CHAPTER 1

FUNDAMENTAL CONCEPTS

1-1. Introduction. The topic of this book is the theory and analysis of electromagnetic phenomena that vary sinusoidally in time, henceforth called a-c (alternating-current) phenomena. The fundamental concepts which form the basis of our study are presented in this chapter. It is assumed that the reader already has some acquaintance with electromagnetic field theory and with electric circuit theory. The vector analysis concepts that we shall need are summarized in Appendix A.

We shall view electromagnetic phenomena from the "macroscopic" standpoint, that is, linear dimensions are large compared to atomic dimensions and charge magnitudes are large compared to atomic charges. This allows us to neglect the granular structure of matter and charge. We assume all matter to be stationary with respect to the observer. No treatment of the mechanical forces associated with the electromagnetic field is given.

The rationalized mksc system of units is used throughout. In this system the unit of length is the meter, the unit of mass is the kilogram, the unit of time is the second, and the unit of charge is the coulomb. We consider these units to be *fundamental units*. The units of all other quantities depend upon this choice of fundamental units, and are called *secondary units*. The mksc system of units is particularly convenient because the electrical units are identical to those used in practice.

The concepts necessary for our study are but a few of the many electromagnetic field concepts. We shall start with the familiar Maxwell equations and specialize them to our needs. New notation and nomenclature, more convenient for our purposes, will be introduced. For the most part, these innovations are extensions of a-c circuit concepts.

1-2. Basic Equations. The usual electromagnetic field equations are expressed in terms of six quantities. These are

$\mathcal{E}$, called the *electric intensity* (volts per meter)
$\mathcal{H}$, called the *magnetic intensity* (amperes per meter)
$\mathcal{D}$, called the *electric flux density* (coulombs per square meter)
$\mathcal{B}$, called the *magnetic flux density* (webers per square meter)
$\mathcal{J}$, called the *electric current density* (amperes per square meter)
q_v, called the *electric charge density* (coulombs per cubic meter)

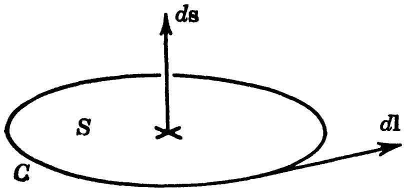

FIG. 1-1. $d\mathbf{l}$ and $d\mathbf{s}$ on an open surface.

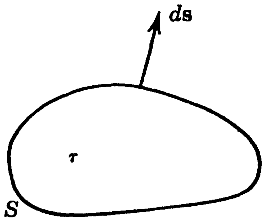

FIG. 1-2. $d\mathbf{s}$ on a closed surface.

We shall call a quantity *well-behaved* wherever it is a continuous function and has continuous derivatives. Wherever the above quantities are well-behaved, they obey the *Maxwell equations*

$$\begin{aligned} \nabla \times \boldsymbol{\mathcal{E}} &= -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} & \qquad \nabla \cdot \boldsymbol{\mathcal{B}} &= 0 \\ \nabla \times \boldsymbol{\mathcal{H}} &= \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} + \boldsymbol{\mathcal{J}} & \qquad \nabla \cdot \boldsymbol{\mathcal{D}} &= q_v \end{aligned} \tag{1-1}$$

These equations include the information contained in the *equation of continuity*

$$\nabla \cdot \boldsymbol{\mathcal{J}} = -\frac{\partial q_v}{\partial t} \tag{1-2}$$

which expresses the conservation of charge. Note that we have used boldface script letters for the various vector quantities, since we wish to reserve the usual boldface roman letters for complex quantities, introduced in Sec. 1-7.

Corresponding to each of Eqs. (1-1) are the integral forms of Maxwell's equations

$$\begin{aligned} \oint \boldsymbol{\mathcal{E}} \cdot d\mathbf{l} &= -\frac{d}{dt} \iint \boldsymbol{\mathcal{B}} \cdot d\mathbf{s} & \qquad \oiint \boldsymbol{\mathcal{B}} \cdot d\mathbf{s} &= 0 \\ \oint \boldsymbol{\mathcal{H}} \cdot d\mathbf{l} &= \frac{d}{dt} \iint \boldsymbol{\mathcal{D}} \cdot d\mathbf{s} + \iint \boldsymbol{\mathcal{J}} \cdot d\mathbf{s} & \qquad \oiint \boldsymbol{\mathcal{D}} \cdot d\mathbf{s} &= \iiint q_v \, d\tau \end{aligned} \tag{1-3}$$

These are actually more general than Eqs. (1-1) because it is no longer required that the various quantities be well-behaved. In the equations of the first column, we employ the usual convention that $d\mathbf{l}$ encircles $d\mathbf{s}$ according to the right-hand rule of Fig. 1-1. In the equations of the last column, we use the convention that $d\mathbf{s}$ points outward from a closed surface, as shown in Fig. 1-2. The circle on a line integral denotes a closed contour; the circle on a surface integral denotes a closed surface. The integral form of Eq. (1-2) is

$$\oiint \boldsymbol{\mathcal{J}} \cdot d\mathbf{s} = -\frac{d}{dt} \iiint q_v \, d\tau \tag{1-4}$$

where the same convention applies. This is the statement of conservation of charge as it applies to a region.

We shall use the name *field quantity* to describe the quantities discussed above. Associated with each field quantity there is a *circuit quantity*, or integral quantity. These circuit quantities are

v, called the *voltage* (volts)
i, called the *electric current* (amperes)
q, called the *electric charge* (coulombs)
ψ, called the *magnetic flux* (webers)
ψ^e, called the *electric flux* (coulombs)
u, called the *magnetomotive force* (amperes)

The explicit relationships of the field quantities to the circuit quantities can be summarized as follows:

$$\begin{aligned} v &= \int \mathcal{E} \cdot d\mathbf{l} & \psi &= \iint \mathcal{B} \cdot d\mathbf{s} \\ i &= \iint \mathcal{J} \cdot d\mathbf{s} & \psi^e &= \iint \mathcal{D} \cdot d\mathbf{s} \\ q &= \iiint q_v \, d\tau & u &= \int \mathcal{H} \cdot d\mathbf{l} \end{aligned} \tag{1-5}$$

All the circuit quantities are algebraic quantities and require reference conditions when designating them. Our convention for a "line-integral" quantity, such as voltage, is positive reference at the start of the path of integration. This is illustrated by Fig. 1-3. Our convention for a "surface-integral" quantity, such as current, is positive reference in the direction of $d\mathbf{s}$. This is shown in Fig. 1-4. Charge is a "net-amount" quantity, being the amount of positive charge minus the amount of negative charge.

We shall call Eqs. (1-1) to (1-4) *field equations*, since all quantities appearing in them are field quantities. Corresponding equations written in terms of circuit quantities we shall describe as *circuit equations*. Equa-

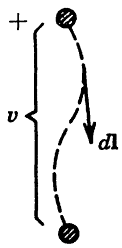

Fig. 1-3. Reference convention for voltage.

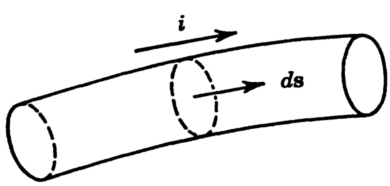

Fig. 1-4. Reference convention for current.

tions (1-3) are commonly written in mixed field and circuit form as

$$\oint \mathcal{E} \cdot d\mathbf{l} = -\frac{d\psi}{dt} \qquad \oiint \mathcal{B} \cdot d\mathbf{s} = 0$$
$$\oint \mathcal{H} \cdot d\mathbf{l} = \frac{d\psi^e}{dt} + i \qquad \oiint \mathcal{D} \cdot d\mathbf{s} = q \tag{1-6}$$

Similarly, the equation of continuity in mixed field and circuit form is

$$\oiint \mathcal{J} \cdot d\mathbf{s} = -\frac{dq}{dt} \tag{1-7}$$

Finally, the various equations can be written entirely in terms of circuit quantities. For this, we shall use the notation that Σ denotes summation over a closed contour for a line-integral quantity, and summation over a closed surface for a surface-integral quantity. In this notation, the circuit forms of Eqs. (1-6) are

$$\sum v = -\frac{d\psi}{dt} \qquad \sum \psi = 0$$
$$\sum u = \frac{d\psi^e}{dt} + i \qquad \sum \psi^e = q \tag{1-8}$$

and the circuit form of Eq. (1-7) is

$$\sum i = -\frac{dq}{dt} \tag{1-9}$$

Note that the first of Eqs. (1-8) is a generalized form of *Kirchhoff's voltage law,* and Eq. (1-9) is a generalized form of *Kirchhoff's current law.*

It is apparent from the preceding summary that many mathematical forms can be used to present a single physical concept. An understanding of the concepts is an invaluable aid to remembering the equations. While an extensive exposition of these concepts properly belongs in an introductory textbook, let us here summarize them. Consider the sets of Eqs. (1-1), (1-3), (1-6), and (1-8). The first equation in each set is essentially *Faraday's law of induction.* It states that a changing magnetic flux induces a voltage in a path surrounding it. The second equation in each set is essentially *Ampère's circuital law,* extended to the time-varying case. It is a partial definition of magnetic intensity and magnetomotive force. The third equation of each set states that magnetic flux has no "flux source," that is, lines of $\mathcal{B}$ can have no beginning or end. The fourth equation in each set is *Gauss' law* and states that lines of $\mathcal{D}$ begin and end on electric charge. It is essentially a partial definition of electric flux. Finally, Eqs. (1-2), (1-4), (1-7), and (1-9) are all forms of the *law of conservation of charge.* They state that charge can be neither created nor destroyed, merely transported. Lines of current must begin and end at points of increasing or decreasing charge density.

1-3. Constitutive Relationships. In addition to the equations of Sec. 1-2 we need equations specifying the characteristics of the medium in which the field exists. We shall consider the domain of $\mathcal{E}$ and $\mathcal{H}$ as the electromagnetic field and express $\mathcal{D}$, $\mathcal{B}$, and $\mathcal{J}$ in terms of $\mathcal{E}$ and $\mathcal{H}$. Equations of the general form

$$\begin{aligned} \mathcal{D} &= \mathcal{D}(\mathcal{E},\mathcal{H}) \\ \mathcal{B} &= \mathcal{B}(\mathcal{E},\mathcal{H}) \\ \mathcal{J} &= \mathcal{J}(\mathcal{E},\mathcal{H}) \end{aligned} \qquad (1\text{-}10)$$

are called *constitutive relationships.* Explicit forms for these can be found by experimentation or deduced from atomic considerations.

The term *free space* will be used to denote vacuum or any other medium having essentially the same characteristics as vacuum (such as air). The constitutive relationships assume the particularly simple forms

$$\left.\begin{aligned} \mathcal{D} &= \epsilon_0\mathcal{E} \\ \mathcal{B} &= \mu_0\mathcal{H} \\ \mathcal{J} &= 0 \end{aligned}\right\} \quad \text{in free space} \qquad (1\text{-}11)$$

where ϵ_0 is the *capacitivity* or *permittivity* of vacuum, and μ_0 is the *inductivity* or *permeability* of vacuum. It is a mathematical consequence of the field equations that $(\epsilon_0\mu_0)^{-1/2}$ is the velocity of propagation of an electromagnetic disturbance in free space. Light is electromagnetic in nature, and this velocity is called the *velocity of light c.* Measurements have established that

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 2.99790 \times 10^8 \approx 3 \times 10^8 \text{ meters per second} \qquad (1\text{-}12)$$

The choice of either ϵ_0 or μ_0 determines a system of electromagnetic units according to our equations. By international agreement, the value of μ_0 has been chosen as

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry per meter} \qquad (1\text{-}13)$$

for the mksc system of units. It then follows from Eq. (1-12) that

$$\epsilon_0 = 8.854 \times 10^{-12} \approx \frac{1}{36\pi} \times 10^{-9} \text{ farad per meter} \qquad (1\text{-}14)$$

for the mksc system of units.

