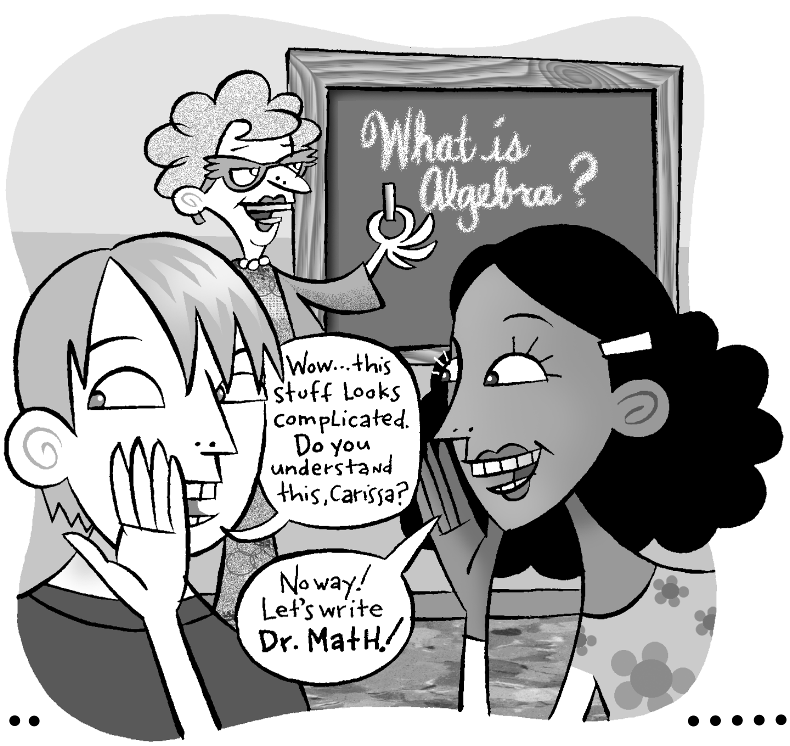
PART 1

Fundamental Operations

Operations are the arithmetic skills introduced and practiced in elementary school. The *fundamental* operations are addition, subtraction, multiplication, and division. Exponentiation is also an operation. In algebra, the fundamental operations are as important as they are in arithmetic. In fact, if you ever want to check your algebraic work by substituting a number for the variable, you'll be reminded of the arithmetic exercises that look more familiar.



Clive and Carissa have a lot of questions about what they're learning. In this part, Dr. Math explains

- Introduction to algebraic thinking
- Variables
- Exponents
- Large and small numbers
- Order of operations
- Distributive property and other properties

1 Introduction to Algebraic Thinking

Algebraic thinking is the bridge between arithmetic and algebra. Representing, analyzing, and generalizing a variety of patterns with tables, graphs, words, and, when possible, symbolic rules are all part of thinking algebraically.

What Is Algebra?

Dear Dr. Math,

What is algebra?

Yours truly,
Clive

Hi, Clive,

Algebra is like arithmetic, but in algebra some of the numbers have names instead of values. For example, if I ask you something like

$$3 + 4 \cdot 5 - 6 \div 3 = ?$$

you can just apply the operations in the correct order to get

$$\begin{aligned} 3 + 4 \cdot 5 - 6 \div 3 &= ? \\ 3 + 20 - 6 \div 3 &= ? \end{aligned}$$

$$3 + 20 - 2 = ?$$
$$23 - 2 = ?$$
$$21 = ?$$

Now, suppose that instead I ask you something like

$$(x + 3) \cdot (x + 4) = 42$$

You can't apply the operations because you don't know the values of the numbers. What's $x + 3$? It depends on the value of x, doesn't it?

In this case, you might start guessing possible values for x that would make the equation true:

$$x = 1? \quad (1 + 3) \cdot (1 + 4) = 4 \cdot 5 = 20 \quad \text{(No.)}$$
$$x = 2? \quad (2 + 3) \cdot (2 + 4) = 5 \cdot 6 = 30 \quad \text{(No.)}$$
$$x = 3? \quad (3 + 3) \cdot (3 + 4) = 6 \cdot 7 = 42 \quad \text{(Yes!)}$$

However, suppose the problem changes to

$$(x + 3) \cdot (x + 4) = 35.75$$

Now it becomes a lot harder to guess an answer. Algebra gives you a set of tools for figuring out the answers to problems like this without having to guess.

