PART 1

Linear Equations

Linear equations (equations that describe a line) are the simplest equations that mathematicians study. The work you've done with integers and coordinate graphing will help you with the topic of linear equations. If you have two points on a graph, you can connect them to form a line. When you see a line on a graph, you might notice some of the points it goes through. You could be precise in naming

those points by identifying both the x- and y-coordinates. You might also notice if the line is slanted to the left or to the right or if it is horizontal or vertical. Both the location of the points on the line and the slant of the line are important characteristics used to describe a line.

To describe a line algebraically, you write *an equation of the line*. When you speak of those equations in general, you are referring to *linear equations*. Once you're familiar with some examples, you can tell a lot about what the graph of an equation will look like just by looking at the equation.

Because linear equations are the most simple examples of the equations that mathematicians study—and because they are so useful for describing things that go on in the world—mathematicians have developed many tricks and shortcuts for dealing with them. In this chapter, we'll learn some of those tricks and shortcuts.

Rx FOR HELP

If you're not clear on integers or coordinate graphing, it's time to go to *Dr. Math Gets You Ready for Algebra.* Work through some of the questions and answers in that book before you get started here.

In this part, Dr. Math explains:

- linear expressions and equations
- slope, intercepts, and slope-intercept form
- graphing linear equations

1 Linear Expressions and Equations

Linear expressions and linear equations are related concepts. A linear expression is an expression with a variable in it; however, the variable is raised only to the first power. For example, $5a$ is a linear expression. Another example is $5a + 2$. If instead you had $5a + 2 = 12$, then you would have an example of a linear equation. What's important to remember is that an *equation* has an equal sign but an *expression* does not.

Linear Expressions

Dear Dr. Math,

What are linear expressions, and how will I use them in my life?

Arturo

Dear Arturo,

Thanks for your question. Let me start with the "what are they" part of your question.

I am going to explain mathematical expressions by comparing them to something you probably understand from studying English grammar in school. When we write, we use sentences to write a complete thought. In a sentence, there must be a noun and a verb, and often there are extras like descriptive words.

In mathematics, we also frequently write in sentences, but we use numbers and symbols to convey a thought. A complete mathematical sentence includes an equal sign or inequality sign (< or >) and at least one term on either side. For instance: $5 + 3 = 8$ is a mathematical sentence, called an **equation,** while $9 < 100$ is also a mathematical sentence, called an inequality.

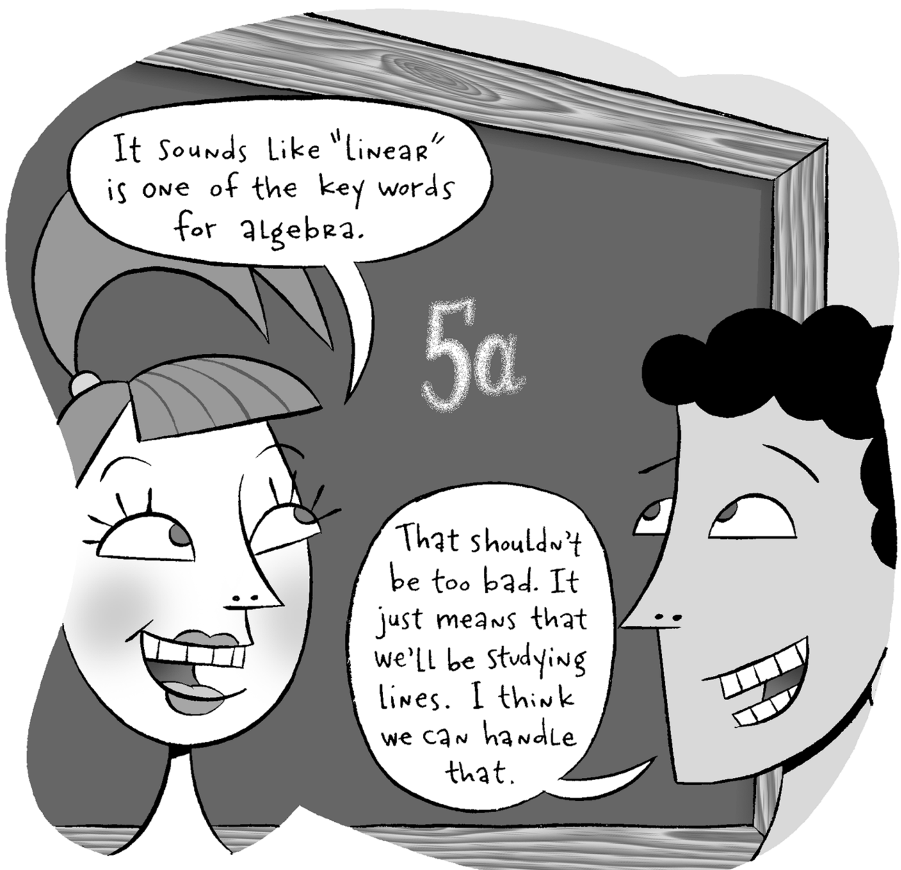
When we write, we may also use phrases, which are groups of words that are not complete sentences, like "in a nutshell" or "good sport." In mathematics, we use phrases too, but they're called **"expressions."** Expressions can be just one number or several numbers and some symbols; however, there is no equal sign or inequality sign between them. For example, 8 is an expression, and so are $\frac{4}{3}$ and $5 + 3$.

Some mathematical expressions include letters that stand for something else. These letters are called **variables** because they represent numbers that can vary. Expressions with variables are used every day in all sorts of situations. Here is an example: Let's say that you have a car that can travel 15 miles for every gallon of gas in the tank. You could represent the total number of miles you can drive based on how much gas you have in the tank using the expression

15g, where g stands for the number of gallons in the tank. You can figure out how far you will be able to go by replacing g with the number of gallons in your tank.

Now for the linear part. A linear expression is an expression with a variable in it, but there is a special condition involving exponents.