Under certain conditions, the constitutive relationships become simple proportionalities for many materials. We say that such matter is linear in the simple sense, and call it *simple matter* for short. Thus

$$\left.\begin{aligned} \mathcal{D} &= \epsilon\mathcal{E} \\ \mathcal{B} &= \mu\mathcal{H} \\ \mathcal{J} &= \sigma\mathcal{E} \end{aligned}\right\} \quad \text{in simple matter} \qquad (1\text{-}15)$$

where, as in the free-space case, ϵ is called the capacitivity of the medium and μ is called the inductivity of the medium. The parameter σ is called the *conductivity* of the medium. We originally made the qualifying statement that Eqs. (1-15) hold "under certain conditions." They may not hold if $\mathcal{E}$ or $\mathcal{H}$ are very large, or if time derivatives of $\mathcal{E}$ or $\mathcal{H}$ are very large.

Matter is often classified according to its values of σ, ϵ, and μ. Materials having large values of σ are called *conductors* and those having small values of σ are called *insulators* or *dielectrics.* For analyses, it is often convenient to approximate good conductors by *perfect conductors,* characterized by $\sigma = \infty$, and to approximate good dielectrics by *perfect dielectrics,* characterized by $\sigma = 0$. The capacitivity ϵ of any material is never less than that of vacuum ϵ_0. The ratio $\epsilon_r = \epsilon/\epsilon_0$ is called the *dielectric constant* or *relative capacitivity.* The dielectric constant of a good conductor is hard to measure but appears to be unity. For most linear matter, the inductivity μ is approximately that of free space μ_0. There is a class of materials, called *diamagnetic,* for which μ is slightly less than μ_0 (of the order of 0.01 per cent). There is a class of materials, called *paramagnetic,* for which μ is slightly greater than μ_0 (again of the order of 0.01 per cent). A third class of materials, called *ferromagnetic,* has values of μ much larger than μ_0, but these materials are often nonlinear. For our purposes, we shall call all materials except the ferromagnetic ones *nonmagnetic* and take $\mu = \mu_0$ for them. The ratio $\mu_r = \mu/\mu_0$ is called the *relative inductivity* or *relative permeability* and is, of course, essentially unity for nonmagnetic matter.

Quite often the restriction on the time rate of change of the field, made on the validity of Eqs. (1-15), can be overcome by extending the definition of linearity. We say that matter is linear in the general sense, and call it *linear matter,* when the constitutive relationships are the following linear differential equations:

$$\left.\begin{aligned} \mathcal{D} &= \epsilon\mathcal{E} + \epsilon_1 \frac{\partial \mathcal{E}}{\partial t} + \epsilon_2 \frac{\partial^2 \mathcal{E}}{\partial t^2} + \cdots \\ \mathcal{B} &= \mu\mathcal{H} + \mu_1 \frac{\partial \mathcal{H}}{\partial t} + \mu_2 \frac{\partial^2 \mathcal{H}}{\partial t^2} + \cdots \\ \mathcal{J} &= \sigma\mathcal{E} + \sigma_1 \frac{\partial \mathcal{E}}{\partial t} + \sigma_2 \frac{\partial^2 \mathcal{E}}{\partial t^2} + \cdots \end{aligned}\right\} \quad \text{in linear matter} \qquad (1\text{-}16)$$

Even more complicated formulas for the constitutive relationships may be necessary in some cases, but Eqs. (1-16) are the most general that we shall consider. Note that Eqs. (1-16) reduce to Eqs. (1-15) when the time derivatives of $\mathcal{E}$ and $\mathcal{H}$ become sufficiently small.

The physical significance of the extended definition of linearity is as follows. The atomic particles of matter have mass as well as charge, so when the field changes rapidly the particles cannot "follow" the field. For example, suppose an electron has been accelerated by the field, and then the direction of $\mathcal{E}$ changes. There will be a time lag before the electron can change direction, because of its momentum. Such a picture holds for $\mathcal{J}$ if the electron is a free electron. It holds for $\mathcal{D}$ if the electron is a bound electron. A similar picture holds for $\mathcal{B}$ except that the magnetic moment of the electron is the contributing quantity. We shall not attempt to give significance to each term of Eqs. (1-16). It will be shown in Sec. 1-9 that all terms of Eqs. (1-16) contribute to an "admittivity" and an "impedivity" of a material in the time-harmonic case.

1-4. The Generalized Current Concept. It was Maxwell who first noted that Ampère's law for statics, $\nabla \times \mathcal{H} = \mathcal{J}$, was incomplete for time-varying fields. He amended the law to include an *electric displacement current* $\partial\mathcal{D}/\partial t$ in addition to the conduction current. He visualized this displacement current in free space as a motion of bound charge in an "ether," an ideal weightless fluid permeating all space. We have since discarded the concept of an ether, for it has proved undetectable and even somewhat illogical in view of the theory of relativity. In dielectrics, part of the term $\partial\mathcal{D}/\partial t$ is a motion of the bound particles and is thus a current in the true sense of the word. However, it is convenient to consider the entire $\partial\mathcal{D}/\partial t$ term as a current. In view of the symmetry of Maxwell's equations, it also is convenient to consider the term $\partial\mathcal{B}/\partial t$ as a *magnetic displacement current.* Finally, to represent sources, we amend the field equations to include *impressed currents,* electric and magnetic. These are the currents we view as the cause of the field. We shall see in the next section that the impressed currents represent energy sources.

The symbols $\mathcal{J}$ and $\mathcal{M}$ will be used to denote electric and magnetic currents in general, with superscripts indicating the type of current. As discussed above, we define total currents

$$\begin{aligned} \mathcal{J}^t &= \frac{\partial\mathcal{D}}{\partial t} + \mathcal{J}^c + \mathcal{J}^i \\ \mathcal{M}^t &= \frac{\partial\mathcal{B}}{\partial t} + \mathcal{M}^i \end{aligned} \qquad (1\text{-}17)$$

where the superscripts t, c, and i denote total, conduction, and impressed currents. The symbols i and k will be used to denote net electric and magnetic currents, and the same superscripts will indicate the type.

Thus, the circuit form corresponding to Eqs. (1-17) is

$$\begin{aligned} i^t &= \frac{d\psi^e}{dt} + i^c + i^i \\ k^t &= \frac{d\psi}{dt} + k^i \end{aligned} \tag{1-18}$$

The i and k are, of course, related to the $\mathcal{J}$ and $\mathfrak{M}$ by

$$i = \iint \mathcal{J} \cdot d\mathbf{s} \qquad k = \iint \mathfrak{M} \cdot d\mathbf{s} \tag{1-19}$$

where these apply to any of the various types of current.

In terms of the generalized current concept, the basic equations of electromagnetism become, in the differential form,

$$\nabla \times \mathcal{E} = -\mathfrak{M}^t \qquad \nabla \times \mathcal{H} = \mathcal{J}^t \tag{1-20}$$

and in the integral form,

$$\oint \mathcal{E} \cdot d\mathbf{l} = -\iint \mathfrak{M}^t \cdot d\mathbf{s} \qquad \oint \mathcal{H} \cdot d\mathbf{l} = \iint \mathcal{J}^t \cdot d\mathbf{s} \tag{1-21}$$

Also, the mixed field-circuit form is

$$\oint \mathcal{E} \cdot d\mathbf{l} = -k^t \qquad \oint \mathcal{H} \cdot d\mathbf{l} = i^t \tag{1-22}$$

and the circuit form is

$$\Sigma v = -k^t \qquad \Sigma u = i^t \tag{1-23}$$

Note that these look simpler than the equations of Sec. 1-2. Actually, we have merely included many concepts in the functions $\mathfrak{M}^t$ and $\mathcal{J}^t$; so some of the information contained in the original Maxwell equations has become hidden. However, our study comprises only a small portion of the general theory of electromagnetism, and the forms of Eqs. (1-20) to (1-23) are well suited to our purposes.

Note that we have omitted the "divergence equations" of Maxwell from our above sets of equations. We have done so to emphasize that this information is included in the above sets. For example, taking the divergence of each of Eqs. (1-20), we obtain

$$\nabla \cdot \mathfrak{M}^t = 0 \qquad \nabla \cdot \mathcal{J}^t = 0 \tag{1-24}$$

for $\nabla \cdot \nabla \times \mathcal{A} = 0$ is an identity. Similarly, Eqs. (1-21) applied to closed surfaces became

$$\oiint \mathfrak{M}^t \cdot d\mathbf{s} = 0 \qquad \oiint \mathcal{J}^t \cdot d\mathbf{s} = 0 \tag{1-25}$$

Thus, the total currents are solenoidal. Lines of total current have no beginning or end but must be continuous.

As an illustration of the generalized current concept, consider the circuits of Figs. 1-5 and 1-6. In Fig. 1-5, the "current source" $\mathcal{J}^i$ produces a conduction current $\mathcal{J}^c$ through the resistor and a displacement current $\mathcal{J}^d = \partial\mathcal{D}/\partial t$ through the capacitor. In Fig. 1-6, the "voltage source" $\mathcal{M}^i$ produces an electric current in the wire which in turn causes the magnetic displacement current $\mathcal{M}^d = \partial\mathcal{B}/\partial t$ in the magnetic core. In these pictures we have used the convention that a single-headed arrow represents an electric current, a double-headed arrow represents a magnetic current.

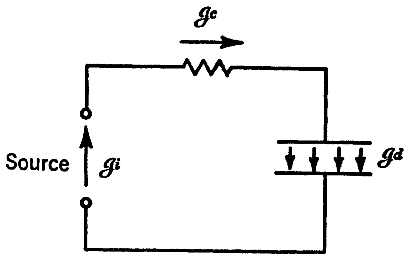

Fig. 1-5. Types of electric current.

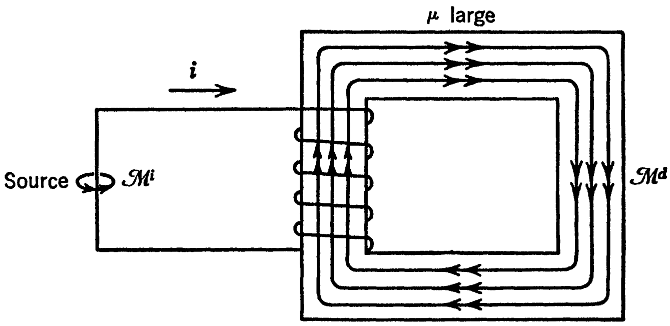

Fig. 1-6. Types of magnetic current.

It is not possible at this time to give the reader a complete picture of the usefulness of impressed currents. Figures 1-5 and 1-6 anticipate one application, namely, that of representing sources. More generally, the impressed currents are those currents we *view* as sources. In a sense, the impressed currents are those currents in terms of which the field is expressed. In one problem, a conduction current might be considered as the source, or impressed, current. In another problem, a polarization or magnetization current might be considered as the source current. Our understanding of the concept will grow as we learn to use it.

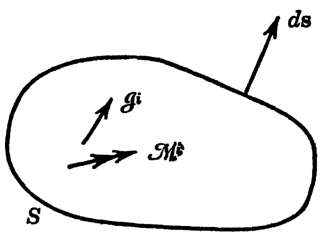

Fig. 1-7. A region containing sources.

1-5. Energy and Power. Consider a region of electromagnetic field, as suggested by Fig. 1-7. The field obeys the Maxwell equations, which in generalized current notation are Eqs. (1-20). As an extension of circuit concepts, it can be shown that a product $\mathcal{E} \cdot \mathcal{J}$ is a power density. This suggests a scalar multiplication of the second of Eqs. (1-20) by $\mathcal{E}$. Also, in view of the vector identity

$$\nabla \cdot (\mathcal{E} \times \mathcal{H}) = \mathcal{H} \cdot \nabla \times \mathcal{E} - \mathcal{E} \cdot \nabla \times \mathcal{H}$$

a scalar multiplication of the first of Eqs. (1-20) by $\mathcal{H}$ is suggested. The difference of the resulting two equations is

$$\nabla \cdot (\mathcal{E} \times \mathcal{H}) + \mathcal{E} \cdot \mathcal{J}^t + \mathcal{H} \cdot \mathcal{M}^t = 0 \tag{1-26}$$

If this equation is integrated throughout a region, and the divergence theorem applied to the first term, there results

$$\oiint \mathcal{E} \times \mathcal{H} \cdot d\mathbf{s} + \iiint (\mathcal{E} \cdot \mathcal{J}^t + \mathcal{H} \cdot \mathcal{M}^t)\, d\tau = 0 \tag{1-27}$$

We shall interpret these as equations for the *conservation of energy*, Eq. (1-26) being the differential form and Eq. (1-27) being the integral form.