This becomes more and more important as you start using more complicated equations involving more than one variable.

— Dr. Math, The Math Forum

Rx In case you've forgotten, here's the correct **order of operations** for any equation:

1. Parentheses or brackets
2. Exponents
3. Multiplication and division (left to right)
4. Addition and subtraction (left to right)

For more about this, see section 5 in this part.

What Is Algebraic Thinking?

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Dear Dr. Math,
How do you start to think algebraically?
Sincerely,
Carissa
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Hi, Carissa,

Well, you already know about multiplication, division, addition, and subtraction. One day (a long, long time ago) somebody—let's call him or her Pat—who knew all of those things was sitting around thinking about addition.

Pat knew that 3 + 4 = 7.

Then Pat asked, "What would happen to the equation if I added 1 to both sides?" Pat tried it and got

$$3 + 4 + 1 = 7 + 1$$

Pat realized right away that this new equation was also true. Then Pat went back to the original equation of 3 + 4 = 7, decided to subtract 3 from both sides, and got

$$3 + 4 - 3 = 7 - 3$$

Pat then did some arithmetic and ended up with 4 = 4.

Right away, Pat realized that this technique could be applied to different types of equations. Pat asked, "What if I didn't know one of the numbers?"

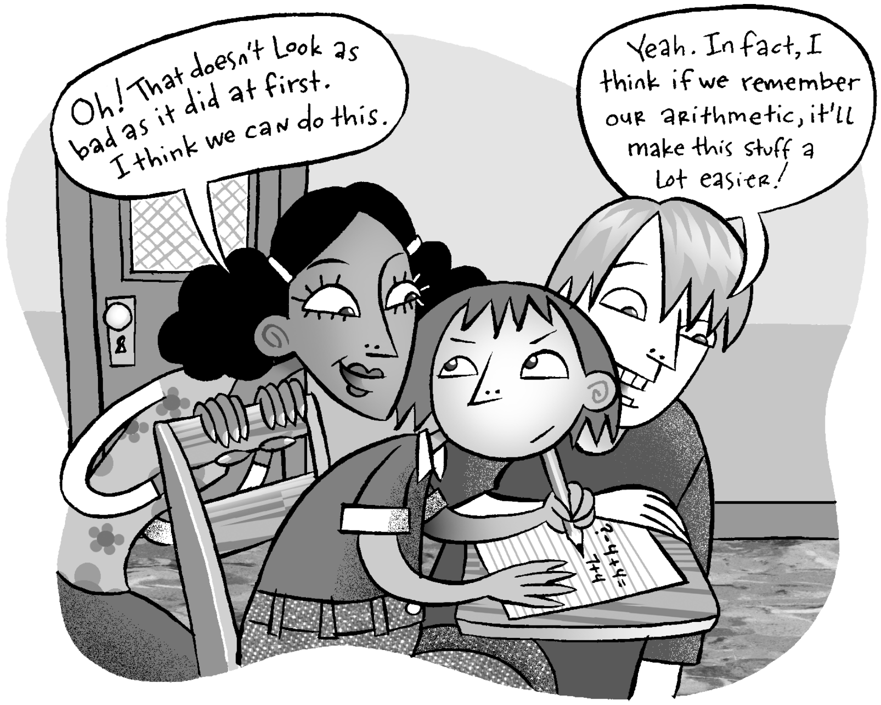
Pat was already familiar with equations like 3 + 4 = ? and knew that you could **solve** those equations.

Pat decided to try something a little different: ? – 4 = 7. Pat knew from before that you can add or subtract the same number from both sides of an equation (see above) and still have a true equation. So Pat added 4 to both sides of this equation and got

$$? - 4 + 4 = 7 + 4$$

After a little bit more arithmetic, Pat ended up with ? = 11. If you keep thinking like this, and instead of using ? you use x or y or a to stand for the missing number, that means that you are starting to think algebraically.

— Dr. Math, The Math Forum



2 Variables

A **variable** is a symbol like *x* or *a* that stands for an unknown quantity in a mathematical expression or equation. If you remember that the word *variable* means *changeable,* then it is a little easier to remember that the value of the *x* or *a* changes depending on the situation.