You may not have learned about exponents yet, so I'll give a brief explanation. We use exponents to symbolize many multiplication operations using the same number. For instance, perhaps you must multiply 3 by 3 by 3 by 3 by 3. We would write this as 3^5, which means the product of five 3's. We read the expression 3^5 as "3 raised to the fifth power." Any number raised to one is simply that number: $4^1 = 4$; $5a^1 = 5a$, and so on. We usually omit writing the "^{1}" when a number is raised to the first power.



A linear expression is an expression with a variable in it; but only when the variable is raised to the first power. For example, $5a$ is a linear expression, because it is understood that a is raised to the first power, but $9t^2 + 8$ is not a linear expression because t is raised to the second power.

There are many possible examples of linear expressions that you might use in your life. Another common one is computing how much money you might earn in a week working at a restaurant where you were paid by the hour. If your salary is $6 per hour, your total pay for a week could be expressed as $6h$, where h represents the number of hours you worked during that week. Can you think of more examples now?

— ***Dr. Math, The Math Forum***

Equation, Function, or Formula?

Dear Dr. Math,

I keep hearing the words "equation," "function," and "formula" in my algebra class, and sometimes it seems as though they all mean the same thing. Am I missing something?

Aimee

Dear Aimee,

Good question! It's not something that gets discussed as much as it should, so I'm glad you asked about it.

An *equation* is simply an assertion that two quantities are equal, for example,

$$3 + 5 = 8$$

When we include a variable in an equation, for example,

$$3 + x = 8$$

we're still asserting that two quantities are equal, but now we're doing something more. We're also asserting that there is some value (or set of values) that can be assigned to the variable to make the equation true. In the equation above, only the assignment $x = 5$ will make the equation true.

When we include a second variable, for example,

$$3 + y = 2x$$

we're asserting that there are *pairs* of assignments that make the equation true. That is, we can *choose* a value for one of the variables, and that choice will *determine* a value (or values) for the other variable.

For example, suppose we choose to assign $x = 5$ in the equation above. Then we have:

$$x = 5: \quad 3 + y = 2(5)$$

which is true only when $y = 7$. So the pair of assignments ($x = 5$, $y = 7$) is one solution to the equation. But there can be other solutions as well. For example, if we choose to assign $y = 5$, then we have:

$$y = 5: \quad 3 + 5 = 2x$$

which is true only when $x = 4$. So ($x = 4$, $y = 5$) is another solution to the equation.

Note that we can manipulate the equation to look like this:

$$y = 2x - 3$$

This has the same meaning, but it's somewhat easier to work with. Now if we choose a value for *x*, we can evaluate the resulting expression, and that gives us the corresponding value for *y*:

Choose	**Evaluate**	**Solution**
$x = 1$	$y = 2(1) - 3 = -1$	$(x = 1, y = -1)$
$x = 2$	$y = 2(2) - 3 = 1$	$(x = 2, y = 1)$
$x = 3$	$y = 2(3) - 3 = 3$	$(x = 3, y = 3)$

When we write the equation in this way, we say that *y* is a **function** of *x*. That is, it's a kind of rule for starting with one value (the "input") and finding a matching value (the "output"). Often, when we turn an equation into a function, we'll simply drop the output variable and use a slightly different notation. Instead of

$$y = 2x - 3$$

we'll write

$$f(x) = 2x - 3$$

But it means the same thing.

So we've talked about equations and about functions. What about formulas? A **formula** is simply an equation that is so useful that we want to share it with other people. It's often written in the form of a function, although it's not restricted to that use. For example, the formula for the volume of a cylinder is:

$$V = \pi r^2 h \quad V = \text{volume},\ r = \text{radius},\ h = \text{height}$$

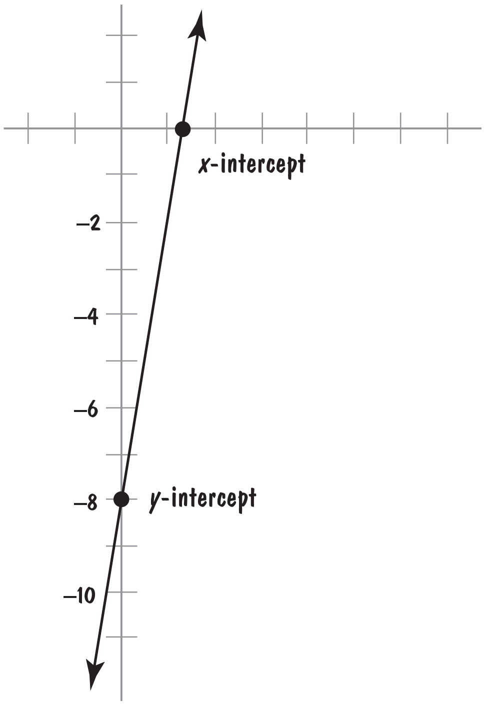
Written this way, it tells us volume as a function of radius and height. But sometimes we already know the volume and the radius, and we need to find the height! We don't bother to make up a new formula, because we can just change this one around to find what we want. So a formula is often a function, but it doesn't have to be, and we don't have to use it that way. Mainly a formula is an equation that is useful enough to write down in a permanent location (like a book or a Web site), so that we can look it up instead of having to figure it out from scratch each time we want to use it.

—Dr. Math, The Math Forum

2 Slope, Intercepts, and Slope-Intercept Form

We talk about "ski slopes" and how steeply a roof "slopes" when we're referring to something that is not horizontal or vertical but at a slant. In math we use the term **slope** similarly. It is a number that tells how steeply a line slants as it goes up or down. Slope is important, because if you know the slope as well as the location of any point on a line, you have everything you need to know to find all of the other points on the line. So often, when you're given some information about a line, the first thing you want to do is figure out what the slope is.

