-5, the energy supplied to the coupling fields may be expressed as

$$W_f = \sum_{j=1}^{J} W_{ej} + \sum_{k=1}^{K} W_{mk} \tag{1.3-17}$$

wherein J electrical and K mechanical inputs exist. The total energy supplied to the coupling field at the electrical inputs is

$$\sum_{j=1}^{J} W_{ej} = \int \sum_{j=1}^{J} e_{fj} i_j\,dt \tag{1.3-18}$$

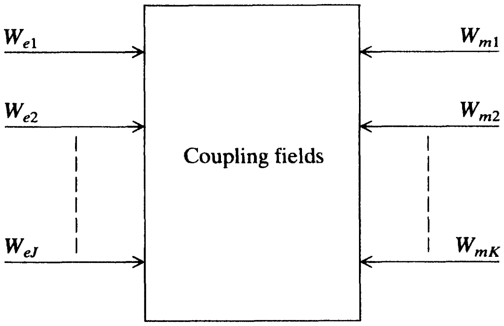

Figure 1.3-5 Multiple electrical and mechanical inputs.

The total energy supplied to the coupling field from the mechanical inputs is

$$\sum_{k=1}^{K} W_{mk} = -\int \sum_{k=1}^{K} f_{ek}\, dx_k \tag{1.3-19}$$

The energy balance equation becomes

$$W_f = \int \sum_{j=1}^{J} e_{fj} i_j\, dt - \int \sum_{k=1}^{K} f_{ek}\, dx_k \tag{1.3-20}$$

In differential form we obtain

$$dW_f = \sum_{j=1}^{J} e_{fj} i_j\, dt - \sum_{k=1}^{K} f_{ek}\, dx_k \tag{1.3-21}$$

Energy in Coupling Fields

Before using (1.3-21) to obtain an expression for the electromagnetic force f_e, it is necessary to derive an expression for the energy stored in the coupling fields. Once we have an expression for W_f, we can take the total derivative to obtain dW_f, which can then be substituted into (1.3-21). When expressing the energy in the coupling fields it is convenient to neglect all losses associated with the electric and magnetic fields whereupon the fields are assumed to be conservative and the energy stored therein is a function of the state of the electrical and mechanical variables. Although the effects of the field losses may be functionally accounted for by appropriately introducing a resistance in the electric circuit, this refinement is generally not necessary because the ferromagnetic material is selected and arranged in laminations so as to minimize the hysteresis and eddy current losses. Moreover, nearly all of the energy stored in the coupling fields is stored in the air gaps of the electromechanical device. Because air is a conservative medium, all of the energy stored therein can be returned to the electrical or mechanical systems. Therefore, the assumption of lossless coupling fields is not as restrictive as it might first appear.

The energy stored in a conservative field is a function of the state of the system variables and not the manner in which the variables reached that state. It is convenient to take advantage of this feature when developing a mathematical expression for the field energy. In particular, it is convenient to fix mathematically the position of the mechanical systems associated with the coupling fields and then excite the electrical systems with the displacements of the mechanical systems held fixed. During the excitation of the electrical systems, W_{mk} is zero even though electromagnetic or electrostatic forces occur. Therefore, with the displacements held fixed the energy stored in the coupling fields during the excitation of the electrical systems is equal to the energy supplied to the coupling fields by the electrical systems. Thus, with $W_{mk} = 0$, the energy supplied from the electrical system may be expressed from (1.3-20) as

$$W_f = \int \sum_{j=1}^{J} e_{fj} i_j \, dt \tag{1.3-22}$$

It is instructive to consider a singly excited electromagnetic system similar to that shown in Fig. 1.3-3. In this case, $e_f = d\lambda/dt$ and (1.3-22) becomes

$$W_f = \int i \, d\lambda \tag{1.3-23}$$

Here $J = 1$; however, the subscript is omitted for the sake of brevity. The area to the left of the λ–i relationship (shown in Fig. 1.3-6) for a singly excited electromagnetic device is the area described by (1.3-23). In Fig. 1.3-6, this area represents the energy stored in the field at the instant when $\lambda = \lambda_a$ and $i = i_a$. The λ–i relationship need not be linear; it need only be single-valued, a property that is characteristic to a conservative or lossless field. Moreover, because the coupling field is conservative, the energy stored in the field with $\lambda = \lambda_a$ and $i = i_a$ is independent of the excursion of the electrical and mechanical variables before reaching this state.

The area to the right of the λ–i curve is called the *coenergy* and is expressed as

$$W_c = \int \lambda \, di \tag{1.3-24}$$

which may also be written as

$$W_c = \lambda i - W_f \tag{1.3-25}$$

Although the coenergy has little or no physical significance, we will find it a convenient quantity for expressing the electromagnetic force. It should be clear that for a linear magnetic system where the λ–i plots are straight-line relationships, $W_f = W_c$.

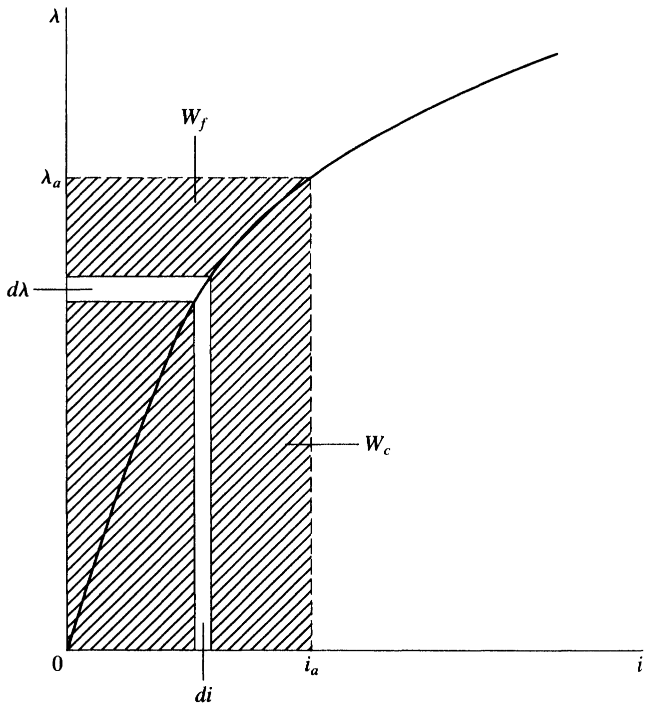

Figure 1.3-6 Stored energy and coenergy in a magnetic field of a singly excited electromagnetic device.

The displacement x defines completely the influence of the mechanical system upon the coupling field; however, because λ and i are related, only one is needed in addition to x in order to describe the state of the electromechanical system. Therefore, either λ and x or i and x may be selected as independent variables. If i and x are selected as independent variables, it is convenient to express the field energy and the flux linkages as

$$W_f = W_f(i, x) \tag{1.3-26}$$

$$\lambda = \lambda(i, x) \tag{1.3-27}$$

With i and x as independent variables we must express $d\lambda$ in terms of di before substituting into (1.3-23). Thus from (1.3-27)

$$d\lambda(i, x) = \frac{\partial\lambda(i, x)}{\partial i}\,di + \frac{\partial\lambda(i, x)}{\partial x}\,dx \tag{1.3-28}$$

In the derivation of an expression for the energy stored in the field, dx is set equal to zero. Hence, in the evaluation of field energy, $d\lambda$ is equal to the first term on the right-hand side of (1.3-28). Substituting into (1.3-23) yields

$$W_f(i,x) = \int i \frac{\partial \lambda(i,x)}{\partial i}\, di = \int_0^i \xi \frac{\partial \lambda(\xi,x)}{\partial \xi}\, d\xi \tag{1.3-29}$$

where ξ is the dummy variable of integration. Evaluation of (1.3-29) gives the energy stored in the field of the singly excited system. The coenergy in terms of i and x may be evaluated from (1.3-24) as

$$W_c(i,x) = \int \lambda(i,x)\, di = \int_0^i \lambda(\xi,x)\, d\xi \tag{1.3-30}$$

With λ and x as independent variables we have

$$W_f = W_f(\lambda,x) \tag{1.3-31}$$

$$i = i(\lambda,x) \tag{1.3-32}$$

The field energy may be evaluated from (1.3-23) as

$$W_f(\lambda,x) = \int i(\lambda,x)\, d\lambda = \int_0^\lambda i(\xi,x)\, d\xi \tag{1.3-33}$$

In order to evaluate the coenergy with λ and x as independent variables, we need to express di in terms of $d\lambda$; thus from (1.3-32) we obtain

$$di(\lambda,x) = \frac{\partial i(\lambda,x)}{\partial \lambda}\, d\lambda + \frac{\partial i(\lambda,x)}{\partial x}\, dx \tag{1.3-34}$$

Because $dx = 0$ in this evaluation, (1.3-24) becomes

$$W_c(\lambda,x) = \int \lambda \frac{\partial i(\lambda,x)}{\partial \lambda}\, d\lambda = \int_0^\lambda \xi \frac{\partial i(\xi,x)}{\partial \xi}\, d\xi \tag{1.3-35}$$

For a linear electromagnetic system the λ–i plots are straight-line relationships; thus for the singly excited system we have

$$\lambda(i,x) = L(x)i \tag{1.3-36}$$

or

$$i(\lambda,x) = \frac{\lambda}{L(x)} \tag{1.3-37}$$

Let us evaluate $W_f(i,x)$. From (1.3-28), with $dx = 0$

$$d\lambda(i,x) = L(x)\,di \tag{1.3-38}$$

Hence, from (1.3-29)

$$W_f(i,x) = \int_0^i \xi L(x)\,d\xi = \frac{1}{2}L(x)\,i^2 \tag{1.3-39}$$

It is left to the reader to show that $W_f(\lambda,x)$, $W_c(i,x)$, and $W_c(\lambda,x)$ are equal to (1.3-39) for this magnetically linear system.

The field energy is a state function, and the expression describing the field energy in terms of system variables is valid regardless of the variations in the system variables. For example, (1.3-39) expresses the field energy regardless of the variations in $L(x)$ and i. The fixing of the mechanical system so as to obtain an expression for the field energy is a mathematical convenience and not a restriction upon the result.

In the case of a multiexcited, electromagnetic system, an expression for the field energy may be obtained by evaluating the following relation with $dx_k = 0$:

$$W_f = \int \sum_{j=1}^{J} i_j\,d\lambda_j \tag{1.3-40}$$

Because the coupling fields are considered conservative, (1.3-40) may be evaluated independently of the order in which the flux linkages or currents are brought to their final values. To illustrate the evaluation of (1.3-40) for a multiexcited system, we will allow the currents to establish their final states one at a time while all other currents are mathematically fixed in either their final or unexcited state. This procedure may be illustrated by considering a doubly excited electric system with one mechanical input. An electromechanical system of this type could be constructed by placing a second coil, supplied from a second electrical system, on either the stationary or movable member of the system shown in Fig. 1.3-3. In this evaluation it is convenient to use currents and displacement as the independent variables. Hence, for a doubly excited electric system we have

$$W_f(i_1,i_2,x) = \int [i_1\,d\lambda_1(i_1,i_2,x) + i_2\,d\lambda_2(i_1,i_2,x)] \tag{1.3-41}$$

In this determination of an expression for W_f, the mechanical displacement is held constant ($dx = 0$); thus (1.3-41) becomes

$$\begin{aligned} W_f(i_1,i_2,x) = \int i_1 &\left[\frac{\partial\lambda_i(i_1,i_2,x)}{\partial i_1}\,di_1 + \frac{\partial\lambda_1\,(i_1,i_2,x)}{\partial i_2}\,di_2\right] \\ + i_2 &\left[\frac{\partial\lambda_2\,(i_1,i_2,x)}{\partial i_1}\,di_1 + \frac{\partial\lambda_2(i_1,i_2,x)}{\partial i_2}\,di_2\right] \end{aligned} \tag{1.3-42}$$

We will evaluate the energy stored in the field by employing (1.3-42) twice. First we will mathematically bring the current i_1 to the desired value while holding i_2 at zero. Thus, i_1 is the variable of integration and $di_2 = 0$. Energy is supplied to the coupling field from the source connected to coil 1. As the second evaluation of (1.3-42), i_2 is brought to its desired current while holding i_1 at its desired value. Hence, i_2 is the variable of integration and $di_1 = 0$. During this time, energy is supplied from both sources to the coupling field because $i_1 d\lambda_1$ is nonzero. The total energy stored in the coupling field is the sum of the two evaluations. Following this two-step procedure the evaluation of (1.3-42) for the total field energy becomes

$$\begin{aligned} W_f(i_1, i_2, x) &= \int i_1 \frac{\partial \lambda_i(i_1, i_2, x)}{\partial i_1} \, di_1 \\ &\quad + \int \left[i_1 \frac{\partial \lambda_1(i_1, i_2, x)}{\partial i_2} \, di_2 + i_2 \frac{\partial \lambda_2(i_1, i_2, x)}{\partial i_2} \, di_2 \right] \end{aligned} \tag{1.3-43}$$

which should be written as

$$\begin{aligned} W_f(i_1, i_2, x) &= \int_0^{i_1} \xi \frac{\partial \lambda_1(\xi, i_2, x)}{\partial \xi} \, d\xi \\ &\quad + \int_0^{i_2} \left[i_1 \frac{\partial \lambda_1(i_1, \xi, x)}{\partial \xi} \, d\xi + \xi \frac{\partial \lambda_2(i_1, \xi, x)}{\partial \xi} \, d\xi \right] \end{aligned} \tag{1.3-44}$$

The first integral on the right-hand side of (1.3-43) or (1.3-44) results from the first step of the evaluation with i_1 as the variable of integration and with $i_2 = 0$ and $di_2 = 0$. The second integral comes from the second step of the evaluation with $i_1 = i_1, di_1 = 0$ and i_2 as the variable of integration. It is clear that the order of allowing the currents to reach their final state is irrelevant; that is, as our first step, we could have made i_2 the variable of integration while holding i_1 at zero $(di_1 = 0)$ and then let i_1 become the variable of integration while holding i_2 at its final variable. The results would be the same. It is also clear that for three electrical inputs the evaluation procedure would require three steps, one for each current to be brought mathematically to its final state.

Let us now evaluate the energy stored in a magnetically linear electromechanical system with two electrical inputs and one mechanical input. For this let

$$\lambda_1(i_1, i_2, x) = L_{11}(x) i_1 + L_{12}(x) i_2 \tag{1.3-45}$$

$$\lambda_2(i_1, i_2, x) = L_{21}(x) i_1 + L_{22}(x) i_2 \tag{1.3-46}$$

With that mechanical displacement held constant $(dx = 0)$, we obtain

$$d\lambda_1(i_1, i_2, x) = L_{11}(x) \, di_1 + L_{12}(x) \, di_2 \tag{1.3-47}$$

$$d\lambda_2(i_1, i_2, x) = L_{12}(x) \, di_1 + L_{22}(x) \, di_2 \tag{1.3-48}$$

It is clear that the coefficients on the right-hand side of (1.3-47) and (1.3-48) are the partial derivatives. For example, $L_{11}(x)$ is the partial derivative of $\lambda_1(i_1, i_2, x)$ with respect to i_1. Appropriate substitution into (1.3-44) gives

$$W_f(i_1, i_2, x) = \int_0^{i_1} \xi L_{11}(x)\, d\xi + \int_0^{i_2} [i_1 L_{12}(x) + \xi L_{22}(x)]\, d\xi \tag{1.3-49}$$

which yields

$$W_f(i_1, i_2, x) = \frac{1}{2} L_{11}(x) i_1^2 + L_{12}(x) i_1 i_2 + \frac{1}{2} L_{22}(x) i_2^2 \tag{1.3-50}$$

The extension to a linear electromagnetic system with J electrical inputs is straightforward whereupon the following expression for the total field energy is obtained:

$$W_f(i_1, \ldots, i_J, x) = \frac{1}{2} \sum_{p=1}^{J} \sum_{q=1}^{J} L_{pq} i_p i_q \tag{1.3-51}$$

It is left to the reader to show that the equivalent of (1.3-22) for a multiexcited electrostatic system is

$$W_f = \int \sum_{j=1}^{J} e_{fj}\, dq_j \tag{1.3-52}$$

Graphical Interpretation of Energy Conversion

Before proceeding to the derivation of expressions for the electromagnetic force, it is instructive to consider briefly a graphical interpretation of the energy conversion process. For this purpose let us again refer to the elementary system shown in Fig. 1.3-3 and let us assume that as the movable member moves from $x = x_a$ to $x = x_b$, where $x_b < x_a$, the λ–i characteristics are given by Fig. 1.3-7. Let us further assume that as the member moves from x_a to x_b the λ–i trajectory moves from point A to point B. It is clear that the exact trajectory from A to B is determined by the combined dynamics of the electrical and mechanical systems. Now, the area $OACO$ represents the original energy stored in field; area $OBDO$ represents the final energy stored in the field. Therefore, the change in field energy is

$$\Delta W_f = \text{area } OBDO - \text{area } OACO \tag{1.3-53}$$

The change in W_e, denoted as ΔW_e, is

$$\Delta W_e = \int_{\lambda_A}^{\lambda_B} i\, d\lambda = \text{area } CABDC \tag{1.3-54}$$

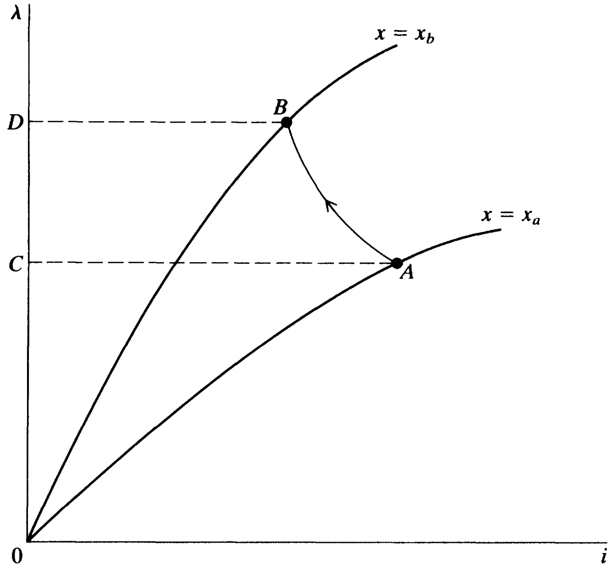

Figure 1.3-7 Graphical representation of electromechanical energy conversion for λ–i path A to B.