For example, what if you are thinking about the number of tires you need for a certain number of cars? You know that 4 tires are needed for each car. You can write $4c = t$, where c is the number of cars, t is the number of tires, and $4c$ means 4 times c. If there are 25 cars, you can figure out that $4(25) = 100$, so you will need 100 tires. If there are 117 cars, you know that $4(117) = 468$ and you will need 468 tires. Because the number of cars can change but the relationship between the cars and tires stays the same, the formula $4c = t$ is a useful way to explain the general situation.

An algebraic **expression** is like a *word* or *phrase,* and an **equation** is like a *sentence.* For example, $12c$ might represent a quantity of eggs. If c is the number of cartons of eggs and there are 12 eggs in each carton, you can see that $12c$ is a useful way of *expressing* the total quantity of eggs.

If you have another number or expression, you can relate it to your first expression in an equation. For example, let $12c = 60$. This is an algebraic equation with one variable. Solving this kind of equation reveals $c = 5$. Thus we have 5 cartons.

Whereas equations can be *solved,* expressions can only be *evaluated* or *rearranged* or *simplified*. Note also the distinction between having $12c$ by itself (an expression) and relating it to another expression using an equal sign: $12c = 60$ (an equation).

By the way, in an expression like $4c - 3$, 3 is called a **constant,** because it doesn't vary. The 4 changes along with the variable it multiplies and is called the coefficient of c.

What Are Variables For?

Dear Dr. Math,

Why is it important to be able to figure out the values of variables? We've been doing that in our math class for more than half a year and I was just wondering why we are doing it.

—Carissa

Hi, Carissa,

This is a very perceptive question.

Variables are important for a couple of reasons, which we might call *planning* and *analysis*.

Think about planning a dinner party. Let's say you know that you'll need one-half of a chicken for each adult and one-quarter of a chicken for each child; you'll need one bottle of wine for every three adults and one bottle of soda for every five children; you'll need a half pound of potatoes for each chicken that you have to cook; you'll need one pie for six adults and one bowl of ice cream for each child; and so on.

But you don't yet know how many people you're going to invite. Variables let you set up a description of the situation (i.e., an equation) such that you can plug in two numbers (the number of adults and the number of children) and get back other numbers that you'll find useful: how many chickens to buy, what the total cost will be, and so on. If you decide at the last minute that you want to add three more guests, you don't have to start your calculations from scratch—you just change the values coming in and the equations will tell you how to change the values at the other end.

This, by the way, is why they are called *variables*—they tell you how some quantities vary in response to changes in other quantities.

Note that a dinner party isn't all that complicated, so it's almost not worth the effort of setting up equations to solve the problems. But when you get to something more complicated—like trying to plan the flight of an airplane or run an entire airline—it becomes absolutely necessary to use variables. A big part of running any business is being able to figure out your potential costs in any situation, because that tells you how much you need to charge for goods and services in order to make enough money to stay in business.

So, that's planning. What about analysis? Well, analysis is just planning in reverse. If you know how many people to invite, you can figure out how much money you'll have to spend. That's planning. If you know how much money you spent, you can figure out how many people you invited. That's analysis. The beauty of variables is that in most cases you can use the same equations to go in either direction—to predict what's going to happen or to understand what already happened.

The planning aspect tends to be more useful in things like business or construction or engineering, where you have to decide what's going to happen. The analysis aspect tends to be more useful in science, where you don't get to decide what happens (the world behaves the way it behaves, whether you like it or not) but would like to understand it anyway, whether due to curiosity or because you'd like to use that understanding to make your planning more accurate.

—Dr. Math, The Math Forum

Using Variables

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Dear Dr. Math,

How come when you use a variable in a
problem sometimes the answer still has a
variable and you cannot get an actual number
answer?

Thanks,

—Clive
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Dear Clive,

I assume you are not talking about making a mistake in solving the problem. If there is only one variable in the original equation, then either you can solve it with a numerical answer or you simply can't solve it—there would be no actual solution that still involved the variable.

But if you are given an equation with two variables in it, like $w = \frac{24}{h}$, and are told to solve it for one of the variables, say $h = \frac{24}{w}$, then the other variable will still be there. In this case, you are simply rearranging a formula for a different use. As given, the formula lets you get the width of a rectangle given its height. After you solve for h, it lets you find the height of a rectangle given its width. You don't know either one yet, but if I gave you a height, you could plug it right into this formula. If you hadn't already solved for h, you would have to put my value into the original equation for w and then solve that for h.