When we talk about intercepts, it makes sense to to look at a graph. In the Cartesian system of graphing, we have the x-axis and the y-axis. When we graph a line, it may cross the x-axis or the y-axis, or both. Any point at which it crosses an axis is called an **intercept.** The point where it crosses the x-axis is known as the ***x*-intercept,** and the point will be in the form $(x, 0)$. The point where it crosses the y-axis is known as the ***y*-intercept,** and the point will be in the form, $(0, y)$.



You can see how useful slope and intercepts are for describing or graphing a line. Because of this, one of the standard forms for linear equations shows slope and y-intercept clearly. It's called the slope-intercept form, and it's written:

$$y = mx + b$$

In this equation, the m stands for the slope of the line, and the b stands for its y-intercept. So this line has slope m and crosses the y-axis at $(0, b)$.

What Is Slope?

Dear Dr. Math,

I am confused about slope. I'm not sure what it means, and I don't know how to use it to draw a line. Can you help?

Arturo

Dear Arturo,

The slope is just a number that tells how steeply a line goes up or down. If the line is perfectly level (it doesn't go up or down at all), the slope is zero. If, as you go to the right, the line gets higher, we say it is sloping up, or has a positive slope. If it goes down as we go to the right, we call that a negative slope.

So that's how to think of slopes. Here's how to measure (or draw) them. Let's first look at specific examples.

Suppose the slope is 1. That means that if you go 1 unit to the right, the line goes up by 1 unit. The units can be whatever you choose, so if you go 1 foot to the right, the line goes up 1 foot. If you go 1 centimeter to the right, the line goes up 1 centimeter, and so on. If you draw this line, you'll find it goes up at a 45-degree angle.

If the slope is 3, the line goes up more steeply. If you go 1 unit to the right, the line goes up 3 units, where "unit" can be inch, foot, centimeter, or whatever.

If the slope is $\frac{1}{2}$, it means that if you go 1 unit to the right, the line goes up $\frac{1}{2}$ unit. If the slope is 1,000, the line is very steep—going 1 unit to the right, the line rises by 1,000 units, and so on.

If the slope is negative, it works the same way, except the line goes down to the right. A slope of -1 means that going 1 unit to the right, the line drops 1 unit. A slope of $-\frac{1}{3}$ means for every unit the line goes to the right, it drops by $\frac{1}{3}$ of a unit, and so on.

There's one nasty problem and that concerns lines that go straight up and down—you can't assign a sensible slope to them, because they never go to the right. We call these slopes **undefined.**

Often you'll get problems like this: What's the slope of a line that goes up 3 units for every 2 units it moves to the right? To get the answer, you just divide the motion up by the motion to the right. That means it goes up 1.5 units for each single unit to the right, so the slope is 1.5. Similarly, you might be asked for the slope of a line that goes up 3 units for every 2 units it moves to the *left*. To get that answer, you divide the motion up by the motion to the left. That means it goes up 1.5 units for each single unit to the left—which is the same as saying it goes *down* 1.5 units for each single unit to the *right,* so the slope is –1.5.

—Dr. Math, The Math Forum

Point-Slope Equations

Dear Dr. Math,

I don't understand how to do point-slope equations. Can you explain them?

Sincerely,

Aimee

Dear Aimee,

The point-slope form of an equation is just one of many ways to write the equation of a line. It's handy to use when you know the slope of a line and one point on it.

A point-slope equation looks like this:

$$y - y_1 = m(x - x_1)$$

where m is the slope and x_1 and y_1 correspond to a point on the line.

In order to solve a problem (that is, write an equation of a line) using the point-slope equation, you need two things: a point on the line (x_1, y_1) and the slope of the line.

For example, to find the equation of a line with a slope of 2 and a point on the line (–1, 3), m would be equal to 2, x_1 would be –1, and y_1 would be 3.

Plugging them into the point-slope equation, you get:

$$y - 3 = 2(x - (-1))$$

Then solve for y to simplify the equation.

$$y - 3 = 2(x + 1)$$

$$y - 3 = 2x + 2$$

$$y = 2x + 5$$

Sometimes you will get a problem that says to write the equation of the line, but you are only given two points and not the slope. For example, find the equation of the line that contains the points (2, 1) and (0, –1). In order to use the point-slope form of the equation, you need to find the slope (m), which is the difference in the y-coordinates divided by the difference in the x-coordinates (i.e., "rise over run"— see Rx on the next page):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Putting in the numbers, you would get

$$m = \frac{-1 - 1}{0 - 2} = \frac{-2}{-2} = 1$$

Therefore the slope of the line is 1.

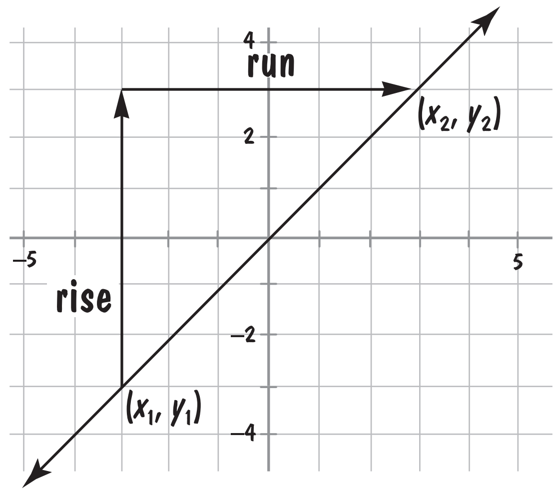
Notice that I subtracted the coordinates of (x_1, y_1) from the coordinates of (x_2, y_2). You might ask what would have happened if I had done it the other way and subtracted the coordinates of (x_2, y_2) from the coordinates of (x_1, y_1), giving

$$\frac{y_1 - y_2}{x_1 - x_2}$$

The result would be the same. Just be sure that you use the same order for both the numerator and the denominator.

