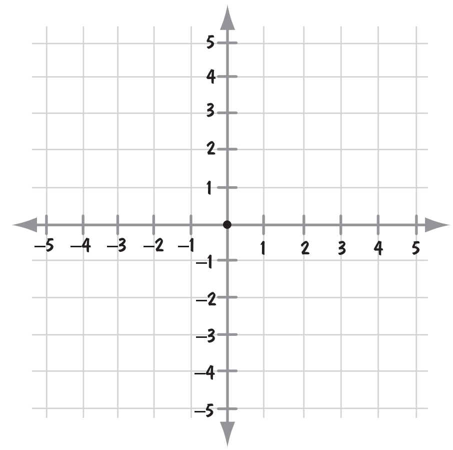
PART 1

Introduction to Two-Dimensional (2-D) Geometric Figures

Two-dimensional geometry, **coordinate plane** geometry, **Cartesian geometry,** and **planar** (pronounced PLANE-er) geometry refer to the same thing: the study of geometric forms in the coordinate plane. Do you remember the coordinate plane? It's a grid system in which two numbers tell you the location of a point—the first, *x*, tells you how far left or right to go from the **origin** (the center point), and the second number, *y*, tells you how far up or down to go. The *y*-axis is **vertical** and the *x*-axis is **horizontal** (like the horizon).

You'll see a lot more of the coordinate plane in geometry, but sometimes all that matters is knowing that a figure is in the plane or two-dimensional without knowing a precise address for it. This part will introduce you to some of the most common figures in two-dimensional geometry and give you some names for their parts and ways to work with them.



In this part, Dr. Math explains

- points, lines, and planes
- angles
- triangles
- quadrilaterals

1 Points, Lines, and Planes

Points, lines, and planes correspond to talking about no dimensions, one dimension, and two dimensions in the coordinate plane. A line is **one-dimensional,** since one number, the distance from zero, tells you where you are. A **plane** is **two-dimensional,** since you need *x* and *y* to locate a point. A **point** is dimensionless. It consists only of location, so it's only possible to be one place if you're on a point—you don't need any extra numbers to tell you where you are. Points, lines, and planes are the foundations of the whole system of geometry.

But point, line, and plane are all **undefined terms.** How can that be? Well, any definition we could give them would depend on the definition of some other mathematical idea that these three terms help define. In other words, the definition would be circular!

Undefined Geometry Terms

Dear Dr. Math,

I know that they call point, line, and plane the undefined terms of geometry, but is there a way to give those terms a definition? I've been thinking, could a line be defined as the joining of two rays going in separate directions? I've never really thought that anything couldn't have a definition, so is it possible for any of these geometric terms to be defined?

Yours truly,

Leon

Dear Leon,

Your definition would require us to first define "ray" and "direction." Can you do that without reference to "point," "line," and "plane"?

Think of it this way: math is a huge building, in which each part is built by a logical chain of reasoning upon other parts below it. What is the foundation? What is everything else built on?

There must be some lowest level that is not based on anything else; otherwise, the whole thing is circular and never really starts anywhere. The undefined terms are part of that foundation, along with rules that tell us how to prove things are true. The goal of mathematicians has not been to make math entirely self-contained, with no undefined terms, but to minimize the number of definitions so that we have to accept only a few basics, and from there we will discover all of math to be well defined. Also, the goal is to make those terms obvious so that we have no trouble accepting them, even though we can't formally prove their existence.

To put it another way, these terms do have a definition in human terms—that is, we can easily understand what they mean. They simply don't have a mathematical definition in the sense of depending only on other previously defined terms.

—Dr. Math, The Math Forum

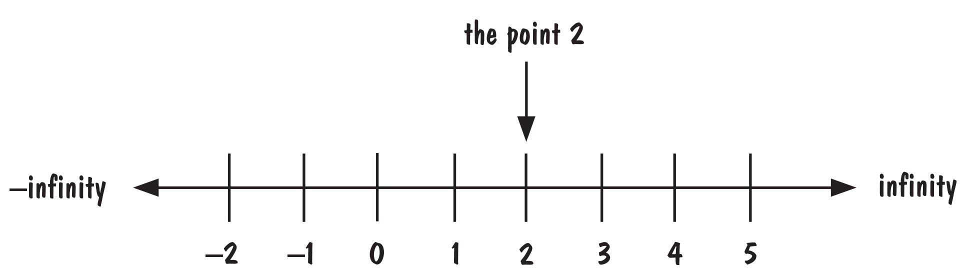
What Is a Point?

Dear Dr. Math,
Define a point, please.
Yours truly,
Lorraine

Dear Lorraine,

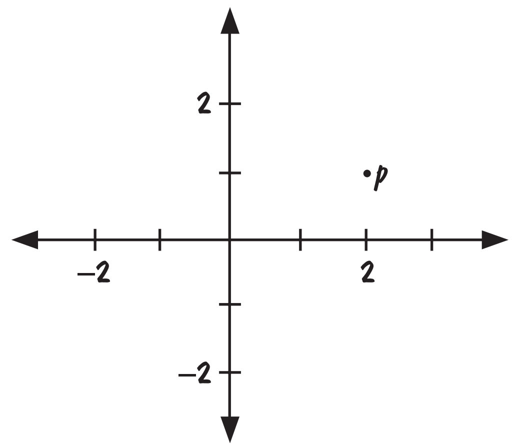
The word "point" is undefined in geometry. But it is pretty easy for us to describe a point, even though it can't be defined. A point is an entity that has only one characteristic: its position. A point has no size, color, smell, or feel. When we talk about points, we are referring to one specific location.

For example, along a number line the number 2 exists at just one point. Points are infinitely small, which means the point at 2 is different from the point at 2.000000001. Here's a picture of a number line:



If you want to distinguish one place along a number line, you "point" at it. You label that place with the corresponding number and refer to it with that number.

Now, how do you distinguish a location in two-dimensional space (e.g., a sheet of paper)? Imagine that we have two number lines: one horizontal and the other vertical. We are pointing at a place *p:*



How do we describe where the point *p* is? We can't just say *p* is at 2 because we don't know which number line that refers to. Is it at 2 along the horizontal number line or the vertical one?

To describe where *p* is, you must talk about where it is both horizontally *and* vertically. So, you can say

p is at 2 horizontally and 1 vertically

However, this is a mouthful. Because describing points in two dimensions is really useful, we have defined some conventions to make life easier. We call the horizontal number line the *x*-axis and the vertical number line the *y*-axis. The convention for talking about points in two dimensions is to write

(position along *x*-axis, position along *y*-axis)

Therefore,

p is at (2, 1)

Points in two dimensions can be described by any pair of numbers. For example, (4, 5), (6.23432, 3.14), and (–12, 4) are all points.

—Dr. Math, The Math Forum

c01.qxd 7/1/04 9:12 AM Page 8

Rays, Line Segments, and Lines

Dear Dr. Math,

I need to know what a ray, a line segment, and a line are.

Sincerely,

Leon

Dear Leon,

In geometry, you can think of a **line** just like a normal straight line, with a couple of special features. The things that make a line in geometry different from a line in any other context—for example, art class—are that it goes on forever in both directions, it's perfectly straight, and it's not thick.

Mathematicians say that their lines have zero thickness, which is pretty hard to imagine. When we draw lines on paper, they always have at least a little bit of width. But when we study lines in geometry, we think of them as having no width at all.

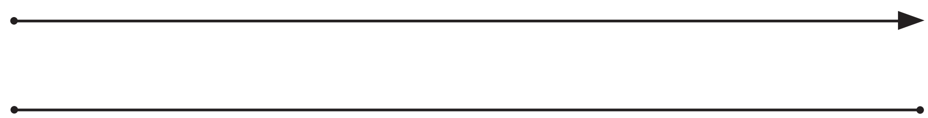
Here's how a lot of people draw lines on paper. The arrows at the ends mean that the line continues forever in both directions:

Rays and line segments are a lot like lines. A **ray** is like a line, except that it only goes on forever in one direction. So it starts at one point and goes on forever in some direction. You can think of the light coming from the sun as an example of a ray: the starting point is at the sun, and the light goes on forever away from the sun.

Here's how we draw rays:

A **line segment** is a little chunk of a line. It starts at one point, goes for a while, and ends at another point. We draw it like this:

Sometimes we like to attach little dots to represent the endpoints of rays and line segments like this:



—*Dr. Math, The Math Forum*

FOR FUTURE REFERENCE

Later in your geometry career, you'll start seeing a notation for lines and segments that will help you tell them apart. Here's a line:

The notation looks like this:

$\overline{AB}$ means the line segment between and including points *A* and *B*; you can also say "segment *AB*."