The generally accepted interpretation of Eqs. (1-26) and (1-27) is as follows. The *Poynting vector*

$$\mathcal{S} = \mathcal{E} \times \mathcal{H} \tag{1-28}$$

is postulated to be a density-of-power flux. The point relationship

$$p_f = \nabla \cdot \mathcal{S} = \nabla \cdot (\mathcal{E} \times \mathcal{H}) \tag{1-29}$$

is then a volume density of power leaving the point, and the integral

$$\mathcal{P}_f = \oiint \mathcal{S} \cdot d\mathbf{s} = \oiint \mathcal{E} \times \mathcal{H} \cdot d\mathbf{s} \tag{1-30}$$

is the total power leaving the region bounded by the surface of integration. The other terms of Eq. (1-26) can then be interpreted as the rate of increase in energy density at a point. Similarly, the other terms of Eq. (1-27) can be interpreted as the rate of increase in energy within the region. Further identification of this energy can be made in particular cases.

For media linear in the simple sense, as defined by Eqs. (1-15), the last two terms of Eq. (1-26) become

$$\begin{aligned} \mathcal{E} \cdot \mathcal{J}^t &= \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon\mathcal{E}^2\right) + \sigma\mathcal{E}^2 + \mathcal{E} \cdot \mathcal{J}^i \\ \mathcal{H} \cdot \mathcal{M}^t &= \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\mathcal{H}^2\right) + \mathcal{H} \cdot \mathcal{M}^i \end{aligned} \tag{1-31}$$

where $\mathcal{J}^i$ and $\mathcal{M}^i$ represent possible source currents. The terms

$$w_e = \tfrac{1}{2}\epsilon\mathcal{E}^2 \qquad w_m = \tfrac{1}{2}\mu\mathcal{H}^2 \tag{1-32}$$

are identified as the electric and magnetic energy densities of static fields, and this interpretation is retained for dynamic fields. The term

$$p_d = \sigma \mathcal{E}^2 \tag{1-33}$$

is identified as the density of power converted to heat energy, called *dissipated power*. Finally, the density of power supplied by the source currents is defined as

$$p_s = -(\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}}^i + \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\mathcal{M}}^i) \tag{1-34}$$

The reference direction for source power is opposite to that for dissipated power, as evidenced by the minus sign of Eq. (1-34). In terms of the above-defined quantities, we can rewrite Eq. (1-26) as

$$p_s = p_f + p_d + \frac{\partial}{\partial t}(w_e + w_m) \tag{1-35}$$

A word statement of this equation is: At any point, the density of power supplied by the sources must equal that leaving the point plus that dissipated plus the rate of increase in stored electric and magnetic energy densities.

A more common statement of the conservation of energy is that which refers to an entire region. Corresponding to the densities of Eqs. (1-32), we define the net electric and magnetic energies within a region as

$$\mathcal{W}_e = \tfrac{1}{2}\iiint \epsilon \mathcal{E}^2 \, d\tau \qquad \mathcal{W}_m = \tfrac{1}{2}\iiint \mu \mathcal{H}^2 \, d\tau \tag{1-36}$$

Corresponding to Eq. (1-33), we define the net power converted to heat energy as

$$\mathcal{P}_d = \iiint \sigma \mathcal{E}^2 \, d\tau \tag{1-37}$$

Finally, corresponding to Eq. (1-34), we define the net power supplied by sources within the region as

$$\mathcal{P}_s = -\iiint (\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}}^i + \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\mathcal{M}}^i) \, d\tau \tag{1-38}$$

In terms of these definitions, Eq. (1-27) can be written as

$$\mathcal{P}_s = \mathcal{P}_f + \mathcal{P}_d + \frac{d}{dt}(\mathcal{W}_e + \mathcal{W}_m) \tag{1-39}$$

Thus, the power supplied by the sources within a region must equal that leaving the region plus that dissipated within the region plus the rate of increase in electric and magnetic energies stored within the region.

If we proceed to the general definition of linearity, Eqs. (1-16), the separation of power into a reversible energy change (storage) and an irreversible energy change (dissipation) is no longer easy. Contributions to energy storage and to energy dissipation may originate from both conduction and displacement currents. However, Eqs. (1-35) and (1-39) still apply to media linear in the general sense. We merely cannot identify the various terms. In Sec. 1-10 we shall see that for a-c fields the division of energy into stored and dissipated components again assumes a simple form.

1-6. Circuit Concepts. The usual equations of circuit theory are specializations of the field equations. Our knowledge of circuit concepts can therefore be of help to us in understanding field concepts. In this section we shall quickly review this relationship of circuits to fields.

Kirchhoff's current law for circuits is an application of the equation of conservation of charge to surfaces enclosing wire junctions. To demonstrate, consider the parallel *RLC* circuit of Fig. 1-8. Let the letter o denote the junction, and the letters a, b, c, d denote the upper terminals of the elements. We apply Eq. (1-7) to a surface enclosing the junction, as represented by the dotted line in Fig. 1-8. The result is

$$i_{oa} + i_{ob} + i_{oc} + i_{od} + i_l + \frac{dq}{dt} = 0$$

where the i_{on} are the currents in the wires, i_l is the leakage current crossing the surface outside of the wires, and q is the charge on the junction. The term dq/dt can be thought of as the current through the stray capacitance between the top and bottom junctions. In most circuit applications both i_l and dq/dt are negligible, and the above equation reduces to

$$i_{oa} + i_{ob} + i_{oc} + i_{od} = 0$$

This is the usual expression of the Kirchhoff current law for the circuit of Fig. 1-8.

Kirchhoff's voltage law for circuits is an application of the first Maxwell equation to closed contours following the connecting wires of the circuit and closing across the terminals of the elements. To demonstrate, consider the series *RLC* circuit of Fig. 1-9. Let the letters a to h denote

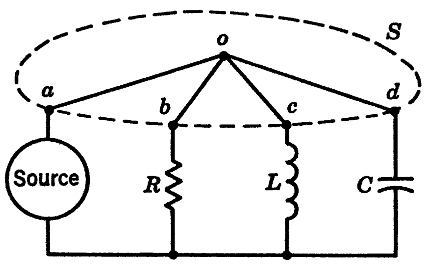

Fig. 1-8. A parallel *RLC* circuit.

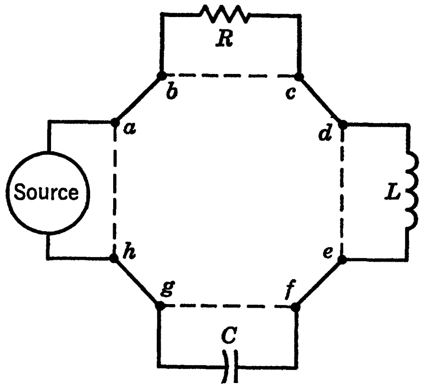

FIG. 1-9. A series RLC circuit.

the terminals of the elements as shown. We apply the first of Eqs. (1-6) to the contour *abcdefgha*, following the dotted lines between terminals. This gives

$$v_{ab} + v_{bc} + v_{cd} + v_{de} + v_{ef} + v_{fg} + v_{gh} + v_{ha} + \frac{d\psi}{dt} = 0$$

where the v_{mn} are the voltage drops along the contour and ψ is the magnetic flux enclosed. The voltages v_{ab}, v_{cd}, v_{ef}, and v_{gh} are due to the resistance of the wire. The term $d\psi/dt$ is the voltage of the stray inductance of the loop. When the wire resistance and the stray inductance can be neglected, the above equation reduces to

$$v_{bc} + v_{de} + v_{fg} + v_{ha} = 0$$

This is the usual form of Kirchhoff's voltage law for the circuit of Fig. 1-9.

In addition to Kirchhoff's laws, circuit theory uses a number of "element laws." Ohm's law for resistors, $v = Ri$, is a specialization of the constitutive relationship $\mathcal{J} = \sigma\mathcal{E}$. The law for capacitors, $q = Cv$, expresses the same concept as $\mathcal{D} = \epsilon\mathcal{E}$. We have from the equation of continuity $i = dq/dt$, so the capacitor law can also be written as $i = C\,dv/dt$. The law for inductors, $\psi = Li$, expresses the same concept as $\mathcal{B} = \mu\mathcal{H}$. From the first Maxwell equation we have $v = d\psi/dt$, so the inductor law can also be written as $v = L\,di/dt$. Finally, the various energy relationships for circuit theory can be considered as specializations of those for field theory. Detailed expositions of the various specializations mentioned above can be found in elementary textbooks. Table 1-1 summarizes the various correspondences between field concepts and circuit concepts.

1-7. Complex Quantities. When the fields are a-c, that is, when the time variation is harmonic, the mathematical analysis can be simplified

TABLE 1-1. CORRESPONDENCES BETWEEN CIRCUIT CONCEPTS AND FIELD CONCEPTS

Circuit concepts	Field concepts
Voltage v	Electric intensity $\mathcal{E}$
Current i	Electric current density $\mathcal{J}$ or magnetic intensity $\mathcal{H}$
Magnetic flux ψ	Magnetic flux density $\mathcal{B}$
Charge q	Charge density q_v or electric flux density $\mathcal{D}$
Kirchhoff's voltage law (generalized) $\sum v_n = -\frac{d\psi}{dt}$	Maxwell-Faraday equation $\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$
Kirchhoff's current law (generalized) $\sum i_n = -\frac{dq}{dt}$	Equation of continuity $\nabla \cdot \mathcal{J}^c = -\frac{\partial q_v}{\partial t}$
Element laws (linear) Resistors $i = \frac{1}{R} v$ Capacitors $q = Cv$ or $i = C\frac{dv}{dt}$ Inductors $\psi = Li$ or $v = L\frac{di}{dt}$	Constitutive relationships (linear in the simple sense) Conductors $\mathcal{J}^c = \sigma\mathcal{E}$ Dielectrics $\mathcal{D} = \epsilon\mathcal{E}$ or $\mathcal{J}^d = \epsilon\frac{\partial \mathcal{E}}{\partial t}$ Magnetic properties $\mathcal{B} = \mu\mathcal{H}$ or $\mathcal{M}^d = \mu\frac{\partial \mathcal{H}}{\partial t}$
Power flow $p_f = vi$	Power flow $\mathcal{S} = \mathcal{E} \times \mathcal{H}$
Power dissipation in resistors $\mathcal{P}_d = vi = \frac{1}{R} v^2$	Power dissipation $p_d = \mathcal{E} \cdot \mathcal{J}^c = \sigma\mathcal{E}^2$
Energy in capacitors $\mathcal{W}_e = \frac{1}{2}qv = \frac{1}{2}Cv^2$	Electric energy $w_e = \frac{1}{2}\mathcal{D} \cdot \mathcal{E} = \frac{1}{2}\epsilon\mathcal{E}^2$
Energy in inductors $\mathcal{W}_m = \frac{1}{2}\psi i = \frac{1}{2}Li^2$	Magnetic energy $w_m = \frac{1}{2}\mathcal{B} \cdot \mathcal{H} = \frac{1}{2}\mu\mathcal{H}^2$

by using complex quantities. The basis for this is Euler's identity

$$e^{j\alpha} = \cos \alpha + j \sin \alpha$$

where $j = \sqrt{-1}$. This gives us a relationship between real sinusoidal functions and the complex exponential function.