RISE OVER RUN

"Rise over run" refers to slope, and it describes the up/down movement on a graph as compared to the right/left movement on a graph. You determine the up/down movement on a graph by comparing the *y*-coordinates of two points on the line. You determine the left/right movement on a graph by comparing the *x*-coordinates of two points on the line. This is also sometimes described as "change in *y* over change in *x*."



Using the slope and one of the points you were given (it doesn't matter which one), you can use the point-slope form (I'll use the first point given):

$$y - 1 = 1(x - 2)$$

Solve for y to simplify.

$$y - 1 = x - 2$$

$$y = x - 1$$

—Dr. Math, The Math Forum

When Is a Slope Zero or Undefined?

Dear Dr. Math,

How do I know when the slope of an equation is zero or undefined (no slope)? Thank you!

Sincerely,

Aimee

Dear Aimee,

Since you didn't say, I'll assume that by "equation" you mean the equation of a line—that is, a linear equation. Talking about slopes gets a great deal more complicated when the equation isn't linear.

Using the slope-intercept form, $y = mx + b$, if the slope of the line is zero, then this means that $m = 0$. But if m is zero, then mx is also zero, since anything times zero must be zero. So we are left with:

$$y = b$$

This is the only way to have a line with slope 0. If we graph this line, what do we get? Let's take a concrete example:

$$y = 1$$

This is a perfectly good line. Notice that x is missing. But we know that y is always equal to 1. So all the points on this line will have 1 as their y-coordinate. Since x is missing, it may be any value at all:

$$(1, 1), (2, 1), (3, 1), (4, 1), (1000, 1), (–2034, 1) \ldots$$

All these are on the line. What does this look like if we graph it? Try it, and you'll see that it is a horizontal line going through (0, 1). The x-value can be any number, but y must always be 1.

So what do we conclude? All horizontal lines have a slope of zero. And if any line has slope of zero, it must be a horizontal line.

This makes sense, given what we know about slope. A line may have a positive slope, negative slope, zero slope, or undefined slope.

The way to tell which slope a line has is to do this: Pretend you are standing on the line, on its graph, and you are going to walk from *left* to *right* along the line. If you are:

walking *up* hill: it's a positive slope

walking *down* hill: it's a negative slope

walking a *flat* line: it's a zero slope with no hill at all

Now, that leaves only the undefined slope. In the case of a line, this means that the line is *vertical*. Remember, a horizontal (flat) line has a slope of zero. A vertical line (one that forms a right angle or is perpendicular to a flat line) has an undefined slope. Think of it using the hill analogy again. If the hill is straight up, we couldn't walk up it. If and only if the line is perfectly vertical do we say it has an undefined slope. If it's almost vertical, but not quite, then it will have a very big (steep) slope but not be undefined.

Why is this so? I'll try to explain.

Imagine you have drawn a vertical line on your graph paper, through the point (1, 0).

To get to the next point on the line, which is (1, 1), what do you have to do? You have to go up 1 and over none. That is, the rise is 1 and the run is 0.

So you have the fraction $\frac{1}{0}$. Aha! Now do you see why a vertical line has an undefined slope? Because you can't divide by zero!

Usually in math, when something is "undefined" it means that somewhere, something is being divided by zero. And that's a no-no. It has no meaning. So we call it undefined.

—Dr. Math, The Math Forum

Rx REMEMBERING *M* AND *B*

Robby Grant, a student, has suggested a way of remembering *m* for slope and *b* for *y*-intercept:

> **I think of *m* as standing for "move" and *b* for "begin." This relates to the way you graph linear equations by hand. You can use the *b* value to plot the "beginning" point (0, *b*). Then the *m* value instructs you where to "move" from point (0, *b*) to plot the next point, thus giving you the line for the equation.**

What Is a *Y*-Intercept?

Dear Dr. Math,

I'm having a problem with finding y-intercepts. How do you know what the y-intercept is for the following equations?

$$2x + y = 3 \text{ or } x - 4y + 8 = 0$$

Arturo

Dear Arturo,

The y-intercept is the place where the graph hits the y-axis. The y-axis represents all the points on the plane where the x value is zero.

Putting this another way, to find a point given the x- and y-coordinates, you move left or right by the x amount, and then up or down by the y amount. To stay on the y-axis, you make no motion in the x direction; in other words, the x-coordinate is zero.

So the y-intercept of $2x + y = 3$, occurs when $x = 0$. This means that $2 \cdot 0 + y = 3$, or $y = 3$, so the y-intercept of this equation is 3. For $x - 4y + 8 = 0$, when you set $x = 0$, you get $4y + 8 = 0$, or $4y = 8$, or $y = 2$. That's all there is to it.

—Dr. Math, The Math Forum

Finding Y-Intercepts

Dear Dr. Math,

I know that an equation for a straight line graph is $y = mx + b$, but how do you find the place where the graph cuts the axes?

Aimee

Dear Aimee,

If you have an equation in the form $y = mx + b$, the slope is m, and it intersects the y-axis at the point $(0, b)$ because the y-axis is where x is zero. To find b, you plug in zero for x and you get $y = m \cdot 0 + b = b$. To find out where the line hits the x-axis, you'd plug in zero for y and solve for x.

—Dr. Math, The Math Forum

Equations of Lines

Dear Dr. Math,

Please, can you help me?

Find the equation of the line through the two given points: $(4, -1)$, $(3, -6)$. The answer given in the textbook is $y = 5x - 21$, but I don't know how they got this answer.