$\overleftrightarrow{AB}$ means the line indicated by those same points; you can also say "line *AB*."

This line could also be called "line *l*"—lowercase letters are sometimes used for this purpose.

Angles

There are angles all around us—between the hands on a clock, the opening created by a door, even the joints of your body. Any time two lines or line segments or rays intersect, they make **angles.**

What makes one angle different from another? Angles differ in how far open their "jaws" are. If you think of opening an angle starting with two line segments on top of each other, you could open it a little bit, or a pretty big amount, or a whole lot; you could bend it back on itself until the line segments are almost on top of each other again. We often measure angles in degrees to describe how far open the angles are.

In this section, we'll talk about the different kinds of angles and the ways we measure them.

What Is a Vertex?

Dear Dr. Math,

What does vertex mean?

Sincerely,

Lorraine

Dear Lorraine,

A **vertex** is the point at which two rays of an angle or two sides of a polygon meet. Vertices (pronounced VER-tih-seez) is the plural of vertex.

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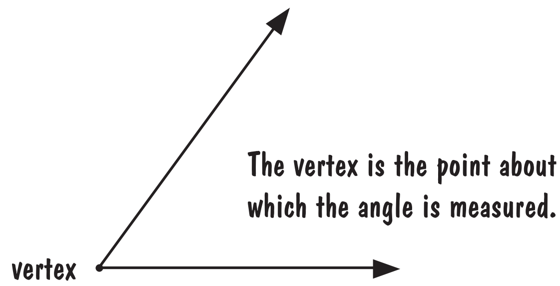
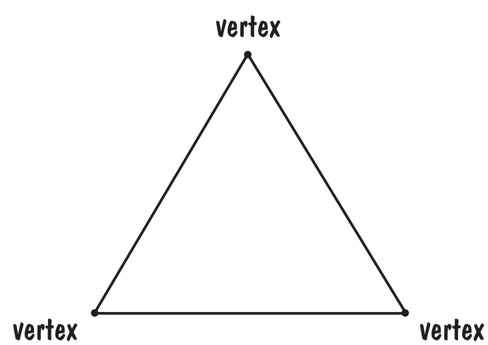


A triangle has three vertices.



—Dr. Math, The Math Forum

Types of Angles: Acute, Right, Obtuse, and Reflex

Dear Dr. Math,

How can I remember what the types of angles mean—for example, acute angle or right angle?

Yours truly,

Leon

Dear Leon,

There are three main types of angles. We tell them apart by how big they are. (Angles are often measured in **degrees**; there are 360 degrees in a circle.) Here they are:

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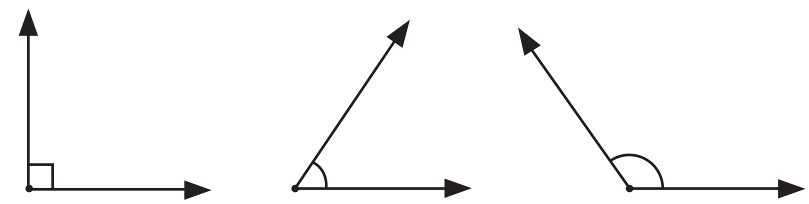
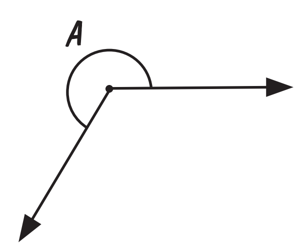


We can start with the right angle: a **right angle** measures exactly 90 degrees. It's called a right angle because it stands upright. Just remember it's an upright angle.

Next is the acute angle. **Acute angles** measure less than 90 degrees. The word "acute" comes from a word that means "sharp." Remember that a sharp pencil or a sharp knife has an acute angle at its point or blade. An acute pain is a sharp pain. Acupuncture uses sharp needles. And, if all else fails, you can remember that an *acute* angle can *cut* you!

Finally, we have the wide-open **obtuse angles,** which measure between 90 and 180 degrees. The word "obtuse" comes from a Latin verb meaning "to make blunt or dull." If a person isn't very sharp (doesn't have an acute intelligence), he may be called obtuse. If that doesn't stick in your mind, just remember that if it isn't right or acute, it's obtuse.

I should mention a fourth kind of angle: the reflex angle. A **reflex angle** is the other side of any other type of angle. Reflex angles measure more than 180 degress. For example, in this diagram, the angle labeled *A* is the reflex angle. (The other angle in the diagram is obtuse.)



One meaning of reflex is "to bend back"; and the angle kind of looks bent back, like an elbow bent too far. Actually, some people can make a reflex angle with their elbow, and some can't. Can you?

I hope the names are memorable for you now.

—Dr. Math, The Math Forum

Complementary and Supplementary Angles

Dear Dr. Math,

In class we're studying complements and supplements of angles. I do not understand any of the terminology behind the problems. Today we took a test, and one of the questions was to find the complement of this angle, c degrees, and I didn't even know where to begin. Another was to find the degrees in the third angle in an isosceles triangle, x degrees, $x - 10$, or something like that. Can you explain this a little better?

Sincerely,

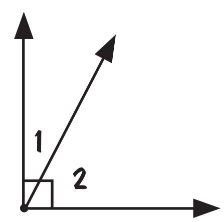
Lorraine

Dear Lorraine,

Part of the problem here is that the names "complement" and "supplement" are kind of confusing, since the literal meanings of these words aren't different enough for us to know which is which, other than by memorizing them.

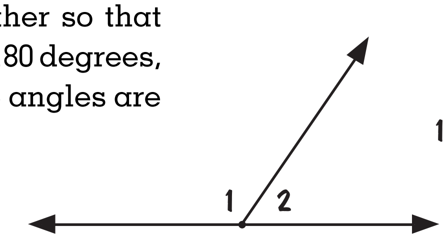
What are complements and supplements?

If you place two angles next to each other so that they add up to 90 degrees, we say that the angles are **complements.**



1 and 2 are complements

If you place two angles next to each other so that they add up to 180 degrees, we say that the angles are **supplements.**



1 and 2 are supplements

Here are some examples of complements and supplements:

Complements	**Supplements**
30 and 60 degrees	30 and 150 degrees
2 and 88 degrees	2 and 178 degrees
14 and 76 degrees	14 and 166 degrees

So what you need to remember is which one adds up to 90 degrees and which one adds up to 180 degrees.

How can you keep them straight? The person who runs the Math Forum's Geometry Problem of the Week tells me that she remembers them this way: c comes before s, and 90 comes before 180. It's the best idea I've heard so far.

If you know that two angles are complements or supplements, you can figure out one given the other. How? Well, if they're supplements, you know that they have to add up to 180:

this + that = 180

So it must be true that

this = 180 – that

and

that = 180 – this

You can do the corresponding calculations for complements using 90 degrees instead of 180 degrees.

So whenever you see the phrase "the supplement of (some angle)°," you can immediately translate it to "180° – (the angle)°." When you have a value for the angle, you end up with something like

the supplement of 26° = (180° – 26°)

which you can just simplify to get a single number. But if you only have a **variable** like *x*, or an **expression** for the angle, like $x - 10$, then you just have to deal with that by substituting the variable or the expression in the equation. For example:

the supplement of $(x° - 10°) = [180° - (x° - 10°)]$

Note that you have to put the expression in parentheses (or brackets), or you can end up with the wrong thing. In this case,

$[180° - (x° - 10°)]$	is *not* the same as	$(180° - x° - 10°)$
$(180° - x° + 10°)$		$(180° - 10° - x°)$
$(190° - x°)$		$(170° - x°)$

Why should you care about complements and supplements?

Well, in geometry you're constantly dividing things into triangles in order to make them easier to work with. And in every triangle, the measures of the interior angles add up to 180 degrees. So if you know two angles, the third is the supplement of the sum of the other two.

The nicest kind of triangle to work with is a right triangle. In a right triangle, you have one right angle and two other angles. Since they all have to add up to 180 degrees, and since the right angle takes up 90 of those degrees, the other two angles must add up to 90 degrees. So if you know one of the acute angles in a right triangle, the other is just the complement of that angle.