Any a-c quantity can be represented by a complex quantity. A scalar quantity is interpreted according to[1]

$$v = \sqrt{2}\,|V| \cos (\omega t + \alpha) = \sqrt{2}\,\text{Re}\,(Ve^{j\omega t}) \tag{1-40}$$

where v is called the *instantaneous quantity* and $V = |V|e^{j\alpha}$ is called the *complex quantity*. The notation Re ( ) stands for "the real part of," that is, the part not associated with j. Other names for V are "phasor quantity" and "vector quantity," the last name causing confusion with space vectors. In our notation v represents a voltage, hence V is a *complex voltage*. Equation (1-40) with v replaced by i and V replaced by I would define a *complex current*, and so on. Note that the complex quantity is *not* a function of time but it may be a function of position. Note also that the magnitude of the complex quantity is the effective (root-mean-square) value of the instantaneous quantity. We have chosen it so because (1) a-c quantities are usually specified or measured in effective values in practice, and (2) equations for complex power and energy retain the same proportionality factors as do their instantaneous counterparts. For example, in circuit theory the instantaneous power is $p = vi$, and complex power is $P = VI^*$. A factor of $\frac{1}{2}$ appears in the equation for complex power if peak values of v and i are used for $|V|$ and $|I|$.

Complex notation can readily be extended to vectors having sinusoidal time variation. A *complex* **E** is defined as related to an *instantaneous* $\boldsymbol{\mathcal{E}}$ according to

$$\boldsymbol{\mathcal{E}} = \sqrt{2}\,\text{Re}\,(\mathbf{E}e^{j\omega t}) \tag{1-41}$$

This means that the spatial components of **E** are related to the spatial components of $\boldsymbol{\mathcal{E}}$ by Eq. (1-40). For example, the x components of **E** and $\boldsymbol{\mathcal{E}}$ are related by

$$\mathcal{E}_x = \sqrt{2}\,\text{Re}\,(E_x e^{j\omega t}) = \sqrt{2}\,|E_x| \cos (\omega t + \alpha_x)$$

where $E_x = |E_x|e^{j\alpha_x}$. Similar equations relate the y and z components of **E** and $\boldsymbol{\mathcal{E}}$. The phase of each component may be different from the phases of the other two components, that is, α_x, α_y, and α_z are not necessarily equal. In our notation $\boldsymbol{\mathcal{E}}$ is an electric intensity, hence **E** is called the *complex electric intensity*. Equation (1-41) with **E** replaced by **H** and $\boldsymbol{\mathcal{E}}$ by $\boldsymbol{\mathcal{H}}$

[1] The convention $v = \sqrt{2}\,\text{Im}\,(Ve^{j\omega t})$ can also be used, where Im ( ) stands for "the imaginary part of." The factor $\sqrt{2}$ can be omitted if it is desired that $|V|$ be the peak value of v.

defines a *complex magnetic intensity* **H**, representing the instantaneous magnetic intensity $\mathcal{H}$, and so on. Note that the magnitude of a component of the complex vector is the effective value of the corresponding component of the instantaneous vector. This choice corresponds to that taken for complex scalars and has essentially the same advantages.

A real vector, such as $\mathcal{E}$ or $\mathcal{H}$, can be thought of as a triplet of real scalar functions, namely, the x, y, and z components. At any instant of time, the vector has a definite magnitude and direction at every point in space and can be represented in three dimensions by arrows. A complex vector, such as **E** or **H**, is a group of six real scalar functions, namely, the real and imaginary parts of the x, y, and z components. It *cannot* be represented by arrows in three-dimensional space except in special cases. One such special case is that for which $\alpha_x = \alpha_y = \alpha_z$, so that the vector has a real direction in space. In this case the instantaneous vector always points in the same direction (or opposite direction), at a point in space, changing only in amplitude. We could define a "complex magnitude" and a "complex direction" for a complex vector as extensions of the corresponding definitions for real vectors, but these would have little use.

Throughout this book we shall use the following notation. Instantaneous quantities are denoted by script letters or lower-case letters. Complex quantities which represent the instantaneous quantities are denoted by the corresponding capital letter. Vectors are denoted by boldface type.

1-8. Complex Equations. The symbol Re ( ) can be considered as a mathematical operator which selects the real part of a complex quantity. A set of rules for manipulating the operator Re ( ) can be formulated from the properties of complex functions. The following are the rules we shall need. Let a capital letter denote a complex quantity and a lower-case letter denote a real quantity. Then

$$\begin{aligned} \operatorname{Re}(A) + \operatorname{Re}(B) &= \operatorname{Re}(A + B) \\ \operatorname{Re}(aA) &= a \operatorname{Re}(A) \\ \frac{\partial}{\partial x} \operatorname{Re}(A) &= \operatorname{Re}\left(\frac{\partial A}{\partial x}\right) \\ \int \operatorname{Re}(A)\, dx &= \operatorname{Re}\left(\int A\, dx\right) \end{aligned} \tag{1-42}$$

The proof of these is left to the reader.

In addition to the above equations we shall need the following lemma. *If A and B are complex quantities, and* $\operatorname{Re}(Ae^{j\omega t}) = \operatorname{Re}(Be^{j\omega t})$ *for all t, then $A = B$.* We can readily show this by first taking $t = 0$, obtaining $\operatorname{Re}(A) = \operatorname{Re}(B)$, and then taking $\omega t = \pi/2$, obtaining $\operatorname{Im}(A) = \operatorname{Im}(B)$. Thus, $A = B$, for the above two equalities are the definition of this.

To illustrate the derivation of an equation for complex quantities from one for instantaneous quantities, consider

$$v = \int \mathcal{E} \cdot d\mathbf{l}$$

Expressing v and $\mathcal{E}$ in terms of their complex counterparts, we have

$$\sqrt{2}\,\text{Re}\,(Ve^{j\omega t}) = \int \sqrt{2}\,\text{Re}\,(\mathbf{E}e^{j\omega t}) \cdot d\mathbf{l}$$

By steps justifiable by Eqs. (1-42), this reduces to

$$\sqrt{2}\,\text{Re}\,(Ve^{j\omega t}) = \sqrt{2}\,\text{Re}\left(e^{j\omega t} \int \mathbf{E} \cdot d\mathbf{l}\right)$$

Cancellation of the $\sqrt{2}$'s and application of the above lemma then gives

$$V = \int \mathbf{E} \cdot d\mathbf{l}$$

Note that this is of the same form as the original instantaneous equation. We have illustrated the procedure with a scalar equation, but the same steps apply to the components of a vector equation.

From our rules for manipulation of the Re ( ) operator, it should be apparent that any equation linearly relating instantaneous quantities and *not* involving time differentiation takes the same form for complex quantities. Thus, the complex circuit quantities V, I, U, and K are related to the complex field quantities **E**, **H**, **J**, and **M** according to

$$\begin{aligned} V &= \int \mathbf{E} \cdot d\mathbf{l} \qquad & U &= \int \mathbf{H} \cdot d\mathbf{l} \\ I &= \iint \mathbf{J} \cdot d\mathbf{s} \qquad & K &= \iint \mathbf{M} \cdot d\mathbf{s} \end{aligned} \tag{1-43}$$

There is no time differentiation explicit in the field equations written in generalized current notation. The complex forms of these must therefore also be the same as the instantaneous forms. For example, the complex form of Eqs. (1-20) is

$$\nabla \times \mathbf{E} = -\mathbf{M}^t \qquad \nabla \times \mathbf{H} = \mathbf{J}^t \tag{1-44}$$

Even though these complex equations look the same as the corresponding instantaneous equations, we should always keep in mind the difference in meaning.

As an illustration of the procedure when the instantaneous equation exhibits a time differentiation, consider the equation

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$$

Again we express the instantaneous quantities in terms of the complex quantities, and obtain

$$\nabla \times [\sqrt{2}\,\mathrm{Re}\,(\mathbf{E}e^{j\omega t})] = -\frac{\partial}{\partial t}[\sqrt{2}\,\mathrm{Re}\,(\mathbf{B}e^{j\omega t})]$$

The time variation is explicit, and the differentiation can be performed. By steps justifiable by Eqs. (1-42), the above equation becomes

$$\sqrt{2}\,\mathrm{Re}\,(\nabla \times \mathbf{E}e^{j\omega t}) = -\sqrt{2}\,\mathrm{Re}\,(j\omega\mathbf{B}e^{j\omega t})$$

By the foregoing lemma, this reduces to

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}$$

It should now be apparent that each time derivative in a linear instantaneous equation is replaced by a $j\omega$ multiplier in the corresponding complex equation. For example, the Maxwell equations in complex form corresponding to Eqs. (1-1) are

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mathbf{B} \qquad & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= j\omega\mathbf{D} + \mathbf{J} \qquad & \nabla \cdot \mathbf{D} &= Q_v \end{aligned} \tag{1-45}$$

The other forms of these can be obtained in a similar fashion.

1-9. Complex Constitutive Parameters. The constitutive relationships for matter linear in the general sense can be specialized to the a-c case by the procedure of the preceding section. To illustrate, consider the first of Eqs. (1-16), which is

$$\mathfrak{D} = \left(\epsilon + \epsilon_1 \frac{\partial}{\partial t} + \epsilon_2 \frac{\partial^2}{\partial t^2} + \cdots\right)\mathcal{E}$$

The complex form of this equation is readily found as

$$\mathbf{D} = (\epsilon + j\omega\epsilon_1 - \omega^2\epsilon_2 + \cdots)\mathbf{E}$$

The quantity $(\epsilon + j\omega\epsilon_1 - \omega^2\epsilon_2 + \cdots)$ is just a complex function of ω, which we shall denote by $\hat{\epsilon}(\omega)$. Thus, the complex equation

$$\mathbf{D} = \hat{\epsilon}(\omega)\mathbf{E}$$

which looks like the form for simple media, is actually valid for media linear in the general sense.

The other two of Eqs. (1-16) simplify in a similar manner; so we have the *a-c constitutive relationships*

$$\begin{aligned} \mathbf{D} &= \hat{\epsilon}(\omega)\mathbf{E} \\ \mathbf{B} &= \hat{\mu}(\omega)\mathbf{H} \\ \mathbf{J}^c &= \hat{\sigma}(\omega)\mathbf{E} \end{aligned} \tag{1-46}$$

for linear media. We call $\hat{\epsilon}$ the *complex permittivity* of the medium, $\hat{\mu}$ the *complex permeability* of the medium, and $\hat{\sigma}$ the *complex conductivity* of the medium. Remember that these parameters are not necessarily the d-c parameters, but

$$\hat{\epsilon}(\omega),\ \hat{\mu}(\omega),\ \hat{\sigma}(\omega) \underset{\omega \to 0}{\longrightarrow} \epsilon,\ \mu,\ \sigma$$

The d-c parameters may apply over a wide range of frequencies for some materials but never over all frequencies (vacuum excepted).

In terms of the generalized current concept, the induced currents (caused by the field) are

$$\begin{aligned} \mathbf{J} &= (\hat{\sigma} + j\omega\hat{\epsilon})\mathbf{E} = \hat{y}(\omega)\mathbf{E} \\ \mathbf{M} &= j\omega\hat{\mu}\mathbf{H} = \hat{z}(\omega)\mathbf{H} \end{aligned} \tag{1-47}$$

The parameter $\hat{y}(\omega)$ has the dimensions of admittance per length and will be called the *admittivity* of the medium. The parameter $\hat{z}(\omega)$ has the dimensions of impedance per length and will be called the *impedivity* of the medium. Note that $\hat{y}$ is a combination of the $\hat{\sigma}$ and $\hat{\epsilon}$ parameters. A measurement of $\hat{y}$ is relatively simple, but it is difficult to separate $\hat{\sigma}$ from $\hat{\epsilon}$. The distinction is primarily philosophical. If the current is due to free charge, we include its effect in $\hat{\sigma}$. If the current is due to bound charge, we include its effect in $\hat{\epsilon}$. Thus, when talking of conductors, the usual convention is to let $\hat{y} = \hat{\sigma} + j\omega\epsilon_0$. When discussing dielectrics, it is common to let $\hat{y} = j\omega\hat{\epsilon}$.