Sincerely,

Arturo

Dear Arturo,

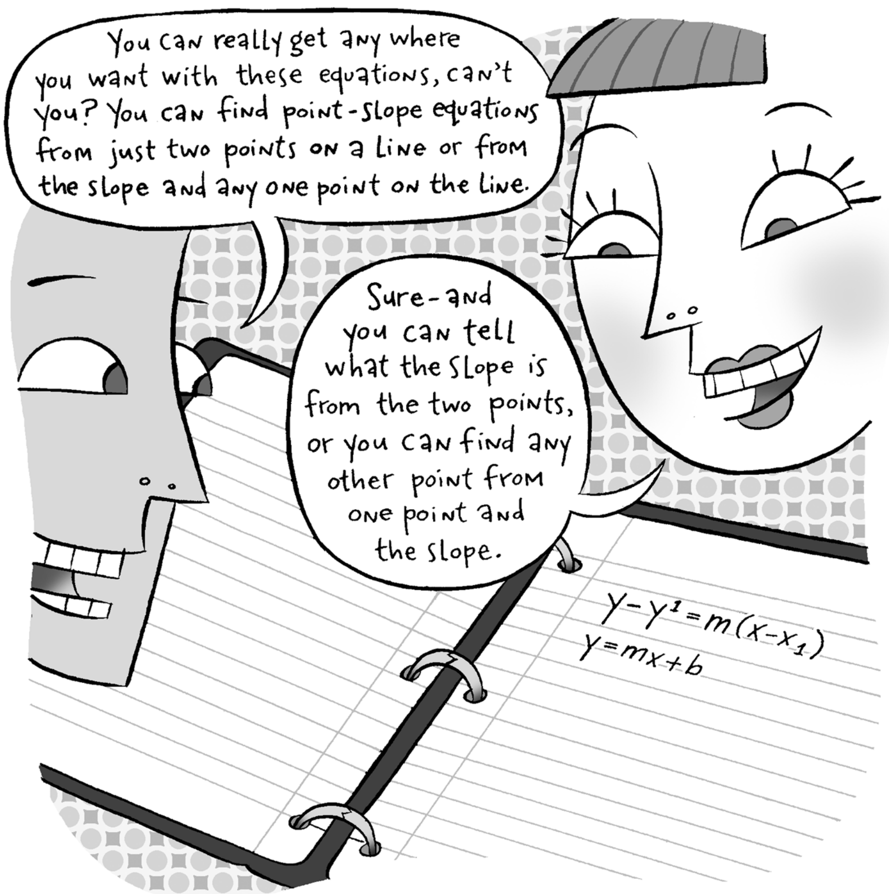
You start by finding the slope of the line. This is a fraction where the numerator is the difference of the *y*-coordinates, and the denominator is the difference of the *x*-coordinates.

Make sure you take the differences in the same order, like this:

$$\frac{-1-(-6)}{4-3} = \frac{-1+6}{4-3} = \frac{5}{1} = 5$$

The slope is usually written as *m,* so here $m = 5$.

The form of the equation of a line you are asking about is called the slope-intercept form. This looks like $y = mx + b$ where b is the *y*-intercept, which is the *y*-value where the line intersects the *y*-axis. That is, $(0, b)$ is on the line.



What we know so far is that the equation looks like $y = 5x + b$. So, what is b? You can find that out by using either of the points that you know is on the line.

Let's try the first one, (4, –1). Since the combination of $x = 4$ and $y = -1$ must be on the line with equation $y = 5x + b$, it must be true that $-1 = 5(4) + b$. That is, –1 equals $20 + b$. The only b satisfying that is $b = -21$. So the equation is $y = 5x + (-21)$ or simply $y = 5x - 21$.

—Dr. Math, The Math Forum

Slope-Intercept Form

Dear Dr. Math,

How come $y = mx + b$ has to be that way?

Aimee

Dear Aimee,

You can write any equation in a lot of different ways. What's nice about the form

$$y = mx + b$$

is that it allows you to sketch the graph of the corresponding line very quickly.

How do you do that? Well, you know that the line (if it's not vertical) has to cross the y-axis somewhere—in particular, where $x = 0$. Since you know $x = 0$,

$$\begin{aligned} y &= m(0) + b \\ &= b \end{aligned}$$

which tells you that $(0, b)$ is a point on the line. You also know that the line (if it's not horizontal) has to cross the x-axis—in particular, where $y = 0$. Since you know $y = 0$,

$$0 = mx + b$$

$$-b = mx$$

$$\frac{-b}{m} = x$$

which tells you that $(-\frac{b}{m}, 0)$ is a point on the line. So just by looking at the equation

$$y = mx + b$$

you can graph two points:

$$(0, b) \text{ and } (-\tfrac{b}{m}, 0)$$

And once you've graphed these two points, you can fill in the rest of the graph with a ruler.

Another nice thing about the form

$$y = mx + b$$

is that you can tell pretty quickly whether two lines are parallel or perpendicular. If they have the same slope and different intercepts, they are parallel. If the slopes can be multiplied to get –1, they are perpendicular.

For example, the following two lines are parallel:

$$y = 2x + 3$$

$$y = 2x + 14$$

And the following two lines are perpendicular:

$$y = 2x + 3$$

$$y = \frac{-1}{2}x - 6$$

—Dr. Math, The Math Forum

Writing in Slope-Intercept Form

Dear Dr. Math,

I don't know how to do this problem.

I need to write in slope-intercept form: $x - y + 2 = 0$. I also need to sketch it.

Sincerely,

Arturo

Dear Arturo,

Slope-intercept form is great; it makes graphing much easier.

Let's take a line, say, $6x - 2y - 4 = 0$, and put it into slope-intercept form. Slope-intercept form is when a line is in the form

$$y = mx + b$$

where m is the slope and b is the y-intercept.