—Dr. Math, The Math Forum

ORDER OF OPERATIONS

In case you've forgotten, here's a quick review of the correct order of operations for any expression:

1. Parentheses or brackets
2. Exponents
3. Multiplication and division (left to right)
4. Addition and subtraction (left to right)

For more about this topic, see Section 5, Part 1 of *Dr. Math Gets You Ready for Algebra.*

Alternate and Corresponding Angles

Dear Dr. Math,

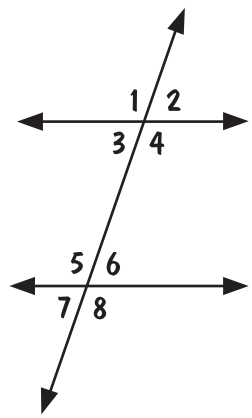
Please explain alternate and corresponding angles.

Sincerely,

Leon

Dear Leon,

Let's first look at a diagram that we can refer to when we define corresponding angles and alternate angles:



There are a lot of numbers in this diagram! Don't worry, though—we'll figure out what everything means.

Assume that the two horizontal lines are **parallel** (that means they have the same slope and never intersect). The diagonal is called a **transversal,** and as you can see, the intersection of the transversal with the horizontal lines makes lots of different angles. I labeled these angles 1 through 8. Whenever you have a setup like this in which you have two parallel lines and a transversal intersecting them, you can think about corresponding angles and alternate angles.

Look at the diagram. Do you see how we could easily split the angles into two groups? Angles 1, 2, 3, and 4 would be the first group—they are the angles the transversal makes with the higher horizontal line. Angles 5, 6, 7, and 8 would be the second group—they are the angles the transversal makes with the lower horizontal line.

Can you see how the bottom set of four angles looks a lot like the top set of four angles? We say that two angles are **corresponding angles** if they occupy corresponding positions in the two sets of angles. For example, 1 and 5 are corresponding angles because they are both in the top left position: 1 is in the top left corner of the set of angles {1, 2, 3, 4}, while 5 is in the top left corner of the set of angles {5, 6, 7, 8}.

Similarly, 3 and 7 are corresponding angles. There are two more pairs of corresponding angles in the diagram. Can you find them?

One neat and helpful fact about corresponding angles is that they are always equal. Can you see why? (Think about the way the nonparallel line intersects the parallel lines.)

Let's move on to alternate angles.

We say that two angles are **alternate angles** if they fulfill three requirements:

1. They must both be on the **interior** (inside or middle part) of the diagram between the parallel lines, or both on the **exterior** (outside or outer part) of the parallel lines. By interior angles, I mean angles 3, 4, 5, and 6; by exterior angles, I mean angles 1, 2, 7, and 8.
2. They must be on opposite sides of the transversal. Hence 3 and 5 cannot be alternate angles because they are both to the left of the transversal.
3. If two angles are alternate, one must be from the group of angles that has the top horizontal line as one of its sides, and the other angle must be from the group of angles that has the bottom horizontal line as one of its sides. In other words, the last requirement says that a pair of alternate angles must consist of one angle from the set {1, 2, 3, 4} and one angle from the set {5, 6, 7, 8}.

This sounds complicated, but if we look at the diagram and apply the three requirements, it will become clear what we mean by alternate angles.

1. The first requirement tells us that 3, 4, 5, and 6 can only be paired with each other and that 1, 2, 7, and 8 can only be paired with each other. That rules out a lot of possibilities.
2. The second requirement tells us that a pair of alternate angles must be on opposite sides of the transversal. So, 2 and 8 cannot be a pair of alternate angles. Similarly, 4 and 6 cannot be a pair of alternate angles.

3. Applying the final constraint, we see that there are exactly four pairs of alternate angles in the diagram. One pair is 3 and 6. Angles 3 and 6 fulfill all the requirements of alternate angles: they are interior angles, they are on opposite sides of the transversal, and they come from different groups of angles. Can you find the other three pairs of alternate angles?

A helpful fact about alternate angles is that they, too, are equal in measure. This fact can make proofs much easier! Can you see why they are equal?

—Dr. Math, The Math Forum

Alternate Exterior Angles

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Dear Dr. Math,

I have been trying to find out what alter-
nate exterior angles are for hours! My
teacher assigned us a vocabulary sheet for
geometry, and the only term I can't find is
alternate exterior angles. I know what an
alternate interior angle is but not an exte-
rior one. I am completely clueless. Please
help!

Sincerely,

Lorraine
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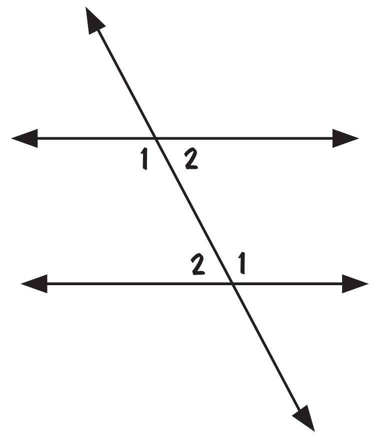
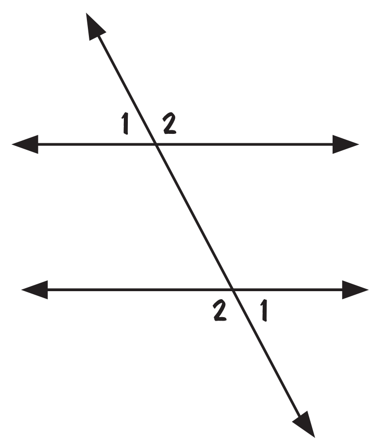
Dear Lorraine,

Here's a clue from everyday English usage: "interior" means "inside," and "exterior" means "outside." (You may see those words on paint can labels, for example.)

So alternate *interior* angles are on opposite sides of the transversal, inside or between the parallel lines, like the pair of angles labeled 1 and the pair labeled 2 here:



Alternate *exterior* angles are also on opposite sides of the transversal but outside the parallel lines:



I suspect a lot of students hear these phrases and never stop to think what the individual words (alternate, interior, exterior) mean because they expect math terms to be incomprehensible and unrelated to real life! Sometimes math makes more sense than you realize at first.

—Dr. Math, The Math Forum

FINDING MATH DEFINITIONS

Maybe I can help so that you do not have to look for hours for the definition of a math term the next time. To find out what a word means, I would first go to a regular English dictionary; then maybe try one of the dictionary or encyclopedia resources listed in our online FAQ, or search our site (mathforum.org); then go to google.com and enter a phrase such as "alternate exterior angles" to see if there is a definition on the Web. You'll find it!

Vertical Angles

Dear Dr. Math,

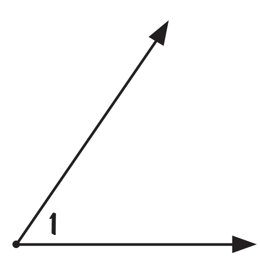
I don't understand vertical angles! I believe that vertical angles are equal. Am I right? How can you tell if angles are vertical?

Sincerely,

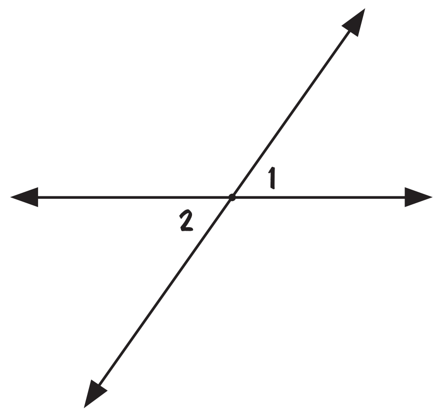
Leon

Dear Leon,

Get a piece of paper and draw an angle, which we'll call angle 1:



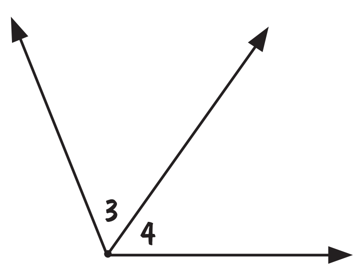
Now take a ruler and extend each ray on the other side of the angle to make two intersecting lines:



You've just made a new angle 2. The angles 1 and 2 form a pair of vertical angles; they are called **vertical angles** because they are on opposite sides of the same vertex.

Think of the handles and blades of a very simple pair of scissors (with no bend to the handles) as another example. Notice that whatever angle you open the scissors to, the handles will be at the same angle, because vertical angles are always **congruent**—that is, they have the same measure.