So, there are two ways a variable can be used. Sometimes it is an unknown, which you want to figure out from the equation. Other times it just stands for a value that you don't know now but will know later, like *w* in my example. Then you just work with it as if it were a value but without being able to do the calculations. When you're done, you can replace it with any value.

—*Dr. Math, The Math Forum*

Writing Expressions with Variables

Dear Dr. Math,

Here's a problem that I'm having trouble with:

> Write an expression that represents a $500 donation plus $5 for every event. Let *n* represent the number of events.

I do not understand what the problem wants me to write.

Yours truly,

Clive

Hi, Clive,

An expression is a collection of numbers and variables connected by arithmetic operations (add, multiply, etc.), so if you worked out all the arithmetic (which is called **evaluating** the expression), you would get a number. In this case, the number would be the total payment.

Let's say you knew there were 4 events. Could you then work out how much to pay? It would be

$$500 + 5 \cdot 4$$

in dollars. This is an expression. You can work it out and get the answer 520.

In fact, you don't know how many events there are. But whatever that number turned out to be, you could put it in place of the 4 in the expression and work it out in just the same way. So we use a letter as a name to stand for whatever number we will end up putting there. This is a variable. It is sort of a placeholder for a number.

You were asked to use the letter *n* to represent the number of events. That means *n* will be our variable and we can put it in place of the 4, like this:

$$500 + 5 \cdot n$$

When multiplying by a variable, you don't need to write the "·". You can just write

$$500 + 5n$$

I hope this helps you work out other problems in writing expressions.

—Dr. Math, The Math Forum

Understanding Variables

Dear Dr. Math,

I have always had a hard time with algebra. It makes no sense to me why someone would replace a number with a letter. Is there some secret pattern that could help me solve algebra problems?

—Carissa

Hi, Carissa,

You're doing something that's a little like algebra whenever you use a pronoun. You could have written this:

> Carissa has always had a hard time with algebra. It makes no sense to Carissa why someone would replace a number with a letter. Is there some secret pattern that could help Carissa solve algebra problems?

When you wrote to us, you used the pronouns *I* and *me* to take the place of your name. A variable is like a pronoun: it's a way of talking about a number without calling it by name.

We don't generally replace a number with a letter; more often we don't know the number yet, so we just give it a nickname (like x) and work with it until we can replace the letter with the right number.

The great discovery that made algebra possible was the realization that even if you don't know what a number is, you can still talk about it and know certain things about its behavior; for example, no matter what the number is, if you add 2 to it and then subtract 2 from the result, you'll have the same number you started with. We can say

$$x + 2 - 2 = x \text{ for any } x$$

Here's one secret that may help you: when you see an equation that confuses you, try putting an actual number in place of the variable and see if it makes sense. For example, in what I just wrote, you could try replacing x with 47:

$$47 + 2 - 2 = 47$$

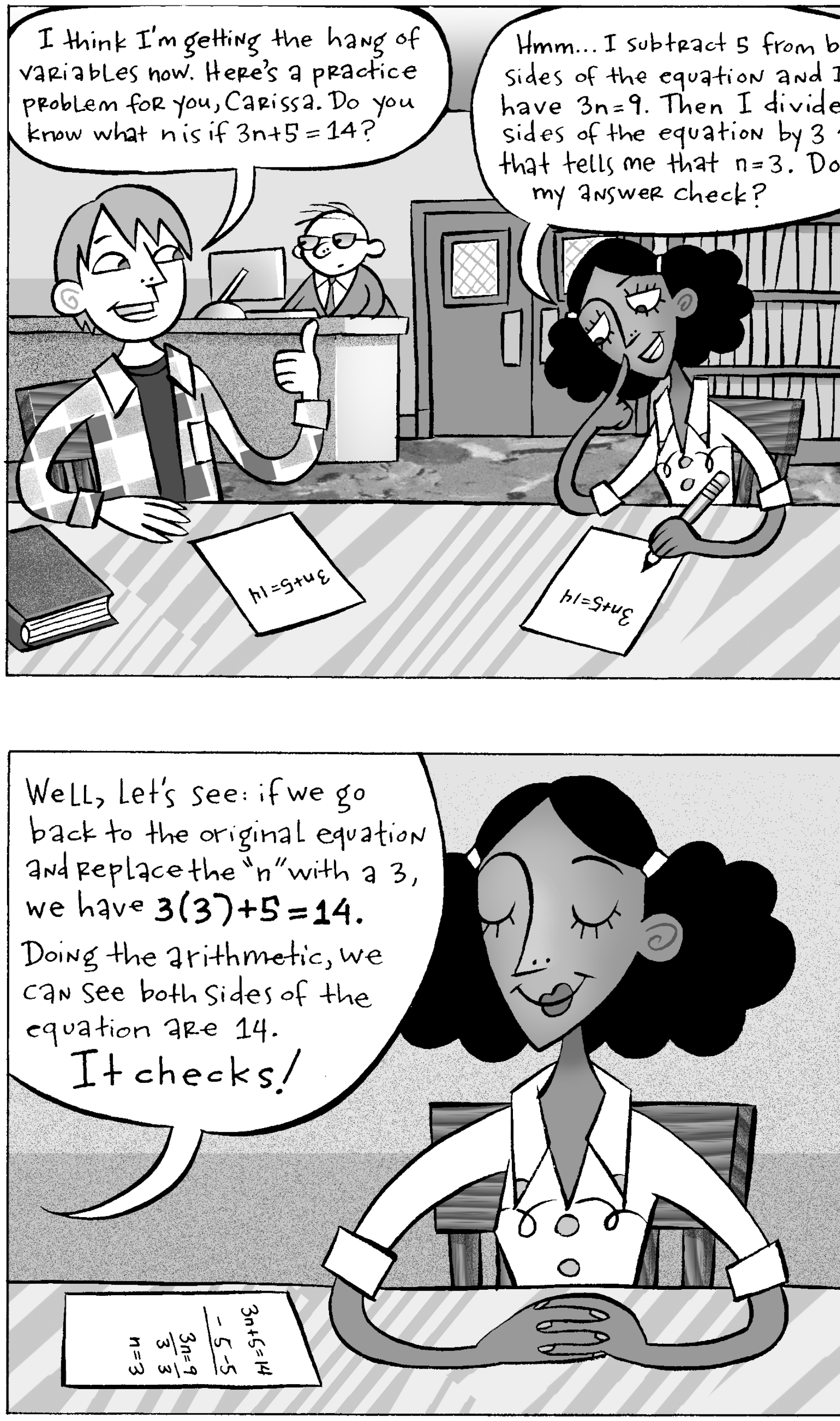
It works! Now think about why it works: 47 plus 2 means you've gone 2 units to the right; minus 2 takes you 2 units back to where you started. It doesn't matter that the place at which you started was 47; adding 2 and subtracting 2 undo one another.

I'll take you one step deeper into algebra and actually solve an equation. Let's say we're told that

$$3x - 2 = 7$$

In words, I can say, "I have a secret number. If I multiply it by 3 and then subtract 2, I get 7. What is it?" (Notice how I used pronouns to stand for the number.) In order to solve this, I can think of it as if the x were a present someone wrapped up for me. First, someone put on some "times 3" paper and then over that some "–2" paper. The package I was given is a 7. I want to unwrap it and see what the x is that's inside.

To take off the "–2," I can add 2 (remember what we said before


I think I'm getting the hang of variables now. Here's a practice problem for you, Carissa. Do you know what n is if 3n+5 = 14?
Hmm... I subtract 5 from both sides of the equation and I have 3n=9. Then I divide both sides of the equation by 3 and that tells me that n=3. Does my answer check?
3n+5=14
3n+5=14
Well, let's see: if we go back to the original equation and replace the "n" with a 3, we have 3(3)+5=14.
Doing the arithmetic, we can see both sides of the equation are 14.
It checks!
3n+5=14
-5 -5
3n=9
3 3
n=3

about adding and subtracting 2 to both sides of the equation). It works like this:

$$3x - 2 = 7$$
$$3x - 2 + 2 = 7 + 2$$
$$3x = 9$$

So I've taken off the "–2" paper and what I found inside was a 9. Now we can take off the "times 3" by dividing both sides of the equation by 3:

$$3x \div 3 = 9 \div 3$$
$$x = 3$$

Now the present is unwrapped and we can see what it is. We were able to do all this because we knew how to handle a number without knowing what it was. Of course, since we can always make mistakes, we should check that we're right; let's wrap it back up and see if it's a 7:

$$3(3) - 2 = 9 - 2 = 7$$

Yup! That's what was in the package. And doing this lets us see what was happening to x by putting a real number (the right one) in its place.