To get a line into that form, we just need to move terms around until it looks like that. So we start with the equation in the form given:

$$6x - 2y - 4 = 0$$

Now we move the x term over to the other side, so it looks like:

$$-2y - 4 = -6x$$

We move the 4 over to the other side too, because remember, we want the y term to be alone on one side. Now our equation looks like:

$$-2y = -6x + 4$$

We're almost there. The y term should have a **coefficient** of 1. (Remember a coefficient is a number that multiplies a variable—so we need to have something like $1y$ or y instead of $-2y$.) We need to divide both sides by -2 to get that, so the result looks like:

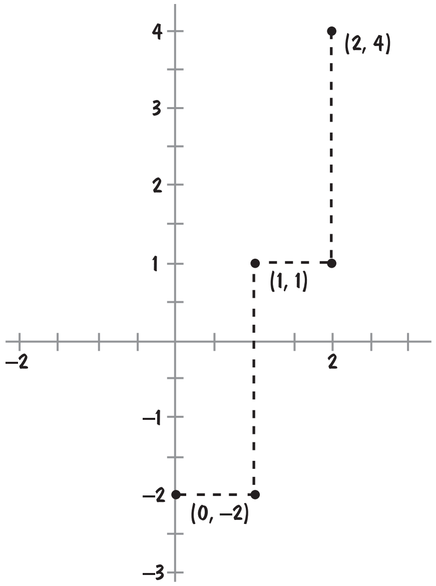
$$y = 3x + -2$$

The line is now in slope-intercept form. Now we need to graph it, and slope-intercept form makes it much easier. We know that if a line looks like $y = mx + b$, then m is the slope and b is the y-intercept. In the case of the line we're using, the slope is 3 and the y-intercept is –2.

The y-intercept is the place on the y-axis where the line intersects it. So, on your graph, plot the point (0, –2), because we know that it lies on the line.

Slope can be defined in many ways, but one way of thinking about it is in terms of rise over run. Rise over run means that the slope is a ratio. In the case of this line, the slope is 3, and that can be expressed as $\frac{3}{1}$. That means for every 3 units you go up, you go 1 unit to the right.

Since you know one point already, you can find another point on the line by starting at that point and using the slope. Start at the point. Count up by the number from the top of the slope fraction, and over by the number from the bottom, and plot your next point. You could go over and then up too—the order doesn't matter as long as you keep the numbers straight! Here's what going over and then up with your slope of 3 from the point (0, –2) looks like on a graph:

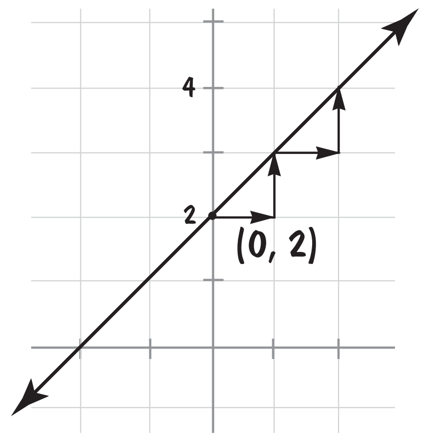


All of these points, (0, –2), (1, 1), (2, 4), and so on, are on the line $y = 3x - 2$. To draw the line, all you need to do is connect the points, and voilà! You're done.

You can use the same method I used to put $x - y + 2 = 0$ into slope-intercept form and then graph it. If you change your equation to slope-intercept form, you have:

$$y = x + 2$$

and so you know that the slope is 1 and the y-intercept is 2. If the y-intercept is 2, one point on your line is (0, 2). Here's the graph that shows that line:



—*Dr. Math, The Math Forum*

Equations of Parallel and Perpendicular Lines

Dear Dr. Math,

How do you tell if the lines of these equations are parallel, perpendicular, or neither without graphing?

$4y - 5 = 3x + 1$ and $12 = -6x + 8y - 3$

Aimee

Dear Aimee,

The best way to get information on how two lines relate to each other (i.e., whether they are parallel, perpendicular, or neither) is to look at their slopes. And the easiest way to find the slopes is to get your equations into slope-intercept form, which means they look like this:

$$y = mx + b$$

The number represented by m in this equation is the slope.

I'll help you get one of your equations into the slope-intercept form:

You were given:	$4y - 5 = 3x + 1$
First add 5 to both sides:	$4y = 3x + 6$
Then divide both sides by 4:	$y = \frac{3}{4}x + \frac{6}{4}$

This means that the slope is $\frac{3}{4}$.

Try the other equation on your own and reduce the slope to the smallest possible fraction.

If the slopes of both equations are the same, it means the two lines are parallel. (This assumes, of course, that the y-intercepts are different. If they're the same *and* the slopes are the same, then the lines are the same!)

If one slope is the negative reciprocal of the other, then the two lines are perpendicular. To get the negative reciprocal of a number, put one over the number and then make it negative. So:

If the number is:	**Its negative reciprocal is:**
2	$-\frac{1}{2}$
$\frac{3}{10}$	$-\frac{10}{3}$
$-\frac{3}{2}$	$\frac{2}{3}$

So if your second equation had a slope of $-\frac{4}{3}$, then the lines of your equations would be perpendicular.

If the slopes are neither the same nor negative reciprocals, the lines are neither parallel nor perpendicular.

—Dr. Math, The Math Forum

Slope-Intercept and Point-Slope Forms

Dear Dr. Math,

My question isn't about how to use the point-slope form but more of what it is used for. Given a problem, I can accurately use: $y - y_1 = m(x - x_1)$ to get the answer, but I don't understand what its specific meaning is. Is it just another way of getting to a $y = mx + b$ style suitable for graphing, or does it serve some other purpose?

Sincerely,

Arturo

Dear Arturo,

Good question! Too often people learn forms but don't stop to ask what they are for.

There are several different ways to write the equation of a line, and each is designed to be used when you have certain pieces of information to start with. You could do everything with one form, such as slope-intercept, but often it's easier to use a form specially designed for one case. For example, there are:

Slope-intercept:	$y = mx + b$
Slope-x-intercept:	$y = m(x - a)$
Point-slope:	$y - y_1 = m(x - x_1)$
Two-point:	$\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$
Two-intercept:	$\dfrac{x}{a} + \dfrac{y}{b} = 1$

These are all equivalent, and which one you use just depends on what you are given. But you don't have to memorize all of them. If you understand how graphs work, you can figure everything out when you need to. All these forms (except for the two-intercept form) are just ways of saying that the slope is constant. Either you are given the slope (m), or you figure out the slope $(y_2 - y_1)/(x_2 - x_1)$, and then you compare the general point (x, y) with either an intercept $(0, b)$ or $(a, 0)$, or another general point (x_1, y_1).