The important thing to remember is that not all congruent angles are vertical; angles are vertical because of where they are, not just because they happen to have the same measure. For example, in this diagram, 3 and 4 are congruent but not vertical angles:



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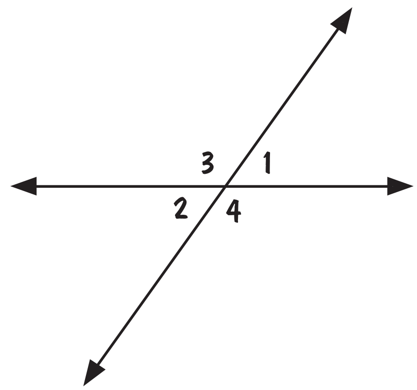
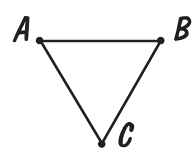
Whenever you see a pair of lines crossing, you will have two pairs of vertical angles. In the diagram below, the vertical angles are 1 and 2, and 3 and 4:

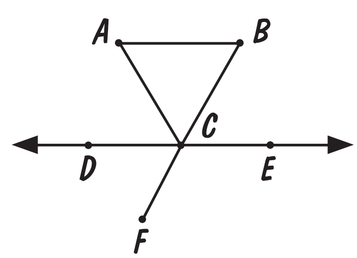


—*Dr. Math, The Math Forum*

NAMING ANGLES



If we wanted to talk about the angles in this diagram, we could call them angles *A, B,* and *C.* But if we add a few more objects, it becomes more difficult to tell which angle is identified by any single letter. For example, in this diagram:



If we referred to angle *C,* which angle would we mean? Mathematicians use an angle symbol, $\angle$, and three letters to name specific angles in diagrams like this. You'll find $\angle BCE$ here:

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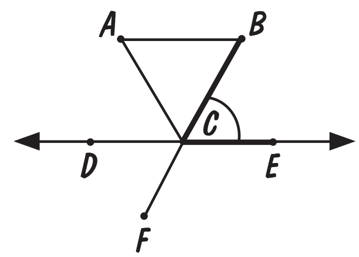
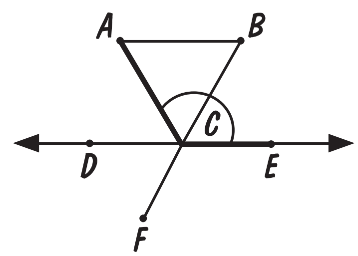
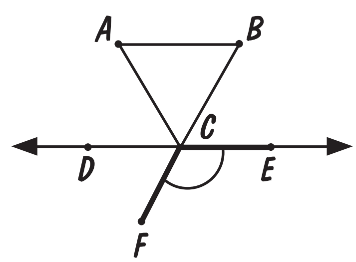


and ∠*ACE* here:



and ∠*FCE* here:



You'll probably see "line *DE*" or "$\overleftrightarrow{DE}$" instead of "∠*DCE*," though. Just keep clear in your mind that there's no bend at *C* in line *DE*!

Measuring Angles with a Protractor

Dear Dr. Math,

I want to know how to measure acute, reflex, and obtuse angles with a protractor.

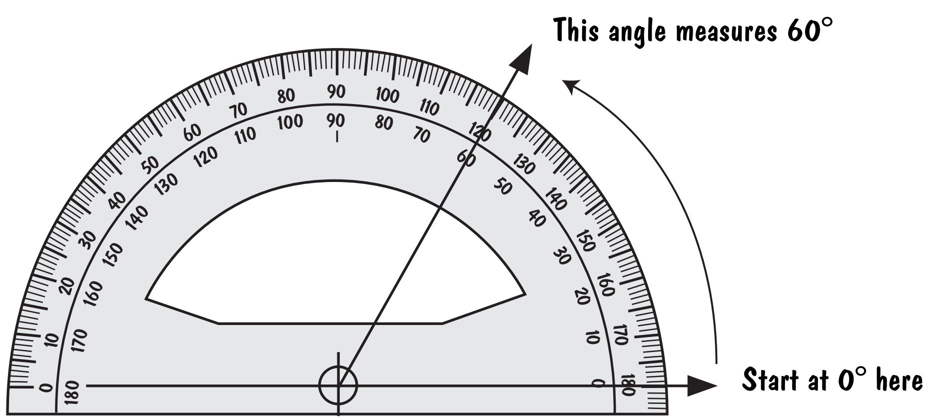
Sincerely,

Lorraine

Dear Lorraine,

As you know, a **protractor** is a tool for measuring angles on paper. The one I'm looking at, which probably looks a lot like yours, looks like this: it's a half-circle of clear plastic, with a line along the straight edge that has a small hole cut out of the middle of it, and a hash mark through the edges of the hole perpendicular to the long line, to help you line up the angle you're measuring. All along the curved edge are little hash marks in degrees, to tell you how big the angle is. On my protractor, there are two sets of numbers: one goes from 0 to 180 clockwise, from left to right, and the other, inner set goes from 0 to 180 counter-clockwise, from right to left. Of course in the middle of the curve where they meet, both sets say 90!

So to measure an acute angle (less than 90 degrees) with this protractor, put the little hole at the vertex of the angle, and align the long line with one of the rays of the angle. Choose the scale that has the zero on the ray of the angle that you lined up with it. Read off the number from this scale at the point where the other side of the angle crosses the protractor. (You may have to use a straightedge to extend this side of the angle if it's not long enough to reach the protractor's marks.)



Measuring obtuse angles (between 90 and 180 degrees) works exactly the same way.

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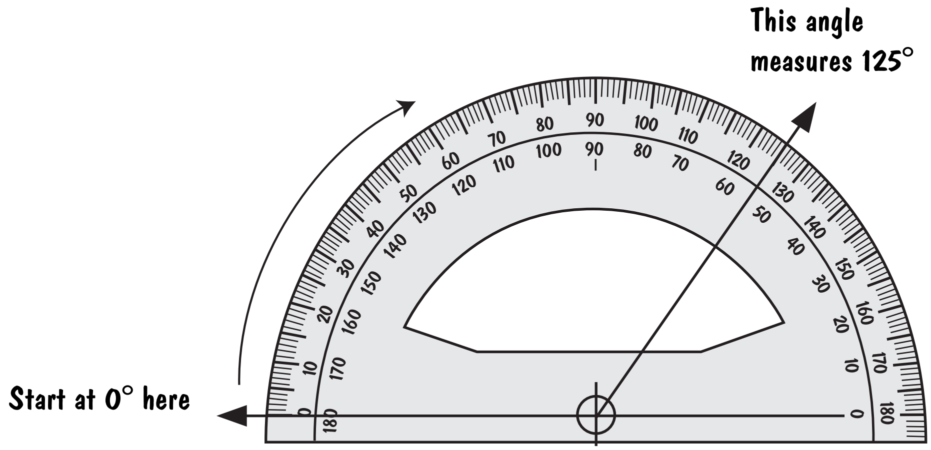
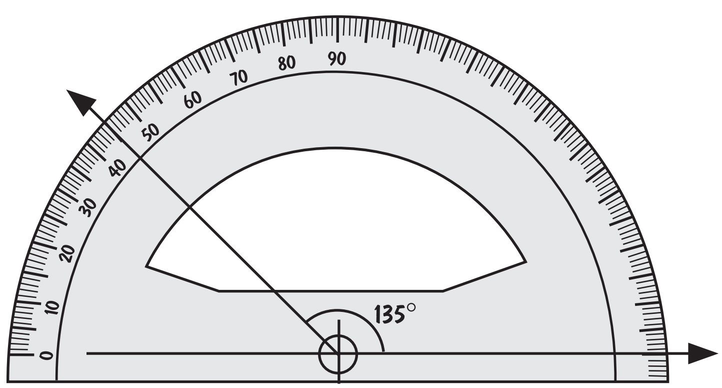


Some protractors have only one scale, with zero on both ends and 90 in the middle. If so, then when you measure an obtuse angle, you'll read a number between 0 and 90, then you'll need to subtract that number from 180 to get the measure of the obtuse angle. For instance:



The protractor reads 45 for this angle, but the angle is really

$$180° - 45° = 135°$$

A reflex angle is the outside of an acute, obtuse, or right angle.