We know that

$$\Delta W_m = \Delta W_f - \Delta W_e \tag{1.3-55}$$

Hence,

$$\Delta W_m = \text{area } OBDO - \text{area } OACO - \text{area } CABDC = -\text{area } OABO \tag{1.3-56}$$

The change in W_m, denoted as ΔW_m, is negative; energy has been supplied to the mechanical system from the coupling field part of which came from the energy stored in the field and part from the electrical system. If the member is now moved back to x_a, the λ–i trajectory may be as shown in Fig. 1.3-8. Hence the ΔW_m is still area $OABO$ but it is positive, which means that energy was supplied from the mechanical system to the coupling field, part of which is stored in the field and part of which is transferred to the electrical system. The net ΔW_m for the cycle from A to B back to A is the shaded area shown in Fig. 1.3-9. Because ΔW_f is zero for this cycle

$$\Delta W_m = -\Delta W_e \tag{1.3-57}$$

For the cycle shown the net ΔW_e is negative, thus ΔW_m is positive; we have generator action. If the trajectory had been in the counterclockwise direction, the net ΔW_e would have been positive and the net ΔW_m would have been negative, which would represent motor action.

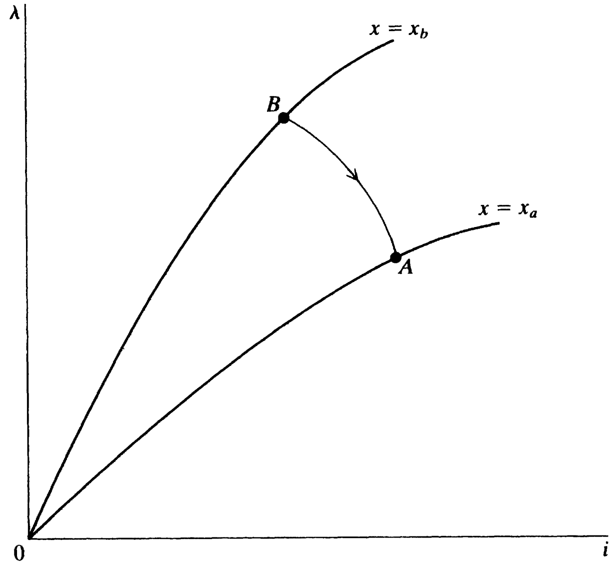

Figure 1.3-8 Graphical representation of electromechanical energy conversion for λ–i path B to A.

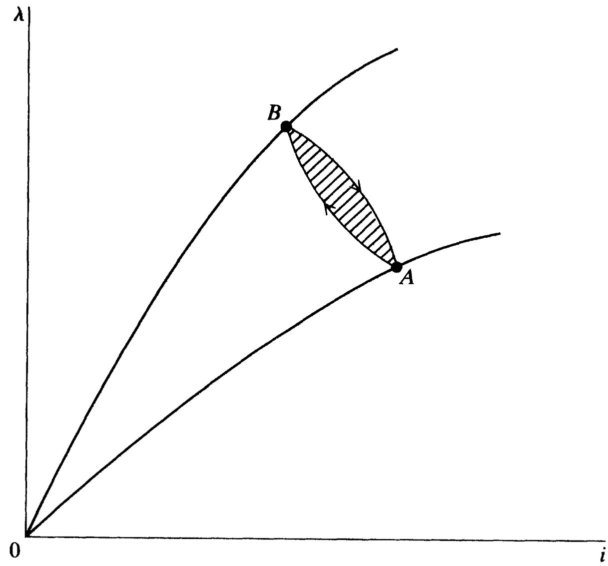

Figure 1.3-9 Graphical representation of electromechanical energy conversion for λ–i path A to B to A.

Electromagnetic and Electrostatic Forces

The energy balance relationships given by (1.3-21) may be arranged as

$$\sum_{k=1}^{K} f_{ek}\,dx_k = \sum_{j=1}^{J} e_{fj}\,i_j\,dt - dW_f \tag{1.3-58}$$

In order to obtain an expression for f_{ek}, it is necessary to first express W_f and then take its total derivative. One is tempted to substitute the integrand of (1.3-22) into (1.3-58) for the infinitesimal change of field energy. This procedure is, of course, incorrect because the integrand of (1.3-22) was obtained with all mechanical displacements held fixed ($dx_k = 0$), where the total differential of the field energy is required in (1.3-58).

The force or torque in any electromechanical system may be evaluated by employing (1.3-58). In many respects, one gains a much better understanding of the energy conversion process of a particular system by starting the derivation of the force or torque expressions with (1.3-58) rather than selecting a relationship from a table. However, for the sake of completeness, derivation of the force equations will be set forth and tabulated for electromechanical systems with K mechanical inputs and J electrical inputs [2].

For an electromagnetic system, (1.3-58) may be written as

$$\sum_{k=1}^{K} f_{ek}\,dx_k = \sum_{j=1}^{J} i_j\,d\lambda_j - dW_f \tag{1.3-59}$$

With i_j and x_k selected as independent variables we have

$$W_f = W_f(i_1, ..., i_J; x_1, ..., x_K) \tag{1.3-60}$$

$$\lambda_j = \lambda_j(i_1, ..., i_J; x_1, ..., x_K) \tag{1.3-61}$$

From (1.3-60) and (1.3-61) we obtain

$$dW_f = \sum_{j=1}^{J} \frac{\partial W_f(i_j, x_k)}{\partial i_j}\,di_j + \sum_{k=1}^{K} \frac{\partial W_f(i_j, x_k)}{\partial x_k}\,dx_k \tag{1.3-62}$$

$$d\lambda_j = \sum_{n=1}^{J} \frac{\partial \lambda_j(i_j, x_k)}{\partial i_n}\,di_n + \sum_{k=1}^{K} \frac{\partial \lambda_j(i_j, x_k)}{\partial x_k}\,dx_k \tag{1.3-63}$$

In (1.3-62) and (1.3-63) and hereafter in this development the functional notation of $(i_1, \ldots, i_J; x_1, \ldots, x_K)$ is abbreviated as (i_j, x_k). The index n is used so as to avoid confusion with the index j because each $d\lambda_j$ must be evaluated for changes in all currents in order to account for mutual coupling between electrical systems.

[Recall that we did this in (1.3-42) for $J = 2$.] Substituting (1.3-62) and (1.3-63) into (1.3-59) yields

$$\sum_{k=1}^{K} f_{ek}(i_j, x_k)\, dx_k = \sum_{j=1}^{J} i_j \left[\sum_{n=1}^{J} \frac{\partial \lambda_j(i_j, x_k)}{\partial i_n} di_n + \sum_{k=1}^{K} \frac{\partial \lambda_j(i_j, x_k)}{\partial x_k} dx_k \right] - \sum_{j=1}^{J} \frac{\partial W_f(i_j, x_k)}{\partial i_j} di_j - \sum_{k=1}^{K} \frac{\partial W_f(i_j, x_k)}{\partial x_k} dx \tag{1.3-64}$$

By gathering terms, we obtain

$$\sum_{k=1}^{K} f_{ek}(i_j, x_k) dx_k = \sum_{k=1}^{K} \left[\sum_{j=1}^{J} i_j \frac{\partial \lambda_j(i_j, x_k)}{\partial x_k} - \frac{\partial W_f(i_j, x_k)}{\partial x_k} \right] dx_k + \sum_{j=1}^{J} \left[\sum_{n=1}^{J} i_j \frac{\partial \lambda_j(i_j, x_k)}{\partial i_n} di_n - \frac{\partial W_f(i_j, x_k)}{\partial i_j} di_j \right] \tag{1.3-65}$$

When we equate coefficients, we obtain

$$f_{ek}(i_j, x_k) = \sum_{j=1}^{J} i_j \frac{\partial \lambda_j(i_j, x_k)}{\partial x_k} - \frac{\partial W_f(i_j, x_k)}{\partial x_k} \tag{1.3-66}$$

$$0 = \sum_{j=1}^{J} \left[\sum_{n=1}^{J} i_j \frac{\partial \lambda_j(i_j, x_k)}{\partial i_n} di_n - \frac{\partial W_f(i_j, x_k)}{\partial i_j} di_j \right] \tag{1.3-67}$$

Although (1.3-67) is of little practical importance, (1.3-66) can be used to evaluate the force at the kth mechanical terminal of an electromechanical system with only magnetic coupling fields and with i_j and x_k selected as independent variables. A second force equation with i_j and x_k as independent variables may be obtained from (1.3-66) by incorporating the expression for coenergy. For a multiexcited system the coenergy may be expressed as

$$W_c = \sum_{j=1}^{J} i_j \lambda_j - W_f \tag{1.3-68}$$

Because i_j and x_k are independent variables, the partial derivative with respect to x is

$$\frac{\partial W_c(i_j, x_k)}{\partial x_k} = \sum_{j=1}^{J} i_j \frac{\partial \lambda_j(i_j, x_k)}{\partial x_k} - \frac{\partial W_f(i_j, x_k)}{\partial x_k} \tag{1.3-69}$$

Hence, substituting (1.3-69) into (1.3-66) yields

$$f_{ek}(i_j, x_k) = \frac{\partial W_c(i_j, x_k)}{\partial x_k} \tag{1.3-70}$$

Table 1.3-1 Electromagnetic Force at kth Mechanical Input[a]

$$f_{ek}(i_j, x_k) = \sum_{j=1}^{J} \left[i_j \frac{\partial \lambda_j(i_j, x_k)}{\partial x_k} \right] - \frac{\partial W_f(i_j, x_k)}{\partial x_k}$$

$$f_{ek}(i_j, x_k) = \frac{\partial W_c(i_j, x_k)}{\partial x_k}$$

$$f_{ek}(\lambda_j, x_k) = -\frac{\partial W_f(\lambda_j, x_k)}{\partial x_k}$$

$$f_{ek}(\lambda_j, x_k) = -\sum_{j=1}^{J} \left[\lambda_j \frac{\partial i_j(\lambda_j, x_k)}{\partial x_k} \right] + \frac{\partial W_c(\lambda_j, x_k)}{\partial x_k}$$

[a]For rotational systems replace f_{ek} with T_{ek} and x_k with θ_k.

It should be recalled that positive f_{ek} and positive dx_k are in the same direction. Also, if the magnetic system is linear, then $W_c = W_f$.

By a procedure similar to that used above, force equations may be developed for magnetic coupling with λ_j and x_k as independent variables. These relations are given in Table 1.3-1 without proof. In Table 1.3-1 the independent variables to be used are designated in each equation by the abbreviated functional notation. Although only translational mechanical systems have been considered, all force relationships developed herein may be modified for the purpose of evaluating the torque in rotational systems. In particular, when considering a rotational system, f_{ek} is replaced with the electromagnetic torque T_{ek}, and x_k is replaced with the angular displacement θ_k. These substitutions are justified because the change of mechanical energy in a rotational system is expressed as

$$dW_{mk} = -T_{ek}\, d\theta_k \tag{1.3-71}$$

The force equation for an electromechanical system with electric coupling fields may be derived by following a procedure similar to that used in the case of magnetic coupling fields. These relationships are given in Table 1.3-2 without explanation.

It is instructive to derive the expression for the electromagnetic force of a singly excited electrical system as shown in Fig. 1.3-3. It is clear that the expressions given in Table 1.3-1 are valid for magnetically linear or nonlinear systems. If we assume that the magnetic system is linear, then $\lambda(i, x)$ is expressed by (1.3-36) and $W_f(i, x)$ is expressed by (1.3-39), which is also equal to the coenergy. Hence, either the first or second entry of Table 1.3-1 can be used to express f_e. In particular,

$$f_e(i, x) = \frac{\partial W_c(i, x)}{\partial x} = \frac{1}{2} i^2 \frac{dL(x)}{dx} \tag{1.3-72}$$

Table 1.3-2 Electrostatic Force at kth Mechanical Input[a]

$$f_{ek}(e_{fj}, x_k) = \sum_{j=1}^{J} \left[e_{fj} \frac{\partial q_j(e_{fj}, x_k)}{\partial x_k} \right] - \frac{\partial W_f(e_{fj}, x_k)}{\partial x_k}$$

$$f_{ek}(e_{fj}, x_k) = \frac{\partial W_c(e_{fj}, x_k)}{\partial x_k}$$

$$f_{ek}(q_j, x_k) = -\frac{\partial W_f(q_j, x_k)}{\partial x_k}$$

$$f_{ek}(q_j, x_k) = -\sum_{j=1}^{J} \left[e_j \frac{\partial e_{fj}(q_j, x_k)}{\partial x_k} \right] + \frac{\partial W_c(q_j, x_k)}{\partial x_k}$$

[a]For rotational systems replace f_{ek} with T_{ek} and x_k with θ_k.

With the convention established, a positive electromagnetic force is assumed to act in the direction of increasing x. Thus with (1.3-15) expressed in differential form as

$$dW_m = -f_e dx \tag{1.3-73}$$

we see that energy is supplied to the coupling field from the mechanical system when f_e and dx are opposite in sign, and energy is supplied to the mechanical system from the coupling field when f_e and dx are the same in sign.

From (1.3-72) it is apparent that when the change of $L(x)$ with respect to x is negative, f_e is negative. In the electromechanical system shown in Fig. 1.3-3 the change $L(x)$ with respect to x is always negative; therefore, the electromagnetic force is in the direction so as to pull the movable member to the stationary member. In other words, an electromagnetic force is set up so as to maximize the inductance of the coupling system, or, since reluctance is inversely proportional to the inductance, the force tends to minimize the reluctance. Because f_e is always negative in the system shown in Fig. 1.3-3, energy is supplied to the coupling field from the mechanical system (generator action) when dx is positive and is supplied from the coupling field to the mechanical system (motor action) when dx is negative.

Steady-State and Dynamic Performance of an Electromechanical System

It is instructive to consider the steady-state and dynamic performance of the elementary electromagnetic system shown in Fig. 1.3-3. The differential equations that describe this system are given by (1.3-7) for the electrical system and by (1.3-8) for the mechanical system. The electromagnetic force f_e is expressed by (1.3-72). If the applied voltage, v, and the applied mechanical force, f, are constant, all derivatives with respect to time are zero during steady-state operation, and the behavior can be predicted by

$$v = ri \tag{1.3-74}$$

$$f = K(x - x_0) - f_e \tag{1.3-75}$$

Equation (1.3-75) may be written as

$$-f_e = f - K(x - x_0) \tag{1.3-76}$$

The magnetic core of the system in Fig. 1.3-3 is generally constructed of ferromagnetic material with a relative permeability on the order of 2000 to 4000. In this case the inductance $L(x)$ can be adequately approximated by

$$L(x) = \frac{k}{x} \tag{1.3-77}$$

In the actual system the inductance will be a large finite value rather than infinity, as predicted by (1.3-77), when $x = 0$. Nevertheless, (1.3-77) is quite sufficient to illustrate the action of the system for $x > 0$. Substituting (1.3-77) into (1.3-72) yields

$$f_e(i, x) = -\frac{ki^2}{2x^2} \tag{1.3-78}$$

A plot of (1.3-76), with f_e replaced by (1.3-78), is shown in Fig. 1.3-10 for the following system parameters:

$$r = 10\,\Omega \qquad x_0 = 3\,\text{mm}$$
$$K = 2667\,\text{N/m} \qquad k = 6.293 \times 10^{-5}\,\text{H}\cdot\text{m}$$

In Fig. 1.3-10, the plot of the negative of the electromagnetic force is for an applied voltage of 5 V and a steady-state current of 0.5 A. The straight lines represent the right-hand side of (1.3-76) with $f = 0$ (lower straight line) and $f = 4\,\text{N}$ (upper straight line). Both lines intersect the $-f_e$ curve at two points. In particular, the upper line intersects the $-f_e$ curve at 1 and 1′; the lower line intersects at 2 and 2′. Stable operation occurs at only points 1 and 2. The system will not operate stably at points 1′ and 2′. This can be explained by assuming that the system is operating at one of these points (1′ and 2′) and then showing that any system disturbance whatsoever will cause the system to move away from these points. If, for example, x increases slightly from its value corresponding to point 1′, the restraining force $f - K(x - x_0)$ is larger in magnitude than $-f_e$, and x will continue to increase until the system reaches operating point 1. If x increases beyond its value corresponding to operating point 1, the restraining force is less than the electromagnetic force. Therefore, the system will establish steady-state operation at 1. If, on the other hand, x decreases

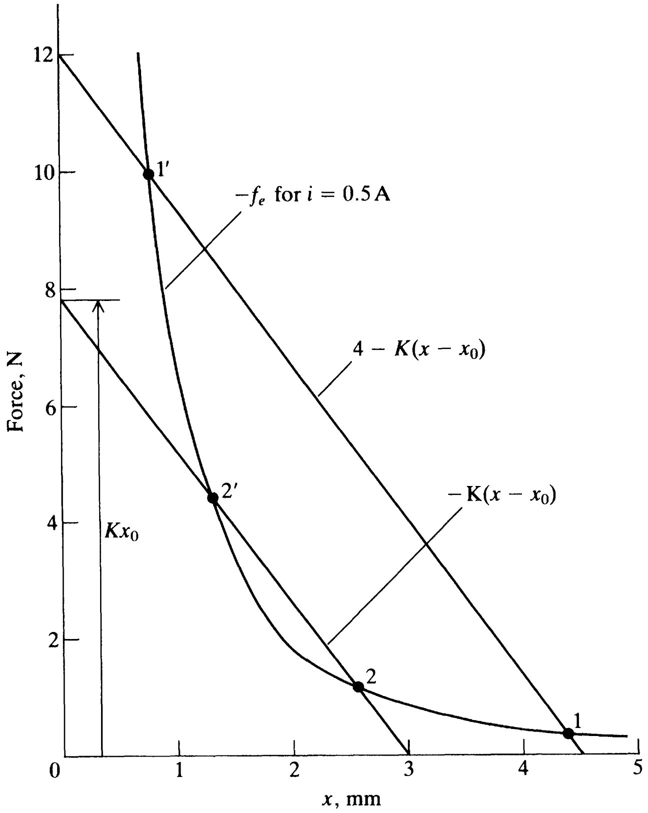

Figure 1.3-10 Steady-state operation of electromechanical system shown in Fig. 1.3-3.

from point 1′, the electromagnetic force is larger than the restraining force. Therefore, the movable member will move until it comes in contact with the stationary member ($x = 0$). It is clear that the restraining force that yields a straight line below the $-f_e$ curve will not permit stable operation with $x > 0$.