To represent sources, impressed currents are added to the induced currents of Eqs. (1-47). Thus, the general form of the a-c field equations is

$$\begin{aligned} -\nabla \times \mathbf{E} &= \hat{z}(\omega)\mathbf{H} + \mathbf{M}^i \\ \nabla \times \mathbf{H} &= \hat{y}(\omega)\mathbf{E} + \mathbf{J}^i \end{aligned} \tag{1-48}$$

The $\hat{z}(\omega)$ and $\hat{y}(\omega)$ specify the characteristics of the media. The $\mathbf{J}^i$ and $\mathbf{M}^i$ represent the sources. Equations (1-48) are therefore two equations for determining the complex field **E, H**. Solutions to these equations are the principal topic of this book.

1-10. Complex Power. In Sec. 1-5 we considered expressions for instantaneous power and energy in terms of the instantaneous field vectors. We shall show now that similar expressions in terms of the complex field vectors represent time-average power and energy in a-c fields. For this, we shall need the concept of complex conjugate quantities, denoted by *, and defined as follows. If $A = a' + ja'' = |A|e^{j\alpha}$, the conjugate of A is $A^* = a' - ja'' = |A|e^{-j\alpha}$. It follows from this that $AA^* = |A|^2$.

Let us first consider any two a-c quantities $\mathcal{A}$ and $\mathcal{B}$, which may be scalars or components of vectors. These are in general of the form

$$\begin{aligned} \mathcal{A} &= \sqrt{2}\,|A| \cos(\omega t + \alpha) = \sqrt{2}\,\mathrm{Re}\,(Ae^{j\omega t}) \\ \mathcal{B} &= \sqrt{2}\,|B| \cos(\omega t + \beta) = \sqrt{2}\,\mathrm{Re}\,(Be^{j\omega t}) \end{aligned}$$

where $A = |A|e^{j\alpha}$ and $B = |B|e^{j\beta}$. The product of two such quantities is

$$\begin{aligned}\mathcal{A}\mathcal{B} &= \sqrt{2}\,|A| \cos(\omega t + \alpha)\,\sqrt{2}\,|B| \cos(\omega t + \beta) \\ &= |A|\,|B|[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)]\end{aligned} \tag{1-49}$$

We shall denote the time average of a quantity by a bar over that quantity. The time average of the above expression is

$$\overline{\mathcal{A}\mathcal{B}} = |A|\,|B| \cos(\alpha - \beta)$$

We also note that

$$AB^* = |A|\,|B|[\cos(\alpha - \beta) + j \sin(\alpha - \beta)]$$

so it is evident that

$$\overline{\mathcal{A}\mathcal{B}} = \operatorname{Re}(AB^*) \tag{1-50}$$

This identity forms the basis of definitions of complex power.

The instantaneous Poynting vector [Eq. (1-28)] can be expanded in rectangular coordinates as

$$\mathcal{S} = \mathbf{u}_x(\mathcal{E}_y\mathcal{H}_z - \mathcal{E}_z\mathcal{H}_y) + \mathbf{u}_y(\mathcal{E}_z\mathcal{H}_x - \mathcal{E}_x\mathcal{H}_z) + \mathbf{u}_z(\mathcal{E}_x\mathcal{H}_y - \mathcal{E}_y\mathcal{H}_x)$$

This is a sum of terms, each of which is the form of Eq. (1-49). It therefore follows that

$$\bar{\mathcal{S}} = \overline{\mathcal{E} \times \mathcal{H}} = \operatorname{Re}(\mathbf{E} \times \mathbf{H}^*)$$

In view of this we define a *complex Poynting vector*

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}^* \tag{1-51}$$

whose real part is the time average of the instantaneous Poynting vector, or

$$\bar{\mathcal{S}} = \operatorname{Re}(\mathbf{S}) \tag{1-52}$$

We shall interpret the imaginary part of S later.

We can obtain an equation in which S appears by operating on the complex field equations in a manner similar to that used in the instantaneous case. Starting from Eqs. (1-44), we scalarly multiply the first by $\mathbf{H}^*$ and the conjugate of the second by E. The difference of the resulting two equations is

$$\mathbf{E} \cdot \nabla \times \mathbf{H}^* - \mathbf{H}^* \cdot \nabla \times \mathbf{E} = \mathbf{E} \cdot \mathbf{J}^{i*} + \mathbf{H}^* \cdot \mathbf{M}^i$$

The left-hand term is $-\nabla \cdot (\mathbf{E} \times \mathbf{H}^*)$ by a mathematical identity; so we have

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) + \mathbf{E} \cdot \mathbf{J}^{i*} + \mathbf{H}^* \cdot \mathbf{M}^i = 0 \tag{1-53}$$

The integral form of this is obtained by integrating throughout a region and applying the divergence theorem. This results in

$$\oiint \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{s} + \iiint (\mathbf{E} \cdot \mathbf{J}^{t*} + \mathbf{H}^* \cdot \mathbf{M}^t)\, d\tau = 0 \tag{1-54}$$

Compare these with Eqs. (1-26) and (1-27). We shall call Eqs. (1-53) and (1-54) expressions for the *conservation of complex power*, the former applying at a point and the latter applying to an entire region.

The various terms of the above equations are interpreted as follows. As suggested by Eqs. (1-29) and (1-52), we define a *complex volume density of power leaving a point* as

$$\hat{p}_f = \nabla \cdot \mathbf{S} = \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) \tag{1-55}$$

The real part of this is a time-average volume density of power leaving a point, or

$$\text{Re}\,(\hat{p}_f) = \bar{p}_f \tag{1-56}$$

where p_f is defined by Eq. (1-29). Similarly, we define the *complex power leaving a region* as

$$P_f = \oiint \mathbf{S} \cdot d\mathbf{s} = \oiint \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{s} \tag{1-57}$$

It is evident from Eqs. (1-30) and (1-52) that the real part of this is the time-average power flow, or

$$\text{Re}\,(P_f) = \bar{\mathcal{P}}_f \tag{1-58}$$

Note that these relationships are quite different from those used to interpret most complex quantities [Eqs. (1-40) and (1-41)]. This is because $\mathcal{S}$, p, and $\mathcal{P}$ are *not* sinusoidal quantities but are formed of products of sinusoidal quantities.

To interpret the other terms of Eq. (1-53), let us first specialize to the case of a source-free field in media linear in the simple sense. We then have

$$\mathbf{J}^t = \hat{y}\mathbf{E} = (\sigma + j\omega\epsilon)\mathbf{E}$$
$$\mathbf{M}^t = \hat{z}\mathbf{H} = j\omega\mu\mathbf{H}$$

so

$$\mathbf{E} \cdot \mathbf{J}^{t*} = \sigma|E|^2 - j\omega\epsilon|E|^2$$
$$\mathbf{H}^* \cdot \mathbf{M}^t = j\omega\mu|H|^2$$

where $|E|^2$ means $\mathbf{E} \cdot \mathbf{E}^*$ and $|H|^2$ means $\mathbf{H} \cdot \mathbf{H}^*$. In terms of the instantaneous energy and power definitions of Eqs. (1-32) and (1-33), we have

$$\left.\begin{aligned} \bar{p}_d &= \sigma|E|^2 \\ \bar{w}_e &= \tfrac{1}{2}\epsilon|E|^2 \\ \bar{w}_m &= \tfrac{1}{2}\mu|H|^2 \end{aligned}\right\} \quad \text{in simple media} \tag{1-59}$$

We can now write Eq. (1-53) as

$$\nabla \cdot \mathbf{S} + \bar{p}_d + j2\omega(\bar{w}_m - \bar{w}_e) = 0 \tag{1-60}$$

Thus, the imaginary part of $\hat{p}_f$ as defined by Eq. (1-55) is 2ω times the difference between the time-average electric and magnetic energy densities. The integral relationships corresponding to Eqs. (1-59) are

$$\left.\begin{aligned} \bar{\mathcal{P}}_d &= \iiint \sigma|E|^2\,d\tau \\ \bar{\mathcal{W}}_e &= \frac{1}{2}\iiint \epsilon|E|^2\,d\tau \\ \bar{\mathcal{W}}_m &= \frac{1}{2}\iiint \mu|H|^2\,d\tau \end{aligned}\right\} \quad \text{in simple media} \tag{1-61}$$

where $\mathcal{P}_d$, $\mathcal{W}_e$, and $\mathcal{W}_m$ are defined by Eqs. (1-36) and (1-37). The specialization of Eq. (1-54) to source-free simple media is therefore

$$\oiint \mathbf{S}\cdot d\mathbf{s} + \bar{\mathcal{P}}_d + j2\omega(\bar{\mathcal{W}}_m - \bar{\mathcal{W}}_e) = 0 \tag{1-62}$$

corresponding to the point relationship of Eq. (1-60). Note that this interpretation of complex power is precisely that chosen in circuit theory.

If sources are present, a *complex power density supplied by the sources* can be defined as

$$\hat{p}_s = -(\mathbf{E}\cdot\mathbf{J}^{i*} + \mathbf{H}^*\cdot\mathbf{M}^i) \tag{1-63}$$

The real part of this is the time-average power density supplied by the sources, or

$$\operatorname{Re}(\hat{p}_s) = \bar{p}_s \tag{1-64}$$

where p_s is defined by Eq. (1-34). We can write Eq. (1-53) in general as

$$\hat{p}_s = \hat{p}_f + \bar{p}_d + j2\omega(\bar{w}_m - \bar{w}_e) \tag{1-65}$$

where all terms have been identified for simple media. Similarly, the *total complex power supplied by sources within a region* can be defined as

$$P_s = -\iiint (\mathbf{E}\cdot\mathbf{J}^{i*} + \mathbf{H}^*\cdot\mathbf{M}^i)\,d\tau \tag{1-66}$$

where, from Eq. (1-38), it is evident that

$$\operatorname{Re}(P_s) = \bar{\mathcal{P}}_s \tag{1-67}$$

Then the form of Eq. (1-65) applicable to an entire region is

$$P_s = P_f + \bar{\mathcal{P}}_d + j2\omega(\bar{\mathcal{W}}_m - \bar{\mathcal{W}}_e) \tag{1-68}$$

The real part of this represents a time-average power balance. The imaginary part is related to time-average energies, and, in conformity with circuit theory nomenclature, is called *reactive power*.

Note that we have never defined $\mathcal{P}_d$, $\mathcal{W}_m$, or $\mathcal{W}_e$ for media linear in the general sense. We can, however, continue to use Eq. (1-68) for the general case of linear media by extending our definitions. This is done as follows. The time-average power dissipation is defined in general as

$$\bar{\mathcal{P}}_d = \text{Re}\left[\iiint (\hat{y}|E|^2 + \hat{z}|H|^2)\, d\tau\right] \tag{1-69}$$

which reduces to the first of Eqs. (1-61) in simple media. The first term of the integrand represents both conduction and dielectric losses, and the second term represents magnetic losses. The time-average electric and magnetic energies are defined in general as

$$\begin{aligned} \bar{\mathcal{W}}_e &= \frac{1}{2\omega}\,\text{Im}\left(\iiint \hat{y}|E|^2\, d\tau\right) \\ \bar{\mathcal{W}}_m &= \frac{1}{2\omega}\,\text{Im}\left(\iiint \hat{z}|H|^2\, d\tau\right) \end{aligned} \tag{1-70}$$

which reduce to the last two of Eqs. (1-61) in simple media. The first of Eqs. (1-70) includes kinetic energy stored by free charges as well as the usual field and polarization energies. More discussion of this concept is given in the next section.

1-11. A-C Characteristics of Matter. In source-free regions, the complex field equations read

$$-\nabla \times \mathbf{E} = \hat{z}(\omega)\mathbf{H} \qquad \nabla \times \mathbf{H} = \hat{y}(\omega)\mathbf{E}$$

In free space, $\hat{z}$ and $\hat{y}$ assume their simplest forms, being

$$\left.\begin{aligned} \hat{y}(\omega) &= j\omega\epsilon_0 \\ \hat{z}(\omega) &= j\omega\mu_0 \end{aligned}\right\} \quad \text{in free space} \tag{1-71}$$

These hold for all frequencies and all field intensities. In metals, the conductivity remains very close to the d-c value for all radio frequencies, that is, up to the infrared frequency spectrum. The permittivity of metals is hard to measure but appears to be approximately that of vacuum. Thus,

$$\left.\begin{aligned} \hat{y}(\omega) &= \sigma + j\omega\epsilon_0 \\ \hat{z}(\omega) &= j\omega\mu_0 \end{aligned}\right\} \quad \text{in nonmagnetic metals} \tag{1-72}$$

In ferromagnetic metals, μ_0 would be replaced by $\hat{\mu}$. We shall consider this case later.