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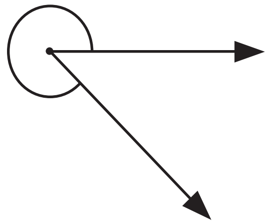
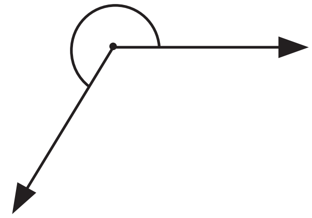
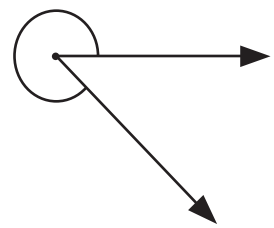


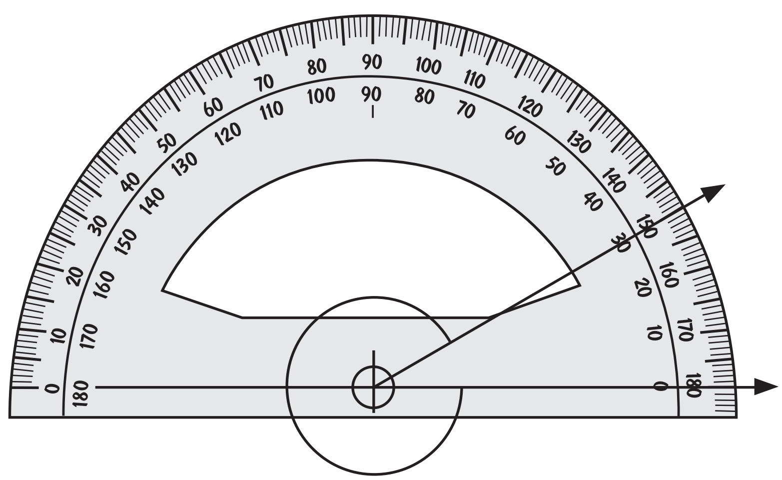


To measure a reflex angle, use the protractor to read the measure of the inside of the angle. Then subtract from 360 to get the measure of the reflex angle.



For example, here is a reflex angle. Note that in this case the other angle is an acute angle:

The sum of the two angles is 360 degrees, right? So if we measure the acute angle instead of the reflex angle and subtract its measure from 360 degrees, we'll have the measure of the reflex angle: 360° – 30° = 330°.



—*Dr. Math, The Math Forum*

Degrees in a Circle

Dear Dr. Math,

I would like to know why a circle measures 360 degrees. Is there any special reason for this, or did the Greeks just kind of pick it out? I'm sure there's a rational explanation, but I just can't seem to figure it out. I hate accepting things that I don't understand, and this is something that really bugs me. Please help!

Sincerely,

Leon

Dear Leon,

A circle has 360 degrees, but it also has 400 gradients and approximately 6.2831853 radians. It all depends on what units you measure your angles with.

Allow me to explain. Say you think 360 is a terrible number, and you think that you want a circle to have 100 "somethings" in it. Well, you divide up the circle into 100 equal angles, all coming out from the center, then you call one of these angles a "zink." Then you've just defined a new way to measure a circle. One hundred zinks are in a circle.

This invented unit, the zink, is much like the degree, except the degree is smaller. (Why? Think of how many quarters it takes to make a dollar and how many pennies. Which is bigger?) They are both angle measures, just as the inch and centimeter are both units of length.

The ancient Babylonians (not the Greeks) decided that a circle should contain 360 degrees. In 1 degree there are 60 minutes (that's the same word as the unit of time, but this means one-sixtieth of a degree). Furthermore, in 1 minute there are 60 seconds (again, although they are the same word, this is a unit of measure for angles, not time).

The French in the early days of the metric system, and the British separately around 1900, chose a different way to divide the circle, specifically into 400 gradients. So 1 **gradient** is a tad bit smaller than a degree.

And what's a **radian?** It's a measurement mathematicians use for angles because it's a way to divide the circle into a number of parts that happen to make certain computations easy. The way they decided it was this: They took a circle, say with radius 1 cm. They took a piece of string and made marks on it, evenly spaced 1 cm apart. Then they took the string and wrapped it around the circle. They then asked how many 1-cm pieces of string fit around the circle, and they got the answer of about 6.2831853 pieces. They decided that the angle that a 1-cm piece of string covers as it is wrapped about the edge of a circle of radius 1 cm should be called 1 radian. Weird but true. So there are about 6.2831853 radians in a circle, which means that radians are a lot bigger than degrees. That funny decimal number just happens to be equal to 2 pi, or 2π. We'll talk about pi later in the book. It's a really important number, especially for circles.

Now, you might be wondering why the Babylonians chose the number 360. The reason is that their number system was based on 60. To compare, we base our number system on 10. For us, 10 is a nice, round number, and we find it very convenient to count in multiples of 10. But the Babylonians liked 60.

Why this system was nice for them, nobody knows, but modern mathematicians agree that 60 is a nice number, too, because $60 = 2 \cdot 2 \cdot 3 \cdot 5$ and $360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$. What's so neat about that, you ask? Well, you will find that 360 is divisible by 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, and 20. There are few other numbers as small as 360 that have so many different factors. This makes the degree a very nice unit to divide the circle into an equal number of parts: 120 degrees is one third of a circle, 90 degrees is one fourth, and so on.

So while a zink, being $\frac{1}{100}$ of a circle, may seem nice and round to us, it isn't so convenient for dividing a pie into thirds. I mean, whoever heard of asking for a $33\frac{1}{3}$ zink piece of pie?

—Dr. Math, The Math Forum

3 Triangles

Just as there are various kinds of angles, which we discussed in the previous section, there are also various kinds of triangles. In this section, we'll talk about what the differences are and how the Pythagorean theorem can help you find the side lengths of one common type.

Types of Triangles

Dear Dr. Math,

What is the difference between an isosceles triangle and a scalene triangle? I always forget which is which!

Yours truly,

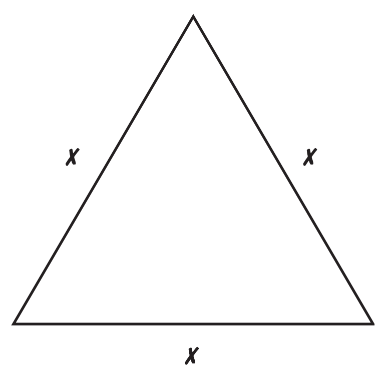
Lorraine

Dear Lorraine,

A useful trick in trying to remember these names and many others is to think about the pieces of words that they're made from.

For example, "lateral" always has to do with sides. The fins on the side of a fish are "lateral fins" (as opposed to "dorsal fins," which are on the back). Trade between two countries is "bilateral trade." In football, a "lateral" is when the quarterback tosses the ball to the side instead of throwing it forward, as in a regular pass. And so on.

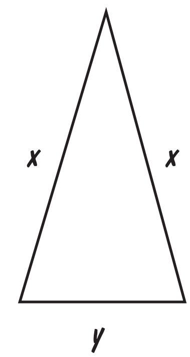
So "equilateral" means "equal sides," and in fact, all the sides of an **equilateral triangle** are equal. (That means its angles are also the same, and figures with sides and angles all the same are called **regular.**



equilateral
triangle

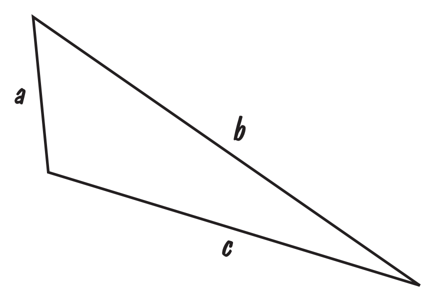
The prefix "iso-" means "same." An "isometric" exercise is one in which the position of the muscles stays the same (as when you press your two hands together). Two things that have the same shape are "isomorphic." And so on.

"Sceles" comes from the Greek "skelos," which means "leg." So an **isosceles triangle** is one that has the "same legs" as opposed to "equal sides." In an equilateral triangle, all the sides are the same; but in an isosceles triangle, only two of the sides, called the **legs,** must have the same measure. The other side is called the **base.**



isosceles triangle

"Scalene" comes from the Greek word for "uneven," and a **scalene** triangle is uneven: no side is the same length as any other. But to be honest, usually I just remember that "scalene" means "not equilateral or isosceles."



scalene triangle

So what can be learned from this? One lesson is that when you're having trouble remembering a word, it's often a good idea to consult a dictionary to find out the history of the word, because understanding how a word was created can help it seem less random. Another lesson is that many of the words that we find in math and science were made up by people who were familiar with Latin and Greek. So studying word roots, prefixes, and suffixes from these languages can make it much easier to learn mathematical and scientific words!