The dynamic behavior of the system during step changes in the source voltage v is shown in Fig. 1.3-11, and Figs. 1.3-12 and 1.3-13 show dynamic behavior during step changes in the applied force f. The following system parameters were used in addition to those given previously:

$$l = 0 \qquad M = 0.055\,\text{kg} \qquad D = 4\,\text{N}\cdot\text{s/m}$$

The computer traces shown in Fig. 1.3-11 depict the dynamic performance of the example system when the applied voltage is stepped from zero to 5 V and then back to zero with the applied mechanical force held equal to zero. The system variables are e_f, λ, i, f_e, x, W_e, W_f, and W_m. The energies are plotted in millijoules (mJ). Initially the mechanical system is at rest with $x = x_0$ (3 mm). When the source

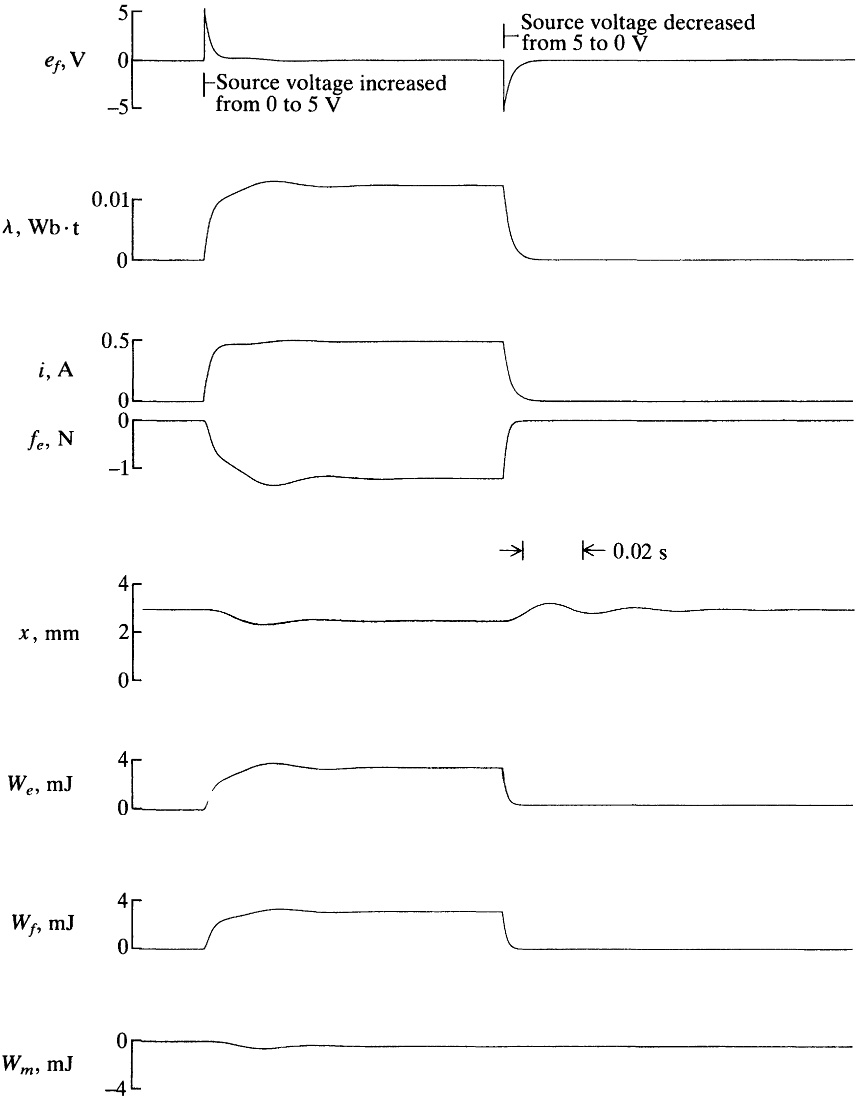

Figure 1.3-11 Dynamic performance of the electromechanical system shown in Fig. 1.3-3 during step changes in the source voltage.

voltage is applied, x decreases; and when steady-state operation is reestablished, x is approximately 2.5 mm. Energy enters the coupling field via W_e. The bulk of this energy is stored in the field (W_f) with a smaller amount transferred to the mechanical system, some of which is dissipated in the damper during the transient period while the remainder is stored in the spring. When the applied voltage is removed, the electrical and mechanical systems return to their original states. The change in W_m

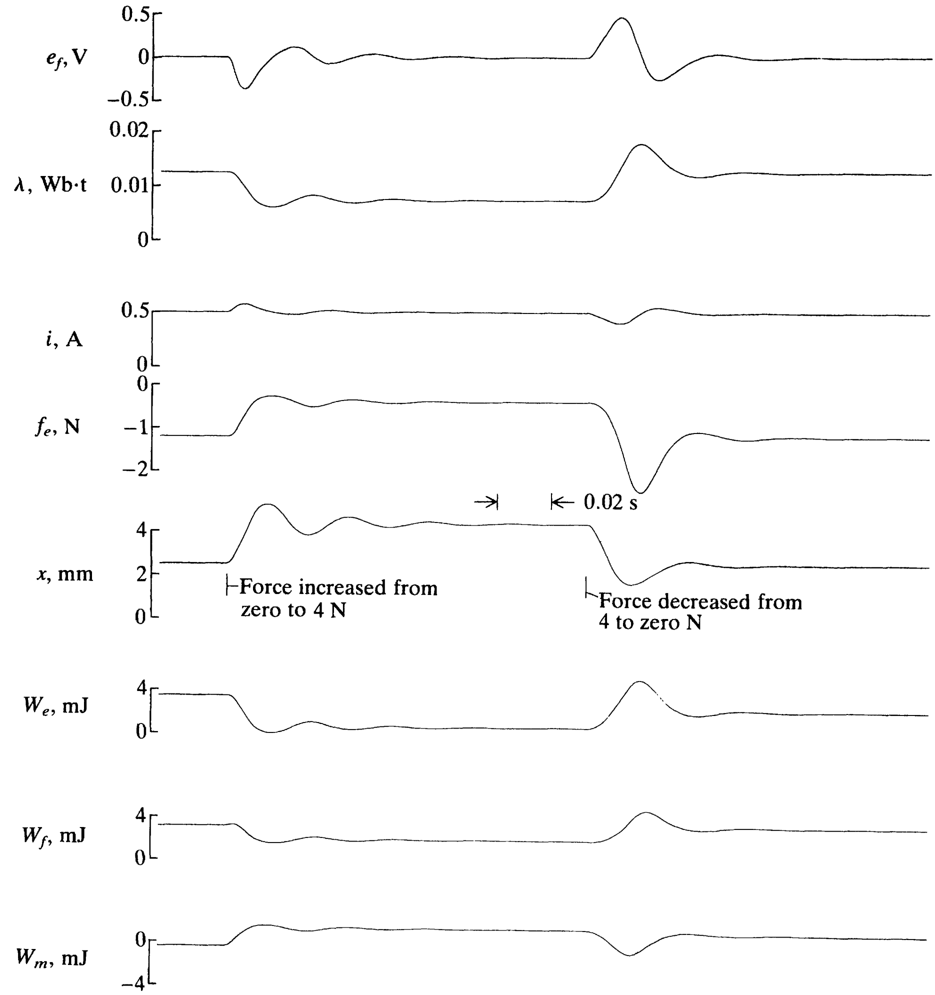

Figure 1.3-12 Dynamic performance of the electromechanical system shown in Fig. 1.3-3 during step changes in the applied force.

is small, increasing only slightly. Hence, during the transient period there is an interchange of energy between the spring and mass which is finally dissipated in the damper. The net change in W_f during the application and removal of the applied voltage is zero; hence the net change in W_e is positive and equal to the negative of the net change in W_m. The energy transferred to the mechanical system during this cycle is dissipated in the damper, because f is fixed at zero, and the mechanical system returns to its initial rest position with zero energy stored in the spring.

In Fig. 1.3-12, the initial state is that shown in Fig. 1.3-11 with 5 V applied to the electrical system. The mechanical force f is increased from zero to 4 N, whereupon energy enters the coupling field from the mechanical system. Energy is transferred

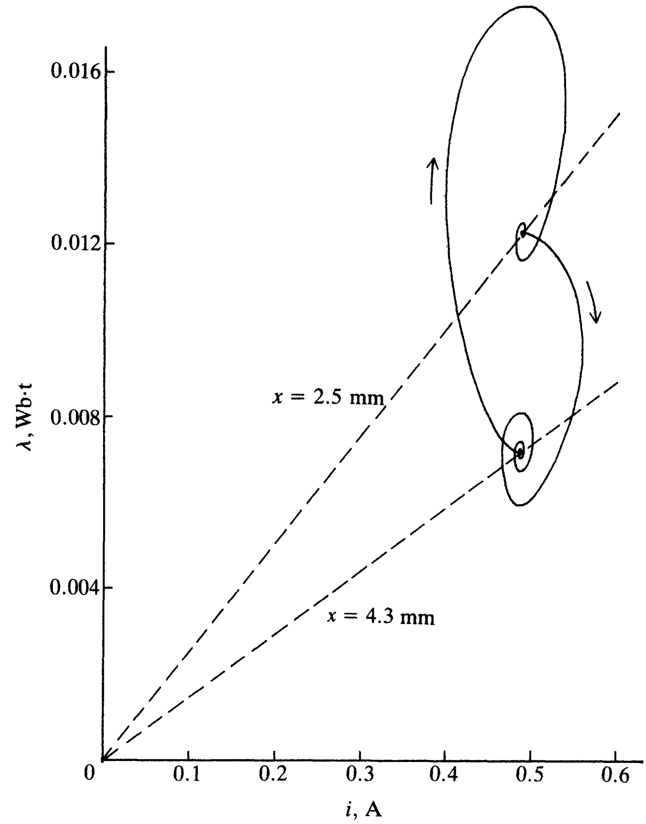

Figure 1.3-13 System response shown in Fig. 1.3-3.

from the coupling field to the electrical system, some coming from the mechanical system and some from the energy originally stored in the magnetic field. Next the force is stepped back to zero from 4 N. The electrical and mechanical systems return to their original states. During the cycle a net energy has been transferred from the mechanical system to the electrical system which is dissipated in the resistance. This cycle is depicted on the λ–i plot shown in Fig. 1.3-13.

Example 1B It is helpful to formulate an expression for the electromagnetic torque of the elementary rotational device shown in Fig. 1B-1. This device consists of two conductors. Conductor 1 is placed on the stationary member (stator); conductor 2 is fixed on the rotating member (rotor). The crossed lines inside a circle indicate that the assumed direction of positive current flow is into the paper (we are seeing the tail of the arrow), whereas a dot inside a circle indicates positive current flow is out of the paper (the point of the arrow). The length of the air gap between the stator and rotor is shown exaggerated relative to the inside diameter of the stator.

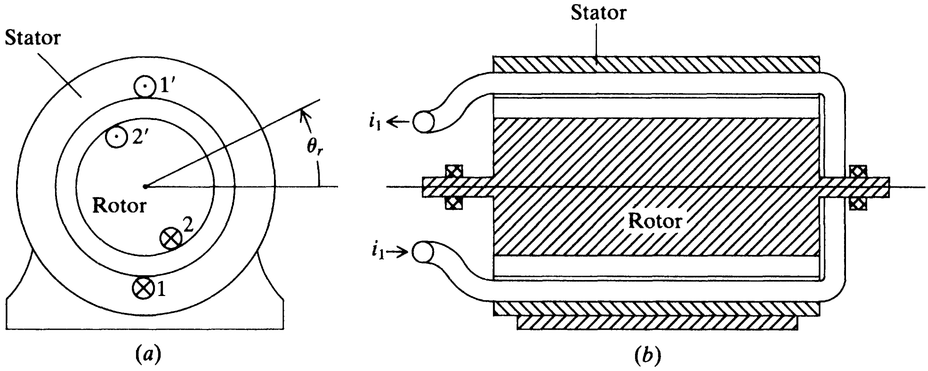

Figure 1B-1 Elementary rotational electromechanical device. (*a*) End view; (*b*) cross-sectional view.

The voltage equations may be written as

$$v_1 = i_1 r_1 + \frac{d\lambda_i}{dt} \tag{1B-1}$$

$$v_2 = i_2 r_2 + \frac{d\lambda_2}{dt} \tag{1B-2}$$

where r_1 and r_2 are the resistances of conductor 1 and 2, respectively. The magnetic system is assumed linear; therefore the flux linkages may be expressed as

$$\lambda_1 = L_{11} i_1 + L_{12} i_2 \tag{1B-3}$$

$$\lambda_2 = L_{21} i_1 + L_{22} i_2 \tag{1B-4}$$

The self-inductances L_{11} and L_{22} are constant. Let us assume that the mutual inductance may be approximated by

$$L_{12} = L_{21} = M \cos \theta_r \tag{1B-5}$$

where θ_r is defined in Fig. 1B-1. The reader should be able to justify the form of (1B-5) by considering the mutual coupling between the two conductors as θ_r varies from 0 to 2π radians.

$$T_e(i_1, i_2, \theta_r) = \frac{\partial W_c(i_1, i_2, \theta_r)}{\partial \theta_r} \tag{1B-6}$$

Because the magnetic system is assumed to be linear, we have

$$W_c(i_1, i_2, \theta_r) = \frac{1}{2} L_{11} i_1^2 + L_{12} i_1 i_2 + \frac{1}{2} L_{22} i_2^2 \tag{1B-7}$$

Substituting into (1B-6) yields

$$T_e = -i_1 i_2 M \sin \theta_r \tag{1B-8}$$

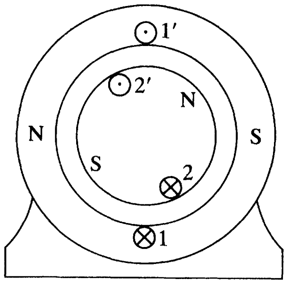

Figure 1B-2 Stator and rotor poles for constant currents.

Consider for a moment the form of the torque if i_1 and i_2 are both constant. For the positive direction of current shown, the torque is of the form

$$T_e = -K \sin \theta_r \tag{1B-9}$$

where K is a positive constant. We can visualize the production of torque by considering the interaction of the magnetic poles produced by the current flowing in the conductors. If both i_1 and i_2 are positive, the poles produced are as shown in Fig. 1B-2. One should recall that flux issues from a north pole. Also, the stator and rotor each must be considered as separate electromagnetic systems. Thus, flux produced by the 1–1′ winding issues from the north pole of the stator into the air gap. Similarly, the flux produced by the 2–2′ winding enters the air gap from the north pole of the rotor. It is left to the reader to justify the fact that the range of θ_r over which stable operation can occur for the expression of electromagnetic torque given by (1B-9) is $-\pi/2 \le \theta_r \le \pi/2$.

1.4 MACHINE WINDINGS AND AIR-GAP MMF

For the purpose of discussing winding configurations in rotating machines and the resulting air-gap MMF as well as the calculations of machine inductances, it is convenient to begin with the elementary 2-pole, 3-phase, wye-connected salient-pole synchronous machine shown in Fig. 1.4-1. Once these concepts are established for this type of a machine, they may be readily modified to account for all types of induction machines and easily extended to include the synchronous machine with short-circuited windings on the rotor (damper windings).