In good dielectrics, it is common practice to neglect $\hat{\sigma}$ and express $\hat{y}$ entirely in terms of $\hat{\epsilon}$. Thus,

$$\left.\begin{aligned} \hat{y}(\omega) &= j\omega\hat{\epsilon} \\ \hat{z}(\omega) &= j\omega\mu_0 \end{aligned}\right\} \quad \text{in nonmagnetic dielectrics} \tag{1-73}$$

Let us now consider $\hat{\epsilon}(\omega)$ in more detail.[1] We can express $\hat{\epsilon}$ in both rec-

[1] A. Von Hipple, "Dielectric Materials and Applications," John Wiley & Sons, Inc., New York, 1954.

tangular and polar form as

$$\hat{\epsilon}(\omega) = \epsilon' - j\epsilon'' = |\hat{\epsilon}|e^{-j\delta} \tag{1-74}$$

where ϵ', ϵ'', and δ are real quantities. We call ϵ' the *a-c capacitivity*, ϵ'' the *dielectric loss factor*, and δ the *dielectric loss angle*. In Sec. 1-13 we shall see that they are related to the capacitance, resistance, and loss angle, respectively, of an ideal circuit capacitor. In terms of power and energy, we have from Eqs. (1-69) and (1-70) that

$$\begin{aligned} \overline{\mathcal{W}}_e &= \frac{1}{2}\iiint \epsilon'|E|^2\,d\tau \\ \overline{\mathcal{P}}_d &= \iiint \omega\epsilon''|E|^2\,d\tau \end{aligned} \tag{1-75}$$

Thus, ϵ' contributes to stored energy (acts like ϵ in simple matter), and $\omega\epsilon''$ contributes to power dissipation (acts like σ in simple matter). Measured values of $\hat{\epsilon}(\omega)$ are usually expressed in terms of ϵ' and $\tan\delta$, or in terms of ϵ' and ϵ''. We shall use the latter representation.

A "perfect dielectric" would be one for which $\epsilon'' = 0$. The only perfect dielectric is vacuum. A "good dielectric" is defined to be one for which ϵ' remains almost constant at all radio frequencies and for which ϵ'' is very small. Examples of good dielectrics are polystyrene, paraffin, and Teflon. Figure 1-10 shows ϵ' and ϵ'' versus frequency for polystyrene to illustrate the characteristics of a good dielectric. There is also a group of "lossy dielectrics," characterized by a varying ϵ' and a large ϵ'' in the radio-frequency range. Examples of lossy dielectrics are Plexiglas, porcelain, and Bakelite. Figure 1-11 shows ϵ' and ϵ'' versus frequency for Plexiglas to illustrate the characteristics of a lossy dielectric. There is a group of dielectrics which have unusually high dielectric constants. The titanate and ferrite ceramics fall into this

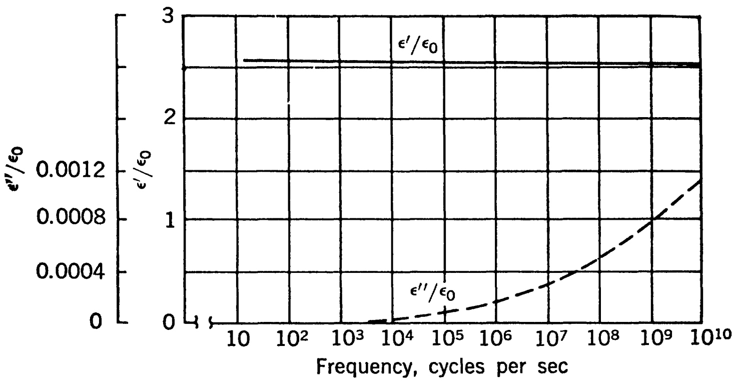

FIG. 1-10. $\hat{\epsilon}(\omega) = \epsilon' - j\epsilon''$ versus frequency for polystyrene at 25°C.

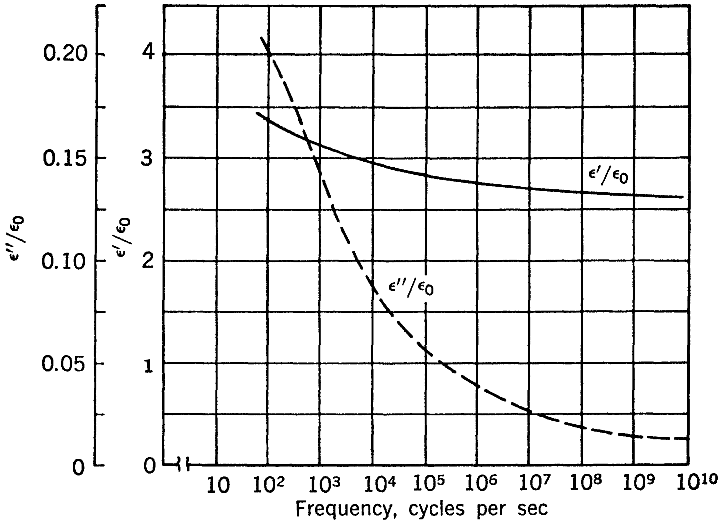

FIG. 1-11. $\hat{\epsilon}(\omega) = \epsilon' - j\epsilon''$ versus frequency for Plexiglas at 25°C.

class (the latter also being ferromagnetic). Such dielectrics are usually lossy. A qualitative explanation of the behavior of $\hat{\epsilon}$ can be made in terms of atomic concepts, but we shall view $\hat{\epsilon}$ as simply a measured parameter. A table of $\hat{\epsilon}$ for some common dielectrics is given in Appendix B.

In ferromagnetic matter, when it can be considered linear, both conduction and dielectric losses may be significant. In addition to these, magnetic losses become important. Thus,

$$\left.\begin{aligned} \hat{y} &= \sigma + j\omega\hat{\epsilon} \\ \hat{z} &= j\omega\hat{\mu} \end{aligned}\right\} \quad \text{in ferromagnetic matter} \tag{1-76}$$

The parameter $\hat{\mu}(\omega)$ can be treated in a manner analogous to the treatment of $\hat{\epsilon}(\omega)$. Thus, we express $\hat{\mu}$ in both rectangular and polar form as

$$\hat{\mu}(\omega) = \mu' - j\mu'' = |\hat{\mu}|e^{-j\delta_m} \tag{1-77}$$

where μ', μ'', and δ_m are real quantities. We call μ' the *a-c inductivity*, μ'' the *magnetic loss factor*, and δ_m the *magnetic loss angle*. In Sec. 1-13 we shall see that they are related to the inductance, resistance, and loss angle, respectively, of an ideal circuit inductor. In terms of power and energy, we have from Eqs. (1-69) and (1-70) that

$$\begin{aligned} \bar{\mathcal{W}}_m &= \frac{1}{2}\iiint \mu'|H|^2\,d\tau \\ \bar{\mathcal{P}}_d &= \iiint \omega\mu''|H|^2\,d\tau \end{aligned} \tag{1-78}$$

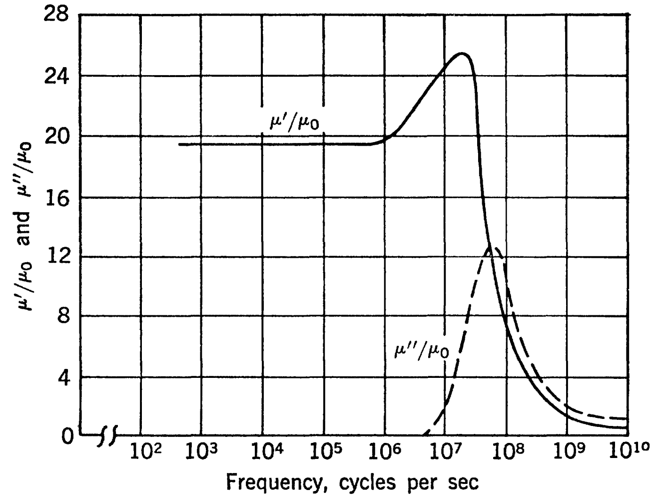

FIG. 1-12. $\hat{\mu}(\omega) = \mu' - j\mu''$ versus frequency for Ferramic A at 25°C.

where the above $\bar{\mathcal{P}}_d$ is only the time-average magnetic power loss, to which must be added the conduction and dielectric losses for the total power dissipation. Thus, μ' contributes to stored energy and μ'' to power dissipation. Measured values of $\hat{\mu}(\omega)$ are usually expressed in terms of μ' and tan δ_m, or in terms of μ' and μ''. We shall use the latter representation.

Ferromagnetic metals are extremely lossy materials (primarily due to σ), and also quite nonlinear with respect to $\hat{\mu}$. They are seldom intentionally used at radio frequencies. However, the ferromagnetic ceramics can be profitably used at radio frequencies to obtain high values of μ'. They are lossy in the magnetic sense, in that they also have appreciable μ''. Figure 1-12 shows μ' and μ'' versus frequency for Ferramic A, to illustrate the characteristics of ferrite ceramics. These materials become even more useful when magnetized by a d-c magnetic field, in which case $\hat{\mu}$ assumes the form of an asymmetrical tensor. Magnetized ferrites can be used to build "nonreciprocal" devices, such as "isolators" and "circulators."[1]

1-12. A Discussion of Current. The concept of current has broadened considerably since its inception. Originally, the term current meant the flow of free charges in conductors. This concept was extended to include displacement current, which was visualized as the displacement of bound charge in matter and in an "ether." The existence of an ether has been disproved, but the concept of displacement current has been retained,

[1] C. L. Hogan, The Ferromagnetic Effect at Microwave Frequencies, *Bell System Tech. J.*, vol. 31, no. 1, January, 1952.

even though it is not entirely a motion of charge. A further generalization was made to include magnetic displacement current as a "dual" concept of the electric displacement current. Finally, impressed currents, both electric and magnetic, have been introduced to represent sources. Because of the breadth of the concept of current, many different phenomena are included, and the nomenclature used is somewhat lengthy. We shall summarize the notation and concepts in complex form in this section.

Consider the complex electric current density. Internal to conductors, the current is, for all practical purposes, due entirely to the motion of free electrons. Such current is called the *conduction current* and is expressed mathematically by $\mathbf{J} = \sigma\mathbf{E}$. (We shall consider $\hat{\sigma} = \sigma$, a real quantity, for this discussion. This is usually true at radio frequencies.) Even in dielectrics there is some conduction current, but it is usually small. In free space there is no motion of charges at all, and we have only a *free-space displacement current*, given by $\mathbf{J} = j\omega\epsilon_0\mathbf{E}$. In matter, in addition to the conduction current and the free-space displacement current, we have a current due to the motion of bound charges. This is called the *polarization current* and is expressed mathematically by $\mathbf{J} = j\omega(\hat{\epsilon} - \epsilon_0)\mathbf{E}$. Because the term $\mathbf{J} = j\omega\hat{\epsilon}\mathbf{E}$ is of the same mathematical form as the free-space displacement current, it is called the *displacement current.* For our purposes, still another division of the electric current is convenient. This involves viewing the current in terms of a component in phase with $\mathbf{E}$, called the *dissipative current*, $\mathbf{J} = (\sigma + \omega\epsilon'')\mathbf{E}$, and a component out of phase with $\mathbf{E}$, called the *reactive current*, $\mathbf{J} = j\omega\epsilon'\mathbf{E}$. This is essentially a generalization of the circuit concept of current, where the dissipative current produces the power loss and the reactive current gives rise to the stored energy. All the currents mentioned are classified as *induced currents*, that is, are caused by the field. *Impressed currents* are used to represent sources or known quantities. In this sense, they are independent of the field and are said to cause the field. The total electric current is the sum of the induced currents plus the impressed currents. The nomenclature used for electric currents is summarized in the first column of Table 1-2.