—Dr. Math, The Math Forum

3 Exponents

Until about four hundred years ago, nobody used exponents, but they were perfectly able to do mathematics. For example, they would write $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$. But between the fourteenth and seventeenth centuries, mathematicians in Europe developed the concept of raised exponential notation. They decided to use exponents to mean how many times they wrote down the number. So, $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ was written quickly as 5^6.

Properties of Exponents

Dear Dr. Math,

I was out of school with the flu and when I came back my class had studied properties of exponents. Will you please help me understand exponents?

Thanks,

Carissa

Hi, Carissa,

I'm going to start from the definition of what an exponent is and show how you can figure out the properties from the definition. So, the first thing to know about **exponents** is that they are just a shorthand notation for a special kind of multiplication. That is, there are times when I need to refer to a number like

$$6 \cdot 6 \cdot 6 \cdot 6 \cdot 6$$

but I don't want to keep writing that down all the time. We have a much nicer notation, which is

$$6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^5$$

Whenever you see something that looks like

$$a^b$$

you know that it means b copies of a, all multiplied together. Here are some more examples:

$$\begin{aligned} 2^3 &= 2 \cdot 2 \cdot 2 \\ 3^2 &= 3 \cdot 3 \\ (a + b)^4 &= (a + b) \cdot (a + b) \cdot (a + b) \cdot (a + b) \\ (4^5)^2 &= (4^5) \cdot (4^5) \end{aligned}$$

It's just an abbreviation, like writing *Dr.* instead of *Doctor* or like writing *MA* instead of *Massachusetts*. You need to be very clear on that or nothing else about exponents is going to make sense.

Multiplying Exponents

When multiplying numbers with exponents that have the same base, it's easy to forget whether to add or multiply those exponents. I use examples to help me remember. If I have

$$(a \cdot a) \cdot (a \cdot a \cdot a)$$

that's $(a^2) \cdot (a^3)$. I know the answer has to be a^5 just from counting the a's. Since I can get that 5 by adding the exponents, that tells me

$$a^b \cdot a^c = a^{(b + c)}$$

So, to multiply powers of the same base, we add the exponents.

Here is another property of multiplying exponents. If I have something like

$$(a \cdot a) \cdot (a \cdot a) \cdot (a \cdot a)$$

that's $(a^2)^3$, and I know the answer has to be a^6 (again, just from counting the a's). Since I can get 6 by multiplying the exponents, I know that

$$(a^b)^c = a^{(b \cdot c)}$$

In other words, to raise a power to a power, we multiply the exponents.

Dividing Exponents

If I have an expression like

$$\frac{a \cdot a \cdot a}{a \cdot a}$$

that's the same as a^3/a^2. If I cancel out as many a's as I can from the original problem, I'm left with a single a, which tells me that

$$\frac{a^b}{a^c} = a^{(b - c)}$$

In other words, to divide powers of the same base, we subtract the exponents.

What happens when you flip that fraction upside down? You get

$$\frac{a \cdot a}{a \cdot a \cdot a}$$

which is a^2/a^3. If I cancel out as many a's as I can from the problem, I'm left with $\frac{1}{a}$. From the property above, I know that $a^2/a^3 = a^{2-3} = a^{-1}$, which tells me that

$$a^{-b} = \frac{1}{a^b}$$

Here's another example:

$$\frac{a \cdot a}{a \cdot a} = ?$$

which is the same as a^2/a^2. From the property above, we know that a^2/a^2 should equal $a^{(2-2)}$, which equals a^0. But I know that it has to be 1, since anything divided by itself must be 1. So,

$$a^0 = 1$$

for any value of a, which seems a little weird. But it follows from the definitions we've been working out, and it doesn't lead to any bizarre consequences, so we just accept it.