The stator windings of the synchronous machine are embedded in slots around the inside circumference of the stationary member. In the 2-pole machine, each phase winding of the 3-phase stator winding is displaced 120° with respect to each other as illustrated in Fig. 1.4-1. The field or *fd* winding is wound on the rotating member. The *as*, *bs*, *cs*, and *fd* axes denote the positive direction of the flux produced by each of the windings. The *as*, *bs*, and *cs* windings are identical in that each winding has the same resistance and the same number of turns. When a machine has three identical stator windings arranged as shown in Fig. 1.4-1, it is often referred to

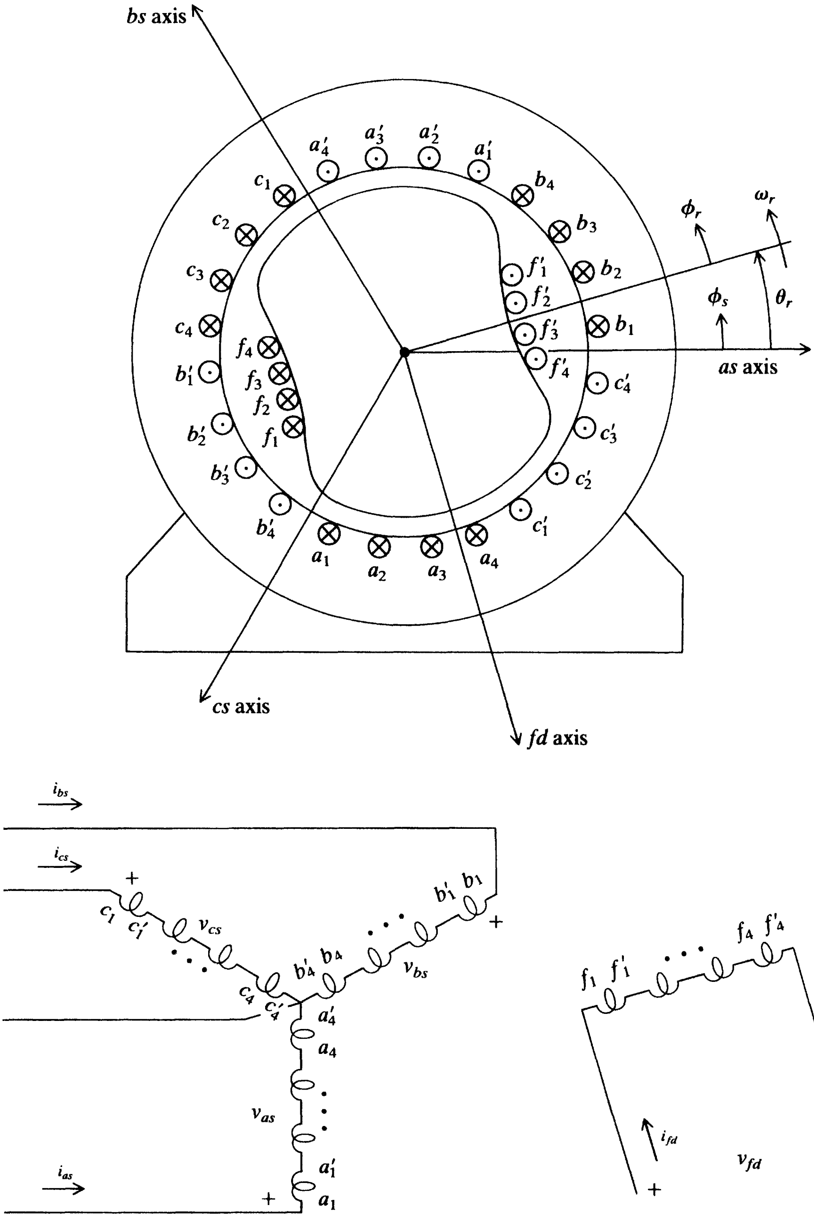

Figure 1.4-1 Elementary, 2-pole, 3-phase, wye-connected salient-pole synchronous machine.

as a machine with symmetrical stator windings. We will find that the symmetrical induction machine has identical multiphase stator windings and identical multiphase rotor windings. An unsymmetrical induction machine has nonidentical multiphase stator windings (generally 2-phase) and symmetrical multiphase rotor windings.

In Fig. 1.4-1, it is assumed that each coil spans π radians of the stator for a 2-pole machine. One side of the coil (coil side) is represented by a $\otimes$ indicating that the assumed positive direction of current is down the length of the stator (into the paper). The $\odot$ indicates that the assumed positive direction of current is out of the paper. Each coil contains n_c conductors. Therefore, in the case of the *as* winding, positive current flows in a conductor of coil a_1, then axially down the length of the stator and back the length of the stator and out at coil side a_1'. This is repeated for n_c conductors. The last conductor of the coil a_1–a_1' is then placed in the appropriate slot so as to start coil a_2–a_2' wherein the current flows down the stator via coil side a_2 and then back through a_2' and so on until a_4'. The *bs* and *cs* windings are arranged similarly, and the last conductors of coil sides a_4', b_4', and c_4' are connected together to form the wye-connected stator. The end turns (looping of the coils) at both ends of the stator so as to achieve the span of π radians are not shown in Fig. 1.4-1. As mentioned, each coil consists of n_c conductors, each of which makes up an individual single conductor coil within the main coil. Thus the number of turns of each winding is determined by the product of n_c and the number of coils or the product of n_c and the number of coil sides carrying current in the same direction. In the case of the *fd* winding, each coil (f_1-f_1', for example) consists of n_f conductors. It should be mentioned that in Example 1B, the stator and rotor coils each consisted of only one coil side with one conductor ($n_c = 1$) in each coil side.

One must realize that the winding configuration shown in Fig. 1.4-1 is an oversimplification of a practical machine. The coil sides of each phase winding are considered to be distributed uniformly over 60° of the stator circumference. Generally, the coil sides of each phase are distributed over a larger area, perhaps as much as 120°, in which case it is necessary for some of the coil sides of two of the phase windings to occupy the same slot. In some cases, the coil sides may not be distributed uniformly over the part of the circumference that it occupies. For example, it would not be uncommon, in the case of the machine shown in Fig. 1.4-1, to have more turns in coil sides a_2 and a_3 than in a_1 and a_4. (Similarly for the *bs* and *cs* windings.) We will find that this winding arrangement produces an air-gap MMF which more closely approximates a sinusoidal air-gap MMF with respect to the angular displacement about the air gap. Another practical consideration is the so-called fractional-pitch winding. The windings shown in Fig. 1.4-1 span π radians for the 2-pole machine. This is referred to as full-pitch winding. In order to reduce voltage and current harmonics, the windings are often wound so that they span slightly less than π radians for a 2-pole machine. This is referred to as a fractional-pitch winding. All of the above-mentioned practical variations from the winding arrangement shown in Fig. 1.4-1 are very important to the machine designer; however, these features are of less importance in machine analysis, where in most cases a simplified approximation of the winding arrangement is sufficient.

A salient-pole synchronous machine is selected for consideration because the analysis of this type of machine may be easily modified to account for other machine types. However, a salient-pole synchronous machine would seldom be a 2-pole machine except in the case of small reluctance machines which are of the synchronous class but which do not have a field winding. Generally, 2- and 4-pole machines are round-rotor machines with the field winding embedded in a solid steel (nonlaminated) rotor. Salient-pole machines generally have a large number of poles composed of laminated steel whereupon the field winding is wound around the poles similar to that shown in Fig. 1.4-1.

For the purposes of deriving an expression for the air-gap MMF, it is convenient to employ the so-called developed diagram of the cross-sectional view of the machine shown in Fig. 1.4-1. The developed diagram is shown in Fig. 1.4-2. The length of the air gap between the stator and rotor is exaggerated in Figs. 1.4-1 and 1.4-2 for clarity. The fact that the air-gap length is small relative to the inside diameter of the stator permits us to employ the developed diagram for analysis purposes. In order to relate the developed diagram to the cross-sectional view of the machine, it is helpful to define a displacement to the left of the origin as positive. The angular displacement along the stator circumference is denoted ϕ_s and ϕ_r along the rotor circumference. The angular velocity of the rotor is ω_r, and θ_r is the angular displacement of the rotor. For a given angular displacement relative to the *as* axis we can relate ϕ_s, ϕ_r, and θ_r as

$$\phi_s = \phi_r + \theta_r \tag{1.4-1}$$

Our analysis may be simplified by considering only one of the stator windings at a time. Figure 1.4-3 is a repeat of Figs. 1.4-1 and 1.4-2 with only the *as* winding shown. Due to the high permeability of the stator and rotor steel the magnetic fields

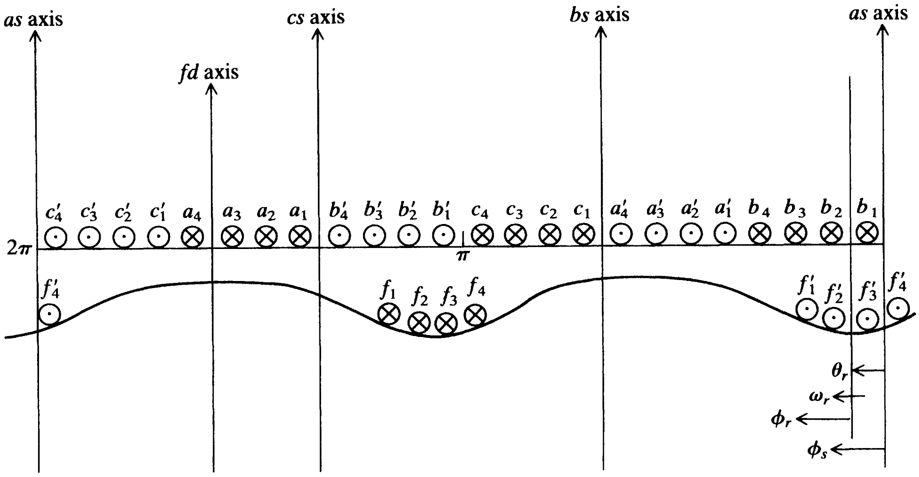

Figure 1.4-2 Development of the machine shown in Fig. 1.4-1.

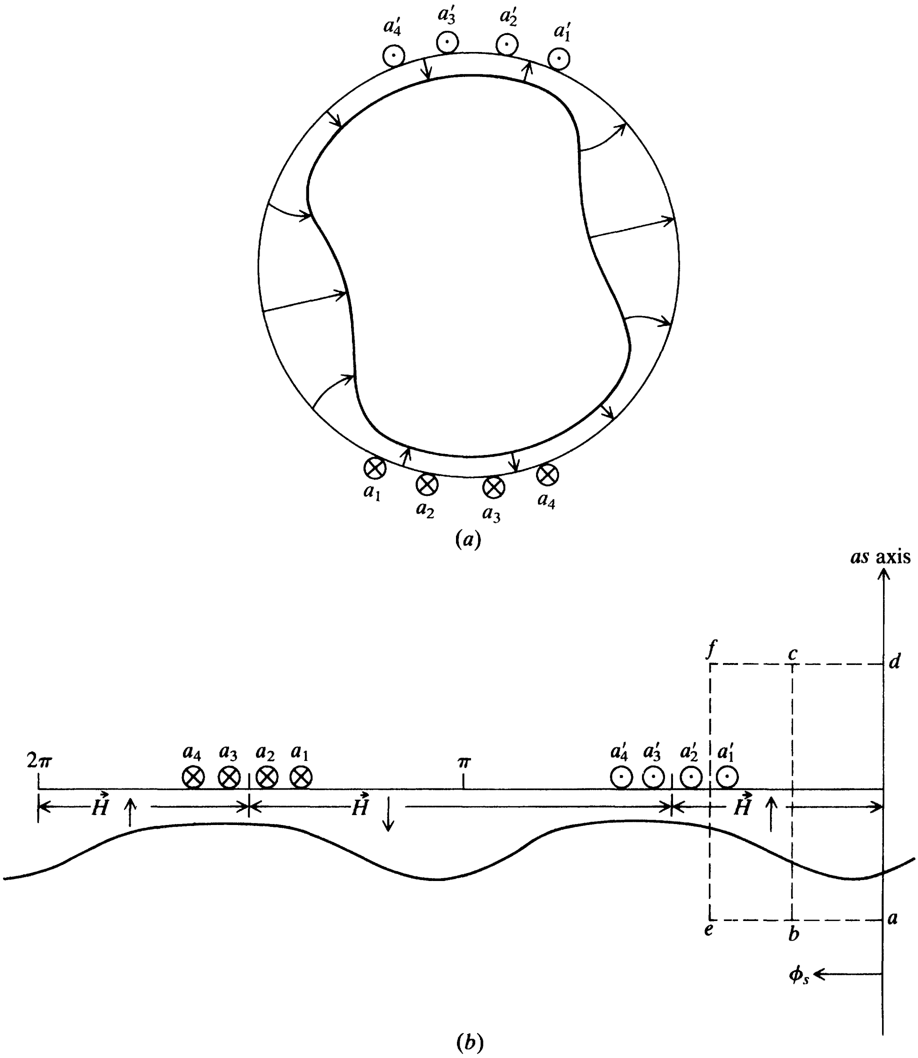

Figure 1.4-3 The *as* winding. (*a*) Direction of $\vec{H}$; (*b*) developed diagram.

essentially exist only in the air gap and tend to be radial in direction due to the short length of the air gap relative to the inside stator diameter. Therefore, in the air gap the magnetic field intensity $\vec{H}$ and the flux density $\vec{B}$ have a component only in the $\vec{a}_r$ direction, and the magnitude is a function of the angle ϕ_s whereupon

$$\vec{H}(r, \phi_s, z) = H_r(\phi_s)\vec{a}_r \tag{1.4-2}$$

and

$$B_r(\phi_s) = \mu_0 H_r(\phi_s) \tag{1.4-3}$$

With the assumed direction of the current i_{as}, the magnetic field intensity $\vec{H}$ in the air gap due to the *as* winding is directed from the rotor to the stator for $-\pi/2 < \phi_s < \pi/2$ and from the stator to the rotor for $\pi/2 < \phi_s < \frac{3}{2}\pi$ (Fig. 1.4-3).

Ampere's law may now be used to determine the form of the air-gap MMF due to the *as* winding. In particular, Ampere's law states that

$$\int \vec{H} \cdot d\vec{L} = i \tag{1.4-4}$$

where i is the net current enclosed within the closed path of integration. Let us consider the closed path of integration depicted in Fig. 1.4-3*b*. Applying Ampere's law around this closed path denoted as *abcda*, where the path *bc* is at $\phi_s = \pi/4$, and neglecting the field intensity within the stator and rotor steel, we can write and evaluate (1.4-4) as

$$\begin{gathered}\int_{r(\pi/4)}^{r(\pi/4)+g(\pi/4)} H\left(\frac{\pi}{4}\right) dL + \int_{r(0)+g(0)}^{r(0)} H(0)\, dL = 0 \\ H_r\left(\frac{\pi}{4}\right) g\left(\frac{\pi}{4}\right) - H_r(0) g(0) = 0\end{gathered} \tag{1.4-5}$$

where $r(\pi/4)$ and $r(0)$ are the radius of the rotor at the respective paths of integration, and $g(\pi/4)$ and $g(0)$ are the corresponding air-gap lengths.

The magnetomotive force is defined as the line integral of $\vec{H}$. Therefore, the terms on the left-hand side of (1.4-5) may be written as MMFs. In particular, (1.4-5) may be written as

$$\text{MMF}\left(\frac{\pi}{4}\right) + \text{MMF}(0) = 0 \tag{1.4-6}$$

In (1.4-6) the MMF includes sign and magnitude; that is, $\text{MMF}(0) = -H_r(0)g(0)$.

Let us now consider the closed path *aefda* where path *ef* occurs at $\phi_s = \frac{7}{16}\pi$; here Ampere's law gives

$$\text{MMF}\left(\frac{7}{16}\pi\right) + \text{MMF}(0) = -n_c i_{as} \tag{1.4-7}$$

The right-hand side of (1.4-7) is negative in accordance with the right-hand or *corkscrew* rule.

We can now start to plot the air-gap MMF due to the *as* winding. For the lack of a better guess let us assume that $\text{MMF}(0) = 0$. With this assumption, (1.4-6) and (1.4-7) tell us that the MMF is zero from $\phi_s = 0$ to where our path of integration encircles the first coil side (a_1'). If we continue to perform line integrations starting and ending at a and each time progressing further in the ϕ_s direction, we will obtain the plot shown in Fig. 1.4-4. Therein, a step change in MMF is depicted at the center

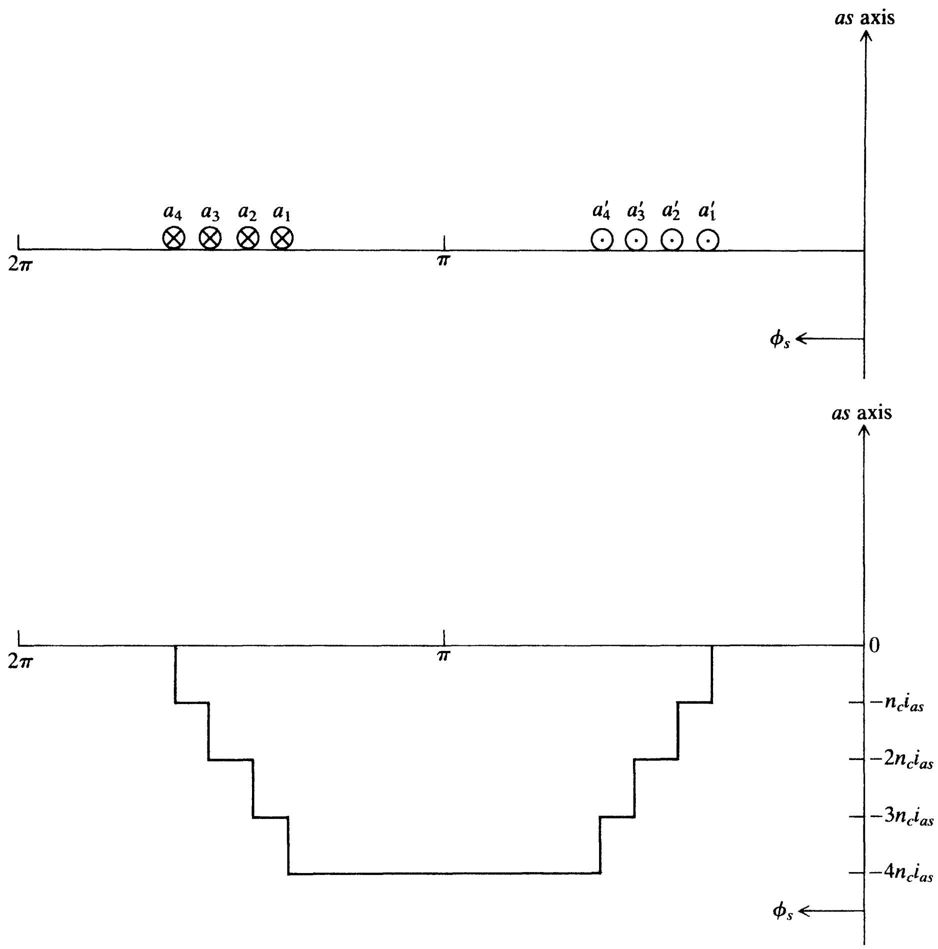

Figure 1.4-4 Plot of the air-gap MMF due to the *as* winding with the assumption that MMF(0) is zero.

of the conductors; actually, there would be a finite slope as the path of integration passes through the conductors.