Both the nomenclature and the concepts of complex magnetic currents are similar to those for electric currents. The one essential difference in the two concepts is the nonexistence of magnetic "charges" in nature. Thus, there is no free magnetic charge and no magnetic conduction current. In absence of matter, we have a *magnetic free-space displacement current*, $\mathbf{M} = j\omega\mu_0\mathbf{H}$, analogous to the electric case. When matter is present, we have magnetic effects due to the motion of the atomic particles, giving rise to an induced magnetic current in addition to the free-space displacement current. We call this the *magnetic polarization*

TABLE 1-2. CLASSIFICATION OF ELECTRIC AND MAGNETIC CURRENTS

Type	Complex electric current density	Complex magnetic current density
Conduction	$\sigma\mathbf{E}$	
Free-space displacement	$j\omega\epsilon_0\mathbf{E}$	$j\omega\mu_0\mathbf{H}$
Polarization	$j\omega(\hat{\epsilon} - \epsilon_0)\mathbf{E}$	$j\omega(\hat{\mu} - \mu_0)\mathbf{H}$
Displacement	$j\omega\hat{\epsilon}\mathbf{E}$	$j\omega\hat{\mu}\mathbf{H}$
Dissipative	$(\sigma + \omega\epsilon'')\mathbf{E}$	$\omega\mu''\mathbf{H}$
Reactive	$j\omega\epsilon'\mathbf{E}$	$j\omega\mu'\mathbf{H}$
Induced	$\hat{y}\mathbf{E} = (\sigma + j\omega\hat{\epsilon})\mathbf{E}$ $= (\sigma + \omega\epsilon'' + j\omega\epsilon')\mathbf{E}$	$\hat{z}\mathbf{H} = j\omega\hat{\mu}\mathbf{H}$ $= (\omega\mu'' + j\omega\mu')\mathbf{H}$
Impressed	$\mathbf{J}^i$	$\mathbf{M}^i$
Total	$\mathbf{J}^t = \hat{y}\mathbf{E} + \mathbf{J}^i$	$\mathbf{M}^t = \hat{z}\mathbf{H} + \mathbf{M}^i$

current, expressed by $\mathbf{M} = j\omega(\hat{\mu} - \mu_0)\mathbf{H}$. The term $\mathbf{M} = j\omega\hat{\mu}\mathbf{H}$ is called the *magnetic displacement current*, being the sum of the free-space displacement current and the polarization current. We find it convenient to divide the magnetic current into a component in phase with $\mathbf{H}$, called the *magnetic dissipative current*, $\mathbf{M} = \omega\mu''\mathbf{H}$, and a component out of phase with $\mathbf{H}$, called the *magnetic reactive current*, $\mathbf{M} = j\omega\mu'\mathbf{H}$. The dissipative magnetic current contributes to the power loss, and the reactive magnetic current contributes to the stored energy. All the aforementioned magnetic currents are *induced currents*, that is, caused by the field. In nonmagnetic matter, the induced magnetic current is simply the free-space displacement current, $\mathbf{M} = j\omega\mu_0\mathbf{H}$, a reactive current. To represent sources or known quantities, we use *impressed currents*. The nomenclature for magnetic currents is summarized in the second column of Table 1-2.

A convenient classification of matter from the electric current standpoint can be made in terms of a *quality factor* Q. This is defined as

$$Q = \frac{\text{magnitude of reactive current density}}{\text{magnitude of dissipative current density}} = \frac{\omega\epsilon'}{\sigma + \omega\epsilon''} \tag{1-79}$$

In nonmagnetic matter, this involves a ratio of stored electric energy to power dissipated. In terms of the energy and power densities, Eq. (1-79) can be written as

$$\begin{aligned} Q &= \frac{\omega\epsilon' |E|^2}{(\sigma + \omega\epsilon'')|E|^2} \\ &= \omega \frac{\text{peak density of electric energy}}{\text{average density of power dissipated}} \\ &= 2\pi \frac{\text{peak density of electric energy}}{\text{density of energy dissipated in one cycle}} \end{aligned} \tag{1-80}$$

Thus, the concept of Q in nonmagnetic matter can be considered as an extension of the concept of Q for capacitors in circuit theory. A good dielectric is a high-Q material, while conductors have an extremely low Q.

When magnetic matter is considered, there is an additional power dissipation due to magnetic hysteresis loss. The interpretation given to Eq. (1-80) must be modified, since it includes only the power loss due to electric effects. In this case, the Q defined above would be called the electric Q, and an analogous magnetic quality factor Q_m could be defined. Since we deal principally with nonmagnetic materials, we shall not expand this concept further.

1-13. A-C Behavior of Circuit Elements. The complex notation used for a-c fields is the extension of the complex notation used for a-c circuits. The complex field equations bear a relationship to the complex circuit equations which is similar to that for the time-varying case, given in Sec. 1-6. Circuit elements (resistors, capacitors, and inductors) are merely configurations of matter and thus have characteristics which depend upon the properties of matter. Insight into the interpretation of the impedivity and admittivity functions of field theory can be gained by considering their relationship to the more familiar characteristics of impedance and admittance of circuit elements.

The basic elements of circuit theory are small[1] two-terminal structures whose fields are largely confined internal to the elements. According to the concepts of Sec. 1-10, the complex power supplied to a circuit element is

$$P = \bar{\mathcal{P}}_d + j2\omega(\bar{\mathcal{W}}_m - \bar{\mathcal{W}}_e) \tag{1-81}$$

In terms of circuit concepts, the power supplied to an element also can be written as

$$P = |I|^2 Z = |V|^2 Y^* \tag{1-82}$$

where Z and Y are the impedance and admittance of the element. In general, an element is called an impedor. When P is primarily real, the

[1] The smallness of an element depends upon the frequency, or wavelength, as we shall see in Chap. 2.

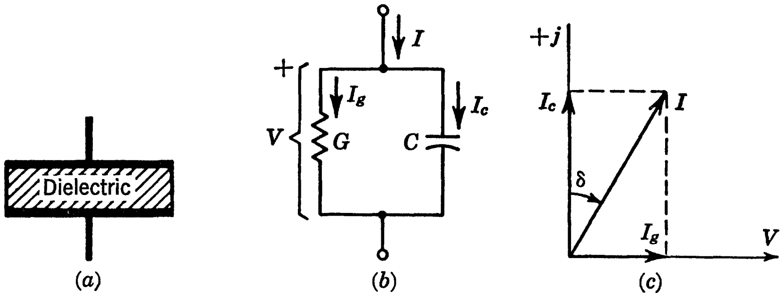

FIG. 1-13. A capacitor according to circuit concepts. (a) Physical capacitor; (b) equivalent circuit; (c) complex diagram.

element is called a resistor, and when P is primarily imaginary, the element is called a reactor. A reactor is called an inductor or capacitor according as Im (Z) is positive or negative, respectively. It should be noted that P, and hence Z, is a function of frequency. Thus, the designation of an element as a resistor, inductor, or capacitor is, to some degree, dependent upon frequency. We usually classify elements according to their low-frequency behavior.

For an explicit discussion, consider the parallel-plate capacitor of Fig. 1-13a. The low-frequency equivalent circuit of this element is shown in Fig. 1-13b, where the conductance G accounts for energy dissipation and the capacitance C accounts for energy storage. The relationship of complex terminal current I to complex terminal voltage V is

$$I = I_g + I_c = YV = (G + j\omega C)V \tag{1-83}$$

Figure 1-13c shows the complex diagram representing this equation. The complex power to the element is[1]

$$P = |V|^2(G - j\omega C)$$

For a "good" capacitor ($\omega C \gg G$) the current leads the voltage by almost 90°, and the power is principally reactive. For a "poor" capacitor ($G \gg \omega C$) the current and voltage are almost in phase, and the power is principally dissipative. The element in this case could be classified as a resistor. The angle between I_c and I is called the loss angle δ, as shown in Fig. 1-13c.

Let us idealize the problem to a capacitor with perfectly conducting plates. Furthermore, we shall approximate the field by

$$E = \frac{V}{d} \qquad J = \frac{I}{A}$$

[1] We are using the convention $P = VI^*$. Some authors define $P = IV^*$, in which case the sign of reactive power is opposite to that which we get.

where A is the area of the plates and d is their separation. The a-c constitutive relationship for the field between the capacitor plates is

$$J = \hat{y}E = (\sigma + \omega\epsilon'' + j\omega\epsilon')E$$

where we have taken $\hat{\sigma} = \sigma$. Substituting for E and J from the preceding equations, we have

$$I = \hat{y}\frac{A}{d}V = (\sigma + \omega\epsilon'' + j\omega\epsilon')\frac{A}{d}V$$

A comparison of this with Eqs. (1-83) shows that

$$Y = \hat{y}\frac{A}{d} \qquad G = (\sigma + \omega\epsilon'')\frac{A}{d} \qquad C = \epsilon'\frac{A}{d}$$

Thus, for our idealized circuit element, the admittance is proportional to the admittivity of the matter between the plates. The equivalency of "field power," Eq. (1-81), to "circuit power," Eq. (1-82), also can be demonstrated. For our idealized element

$$P = \iiint \hat{y}^*|E|^2\,d\tau = \hat{y}^*|E|^2Ad = |V|^2Y^*$$

We can use this result to define the admittance of a cube and then view admittivity $\hat{y}$ as the admittance of a unit cube.

The magnetic properties of matter are similarly related to the circuit behavior of an inductor. To demonstrate this, consider the toroidal inductor of Fig. 1-14*a*. The low-frequency equivalent circuit of this element is shown in Fig. 1-14*b*, where the resistance R accounts for energy dissipation and the inductance L accounts for energy storage. The relationship of complex terminal voltage V to complex terminal current I is

$$V = V_r + V_l = ZI = (R + j\omega L)I \tag{1-84}$$

The complex diagram representing this equation is shown in Fig. 1-14*c*.

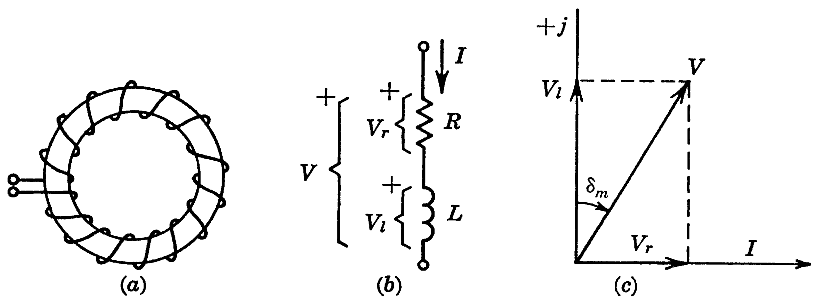

FIG. 1-14. An inductor according to circuit concepts. (*a*) Toroidal inductor; (*b*) equivalent circuit; (*c*) complex diagram.

The complex power to the element is

$$P = |I|^2(R + j\omega L)$$

For a good inductor ($\omega L \gg R$) the current lags the voltage by almost 90°, and the power is principally reactive. For a poor inductor ($R \gg \omega L$) the current and voltage are almost in phase, and the power is principally dissipative. The element in this case could be classified as a resistor. The angle between V_l and V is called the magnetic loss angle δ_m, as shown on Fig. 1-14c.

We now idealize the problem to an inductor of perfectly conducting wire and approximate the field by

$$H = \frac{NI}{l} \qquad M = \frac{V}{NA}$$

where N is the number of turns, l is the average circumference, and A is the cross-sectional area. The magnetic constitutive relationship for the field in the core is

$$M = \hat{z}H = (\omega\mu'' + j\omega\mu')H$$

A substitution for H and M from the preceding equations gives

$$V = \hat{z}\frac{N^2A}{l}I = (\omega\mu'' + j\omega\mu')\frac{N^2A}{l}I$$

Comparing this with Eq. (1-84), we see that

$$Z = \hat{z}\frac{N^2A}{l} \qquad R = \omega\mu''\frac{N^2A}{l} \qquad L = \mu'\frac{N^2A}{l}$$

Thus, for the idealized inductor, the impedance is proportional to the impedivity of the matter. From Eq. (1-82), the power supplied to the inductor is

$$P = \iiint \hat{z}|H|^2\,d\tau = \hat{z}|H|^2Al = |I|^2Z$$

which is consistent with Eq. (1-82). Using this result to define the impedance of a cube, we can think of impedivity as the impedance of a unit cube.