You can do a little more with exponents, too. According to the rules that we just figured out, note that

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{(\frac{1}{2}+\frac{1}{2})} = a^1 = a$$

To summarize, here are the interesting properties of exponents. If a and b are positive integers, then

$$a^b \cdot a^c = a^{(b+c)}$$

$$\frac{a^b}{a^c} = a^{(b-c)}$$

$$(a^b)^c = a^{(b \cdot c)}$$

$$a^{-b} = \frac{1}{a^b}$$

$$a^1 = a$$

$$a^0 = 1$$

$$a^{\frac{1}{n}} = \sqrt[n]{a} = \text{the } n\text{th root of } a$$

Don't forget that these properties apply only if the bases are the same—you can't do anything with $5^4 \cdot 3^2$ except work it out numerically. You can't do anything with $x^3 \cdot y^7$ as it stands.

So, $a^{\frac{1}{2}}$ multiplied by itself is a . . . , which tells me that

$$a^{\frac{1}{2}} = \sqrt{a}$$

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

and so on.

The main thing you need to do is avoid the feeling that you have to memorize these properties. As you can see, they are all just *consequences* that follow from the definition of an exponent, which is just something that mathematicians invented so that they could be lazy about writing things down. If you really understand the definition, you can make up little examples like I've done to rediscover the properties whenever you need them.

Instead of trying to memorize the various properties, if you take the time to make sure you can really follow these examples—and even explain them to someone else—then you should have no problems with exponents.

—Dr. Math, The Math Forum

Using Scientific Notation

Dear Dr. Math,

How would you solve a problem using scientific notation? I know it's used to multiply large numbers to get a correct answer, but I don't understand how to do it. Here's what I've tried:

4,567,839 · 5,493,711 =
4.567839 · 10 to the sixth power ·
5.493711 · 10 to the sixth power = ?

I don't know where to go from here. Do I multiply the number by the decimal by the exponent, or ignore the exponent and simply multiply? Please give me an example.

I also understand that you do different things with the problem depending on the operation being used. Help!

From,

Carissa

Hello, Carissa,

The thing to remember when working with **scientific notation** to do multiplication is the law of exponents that says

$$a^m \cdot a^n = a^{(m+n)}$$

For example,

$$\begin{aligned} 10^6 \cdot 10^6 &= 10^{(6+6)} \\ &= 10^{12} \end{aligned}$$

Notice that the base has to be the same in both numbers! You cannot apply this law of exponents to $10^6 \cdot 5^6$ because the bases are different—the first number has a base of 10 and the second has a base of 5.

Getting back to your problem, you already know that 4,567,839 = $4.567839 \cdot 10^6$ and that 5,493,711 = $5.493711 \cdot 10^6$. This makes the problem $(4.567839 \cdot 10^6) \cdot (5.493711 \cdot 10^6)$.

Applying the Associative Property of Multiplication, which says

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

and the Commutative Property of Multiplication, which says

$$a \cdot b = b \cdot a$$

we get

$$(4.567839 \cdot 5.493711) \cdot (10^6 \cdot 10^6)$$

By multiplying the two left-most factors and applying the above law of exponents to the right-most factors we get

$$25.094387 \cdot 10^{12}$$

So, in a nutshell, we converted our large numbers to scientific notation, added together the exponents where we had common bases, then multiplied the numbers without exponents. You could leave the answer in exponential form or you could expand it to

25,094,387,000,000

The format you choose depends on what you have been asked to use for your answer.

— ***Dr. Math, The Math Forum***

4 Large and Small Numbers

When we deal with numbers in real life, we often talk about things that can only be described using very large numbers or very small numbers. For example, if we are talking about space travel and the distances between the earth and other planets, the numbers we use are quite large. Small numbers are commonly used when we talk about microscopic things. Very large or very small numbers are often most easily expressed using exponents, usually with a technique called scientific notation.

When do you use scientific notation and when do you use exponents? The general answer is that when you have a big number in base 10, you can use scientific notation to make it easier to deal with. But sometimes you don't get your big number in base 10! You get it indirectly from a simulation of some kind of process, like the accumulation of interest, or successive divisions of bacteria, which naturally leads to the use of exponents. For example, if you multiply 186,000 miles per second by 86,400 seconds per day, then multiply by 365.25 days per year, you get 5,869,713,600,000, or $5.87 \cdot 10^{12}$, which is the number of miles in a light-year. But if you let 1,000 dollars accumulate for 120 years at 5 percent interest, you end up with $1{,}000 \cdot 1.05^{120}$ dollars.