There are two items left to be considered. First, Gauss's law states that

$$\int_s \vec{B} \cdot d\vec{S} = 0 \tag{1.4-8}$$

Hence, no net flux may travel across the air gap because the flux density is assumed to exist only in the radial direction:

$$\int_0^{2\pi} B_r(\phi_s) rl \, d\phi_s = 0 \tag{1.4-9}$$

where l is the axial length of the air gap of the machine and r is the mean radius of the air gap. It follows that $rl\, d\phi_s$ is the incremental area of an imaginary cylindrical surface within the air gap of the machine. This fact in itself makes us suspicious of our assumption that MMF(0) = 0 because the air-gap MMF shown in Fig. 1.4-4 is unidirectional, which would give rise to a flux in only one direction across the air gap. The second fact is one that is characteristic of electrical machines. In any practical electric machine, the air-gap length is either (a) constant as in the case of a round-rotor machine or (b) a periodic function of displacement about the air gap as in the case of salient-pole machine. In particular, for a 2-pole machine

$$g(\phi_r) = g(\phi_r + \pi) \tag{1.4-10}$$

or

$$g(\phi_s - \theta_r) = g(\phi_s - \theta_r + \pi) \tag{1.4-11}$$

Equations (1.4-8) and (1.4-10) are satisfied if the air-gap MMF has zero average value and, for a 2-pole machine,

$$\text{MMF}(\phi_s) = -\text{MMF}(\phi_s + \pi) \tag{1.4-12}$$

Hence, the air-gap MMF wave for the *as* winding, which is denoted as MMF_{as}, is shown in Fig. 1.4-5. It is clear that the MMF due to the *bs* winding, MMF_{bs}, is identical to MMF_{as} but displaced to the left by 120°. MMF_{cs} is also identical but displaced 240° to the left. The significance of the *as*, *bs*, and *cs* axes is now apparent. These axes are positioned at the center of maximum positive MMF corresponding to positive current into the windings.

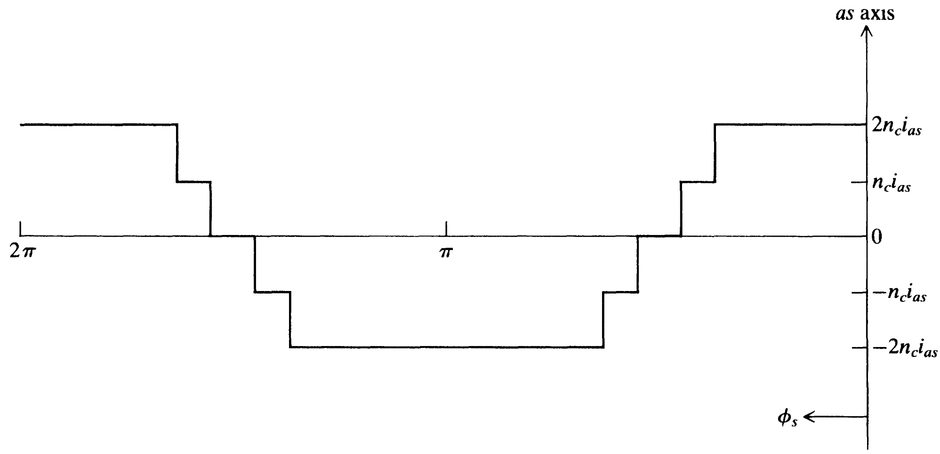

Figure 1.4-5 Air-gap MMF due to *as* winding.

The waveform of the MMF's produced by the stator phase windings shown in Fig. 1.4-1 may be considered as coarse approximations of sinusoidal functions of ϕ_s. Actually most electric machines, especially large machines, are designed so that the stator windings produce a relatively good approximation of a sinusoidal air-gap MMF with respect to ϕ_s so as to minimize the voltage and current harmonics. In order to establish a truly sinusoidal MMF waveform (often referred to as a space sinusoid) the winding must also be distributed sinusoidally. Except in cases where the harmonics due to the winding configuration are of importance, it is typically assumed that all windings may be approximated as sinusoidally distributed windings. We will make this same assumption in our analysis.

A sinusoidally distributed *as* winding and a sinusoidal air-gap MMF_{as} are depicted in Fig. 1.4-6. The distribution of the *as* winding may be written as

$$N_{as} = N_p \sin\phi_s, \qquad 0 \le \phi_s \le \pi \tag{1.4-13}$$

$$N_{as} = -N_p \sin\phi_s, \qquad \pi \le \phi_s \le 2\pi \tag{1.4-14}$$

where N_p is the maximum turn or conductor density expressed in turns per radian. If N_s represents the number of turns of the equivalent sinusoidally distributed

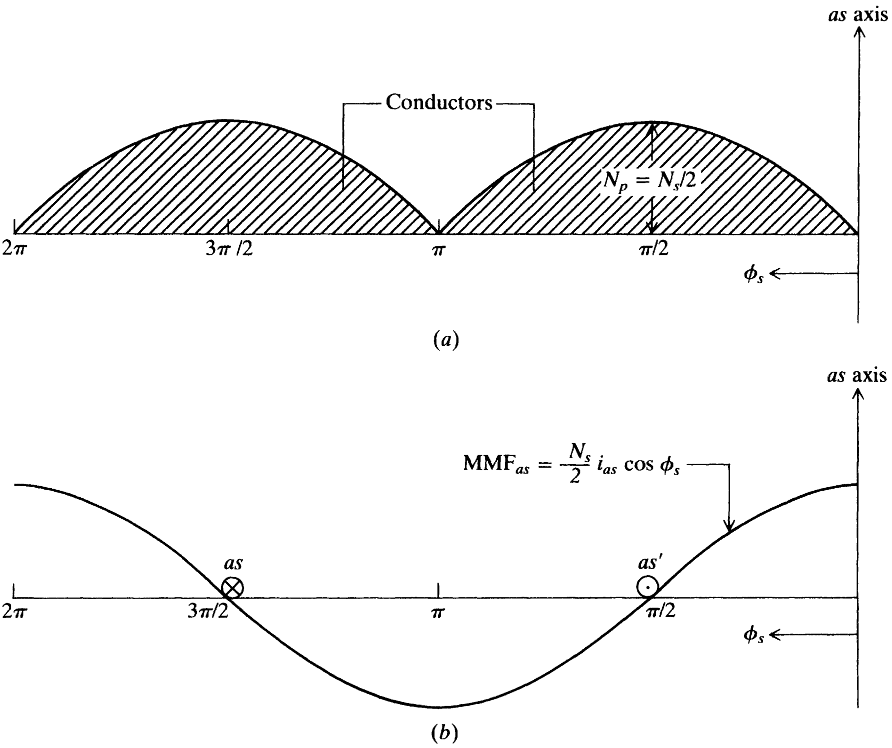

Figure 1.4-6 Sinusoidal distribution. (*a*) Equivalent distribution of *as* winding; (*b*) MMF due to equivalent *as* winding.

winding, then

$$N_s = \int_0^{\pi} N_p \sin\phi_s \, d\phi_s = 2N_p \tag{1.4-15}$$

It is important to note that the number of turns is obtained by integrating N_{as} over π radians rather than 2π radians. Also, N_s is not the total number of turns of the winding; instead, it is an equivalent number of turns of a sinusoidally distributed winding which would give rise to the same fundamental component as the actual winding distribution. For example, if we represent the air-gap MMF wave given in Fig. 1.4-5 as a sinusoidal air-gap MMF as given in Fig. 1.4-6, it can be readily shown by Fourier analysis that $N_s/2$ is equal to $2.43n_c$. It is clear that the influence of design features that have not been considered in the analysis such as fractional-pitch windings are accounted for in N_s. (See problem at end of chapter.) Information regarding machine windings may be found in reference 3.

Current flowing out of the paper is assumed negative so as to comply with the convention established earlier when applying Ampere's law. The circles, which are labeled *as* and *as'* and positioned at the point of maximum turn density in Fig. 1.4-6*b*, will be used hereafter to signify schematically a sinusoidally distributed winding. It is interesting to note in passing that some authors choose to incorporate the direction of current flow in the expression for the turns whereupon the turns for the *as* winding are expressed as

$$N_{as} = -\frac{N_s}{2}\sin\phi_s \tag{1.4-16}$$

The air-gap MMF waveform is also shown in Fig. 1.4-6*b*. This waveform is readily established by applying Ampere's law. The MMF waveform of the equivalent *as* winding is

$$\mathrm{MMF}_{as} = \frac{N_s}{2} i_{as} \cos\phi_s \tag{1.4-17}$$

It follows that

$$\mathrm{MMF}_{bs} = \frac{N_s}{2} i_{bs} \cos\left(\phi_s - \frac{2\pi}{3}\right) \tag{1.4-18}$$

$$\mathrm{MMF}_{cs} = \frac{N_s}{2} i_{cs} \cos\left(\phi_s + \frac{2\pi}{3}\right) \tag{1.4-19}$$

Let us now express the total air-gap MMF produced by the stator currents. This can be obtained by adding the individual MMFs, given by (1.4-17)–(1.4-19):

$$\mathrm{MMF}_s = \frac{N_s}{2}\left[i_{as}\cos\phi_s + i_{bs}\cos\left(\phi_s - \frac{2\pi}{3}\right) + i_{cs}\cos\left(\phi_s + \frac{2\pi}{3}\right)\right] \tag{1.4-20}$$

For balanced, steady-state conditions the stator currents may be expressed as

$$I_{as} = \sqrt{2}I_s \cos\left[\omega_e t + \theta_{ei}(0)\right] \tag{1.4-21}$$

$$I_{bs} = \sqrt{2}I_s \cos\left[\omega_e t - \frac{2\pi}{3} + \theta_{ei}(0)\right] \tag{1.4-22}$$

$$I_{cs} = \sqrt{2}I_s \cos\left[\omega_e t + \frac{2\pi}{3} + \theta_{ei}(0)\right] \tag{1.4-23}$$

where $\theta_{ei}(0)$ is the phase angle at time zero. Substituting the currents into (1.4-20) and using the trigonometric relations given in Appendix A yields

$$\text{MMF}_s = \left(\frac{N_s}{2}\right)\sqrt{2}I_s\left(\frac{3}{2}\right)\cos\left[\omega_e t + \theta_{ei}(0) - \phi_s\right] \tag{1.4-24}$$

If the argument is set equal to a constant while the derivative is taken with respect to time, we see that the above expression describes a sinusoidal air-gap MMF wave with respect to ϕ_s, which rotates about the stator at an angular velocity of ω_e in the counterclockwise direction and which may be thought of as a rotating magnetic pole pair. If, for example, the phase angle $\theta_{ei}(0)$ is zero, then at the instant $t = 0$ the rotating air-gap MMF is positioned in the *as* axis with the north pole at $\phi_s = 180°$ and the south pole at $\phi_s = 0$. (The north pole is by definition the stator pole from which the flux issues into the air gap.)

The rotating air-gap MMF of a P-pole machine can be determined by considering a 4-pole machine. The arrangement of the windings is shown in Fig. 1.4-7. Each phase winding consists of two series connected windings, which will be considered as sinusoidally distributed windings. The air-gap MMF established by each phase is now a sinusoidal function of $2\phi_s$ for a 4-pole machine or, in general, $(P/2)\phi_s$ where P is the number of poles. In particular,

$$\text{MMF}_{as} = \frac{N_s}{P} i_{as} \cos\frac{P}{2}\phi_s \tag{1.4-25}$$

$$\text{MMF}_{bs} = \frac{N_s}{P} i_{bs} \cos\left(\frac{P}{2}\phi_s - \frac{2\pi}{3}\right) \tag{1.4-26}$$

$$\text{MMF}_{cs} = \frac{N_s}{P} i_{cs} \cos\left(\frac{P}{2}\phi_s + \frac{2\pi}{3}\right) \tag{1.4-27}$$

where N_s is the total equivalent turns per phase. With balanced steady-state stator currents as given previously, the air-gap MMF becomes

$$\text{MMF}_s = \left(\frac{2N_s}{P}\right)\sqrt{2}I_s\left(\frac{3}{2}\right)\cos\left[\omega_e t + \theta_{ei}(0) - \frac{P}{2}\phi_s\right] \tag{1.4-28}$$

Here we see that the MMF produced by balanced steady-state stator currents rotates about the air gap in the counterclockwise direction at a velocity of $(2/P)\omega_e$. It may

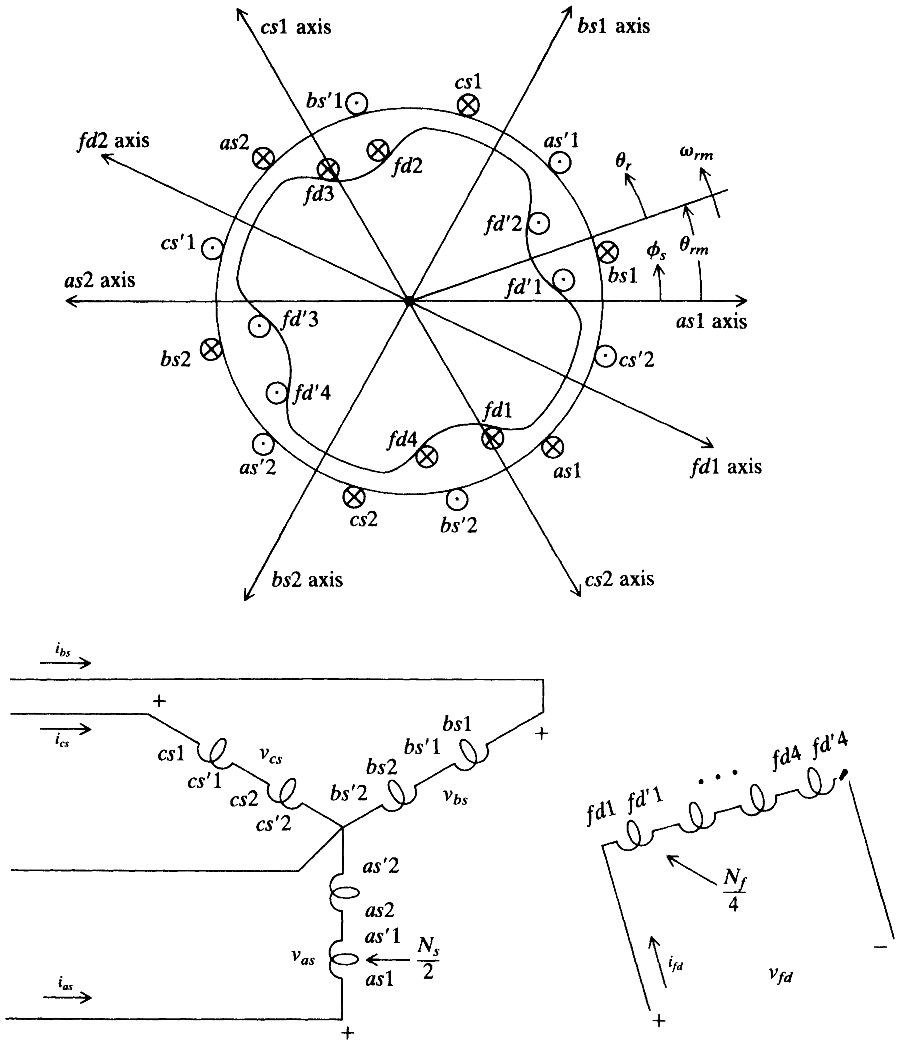

Figure 1.4-7 Winding arrangement of an elementary 4-pole, 3-phase, wye-connected, salient-pole synchronous machine.

at first appear necessary to modify extensively the analysis of a 2-pole machine in order to analyze a P-pole machine. Fortunately, we will find that the modification amounts to a simple change of variables.

We can now start to see the mechanism by which torque is produced in a synchronous machine. The stator windings are arranged so that with balanced steady-state currents flowing in these windings, an air-gap MMF is produced which rotates about the air gap as a set of magnetic poles at an angular velocity corresponding to the frequency of the stator currents and the number of poles. During steady-state operation the voltage applied to the field winding is constant. The resulting constant field current produces a set of magnetic poles, stationary with respect to the rotor. If the rotor rotates at the same speed, or in synchronism with the rotating MMF established by the stator currents, torque is produced due to the interaction of these poles. It follows that a steady-state average torque is produced only when the rotor and the stator air-gap MMFs are rotating in synchronism, thus the name synchronous machine.

Example 1C When the stator currents of a multiphase electric machine, which is equipped with symmetrical stator windings, are unbalanced in amplitude and/or in phase, the total air-gap MMF consists of two oppositely rotating MMFs. This can be demonstrated by assuming that during steady-state operation, the 3-phase stator currents are unbalanced in amplitude. In particular,

$$I_{as} = \sqrt{2} I_a \cos \omega_e t \tag{1C-1}$$

$$I_{bs} = \sqrt{2} I_b \cos\left(\omega_e t - \frac{2\pi}{3}\right) \tag{1C-2}$$

$$I_{cs} = \sqrt{2} I_c \cos\left(\omega_e t + \frac{2\pi}{3}\right) \tag{1C-3}$$

where I_a, I_b, and I_c are unequal constants. Substituting into (1.4-20) yields

$$\begin{aligned} \mathrm{MMF}_s = \frac{N_s}{2}\sqrt{2}\Bigg[&\left(\frac{I_a + I_b + I_c}{2}\right)\cos(\omega_e t - \phi_s) \\ &+ \left(\frac{2I_a - I_b - I_c}{4}\right)\cos(\omega_e t + \phi_s) \\ &+ \frac{\sqrt{3}}{4}(I_b - I_c)\sin(\omega_e t + \phi_s)\Bigg] \end{aligned} \tag{1C-4}$$

The first term is an air-gap MMF rotating in the counterclockwise direction at ω_e. The last two terms are air-gap MMFs (which may be combined into one) that rotate in the clockwise direction at ω_e. Note that when the steady-state currents are balanced, I_a, I_b, and I_c are equal whereupon the last two terms of (1C-4) disappear.

1.5 WINDING INDUCTANCES AND VOLTAGE EQUATIONS

Once the concepts of the sinusoidally distributed winding and the sinusoidal air-gap MMF have been established, the next step is to determine the self- and mutual inductances of the machine windings. As in the previous section, it is advantageous to use the elementary 2-pole, 3-phase synchronous machine to develop these inductance relationships. This development may be readily modified to account for additional windings (damper windings) placed on the rotor of the synchronous machine or for a synchronous machine with a uniform air gap (round rotor). Also, we will show that these inductance relationships may be easily altered to describe the winding inductances of an induction machine.