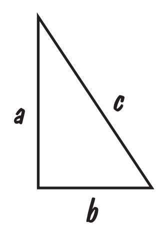
—Dr. Math, The Math Forum

The Pythagorean Theorem

Dear Dr. Math,
What is the Pythagorean theorem?
Yours truly,
Leon

Dear Leon,

Pythagoras was a Greek mathematician who lived around 569–475 B.C. The Babylonians came up with this idea a thousand years earlier, but Pythagoras might have been the first to prove it, so it was named for him. The **Pythagorean theorem** has to do with the lengths of the sides of a right triangle. A right triangle is any triangle that has one right angle (an angle of 90 degrees)—like this:



right triangle

If the sides next to the right angle are of lengths a and b, and the third side is of length c, then the Pythagorean theorem says that $a^2 + b^2 = c^2$. That is, $(a \cdot a) + (b \cdot b) = (c \cdot c)$. When people say this, they say, "a squared plus b squared equals c squared":

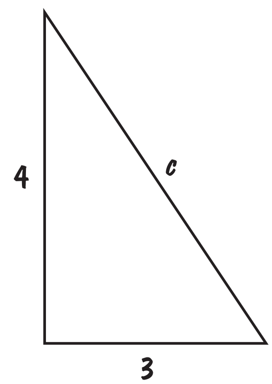
$$a^2 + b^2 = c^2$$

REMINDER: SQUARES AND SQUARE ROOTS

When you multiply a number by itself, such as $a \cdot a$, you call it "squaring the number," and you write it as a^2. The reverse of that process is called "taking the square root of a number." The square root of a^2 is written $\sqrt{a^2}$ and is equal to a. The square root sign, $\sqrt{\ }$, is also called a radical.

c01.qxd 7/1/04 9:12 AM Page 35

Some numbers that work in this equation are 3, 4, and 5; and 5, 12, and 13.

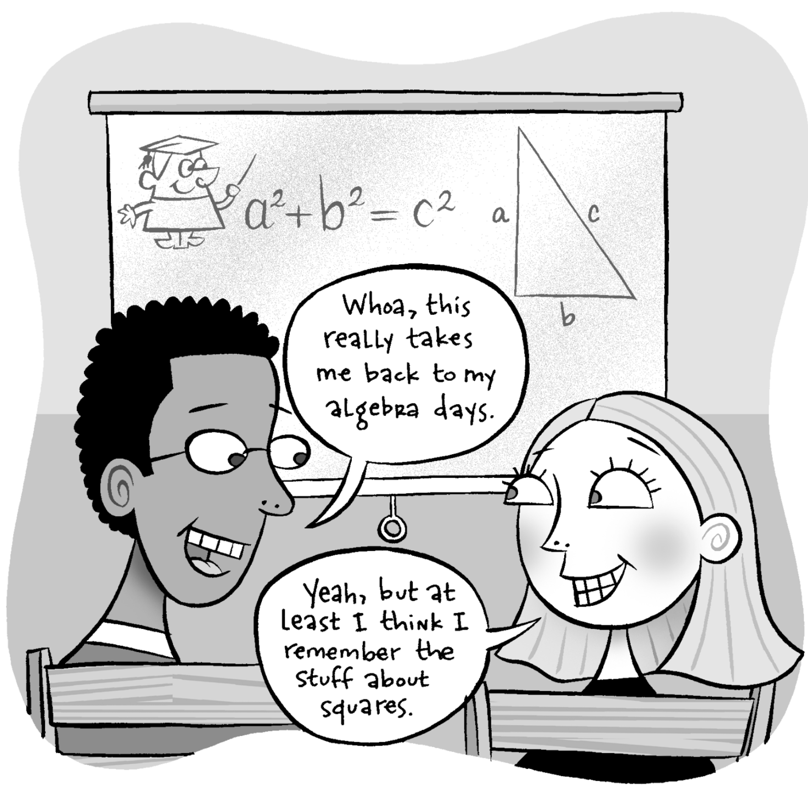


So if you are told that you have a right triangle whose legs are 3 and 4 units, as in this diagram, then you can use this theorem to find out the length of the third side. The third side (the side opposite the right angle in a right triangle) is called the **hypotenuse.**

$$3^2 + 4^2 = 9 + 16$$
$$= 25$$
$$\text{If } c^2 = 25$$
$$c = \sqrt{25}$$

So $c = 5$.

—*Dr. Math, The Math Forum*



ASK DR. MATH FAQ THE PYTHAGOREAN THEOREM

When would I use the Pythagorean theorem?

The Pythagorean theorem is used any time we have a right triangle, we know the length of two sides, and we want to find the third side.

For example, I was in the furniture store the other day and saw a nice entertainment center on sale at a good price. The space for the TV set measured 17×21 inches. I didn't want to take the time to go home to measure my TV set or get the cabinet home only to find that it was too small.

I knew my TV set had a 27-inch screen, and TV screens are measured on the diagonal. To figure out whether my TV would fit, I calculated the diagonal of the TV space in the entertainment center using the Pythagorean theorem:

$$\begin{aligned} 17^2 + 21^2 &= 289 + 441 \\ &= 730 \end{aligned}$$

So the diagonal of the entertainment center is the square root of 730, which is about 27.02 inches.

It seemed like my TV would fit, but the 27-inch diagonal on the TV set measures the screen only, not the housing, speakers, and control buttons. These extend the TV set's diagonal several inches, so I figured that my TV would not fit in the cabinet. When I got home, I measured my TV set and found that the entire set was 21×27.5 inches, so it was a good decision not to buy the entertainment center.

The Pythagorean theorem is also frequently used in more advanced math. The applications that use the Pythagorean theorem include computing the distance between points on a plane; computing perimeters, surface areas, and volumes of various geometric shapes; and calculating the largest and smallest possible perimeters of objects, or surface areas and volumes of various geometric shapes.

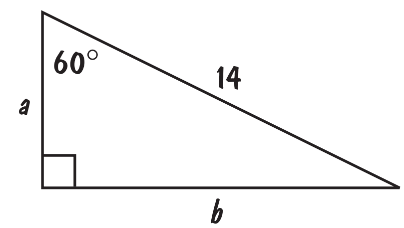
c01.qxd 7/1/04 9:12 AM Page 37

Special Right Triangles

Dear Dr. Math,

I need help figuring out something my teacher calls “special right triangles.” I’ve tried to start with *a* and *b*, but I get confused. I don’t know the next step.

For example, one problem is: find *a* and *b*. Simplify radicals whenever possible.



I need help with 45–45–90 triangles and 30–60–90 triangles. If you have any available notes to show me step by step, I’ll be very grateful.

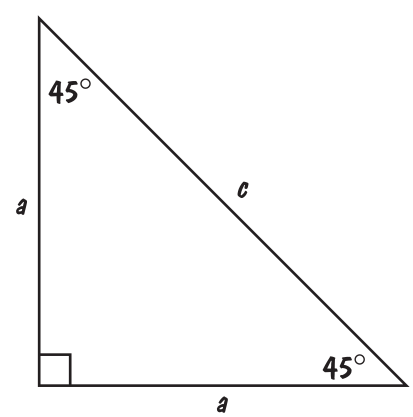
Yours truly,

Lorraine

Dear Lorraine,

Thanks for a carefully explained question!

The two special kinds of triangles you describe are special because two sides are related in a simple way. For the 45–45–90 triangle,

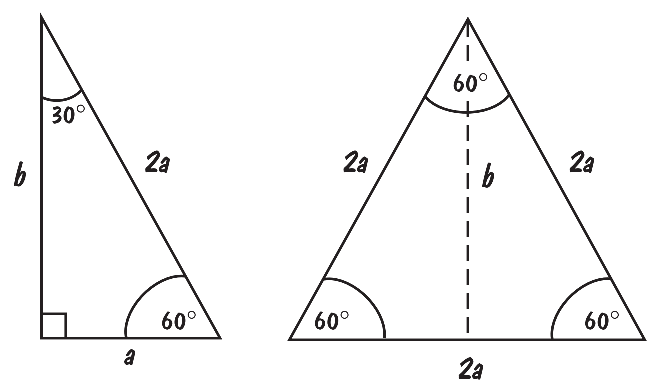


Since the two base angles, the 45-degree angles, are equal, it's an isosceles triangle, and therefore the two sides opposite the 45-degree angles are equal. You can get the length of the other side using the Pythagorean theorem:

$$\begin{aligned} c^2 &= a^2 + a^2 \\ c &= \sqrt{a^2 + a^2} \\ &= \sqrt{2a^2} \\ &= a \cdot \sqrt{2} \end{aligned}$$

That is, the hypotenuse is the square root of 2 times the length of the other sides.