So the "meaning" of the point-slope form is simply that you can get the rise $(y - y_1)$ by multiplying the run $(x - x_1)$ by the given slope. You can graph it as it stands (by identifying the point and the slope), or you can change it into any other form you want. It's yours to use any way you like.

—Dr. Math, The Math Forum

3 Graphing Linear Equations

A graph of an equation is a picture of its solutions, all at once. In the case of linear equations, we can tell how big or small the slope of an equation is just by looking at the picture of the line it makes in the Cartesian plane.

Graphing $y = mx + b$

Dear Dr. Math,

I am having trouble understanding $3x + 2y = 5$. This has to be in $y = mx + b$ form. After I get it in the proper form, how do I graph it?

Aimee

Dear Aimee,

Let's take a look at your question step-by-step. The first thing you told me is that you need to get the equation into $y = mx + b$ form. So, to begin, you will work on solving this equation for y. As you may recall, what that means is that you want to manipulate the equation so you get y by itself on one side of the equation. This is the process I would use:

$$3x + 2y = 5$$

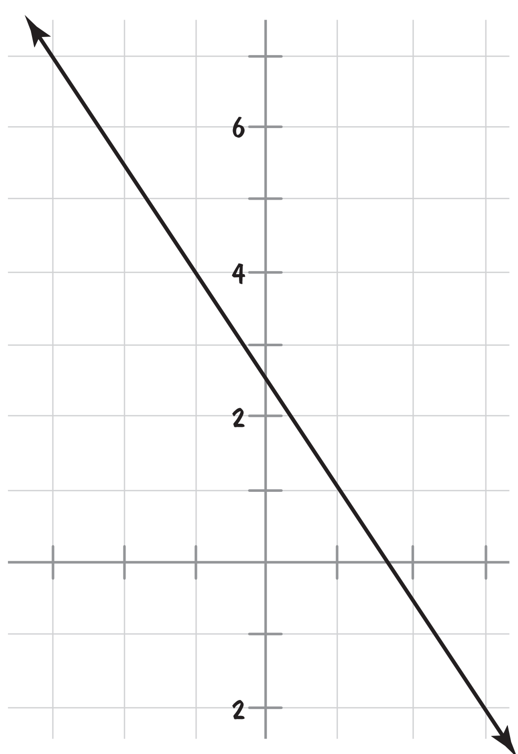
$$2y = -3x + 5$$ Subtract $3x$ from each side to get $2y$ by itself.

$$y = -\frac{3}{2}x + \frac{5}{2}$$ Divide both sides by 2 to get y by itself.

Now that the equation is in the desired form, we're ready to graph it.

Remember that $y = mx + b$ is called "slope-intercept form." If you have an equation in this form, you will have the slope and the y-intercept for the graph. The slope is "m" (the coefficient of x) and is the rise over the run (or the change in y over the change in x). The y-intercept is "b" and is where the graph crosses the y-axis.

In this problem, $m = -\frac{3}{2}$ and $b = \frac{5}{2}$. I would first locate the y-intercept on the graph. Since $b = \frac{5}{2}$, I would find where $\frac{5}{2}$ (or $2\frac{1}{2}$) is on the y-axis and plot a point. This is where the graph crosses the y-axis.



From there, I would use the slope to find other points on the graph. Slope is "rise over run." That means, in fraction form, the numerator indicates the distance you move in the vertical or y direction to find the next point, and the denominator indicates the distance you move in the horizontal or x direction. Keep in mind that a positive y direction is up, a negative y direction is down, a positive x direction is right, and a negative x direction is left. When you have a positive slope, you will use either a positive y and a positive x direction, or a negative y and a negative x direction. This is because a positive number divided by a positive number is positive, and a negative number divided by a negative number is also positive. Since we have a negative slope, we need one positive number and one negative number.

As I describe the graphing process below, I am assuming that you are using graph paper. This process will work with any scale on the graph (1 square could equal 2 units or 10 units, etc.), but I'm assuming 1 square equals 1 unit.

Our slope is $-\frac{3}{2}$. This means I can do one of two things: either go up 3 squares (positive direction) and left 2 squares (negative direction) or go down 3 squares (negative direction) and right 2 squares (positive direction). Be careful with this problem because you are beginning with a fraction (the $\frac{5}{2}$ from above)—you'll have to either estimate where halfway is or make each mark equal to $\frac{1}{2}$ (in other words, 2 marks on the graph = 1 unit).

Go to the $\frac{5}{2}$ you marked earlier. You can go up 3 squares from the $\frac{5}{2}$ and then move over left 2 squares and mark your next point. You can also go down 3 squares from the $\frac{5}{2}$ and then move over right 2 squares and mark your next point. If you do both of those operations, you will have three points and can connect the dots to make a line. (Although by definition, you need only two points in order to graph a line, adding a third point—or more—can be a good check that you're doing it correctly!) You can count out as many points as you need to make the line, but I suggest a minimum of three points for accuracy (although some teachers want you to use five points). The more points you have, the more accurate the line will appear on your graph.