Synchronous Machine

Figure 1.5-1 is Fig. 1.4-1 redrawn with the windings portrayed as sinusoidally distributed windings. In a magnetically linear system the self-inductance of a winding is the ratio of the flux linked by a winding to the current flowing in the winding with all other winding currents zero. Mutual inductance is the ratio of flux linked by one winding due to current flowing in a second winding with all other winding currents zero including the winding for which the flux linkages are being determined. For this analysis it is assumed that the air-gap length may be approximated

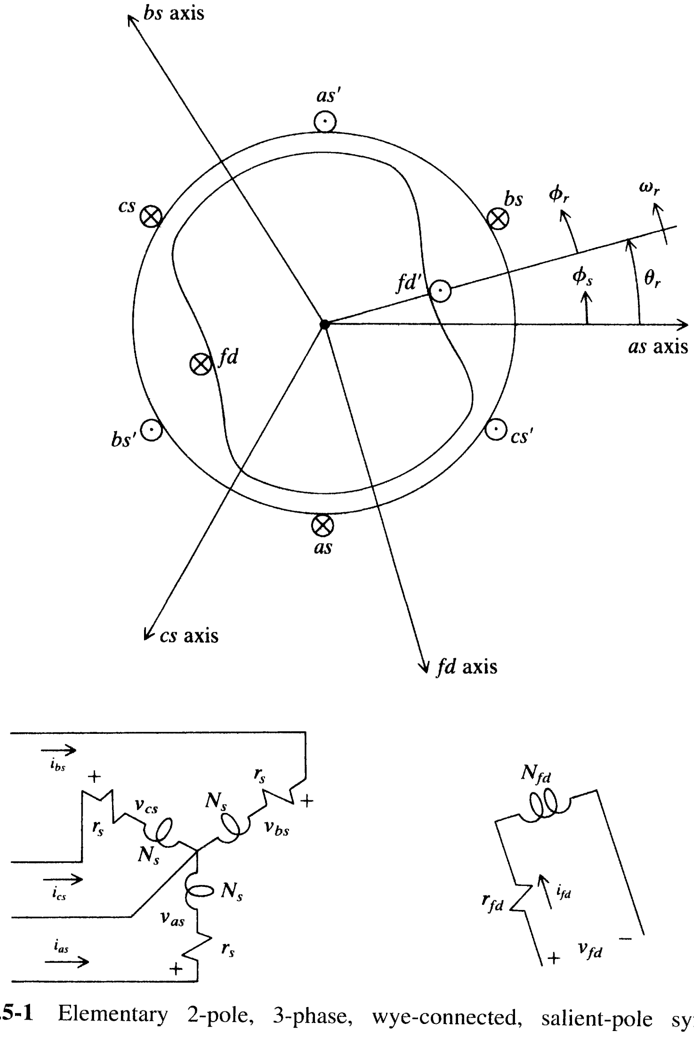

Figure 1.5-1 Elementary 2-pole, 3-phase, wye-connected, salient-pole synchronous machine.

as (Fig. 1.4-2)

$$g(\phi_r) = \frac{1}{\alpha_1 - \alpha_2 \cos 2\phi_r} \tag{1.5-1}$$

or

$$g(\phi_s - \theta_r) = \frac{1}{\alpha_1 - \alpha_2 \cos(2\phi_s - \theta_r)} \tag{1.5-2}$$

where the minimum air-gap length is $(\alpha_1 + \alpha_2)^{-1}$ and the maximum is $(\alpha_1 - \alpha_2)^{-1}$.

Recall from (1.4-5) and (1.4-6) that MMF is defined as the line integral of $\vec{H}$. Thus, from (1.4-3)

$$B_r = \mu_0 \frac{\text{MMF}}{g} \tag{1.5-3}$$

or, because it is convenient to express the stator MMF in terms of $\text{MMF}(\phi_s)$, we can write

$$B_r(\phi_s, \theta_r) = \mu_0 \frac{\text{MMF}(\phi_s)}{g(\phi_s - \theta_r)} \tag{1.5-4}$$

The air-gap flux density due to current in the *as* winding (i_{as}) with all other currents zero may be obtained by substituting (1.4-17) and (1.5-2) into (1.5-4). In particular, the flux density with all currents zero except i_{as} may be expressed as

$$B_r(\phi_s, \theta_r) = \mu_0 \frac{\text{MMF}_{as}(\phi_s)}{g(\phi_s - \theta_r)} = \mu_0 \frac{N_s}{2} i_{as} \cos\phi_s [\alpha_1 - \alpha_2 \cos 2(\phi_s - \theta_r)] \tag{1.5-5}$$

Similarly, the flux density with all currents zero except i_{bs} is

$$B_r(\phi_s, \theta_r) = \mu_0 \frac{N_s}{2} i_{bs} \cos\left(\phi_s - \frac{2\pi}{3}\right)[\alpha_1 - \alpha_2 \cos 2(\phi_s - \theta_r)] \tag{1.5-6}$$

With all currents zero except i_{cs} we have

$$B_r(\phi_s, \theta_r) = \mu_0 \frac{N_s}{2} i_{cs} \cos\left(\phi_s + \frac{2\pi}{3}\right)[\alpha_1 - \alpha_2 \cos 2(\phi_s - \theta_r)] \tag{1.5-7}$$

In the case of salient-pole synchronous machines the field winding is generally uniformly distributed and the poles are shaped to approximate a sinusoidal distribution of air-gap flux due to current flowing in the field winding. In the case of round-rotor synchronous machines the field winding is arranged to approximate more closely a sinusoidal distribution [4–6]. In any event it is sufficient for our purposes to assume as a first approximation that the field winding is sinusoidally distributed with N_f equivalent turns. Later we will assume that the damper windings (additional rotor windings) may also be approximated by sinusoidally distributed windings. Thus, the air-gap MMF due to current i_{fd} flowing in the fd winding may be expressed from Fig. 1.4-2 as

$$\mathrm{MMF}_{fd} = -\frac{N_f}{2} i_{fd} \sin\phi_r \tag{1.5-8}$$

Hence the air-gap flux density due to i_{fd} with all other currents zero may be expressed from (1.5-3) as

$$B_r(\phi_r) = -\mu_0 \frac{N_f}{2} i_{fd} \sin\phi_r(\alpha_1 - \alpha_2 \cos 2\phi_r) \tag{1.5-9}$$

In the determination of self-inductance it is necessary to compute the flux linking a winding due to its own current. To determine mutual inductance it is necessary to compute the flux linking one winding due to current flowing in another winding. Let us consider the flux linkages of a single turn of a stator winding which spans π radians and which is located at an angle ϕ_s. In this case the flux is determined by performing a surface integral over the open surface of the single turn. In particular,

$$\Phi(\phi_s, \theta_r) = \int_{\phi_s}^{\phi_s+\pi} B_r(\xi, \theta_r) rl\, d\xi \tag{1.5-10}$$

where Φ is the flux linking a single turn oriented ϕ_s from the as axis, l is the axial length of the air gap of the machine, r is the radius to the mean of the air gap (essentially to the inside circumference of the stator), and ξ is a dummy variable of integration. In order to obtain the flux linkages of an entire winding, the flux linked by each turn must be summed. Because the windings are considered to be sinusoidally distributed and the magnetic system is assumed to be linear, this summation may be accomplished by integrating over all coil sides carrying current in the same direction. Hence, computation of the flux linkages of an entire winding involves a double integral. As an example, let us determine the total flux linkages of the as winding due to current flowing only in the as winding. Here

$$\lambda_{as} = L_{ls} i_{as} + \int N_{as}(\phi_s)\Phi(\phi_s, \theta_r)\, d\phi = L_{ls} i_{as} + \int N_{as}(\phi_s) \int_{\phi_s}^{\phi_s+\pi} B_r(\xi, \theta_r) rl\, d\xi\, d\phi_s \tag{1.5-11}$$

In (1.5-11), L_{ls} is the stator leakage inductance due primarily to leakage flux at the end turns. Generally, this inductance accounts for 5% to 10% of the maximum self-inductance. Substituting (1.4-14) (with N_p replaced by $N_s/2$) and (1.5-5) for $B_r(\xi, \theta_r)$ into (1.5-11) yields

$$\begin{aligned} \lambda_{as} &= L_{ls} i_{as} - \int_{\pi}^{2\pi} \frac{N_s}{2} \sin\phi_s \int_{\phi_s}^{\phi_s+\pi} \mu_0 \frac{N_s}{2} i_{as} \cos\xi[\alpha_1 - \alpha_2 \cos 2(\xi - \theta_r)] rl\, d\xi\, d\phi_s \\ &= L_{ls} i_{as} + \left(\frac{N_s}{2}\right)^2 \pi\mu_0 rl\left(\alpha_1 - \frac{\alpha_2}{2}\cos 2\theta_r\right) i_{as} \end{aligned} \tag{1.5-12}$$

The interval of integration is taken from π to 2π so as to comply with the convention that positive flux linkages are obtained in the direction of the positive *as* axis by circulation of the assumed positive current in the clockwise direction about the coil (right-hand rule). The self-inductance of the *as* winding is obtained by dividing (1.5-12) by i_{as}. Thus

$$L_{asas} = L_{ls} + \left(\frac{N_s}{2}\right)^2 \pi\mu_0 rl\left(\alpha_1 - \frac{\alpha_2}{2}\cos 2\theta_r\right) \tag{1.5-13}$$

The mutual inductance between the *as* and *bs* windings may be determined by first computing the flux linking the *as* winding due to current flowing only in the *bs* winding. In this case it is assumed that the magnetic coupling that might occur at the end turns of the windings may be neglected. Thus

$$\lambda_{as} = \int N_{as}(\phi_s)\int_{\phi_s}^{\phi_s+\pi} B_r(\xi,\theta_r)rl\, d\xi\, d\phi_s \tag{1.5-14}$$

Substituting (1.4-14) and (1.5-6) into (1.5-14) yields

$$\lambda_{as} = -\int_{\pi}^{2\pi}\frac{N_s}{2}\sin\phi_s\int_{\phi_s}^{\phi_s+\pi}\mu_0\frac{N_s}{2}i_{bs}\cos\left(\xi - \frac{2\pi}{3}\right)[\alpha_1 - \alpha_2\cos 2(\xi-\theta_r)]rl\, d\xi\, d\phi \tag{1.5-15}$$

Therefore, the mutual inductance between the *as* and *bs* windings is obtained by evaluating (1.5-15) and dividing the result by i_{bs}. This gives

$$L_{asbs} = -\left(\frac{N_s}{2}\right)^2\frac{\pi}{2}\mu_0 rl\left[\alpha_1 + \alpha_2\cos 2\left(\theta_r - \frac{\pi}{3}\right)\right] \tag{1.5-16}$$

The mutual inductance between the *as* and *fd* windings is determined by substituting (1.5-9), expressed in terms of $\phi_s - \theta_r$, into (1.5-14). Thus

$$\lambda_{as} = \int_{\pi}^{2\pi}\frac{N_s}{2}\sin\phi_s\int_{\phi_s}^{\phi_s+\pi}\mu_0\frac{N_f}{2}i_{fd}\sin(\xi-\theta_r)[\alpha_1 - \alpha_2\cos 2(\xi-\theta_r)]rl\, d\xi\, d\phi_s \tag{1.5-17}$$

Evaluating and dividing by i_{fd} yields

$$L_{asfd} = \left(\frac{N_s}{2}\right)\left(\frac{N_f}{2}\right)\pi\mu_0 rl\left(\alpha_1 + \frac{\alpha_2}{2}\right)\sin\theta_r \tag{1.5-18}$$

The self-inductance of the field winding may be obtained by first evaluating the flux linking the *fd* winding with all currents equal to zero except i_{fd}. Thus, with the *fd* winding considered as sinusoidally distributed and the air-gap flux density expressed by (1.5-9) we can write

$$\lambda_{fd} = L_{lfd} i_{fd} + \int_{\pi/2}^{3\pi/2} \frac{N_f}{2} \cos\phi_r \int_{\phi_r}^{\phi_r+\pi} \mu_0 \frac{N_f}{2} i_{fd} \sin\xi(\alpha_1 - \alpha_2 \cos 2\xi) rl \, d\xi \, d\phi_r \tag{1.5-19}$$

from which

$$L_{fdfd} = L_{lfd} + \left(\frac{N_f}{2}\right)^2 \pi\mu_0 rl\left(\alpha_1 + \frac{\alpha_2}{2}\right) \tag{1.5-20}$$

where L_{lfd} is the leakage inductance of the field winding.

The remaining self- and mutual inductances may be calculated using the same procedure. We can express these inductances compactly by defining

$$L_A = \left(\frac{N_s}{2}\right)^2 \pi\mu_0 rl\alpha_1 \tag{1.5-21}$$

$$L_B = \frac{1}{2}\left(\frac{N_s}{2}\right)^2 \pi\mu_0 rl\alpha_2 \tag{1.5-22}$$

$$L_{sfd} = \left(\frac{N_s}{2}\right)\left(\frac{N_f}{2}\right) \pi\mu_0 rl\left(\alpha_1 + \frac{\alpha_2}{2}\right) \tag{1.5-23}$$

$$L_{mfd} = \left(\frac{N_f}{2}\right)^2 \pi\mu_0 rl\left(\alpha_1 + \frac{\alpha_2}{2}\right) \tag{1.5-24}$$

The machine inductances may now be expressed as

$$L_{asas} = L_{ls} + L_A - L_B \cos 2\theta_r \tag{1.5-25}$$

$$L_{bsbs} = L_{ls} + L_A - L_B \cos 2\left(\theta_r - \frac{2\pi}{3}\right) \tag{1.5-26}$$

$$L_{cscs} = L_{ls} + L_A - L_B \cos 2\left(\theta_r + \frac{2\pi}{3}\right) \tag{1.5-27}$$

$$L_{fdfd} = L_{lfd} + L_{mfd} \tag{1.5-28}$$

$$L_{asbs} = -\frac{1}{2} L_A - L_B \cos 2\left(\theta_r - \frac{\pi}{3}\right) \tag{1.5-29}$$

$$L_{ascs} = -\frac{1}{2} L_A - L_B \cos 2\left(\theta_r + \frac{\pi}{3}\right) \tag{1.5-30}$$

$$L_{bscs} = -\frac{1}{2} L_A - L_B \cos 2(\theta_r + \pi) \tag{1.5-31}$$

$$L_{asfd} = L_{sfd} \sin\theta_r \tag{1.5-32}$$

$$L_{bsfd} = L_{sfd} \sin\left(\theta_r - \frac{2\pi}{3}\right) \tag{1.5-33}$$

$$L_{csfd} = L_{sfd} \sin\left(\theta_r + \frac{2\pi}{3}\right) \tag{1.5-34}$$

In later chapters we will consider a practical synchronous machine equipped with short-circuited rotor windings (damper windings). The expressions for the inductances of these additional windings can be readily ascertained from the work presented here. Also, high-speed synchronous machines used with steam turbines are round rotor devices. The $2\theta_r$ variation is not present in the inductances of a uniform air-gap machine. Therefore, the winding inductances may be determined from the above relationships by simply setting $\alpha_2 = 0$ in (1.5-22)–(1.5-24). It is clear that with $\alpha_2 = 0$, $L_B = 0$.

The voltage equations for the elementary synchronous machine shown in Fig. 1.5-1 are

$$v_{as} = r_s i_{as} + \frac{d\lambda_{as}}{dt} \tag{1.5-35}$$

$$v_{bs} = r_s i_{bs} + \frac{d\lambda_{bs}}{dt} \tag{1.5-36}$$

$$v_{cs} = r_s i_{cs} + \frac{d\lambda_{cs}}{dt} \tag{1.5-37}$$

$$v_{fd} = r_{fd} i_{fd} + \frac{d\lambda_{fd}}{dt} \tag{1.5-38}$$

where r_s is the resistance of the stator winding and r_{fd} the resistance of the field winding. The flux linkages are expressed as

$$\lambda_{as} = L_{asas} i_{as} + L_{asbs} i_{bs} + L_{ascs} i_{cs} + L_{asfd} i_{fd} \tag{1.5-39}$$

$$\lambda_{bs} = L_{bsas} i_{as} + L_{bsbs} i_{bs} + L_{bscs} i_{cs} + L_{bsfd} i_{fd} \tag{1.5-40}$$

$$\lambda_{cs} = L_{csas} i_{as} + L_{csbs} i_{bs} + L_{cscs} i_{cs} + L_{csfd} i_{fd} \tag{1.5-41}$$

$$\lambda_{fd} = L_{fdas} i_{as} + L_{fdbs} i_{bs} + L_{fdcs} i_{cs} + L_{fdfd} i_{fd} \tag{1.5-42}$$

One starts to see the complexity of the voltage equations due to the fact that some of the machine inductances are functions of θ_r and therefore a function of the rotor speed ω_r. Hence, the coefficients of the voltage equations are time-varying except when the rotor is stalled. Moreover, rotor speed is a function of the electromagnetic torque, which we will find to be a product of machine currents. Clearly, the solution of the voltage equations is very involved. In fact, a computer is required in order to solve for the transient response of the combined electrical and mechanical systems. In Chapter 3 we will see that a change of variables allows us to eliminate the time-varying inductances and thereby markedly reduce the complexity of the voltage equations. However, the resulting differential equations are still nonlinear, requiring a computer to solve for the electromechanical transient behavior.