For the 30–60–90 triangle, the important thing to know is that it's exactly half of an equilateral triangle:



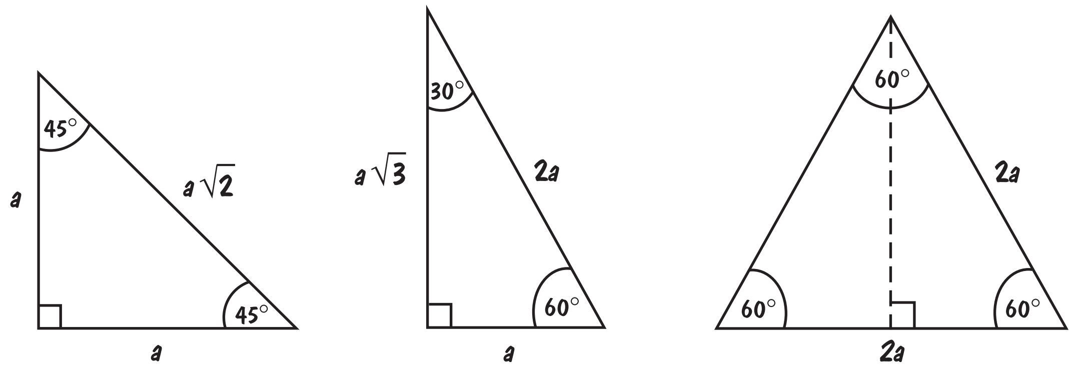
This means that the side opposite the 30-degree angle is half the length of the hypotenuse. Again, you can get the length of the third side by using the Pythagorean theorem:

$$b = \sqrt{(2a)^2 - (a)^2} = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = a \cdot \sqrt{3}$$

So in your first problem, we're dealing with another 30–60–90 triangle, and a must be half of 14, or 7 (just imagine completing the equilateral triangle by reflecting the triangle over side b if this isn't clear). You can use the Pythagorean theorem directly on these numbers, or multiply 7 by the square root of 3 from the formula above to get the answer.

If you prefer, just try memorizing these diagrams:

c01.qxd 7/1/04 9:12 AM Page 39



If you have trouble remembering which triangle uses the $\sqrt{3}$ and which uses the $\sqrt{2}$, just remember that the 45–45–90 triangle has two different edge lengths, and it gets the $\sqrt{2}$. The 30–60–90 triangle has three different edge lengths, and it gets the $\sqrt{3}$.

—***Dr. Math, The Math Forum***

Why Do the Angles of a Triangle Add Up to 180 Degrees?

Dear Dr. Math,

We were wondering why all the angles in a triangle add up to 180 degrees. Several of us are trying to prove our teacher wrong and draw a triangle differently!

Yours truly,

Leon

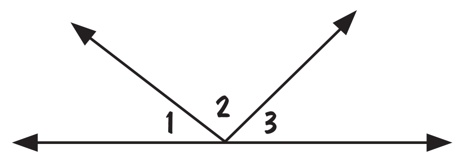
Dear Leon,

There are several ways that you can show the angles of a triangle add up to 180 degrees. The first example I'll give you involves paper and pencil. The other three examples are more formal in that you construct a figure and use some of the rules of geometry. If you don't understand the last few examples all the way, don't worry about it. You'll cover that stuff in more detail later on.

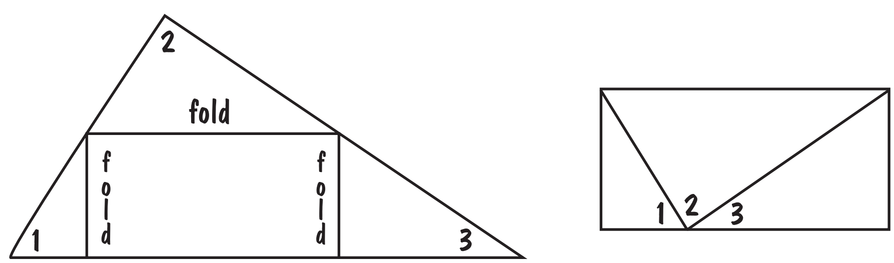
1. Here's a simple way to demonstrate that the three angles of a triangle add up to 180 degrees: the angles can be put together

c01.qxd 7/1/04 9:12 AM Page 40

to form a straight angle (a line). Make a triangle out of paper, tear off the three corners, and fit them together at the points. They should always form a straight line:

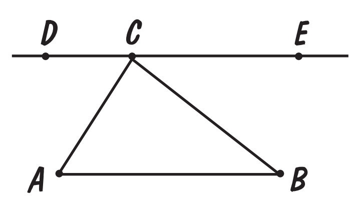


Here's another way to do pretty much the same thing: make a paper triangle in which angles 1 and 3 are both acute (2 may be obtuse) and fold the corners in (dividing two edges in half) so that they all meet on the remaining edge:



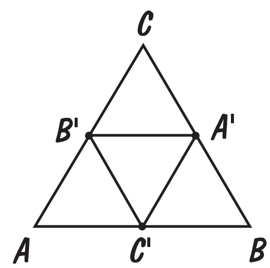
You will end up with a rectangle surrounding three angles that together form its bottom edge: 180 degrees.

2. Let *ABC* be a triangle. Draw a line through C parallel to *AB* (we'll label two points on this line *D* and *E* for clarity):



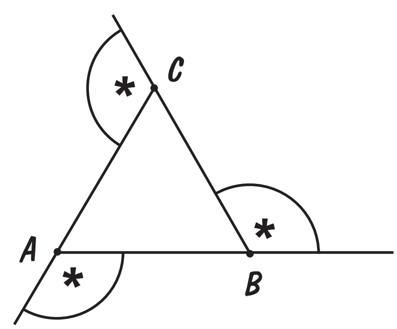
Because line *BC* cuts across two parallel lines, that makes it a transversal. So $\angle BCE = \angle CBA$ and $\angle ACD = \angle CAB$ because they are alternate interior angles. Since $\angle BCE + \angle BCA + \angle ACD = 180°$ (they form a straight angle), the same goes for the angles of triangle *ABC*.

3. Let ABC be a triangle. Let A' be the midpoint of BC, B' the midpoint of AC, and C' the midpoint of AB:



In this way we form four congruent triangles $A'B'C'$, $AB'C'$, $A'BC'$, and $A'B'C$, of which the sum of the angles is equal to the sum of the angles of ABC. If we leave out the angles of triangle ABC, three straight angles are left for the sum of three of the triangles. So each triangle must have a sum equal to one straight angle: 180 degrees.

4. Let ABC be a triangle, and consider the following figure:



Note that the three angles marked with * add up to one complete turn—that is, 360 degrees. Note also that each of the angles marked with * makes a straight angle when added to one of the angles of ABC. So the three angles marked with * added to the angles of ABC add up to $3 \cdot 180° = 540°$. That leaves $540° - 360° = 180°$ for the angles of ABC.

—*Dr. Math, The Math Forum*

4 Quadrilaterals

A **polygon** (any figure made up of connected straight line segments) that has four sides is called a **quadrilateral**—remember "lateral" means "side," as in "equilateral," and "quad" means "four." But just knowing something's a quadrilateral doesn't tell you much about its angles or sides except that there are four of them. In this section, we'll discuss the various types of quadrilaterals.

The Seven Quadrilaterals

Dear Dr. Math,

I really need to know the seven types of quadrilaterals. Please give me a hand!