—Dr. Math, The Math Forum

How to Draw Linear Equations

Dear Dr. Math,

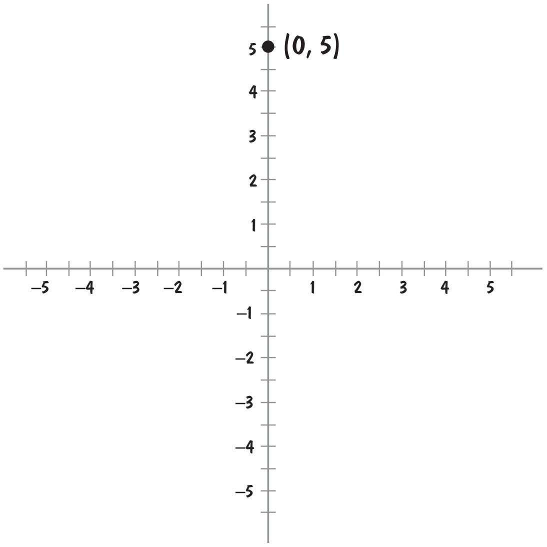
I have a lot of trouble drawing linear equations. Under the $y = mx + b$ formula, say I had $y = (\frac{2}{3})x + 5$. How would I draw that? How would I go about finding other points?

Arturo

Dear Arturo,

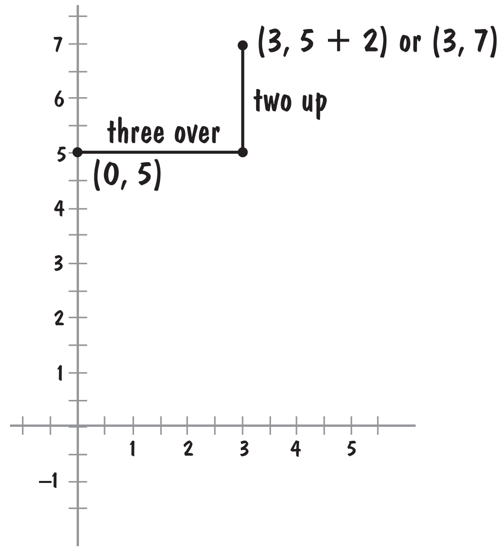
Getting a mental picture of what an equation looks like is not always easy. But there are clues to help us.

The point-slope formula, $y = mx + b$, is used for straight lines, so anything that fits that equation is a straight line. Let's use what we know from the equation you were given to draw the line on a graph. The value of b in any point-slope equation is the y-intercept, and because $b = 5$ in your equation, that means the line crosses the y-axis at $y = 5$. So we will put a mark on our graph at 5 on the y-axis.



The value of m in any point-slope equation is the line's slope. In your equation, the slope is $\frac{2}{3}$. Slope is measured in rise over run. Since the slope is 2 over 3, then each step is a rise of 2 and a run of 3. So the second point will be 3 steps over to the right of and 2 steps above the first point.

Now that you have two points, you can take a ruler and draw a straight line that touches both points, put arrows on the ends, and the graph is done. Any other point that the line touches will also work in your equation. Try it out and see.



—*Dr. Math, The Math Forum*

Rewriting an Equation

Dear Dr. Math,

I have this algebra problem: $-3x + 2y = -6$.

I need to sketch it using the slope and y-intercept. Before I do that, how do I rearrange the problem so I can sketch it?

Sincerely,

Aimee

Dear Aimee,

Linear equations with two variables, such as the one you've asked about, can be written in a number of ways. The best way to write a linear equation depends on the information you want to get out of it. Here are three types:

1. General form: $ax + by = c$.

   This is the form of the equation you've asked about. Writing equations in this form allows them to be ordered with other more complicated equations—in much the same way that putting words in alphabetical order in a dictionary makes them easier to look up.

2. Slope-intercept form: $y = mx + b$.

   This is the form you want because it will let you easily see the slope, m, and the y-intercept, b.

3. Point-slope form: $y - y_1 = m(x - x_1)$.

   The slope is m and the point is $(x_1 - y_1)$. This is a variation of the slope-intercept form, which is good to use if instead of having the intercept point $(0, b)$, you have a specific point not on the y-axis. It's an easy form to use because you can plot the point and use the slope of the line to draw the line through that point.

To get from the general form to the slope-intercept form, you need to isolate the y term on one side of the equation. Here's how to do this when you start with the general form:

$$ax + by = c$$

Subtract the ax term from both sides:

$$by = c - ax$$

Rewrite the right side, putting the x terms first (this isn't necessary, but it looks better):

$$by = -ax + c$$

Divide both sides by b to get the y alone on one side:

$$\frac{by}{b} = \frac{-ax}{b} + \frac{c}{b}$$

$$y = \frac{-ax}{b} + \frac{c}{b}$$

Now you have an equation in the slope-intercept form, so you can see that the slope is $-\frac{a}{b}$ and the y-intercept is $\frac{c}{b}$.

Use this procedure to rewrite your equation, and you should be all set.

—*Dr. Math, The Math Forum*

Resources on the Web

Learn more about linear equations at these sites:

Math Forum: Chameleon Graphing: Lines and Slope

mathforum.org/cgraph/cslope/

A Web unit for middle school and early high school students, in which Joan the Chameleon introduces and explores lines and slope.

Math Forum: Graphing Linear Functions

mathforum.org/alejandre/linear.graph.html

Step-by-step directions to create graphs of linear functions using a ClarisWorks spreadsheet file.

Math Forum: Middle School Algebra Links

mathforum.org/alejandre/frisbie/math/algebra.html

A variety of resources on the Web emphasizing algebraic thought and addressing each of the particular NCTM standards for grades 6, 7, and 8.

Shodor Organization: Project Interactivate: Slope Slider

shodor.org/interactivate/activities/slopeslider/

This activity allows the manipulation of a linear function of the form $f(x) = mx + b$ and encourages the user to explore the relationship between slope and intercept in the Cartesian coordinate system.

Shodor Organization: Project Interactivate: Linear Function Machine

shodor.org/interactivate/activities/lfm/

Students investigate linear functions by trying to guess the slope and intercept from inputs and outputs.

Shodor Organization: Project Interactivate: Positive Linear Function Machine

shodor.org/interactivate/activities/plfm/

Students investigate linear functions with positive slopes by trying to guess the slope and intercept from inputs and outputs.