Induction Machine

The winding arrangement of a 2-pole, 3-phase, wye-connected symmetrical induction machine is shown in Fig. 1.5-2. The stator windings are identical with

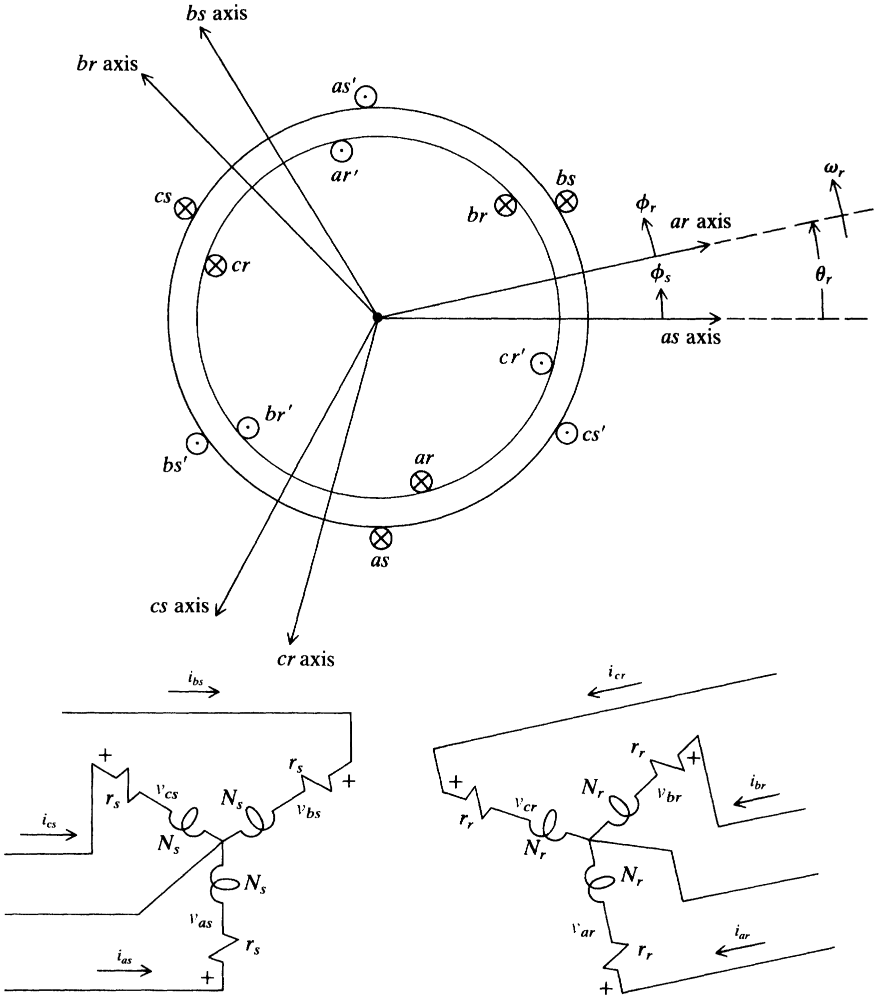

Figure 1.5-2 Two-pole, 3-phase, wye-connected symmetrical induction machine.

equivalent turns N_s and resistance r_s. The rotor windings, which may be wound or forged as a squirrel cage winding, can be approximated as identical windings with equivalent turns N_r and resistance r_r. The air gap of an induction machine is uniform and it is assumed that the stator and rotor windings may be approximated as sinusoidally distributed windings.

In nearly all applications the induction machine is operated as a motor with the stator windings connected to a balanced 3-phase source and the rotor windings short-circuited. The principle of operation in this mode is quite easily deduced [4–6]. With balanced 3-phase currents flowing in the stator windings, a rotating air-gap MMF is established, as in the case of the synchronous machine, which rotates about the air gap at a speed determined by the frequency of the stator currents and the number of poles. If the rotor speed is different from the speed of this rotating MMF, balanced 3-phase currents will be induced (thus the name induction) in the short-circuited rotor windings. The frequency of the rotor currents corresponds to the difference in the speed of the rotating MMF due to the stator currents and the speed of the rotor. The induced rotor currents will, in turn, produce an air-gap MMF that rotates relative to the rotor at a speed corresponding to the frequency of the rotor currents. The speed of the rotor air-gap MMF superimposed upon the rotor speed is the same speed as that of the air-gap MMF established by the currents flowing in the stator windings. These two air-gap MMFs rotating in unison may be thought of as two synchronously rotating sets of magnetic poles. Torque is produced due to an interaction of the two magnetic systems. It is clear, however, that torque is not produced when the rotor is running in synchronism with the air-gap MMF due to the stator currents because in this case currents are not induced in the short-circuited rotor windings.

The winding inductances of the induction machine may be expressed from the inductance relationships given for the salient-pole synchronous machine. In the case of the induction machine the air gap is uniform. Thus, $2\theta_r$ variations in the self- and mutual inductances do not occur. This variation may be eliminated by setting $\alpha_2 = 0$ in the inductance relationship given for the salient-pole synchronous machine. All stator self-inductances are equal; that is, $L_{asas} = L_{bsbs} = L_{cscs}$ with

$$L_{asas} = L_{ls} + L_{ms} \tag{1.5-43}$$

where L_{ms} is the stator magnetizing inductance that corresponds to L_A in (1.5-25). It may be expressed as

$$L_{ms} = \left(\frac{N_s}{2}\right)^2 \frac{\pi\mu_0 rl}{g} \tag{1.5-44}$$

where g is the length of the uniform air gap. Likewise all stator-to-stator mutual inductances are the same. For example,

$$L_{asbs} = -\frac{1}{2}L_{ms} \tag{1.5-45}$$

which corresponds to (1.5-29) with $L_B = 0$.

It follows that the rotor self-inductances are equal with

$$L_{arar} = L_{lr} + L_{mr} \tag{1.5-46}$$

where the rotor magnetizing inductance may be expressed as

$$L_{mr} = \left(\frac{N_r}{2}\right)^2 \frac{\pi\mu_0 rl}{g} \tag{1.5-47}$$

The rotor-to-rotor mutual inductances are

$$L_{arbr} = -\frac{1}{2}L_{mr} \tag{1.5-48}$$

Expressions for the mutual inductances between stator and rotor windings may be written by noting the form of the mutual inductances between the field and stator windings of the synchronous machine given by (1.5-32)–(1.5-34). Here we see that L_{asar}, L_{bsbr}, and L_{cscr} are equal with

$$L_{asar} = L_{sr}\cos\theta_r \tag{1.5-49}$$

Also, L_{asbr}, L_{bscr}, and L_{csar} are equal with

$$L_{asbr} = L_{sr}\cos\left(\theta_r + \frac{2\pi}{3}\right) \tag{1.5-50}$$

Finally, L_{ascr}, L_{bsar}, and L_{csbr} are equal with

$$L_{ascr} = L_{sr}\cos\left(\theta_r - \frac{2\pi}{3}\right) \tag{1.5-51}$$

where

$$L_{sr} = \left(\frac{N_s}{2}\right)\left(\frac{N_r}{2}\right)\frac{\pi\mu_0 rl}{g} \tag{1.5-52}$$

The voltage equations for the induction machine shown in Fig. 1.5-2 are

$$v_{as} = r_s i_{as} + \frac{d\lambda_{as}}{dt} \tag{1.5-53}$$

$$v_{bs} = r_s i_{bs} + \frac{d\lambda_{bs}}{dt} \tag{1.5-54}$$

$$v_{cs} = r_s i_{cs} + \frac{d\lambda_{cs}}{dt} \tag{1.5-55}$$

$$v_{ar} = r_r i_{ar} + \frac{d\lambda_{ar}}{dt} \tag{1.5-56}$$

$$v_{br} = r_r i_{br} + \frac{d\lambda_{br}}{dt} \tag{1.5-57}$$

$$v_{cr} = r_r i_{cr} + \frac{d\lambda_{cr}}{dt} \tag{1.5-58}$$

where r_s is the resistance of each stator phase winding and r_r is the resistance of each rotor phase winding. The flux linkages may be written following the form of λ_{as}:

$$\lambda_{as} = L_{asas}i_{as} + L_{asbs}i_{bs} + L_{ascs}i_{cs} + L_{asar}i_{ar} + L_{asbr}i_{br} + L_{ascr}i_{cr} \tag{1.5-59}$$

Here again we see the complexity of the voltage equations due to the time-varying mutual inductances between stator and rotor circuits (circuits in relative motion). We will see in Chapter 3 that a change of variables eliminates the time-varying inductances resulting in voltage equations which are still nonlinear but much more manageable.

Example 1D The winding inductances of a P-pole machine may be determined by considering the elementary 4-pole, 3-phase synchronous machine shown in Fig. 1.4-7. Let us determine the self-inductance of the *as* winding. The air-gap flux density due to current only in the *as* winding may be expressed from (1.5-4); in particular,

$$B_r(\phi_s, \theta_{rm}) = \mu_0 \frac{\mathrm{MMF}_{as}}{g(\phi_s - \theta_{rm})} \tag{1D-1}$$

where for a P-pole machine, MMF_{as} is given by (1.4-25) and θ_{rm} is defined in Fig. 1.4-7. From (1.5-2) and Fig. 1.4-7 we obtain

$$g(\phi_s - \theta_{rm}) = \frac{1}{\alpha_1 - \alpha_2 \cos(P/2)2(\phi_s - \theta_{rm})} \tag{1D-2}$$

Substituting into (1D-1) yields

$$B_r(\phi_s, \theta_{rm}) = \mu_0 \frac{N_s}{P} i_{as} \cos\frac{P}{2}\phi_s \left[\alpha_1 - \alpha_2 \cos\left(\frac{P}{2}\right) 2(\phi_s - \theta_{rm})\right] \tag{1D-3}$$

Following (1.5-11) for a P-pole machine we have

$$\lambda_{as} = L_{ls} i_{as} - \frac{P}{2} \int_{2\pi/P}^{4\pi/P} N_{as}(\phi_s) \int_{\phi_s}^{\phi_s + 2\pi/P} B_r(\xi, \theta_{rm}) rl \, d\xi \, d\phi_s \tag{1D-4}$$

where

$$N_{as}(\phi_s) = -\frac{N_s}{P} \sin\frac{P}{2}\phi_s, \qquad \frac{2\pi}{P} \le \phi_s \le \frac{4\pi}{P} \tag{1D-5}$$

The double integral is multiplied by $P/2$ to account for the flux linkages of the complete *as* winding. Evaluating (1D-4) and dividing by i_{as} yields

$$L_{asas} = L_{ls} + \frac{N_s^2}{2P} \pi \mu_0 rl \left[\alpha_1 - \frac{\alpha_2}{2} \cos 2\left(\frac{P}{2}\right)\theta_{rm}\right] \tag{1D-6}$$

If, in the above relation, we substitute

$$\theta_r = \frac{P}{2}\theta_{rm} \tag{1D-7}$$

then (1D-3) is of the same form as for the 2-pole machine, (1.5-13). Moreover, if we evaluate all of the machine inductances for the P-pole machine and make this same substitution, we will find that the winding inductances of a P-pole machine are of the same form as those of a 2-pole machine. Also, ω_r is defined as

$$\omega_r = \frac{P}{2}\omega_{rm} \tag{1D-8}$$

and the substitute variables θ_r and ω_r are referred to as the electrical angular displacement and the electrical angular velocity of the rotor, respectively. This connotation stems from the fact that $\theta_r(\omega_r)$ refers a rotor displacement (velocity) to an electrical displacement (velocity). It is clear that, at synchronous speed, ω_r is equal to the angular velocity of the electrical system connected to the stator windings regardless of the number of poles.

Because the winding inductances of a P-pole machine are identical in form to those of a 2-pole machine if θ_r and ω_r are defined by (1D-7) and (1D-8), respectively, all machines can be treated as if they were 2-pole devices as far as the voltage equations are concerned. We will find that it is necessary to multiply the torque equation of a 2-pole machine by $P/2$ in order to express the electromagnetic torque of a P-pole machine correctly.

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PROBLEMS

1 A two-winding, iron-core transformer is shown in Fig. 1.P-1. $N_1 = 50$ turns, $N_2 = 100$ turns, and $\mu_R = 4000$. Calculate L_{m1}, L_{m2}, and L_{12}. The resistance of coils 1 and 2 is $2\,\Omega$ and $4\,\Omega$, respectively. If the voltage applied to coil 1 is $v_1 = 10\cos 400t$, calculate the voltage appearing at the terminals of coil 2 when coil 2 is open-circuited.

2 Repeat Problem 1 if the iron core has an air gap of 0.2 cm in length and is cut through the complete cross section. Assume that fringing does not occur in the air gap, that is, the effective cross-sectional area of the air gap is $25\,\text{cm}^2$.

3 Two coupled coils have the following parameters

$$\begin{aligned} L_{11} &= 100\,\text{mH} & r_1 &= 10\ \Omega \\ L_{22} &= 25\,\text{mH} & r_2 &= 2.5\ \Omega \\ N_1 &= 1000\,\text{turns} & N_2 &= 500\,\text{turns} \\ L_{l1} &= 0.1L_{11} & L_{l1} &= 0.1L_{11} \end{aligned}$$

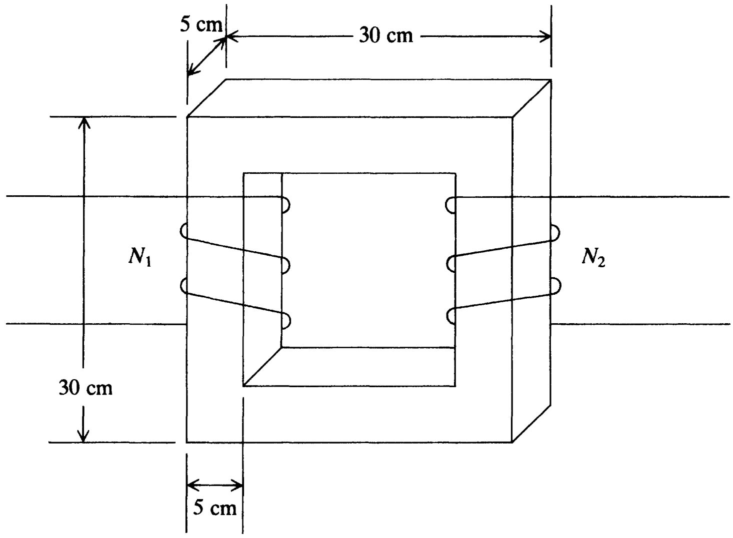

Figure 1.P-1 Two-winding, iron-core transformer.

(*a*) Develop a T equivalent circuit with coil 1 as the reference coil. (*b*) Repeat with coil 2 as the reference coil. Determine the input impedance of the coupled circuits if the applied frequency to coil 1 is 60 Hz with coil 2 (*c*) open-circuited; (*d*) short-circuited. Repeat (*d*) with the current flowing in the magnetizing reactance neglected.

4 A constant 10 V is suddenly applied to coil 1 of the coupled circuits given in Problem 3. Coil 2 is short-circuited. Calculate the transient and steady-state current flowing in each coil.

5 Consider the coils given in Problem 3. Assume that the coils are wound so that, with the assigned positive direction of currents i_1 and i_2, the mutual inductance is negative. (*a*) If the coils are connected in series so that $i_1 = i_2$, calculate the self-inductance of the series combination. (*b*) Repeat (*a*) with the coils connected in series so that $i_1 = -i_2$.

6 A third coil is wound on the ferromagnetic core shown in Fig. 1.2-1. The resistance is r_3 and the leakage and magnetizing inductances are L_{l3} and L_{m3}, respectively. The coil is wound so that positive current (i_3) produces Φ_{m3} in the opposite direction as Φ_{m1} and Φ_{m2}. Derive the T equivalent circuit for this three-winding transformer. Actually one should be able to develop the equivalent circuit without derivation.

7 Use Σ and $1/p$ to denote summation and integration, respectively. Draw a time-domain block diagram for two coupled windings with saturation.

8 A resistor and an inductor are connected as shown in Fig. 1.P-2 with $R = 15\,\Omega$ and $L = 250\,\text{mH}$. Determine the energy stored in the inductor W_{eS} and the energy dissipated by the resistor W_{eL} for $i > 0$ if $i(0) = 10$ A.

9 Consider the spring–mass–damper system shown in Fig. 1.P-3. At $t = 0$, $x(0) = x_0$ (rest position) and $dx/dt = 1.5\,\text{m/s}$. $M = 0.8\,\text{kg}$, $D = 10\text{N} \cdot \text{s/m}$, and $K = 120\,\text{N} \cdot \text{m}$. For $t > 0$, determine the energy stored in the spring W_{mS1}, the kinetic energy of the mass W_{mS2}, and the energy dissipated by the damper W_{mL}.

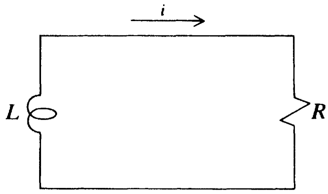

Figure 1.P-2 *R–L* circuit.

10 Express $W_f(i,x)$ and $W_c(i,x)$ for (*a*) $\lambda(i,x) = i^{3/2}x^3$; (*b*) $\lambda(i,x) = xi^2 + ki\cos x$.

11 The energy stored in the coupling field of a magnetically linear system with two electrical inputs may be expressed as

$$W_f(\lambda_1, \lambda_2, x) = \frac{1}{2}B_{11}\lambda_1^2 + B_{12}\lambda_1\lambda_2 + \frac{1}{2}B_{22}\lambda_2^2$$

Express B_{11}, B_{12}, and B_{22} in terms of inductances L_{11}, L_{12}, and L_{22}.

12 An electromechanical system has two electrical inputs. The flux linkages may be expressed as

$$\lambda_1(i_1, i_2, x) = x^2 i_1^2 + x i_2$$
$$\lambda_2(i_1, i_2, x) = x^2 i_2^2 + x i_1$$

Express $W_f(i_1, i_2, x)$ and $W_c(i_1, i_2, x)$.

13 Express $f_e(i,x)$ for the electromechanical systems described by the relations given in Problem 10.

14 Express $f_e(i_1, i_2, x)$ for the electromechanical system given in Problem 12.

15 Refer to Fig. 1.3-7. As the system moves from x_a to x_b the λ–i trajectory moves from *A* to *B*. Does the voltage v increase or decrease? Does the applied force f increase or decrease? Explain.