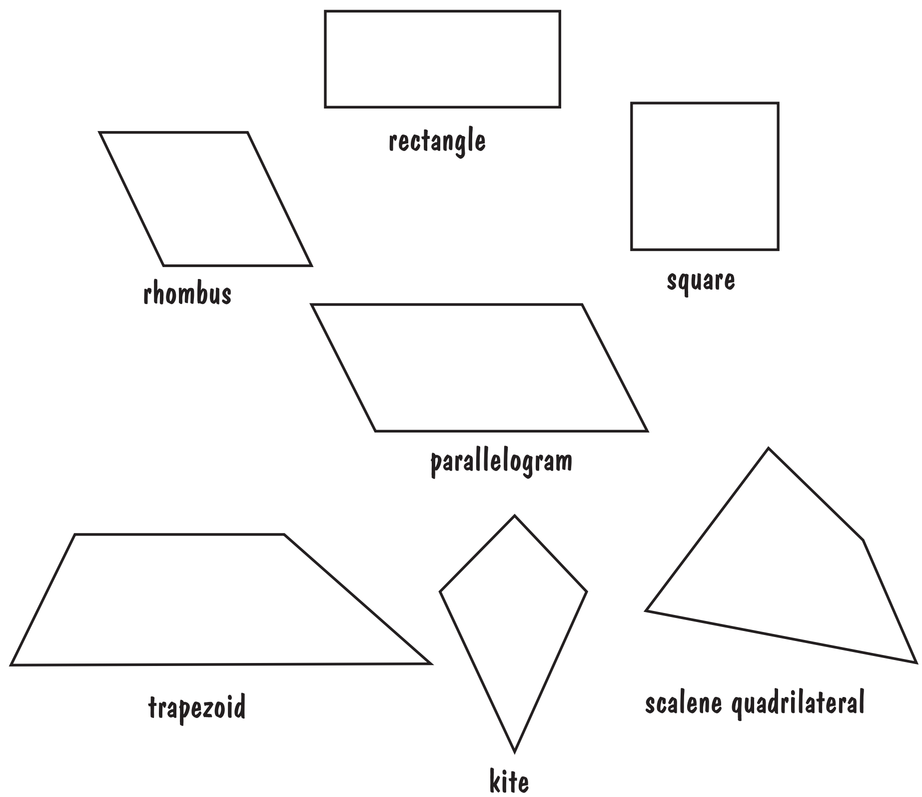
Yours truly,

Lorraine

c01.qxd 7/1/04 9:12 AM Page 43

Dear Lorraine,

I think you're talking about these:



Here are the things you ought to know:

1. A **rhombus** is an equilateral quadrilateral (all sides have the same length).
2. A **rectangle** is an equiangular quadrilateral (all angles have the same measure).
3. A **square** is an equilateral, equiangular quadrilateral, or

simply a regular quadrilateral. Every square is also a rhombus (because it's equilateral) and a rectangle (because it's equiangular).

4. A **parallelogram** is a quadrilateral with exactly two pairs of parallel sides. Every rhombus is a parallelogram and so is every rectangle. And if every rectangle is a parallelogram, then so is every square.

5. There are two definitions commonly used for **trapezoid.** The traditional American definition is a quadrilateral with *exactly* one pair of parallel sides. The British and "new" American definition is a quadrilateral with *at least* one pair of parallel sides. In this book we will use the second definition, which means that all parallelograms (including rhombuses, rectangles, and squares) may be considered trapezoids, because they all have at least one pair of parallel sides. (If the trapezoid is isosceles, then the nonparallel sides have the same length and the base angles are equal.)

6. A **kite** may or may not have parallel sides; it does have two pairs of adjacent sides with equal lengths—that is, instead of being across from each other, the sides with equal lengths are next to each other. So a kite can look like the kind of toy you'd fly in a field on a windy day. But a rhombus and a square are also special cases of a kite: while they do have two pairs of adjacent sides that have equal lengths, those lengths are also equal to each other.

   Just as there are two definitions for the trapezoid, there are two definitions for the kite. We use the one given above; some people use one that says the two pairs of congruent sides must have different lengths, so for them, a rhombus (and therefore a square) is not a kite.

7. A **scalene** quadrilateral has four unequal sides that are not parallel.

—Dr. Math, The Math Forum

The Venn Diagram to Classify Quadrilaterals

Dear Dr. Math,

I am looking for a diagram that will accurately display the relation among trapezoids, parallelograms, kites, rhombuses, rectangles, and squares. Is a square also a kite? Is a kite defined as "a quadrilateral having at least two pairs of adjacent sides congruent, with no sides used twice in the pairs"? Why the "at least two pairs" and the "no sides used twice"?

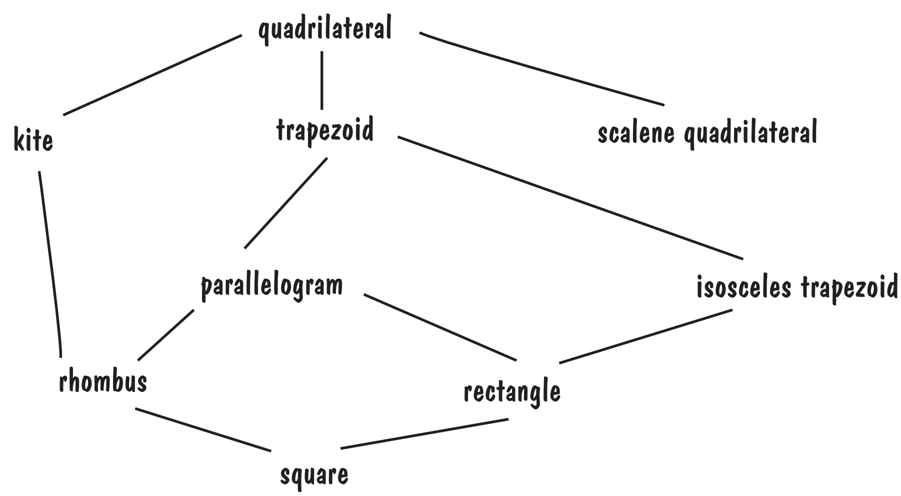
Yours truly,

Leon

Dear Leon,

Your definition of a kite seems awkward. The people who wrote the definition want to make sure you don't count three consecutive congruent sides as two pairs, so they say you can't use the same side twice. I can't imagine why they bother saying "at least two pairs," since once you've chosen two separate pairs, you've used up all the sides. Maybe they want to make sure that they allow for the square, in which there are four pairs of congruent sides, giving two ways to choose two sides that are disjoint to fit the other rule. In any case, the square is a kite by the definition we use. (If you are studying kites in school, check your math book and with your teacher to be sure they both use the same definition, since there's another definition by which the square is not a kite.)

Now for your other question. Here's a diagram showing the relationships between shapes. The lines indicate that the lower term is a subset, or special case, of the upper term. For example, a rectangle is a type of parallelogram *and* a type of isosceles trapezoid, but a rectangle is not a type of rhombus.



Keep your quadrilateral definitions handy, and check to see if the diagram makes sense to you. Here are some of the things it should tell you. Some quadrilaterals are kites, some are trapezoids, and some are scalene quadrilaterals. Some trapezoids are parallelograms, some are isosceles, and some are neither. Parallelograms that are also isosceles trapezoids are rectangles; those that are both isosceles trapezoids and rhombuses are squares.

Not only are all rectangles parallelograms, but all of the properties of parallelograms are true for rectangles. Two properties of parallelograms are that the opposite sides are parallel and the diagonals bisect each other. Since rectangles and rhombuses are parallelograms, then they also have opposite sides that are parallel and diagonals that bisect each other.

Note that I am using the definition of a trapezoid that says that *at least* one pair of the sides must be parallel. If we have two sides that are parallel, then it's also a parallelogram. Some math books use a different definition in which *exactly* one pair of sides is parallel.

—Dr. Math, The Math Forum

Resources on the Web

Learn more about two-dimensional geometric figures at these sites:

Math Forum: Ask Dr. Math: Point and Line

mathforum.org/library/drmath/view/55297.html

A point has no dimension (I'm assuming), and a line, which has dimension, is a bunch of points strung together. How does something without dimension create something with dimension?

Math Forum: Problems of the Week: Middle School: Back Yard Trees

mathforum.org/midpow/solutions/solution.ehtml?puzzle=35

How many different quadrilaterals can be formed by joining any four of the nine trees in my backyard?

Math Forum: Problems of the Week: Middle School: Picture-Perfect Geometry

mathforum.org/midpow/solutions/solution.ehtml?puzzle=97

Graph four points and name the figure that you have drawn.

Math Forum: Problems of the Week: Middle School: Shapes Rock

mathforum.org/midpow/solutions/solution.ehtml?puzzle=93

Find the number of diagonals in a polygon of forty sides.

Math Forum: Sketchpad for Little Ones

mathforum.org/sketchpad/littleones/

A variety of introductory Geometer's Sketchpad activities originally written for second through sixth grades but that older students have also found useful.

Shodor Organization: Project Interactivate: Angles

shodor.org/interactivate/activities/angles/

Students practice their knowledge of acute, obtuse, and alternate angles.

Shodor Organization: Project Interactivate: Triangle Explorer

shodor.org/interactivate/activities/triangle/

Students learn about areas of triangles and about the Cartesian coordinate system through experimenting with triangles drawn on a grid.

Shodor Organization: Project Interactivate: Pythagorean Explorer

shodor.org/interactivate/activities/pyth2/

Students find the length of a side of a right triangle by using the Pythagorean theorem, then check their answers.