This development serves to illustrate the close correspondences between a-c circuit concepts and a-c field concepts. A summary of the various concepts is given in Table 1-3.

1-14. Singularities of the Field. A field is said to be *singular* at a point for which the function or its derivatives are discontinuous. Most of our discussion so far has been about well-behaved fields, but we have meant to include by implication certain types of allowable singularities.

TABLE 1-3. CORRESPONDENCES BETWEEN A-C CIRCUIT CONCEPTS AND A-C FIELD CONCEPTS

A-C circuit concepts	A-C field concepts
Complex voltage V	Complex electric intensity $\mathbf{E}$ Complex magnetic current density $\mathbf{M}$
Complex current I	Complex electric current density $\mathbf{J}$ Complex magnetic intensity $\mathbf{H}$
Complex power flow VI^*	Density of complex power flow $\mathbf{E} \times \mathbf{H}^*$
Impedance $Z(\omega)$	Impedivity $\hat{z}(\omega)$
Admittance $Y(\omega)$	Admittivity $\hat{y}(\omega)$
Resistors: Admittance, $Y(\omega) = \frac{1}{R}$ Current, $I = \frac{1}{R} V$ Power dissipation $\frac{1}{R} VV^*$	Conductors ($\sigma \gg \omega\epsilon_0$): Admittivity, $\hat{y}(\omega) \approx \sigma$ Current density, $\mathbf{J} \approx \sigma\mathbf{E}$ Density of power dissipation, $\sigma\mathbf{E} \cdot \mathbf{E}^*$
Capacitors: Admittance, $Y(\omega) = \frac{1}{R} + j\omega C$ Current, $I = \left(\frac{1}{R} + j\omega C\right) V$ Stored energy $\frac{1}{2}CVV^*$ Power dissipation $\frac{1}{R} VV^*$	Dielectrics ($\omega\epsilon'' \gg \sigma$): Admittivity, $\hat{y}(\omega) \approx \omega\epsilon'' + j\omega\epsilon'$ Current density, $\mathbf{J} \approx (\omega\epsilon'' + j\omega\epsilon')\mathbf{E}$ Density of stored energy, $\frac{1}{2}\epsilon'\mathbf{E} \cdot \mathbf{E}^*$ Density of power dissipation, $\omega\epsilon''\mathbf{E} \cdot \mathbf{E}^*$
Inductors: Impedance, $Z(\omega) = R + j\omega L$ Voltage, $V = (R + j\omega L)I$ Stored energy, $\frac{1}{2}LII^*$ Power dissipation, RII^*	Magnetic properties: Impedivity, $\hat{z}(\omega) = \omega\mu'' + j\omega\mu'$ Magnetic current, $\mathbf{M} = (\omega\mu'' + j\omega\mu')\mathbf{H}$ Density of stored energy, $\frac{1}{2}\mu'\mathbf{H} \cdot \mathbf{H}^*$ Density of power dissipation, $\omega\mu''\mathbf{H} \cdot \mathbf{H}^*$

These can occur at material boundaries (discontinuous $\hat{z}$ and $\hat{y}$) and at singular source distributions, such as sheets and filaments of currents.

As evidenced by Eqs. (1-44), the total electric and magnetic currents are vortices of $\mathbf{H}$ and $-\mathbf{E}$, respectively. Suppose we have a surface distribution of currents $\mathbf{J}_s$ and $\mathbf{M}_s$, as represented by Fig. 1-15. By applying

$$\oint \mathbf{H} \cdot d\mathbf{l} = I^t \qquad \oint \mathbf{E} \cdot d\mathbf{l} = -K^t \tag{1-85}$$

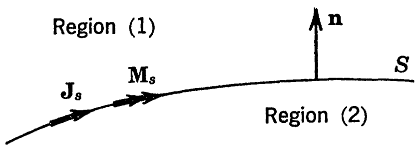

FIG. 1-15. Surface currents.

to rectangular paths enclosing a portion of the surface currents, we obtain[1]

$$\mathbf{n} \times [\mathbf{H}^{(1)} - \mathbf{H}^{(2)}] = \mathbf{J}_s \qquad [\mathbf{E}^{(1)} - \mathbf{E}^{(2)}] \times \mathbf{n} = \mathbf{M}_s \tag{1-86}$$

where **n** is the unit vector normal to the surface and pointing into region (1). The superscripts (1) and (2) denote the side of S on which **E** or **H** is evaluated. Equations (1-86) are essentially the field equations at sheets of currents. They express at current sheets the same concept as Eqs. (1-44) express at volume distributions of currents. If $\mathbf{J}_s$ and $\mathbf{M}_s$ are impressed currents, Eqs. (1-86) are the "boundary conditions" to be satisfied at the source.

Equations (1-86) apply regardless of whether or not a discontinuity in media exists on S. Whenever $\mathbf{J}_s$ and $\mathbf{M}_s$ are zero, Eqs. (1-86) state that the tangential components of **E** and **H** are continuous across the surface. If $\hat{z}$ and $\hat{y}$ are finite in both regions 1 and 2, no induced surface current can result. Thus, *tangential components of* **E** *and* **H** *are continuous across any material boundary, perfect conductors excepted.* If one side of S is a perfect electric conductor, say region 2, a surface conduction current $\mathbf{J}_s$ can exist even though **E** is zero, since $\hat{y} = \sigma$ is infinite. In this case, Eqs. (1-86) reduce to

$$\left.\begin{aligned} \mathbf{n} \times \mathbf{H} &= \mathbf{J}_s \\ \mathbf{n} \times \mathbf{E} &= 0 \end{aligned}\right\} \quad \text{at a perfect conductor} \tag{1-87}$$

where **n** points into the region of field. Thus, the "boundary condition" at a perfect electric conductor is vanishing tangential components of **E**. The *perfect magnetic conductor* is defined to be a material for which the tangential components of **H** are zero at its surface. This is, however, purely a mathematical concept. The necessary "magnetic conduction current" on its surface has no physical significance.

Finally, at a filament of current, the field must be singular such that Eqs. (1-85) yield the current enclosed, no matter how small the contour. For example, at a filament of electric current I, the boundary condition for **H** is

$$\oint_C \mathbf{H} \cdot d\mathbf{l} \xrightarrow[\text{radius of } C \to 0]{} I \tag{1-88}$$

A similar limit of the second of Eqs. (1-85) must be satisfied at a filament of magnetic current.

[1] R. F. Harrington, "Introduction to Electromagnetic Engineering," McGraw-Hill Book Company, Inc., p. 74, 1958.

It is often convenient for mathematical and discussional purposes to consider the various singular quantities as limits of nonsingular quantities. For example, we can think of an abrupt material boundary as the limit of a continuous, but rapid, change in $\hat{y}$ and $\hat{z}$. Similarly, a sheet of current can be thought of as a volume distribution of current having a large magnitude and confined to a thin shell. By such expediencies we can avoid much tedium in the exposition of the theory.

PROBLEMS

1-1. Using Stokes' theorem and the divergence theorem, show that Eqs. (1-1) are equivalent to Eqs. (1-3).

1-2. The conduction current in conductors is affected by the magnetic field as well as by the electric field (Hall effect). Using an atomic model, justify that

$$\mathcal{J} \approx \sigma\mathcal{E} + \sigma^2 h \mathcal{E} \times \mathcal{B}$$

where h is the Hall constant. For copper ($h = -5.5 \times 10^{-11}$), determine the $\mathcal{B}$ for which the second term of the above equation is 1 per cent of the first term.

1-3. Given $\mathcal{E} = \mathbf{u}_x y^2 \sin \omega t$ and $\mathcal{H} = \mathbf{u}_y x \cos \omega t$, determine $\mathcal{J}^t$ and $\mathcal{M}^t$. Determine i^t and k^t through the disk $z = 0$, $x^2 + y^2 = 1$.

1-4. For the field of Prob. 1-3, determine the Poynting vector. Show that Eq. (1-26) is satisfied for this field.

1-5. Starting from Maxwell's equations, derive the circuit law for capacitors, $i = C\,dv/dt$, and the circuit law for inductors, $v = L\,di/dt$.

1-6. Determine the instantaneous quantities corresponding to (*a*) $I = 10 + j5$, (*b*) $\mathbf{E} = \mathbf{u}_x(5 + j3) + \mathbf{u}_y(2 + j3)$, (*c*) $\mathbf{H} = (\mathbf{u}_x + \mathbf{u}_y)e^{j(x+y)}$.

1-7. Prove Eqs. (1-42).

1-8. Given $\mathbf{H} = \mathbf{u}_x \sin y$ in a source-free region of Plexiglas, determine $\mathbf{E}$ and $\mathcal{E}$ at a frequency of (*a*) 1 megacycle, (*b*) 100 megacycles.

1-9. Show that $Q_v = 0$ (complex charge density vanishes) in a source-free region of homogeneous matter, linear in the general sense.

1-10. Show that the instantaneous Poynting vector is given by

$$\mathcal{S} = \text{Re}\,(\mathbf{S} + \mathbf{E} \times \mathbf{H}e^{j2\omega t})$$

Why is $\mathcal{S}$ not related to $\mathbf{S}$ by Eq. (1-41)?

1-11. Consider the unit cube shown in Fig. 1-16 which has all sides except the face $x = 0$ covered by perfect conductors. If $E_z = 100 \sin(\pi y)$ and $H_y = e^{j\pi/6} \sin(\pi y)$

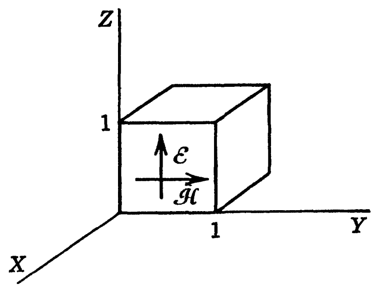

FIG. 1-16. Unit cube for Prob. 1-11.

over the open face and no sources exist within the cube, determine (*a*) the time-average power dissipated within the cube, (*b*) the difference between the time-average electric and magnetic energies within the cube.

1-12. Suppose a filament of z-directed electric current $I^i = 10$ is impressed along the z axis from $z = 0$ to $z = 1$. If $\mathbf{E} = \mathbf{u}_z(1 + j)$, determine the complex power and the time-average power supplied by this source.

1-13. Suppose we have a 10-megacycle field $\mathbf{E} = \mathbf{u}_x 5$, $\mathbf{H} = \mathbf{u}_y 2$, at some point in a material having $\sigma = 10^{-4}$, $\hat{\epsilon} = (8 - j10^{-2})\epsilon_0$, and $\hat{\mu} = (14 - j)\mu_0$ at the operating frequency. Determine each type of current (except impressed) listed in Table 1-2.

1-14. A small capacitor has a d-c capacitance of 300 micromicrofarads when air-filled. When it is oil-filled, it is found to have an impedance of $(500 - j) \times 10^3$ at $\omega = 10^6$. Determine $\hat{y}$, ϵ', and ϵ'' of the oil, neglecting conductor losses.

1-15. For a practical toroidal inductor of the type shown in Fig. 1-14*a*, show that the power loss in the wire will usually be much larger than that in a core of low-loss ferromagnetic material.

1-16. Assume that $\hat{\epsilon} = \epsilon' - j\epsilon''$ is an analytic function of ω and show that

$$\epsilon'(\omega) = \epsilon_0 + \frac{2}{\pi}\int_0^\infty \frac{w\epsilon''(w)\,dw}{w^2 - \omega^2}$$

$$\epsilon''(\omega) = -\frac{2}{\pi}\int_0^\infty \frac{w[\epsilon'(w) - \epsilon_0]\,dw}{w^2 - \omega^2}$$

(Equations of this type are valid for any analytic function regular in the lower half plane.)

1-17. Derive Eqs. (1-86).