16 Refer to Fig. 1.3-11. Following the system transients due to the application of the source voltage ($v = 5$V), the system assumes steady-state operation. During this steady-state operation calculate W_{eS}, W_f, W_c, and W_{mS}.

17 Refer to Fig. 1.3-12. Repeat Problem 16 for steady-state operation following the application of $f = 4N$.

18 Refer to Fig. 1.3-13. Identify the area corresponding to ΔW_m when (*a*) x moves from 2.5 mm to 4.3 mm, and (*b*) x moves from 4.3 mm to 2.5 mm.

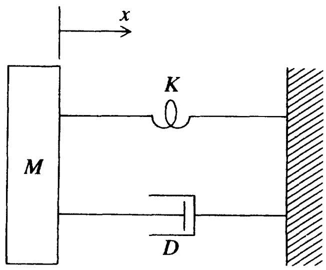

Figure 1.P-3 Spring–mass–damper system.

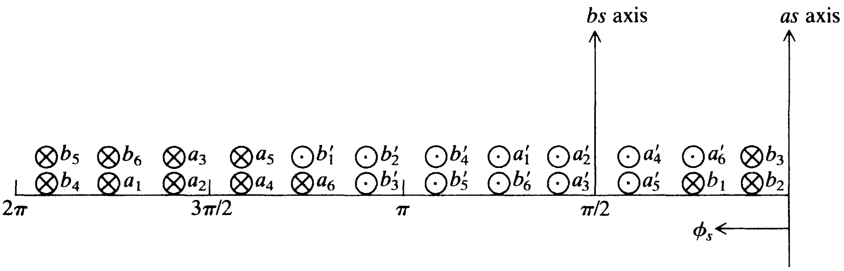

Figure 1.P-4 Stator winding arrangement of a 2-pole, 2-phase machine.

19 Assume the steady-state currents flowing in the conductors of the device shown in Fig. 1B-1 are

$$I_1 = I_s \cos[\omega_1 t + \theta_1(0)]$$
$$I_2 = I_r \cos[\omega_2 t + \theta_2(0)]$$

where $\theta_1(0)$ and $\theta_2(0)$ are the time-zero displacements of I_1 and I_2, respectively. Assume also that during steady-state operation the rotor speed is constant; thus

$$\theta_r = \omega_r t + \theta_r(0)$$

where $\theta_r(0)$ is the rotor displacement at time zero. Determine the rotor speeds at which the device produces a nonzero average torque during steady-state operation if (*a*) $\omega_1 = \omega_2 = 0$; (*b*) $\omega_1 = \omega_2 \neq 0$; (*c*) $\omega_2 = 0$. Express the instantaneous torque at each of these rotor speeds.

20 The developed diagram shown in Fig. 1.P-4 shows the arrangement of the stator windings of a 2-pole, 2-phase machine. Each coil side has n_c conductors, and i_{as} flows in the *as* winding and i_{bs} flows in the *bs* winding. Draw the air-gap MMF (*a*) due to $i_{as}(\text{MMF}_{as})$ and (*b*) due to $i_{bs}(\text{MMF}_{bs})$.

21 Assume that each coil side of the winding shown in Fig. 1.P-4 contains n_c conductors. The windings are to be described as sinusoidally distributed windings with N_p the maximum turns density and N_s the number of equivalent turns in each winding, which is a function of n_c. Express (*a*) N_{as} and N_{bs}, (*b*) MMF_{as} and MMF_{bs}, and (*c*) the total air-gap MMF (MMF_s) produced by the stator windings.

22 A fractional-pitch stator winding (*a* phase) is shown in developed form in Fig. 1.P-5. Each coil side contains n_c conductors. The winding is to be described as a sinusoidally distributed winding. Express the number of equivalent turns N_s in terms of n_c.

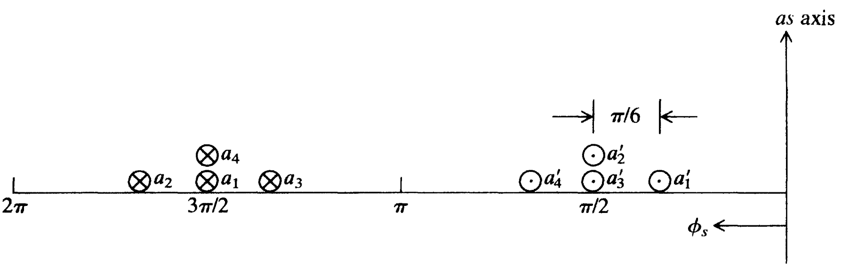

Figure 1.P-5 Stator winding arrangement for fractional-pitch winding.

23 Consider the 2-phase stator windings described in Problem 21. (*a*) With $I_{as} = \sqrt{2}I_s \cos[\omega_e t + \theta_{es}(0)]$ determine I_{bs} in order to obtain a constant amplitude MMF (MMF_s) which rotates clockwise around the air gap of the machine. Repeat (*a*) with (*b*) $I_{as} = -\sqrt{2}I_s \cos\omega_e t$, (*c*) $I_{as} = \sqrt{2}I_s \sin\omega_e t$, and (*d*)$I_{as} = -\sqrt{2}I_s \sin[\omega_e t + \theta_{es}(0)]$.

24 A single-phase source is connected between terminals *as* and *bs* of the 4-pole, 3-phase machine shown in Fig. 1.4-7. The *cs* terminal is open-circuited. The current into the *as* winding is $I_{as} = \sqrt{2}I\cos\omega_e t$. Express the total air-gap MMF.

25 An elementary 2-pole, 2-phase, salient-pole synchronous machine is shown in Fig. 1.P-6. The winding inductances may be expressed as

$$\begin{aligned}
L_{asas} &= L_{ls} + L_A - L_B \cos 2\theta_r \\
L_{bsbs} &= L_s + L_A + L_B \cos 2\theta_r \\
L_{asbs} &= -L_B \sin 2\theta_r \\
L_{fdfd} &= L_{lfd} + L_{mfd} \\
L_{asfd} &= L_{sfd} \sin\theta_r \\
L_{bsfd} &= -L_{sfd} \cos\theta_r
\end{aligned}$$

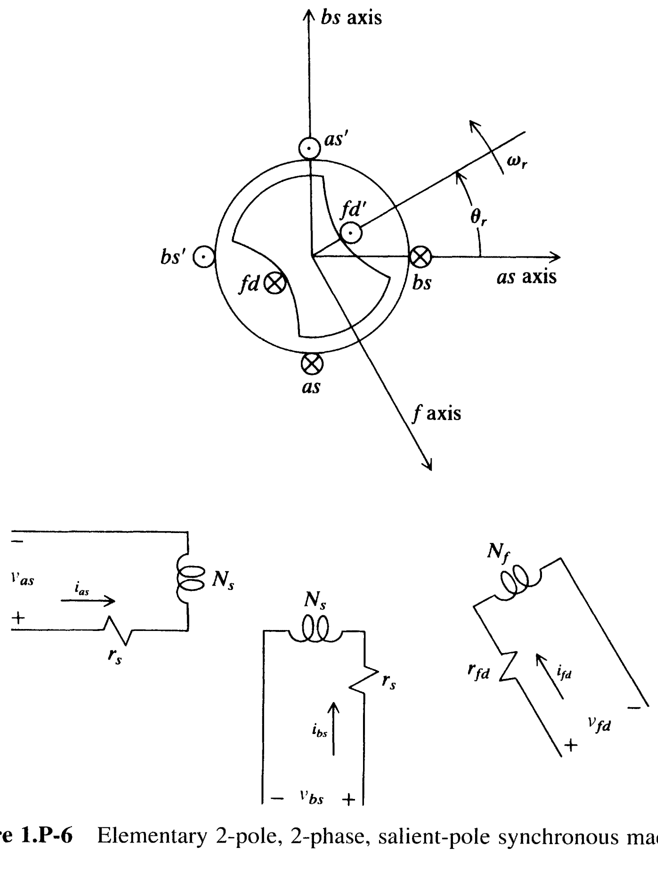

Figure 1.P-6 Elementary 2-pole, 2-phase, salient-pole synchronous machine.

Verify these relationships and give expressions for the coefficients L_A, L_B, L_{mfd}, and L_{sfd} similar in form to those given for a 3-phase machine. Modify these inductance relationships so that they will describe a 2-phase, uniform air-gap synchronous machine.

26 Write the voltage equations for the elementary 2-pole, 2-phase, salient-pole synchronous machine shown in Fig. 1.P-6 and derive the expression for $T_e(i_{as}, i_{bs}, i_{fd}, \theta_r)$.

27 An elementary 4-pole, 2-phase, salient-pole synchronous machine is shown in Fig. 1.P-7. Use this machine as a guide to derive expressions for the winding inductances of a P-pole synchronous machine. Show that these inductances are of the same form as those given in Problem 25 if $(P/2)\theta_{rm}$ is replaced by θ_r.

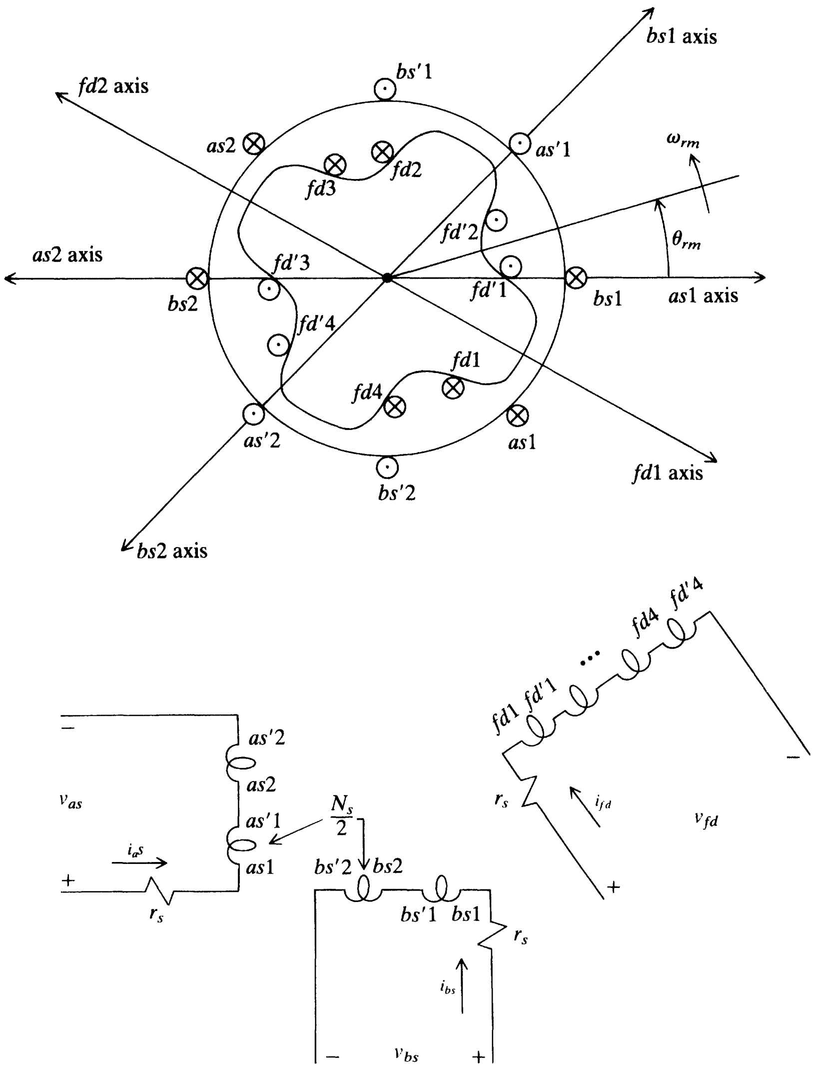

Figure 1.P-7 Elementary 4-pole, 2-phase, salient-pole synchronous machine.

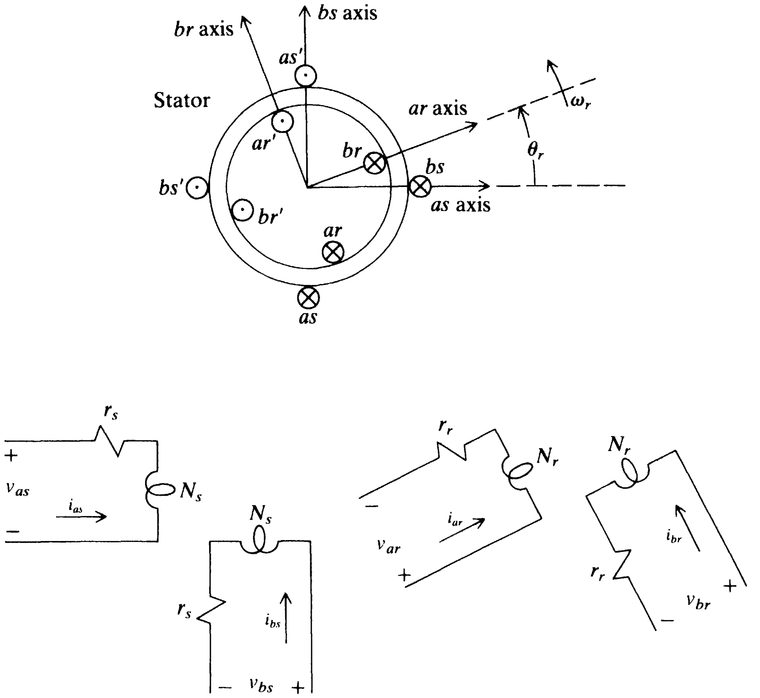

Figure 1.P-8 Elementary 2-pole, 2-phase symmetrical induction machine.

28 Derive an expression for the electromagnetic torque, $T_e(i_{as}, i_{bs}, i_{fd}, \theta_r)$, for a P-pole, 2-phase, salient-pole synchronous machine. This expression should be identical in form to that obtained in Problem 26 multiplied by $P/2$.

29 A reluctance machine has no field winding on the rotor. Modify the inductance relationships given in Problem 25 so as to describe the winding inductances of a 2-pole, 2-phase, reluctance machine. Write the voltage equations and derive an expression for $T_e(i_{as}, i_{bs}, \theta_r)$.

30 An elementary 2-pole, 2-phase, symmetrical induction machine is shown in Fig. 1.P-8. Derive expressions for the winding inductances. If you have worked Problem 25, you may modify those results accordingly.

31 Write the voltage equations for the induction machine shown in Fig. 1.P-8 and derive an expression for the electromagnetic torque $T_e(i_{as}, i_{bs}, i_{ar}, i_{br}, \theta_r)$.

32 An elementary 4-pole, 2-phase, symmetrical induction machine is shown in Fig. 1.P-9. Use this machine as a guide to derive expressions for the winding inductance of a P-pole induction machine. Show that these inductances are of the same form as those given in Problem 30 if $(P/2)\theta_{rm}$ is replaced by θ_r.

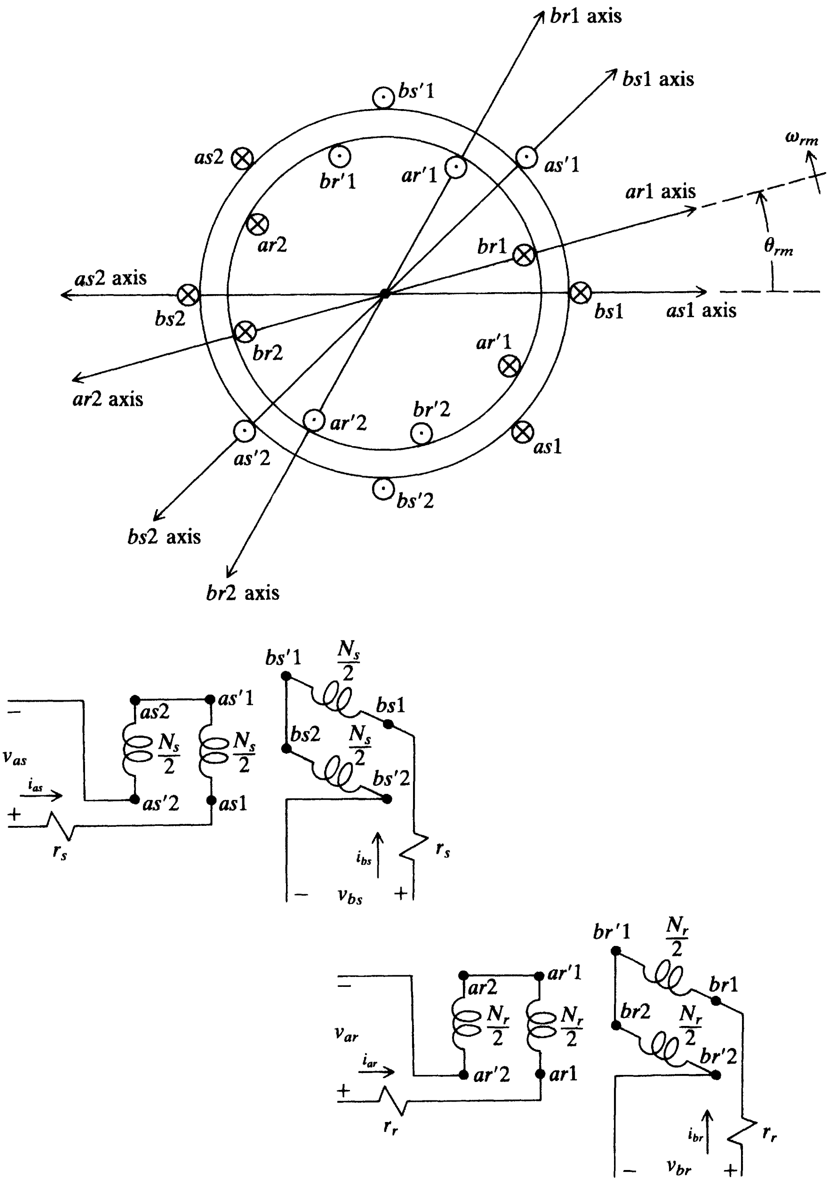

Figure 1.P-9 Elementary 4-pole, 2-phase symmetrical induction machine.