1

Continuous and Discrete Signals

In this chapter we shall review several concepts concerning analog and digital signals, namely the Fourier, $\mathcal{Z}$, and Laplace transforms, the sampling theorem, and the aliasing problem. These topics are presented in order to establish notation that we will use in mixed signal circuits. We will also present exponential, Euler, and bilinear mappings from the s domain to the z domain, as well as transfer functions describing two-dimensional systems in both domains. Finally, we will describe the discrete cosine transform, which is very important in image compression and will be used in the second part of the book.

1.1 FOURIER, $\mathcal{Z}$, AND LAPLACE TRANSFORMS

A discrete-time signal is defined as a sequence $\{x(k)\}$ resulting from sampling a continuous-time signal $x(t)$. The symbol $x(k)$ denotes the element of the sequence that is equal to the value of the function $x(t)$ for $t = kT$, where T is the sampling interval. The relation

$$x(k) = \int_{-\infty}^{\infty} \hat{x}_k(t)dt \tag{1.1}$$

describes the sampling operation, where

$$\hat{x}_k(t) = x(t)\delta(t - kT) \tag{1.2}$$

$\delta(t)$ is the delta function or distribution function. The function obtained as a sum of (1.2) for all indices k

$$\hat{x}(t) = \sum_k \hat{x}_k(t) = x(t)\sum_k \delta(t - kT) = \sum_k x(k)\delta(t - kT) \tag{1.3}$$

is called a continuous-time PAM (pulse amplitude modulation) representation of a discrete-time signal.

The periodic function $x(t)$ with a period T_p can be expanded in a Fourier series in the following complex form

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j(2\pi/T_p)nt} \tag{1.4}$$

where

$$c_n = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)e^{-j(2\pi/T_p)nt}dt \tag{1.5}$$

and $j = \sqrt{-1}$. The coefficients c_n fulfill the relation

$$c_n = c^*_{-n} = \frac{a_n - jb_n}{2} \tag{1.6}$$

where a_n, b_n, $n = 1, 2, 3, \ldots,$ denote the coefficients of the Fourier series in a trigonometric form.

An extension of the formulae (1.4) and (1.5) for $T_p \to \infty$ gives the Fourier transform pair of an arbitrary continuous-time signal $x(t)$ in the form

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \qquad x(t) = \frac{1}{2}\pi\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \tag{1.7}$$

For the signal $\{x(k)\}$ the Fourier transform is called a discrete-time one (DTFT) and takes the form

$$X(e^{j\omega T}) = \sum_k x(k)e^{-j\omega kT}, \qquad x(k) = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} X(e^{j\omega T})e^{j\omega kT}d\omega \tag{1.8}$$

If, however, in the above equations only N samples $x(k)$ are taken for $k = 0, 1, \ldots,$ $N - 1$ and only N samples of $X(e^{j\omega T})$ are calculated for $\omega = n\omega_0$, $n = 0, 1, \ldots, N - 1$, where $\omega_0 = (2\pi/T)/N = \omega_s/N$, the discrete Fourier transform (DFT) is defined as

$$X_N(n) = X(e^{jn\omega_0 T}) = \sum_{k=0}^{N-1} x(k)e^{-j2\pi nk/N} \tag{1.9}$$

We see that the Fourier transform (1.7) gives relations between functions of real variables t and k and the frequency variable ω, which is also a real variable. In the case of the Laplace and $\mathcal{Z}$ transforms, after transformation of $x(t)$ and $x(k)$ we obtain functions

$$X(s) = \int_0^{\infty} x(t)e^{-st}dt \tag{1.10}$$

and

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} \tag{1.11}$$

of complex variables s and z, respectively. We assume that the functions $x(t)$ and $x(k)$ in (1.10), (1.11), are equal to zero for negative arguments t and $k[x(t) = 0$ for $t < 0$ and $x(k) = 0$ for $k < 0]$, i.e., that they are causal functions. For causal functions, the Laplace transform is equivalent to the Fourier transform for $s = j\omega$ and the $\mathcal{Z}$ transform to DTFT for $z = e^{j\omega T}$. It means that the variable ω is represented by the imaginary axis on the s plane and by the unit circle on the z plane. The Laplace transform (1.10) of the PAM representation of $x(k)$ described by (1.3) is as follows:

$$\hat{X}(s) = \sum_{k=0}^{\infty} x(k)e^{-skT} \tag{1.12}$$

For

$$z = e^{sT} \tag{1.13}$$

it gives the equivalence between the Laplace and $\mathcal{Z}$ transforms and the relation

$$\hat{X}(j\omega) = X(e^{j\omega T}) \tag{1.14}$$

Equation (1.13) shows that the imaginary axis on the s plane is transformed into the unit circle with the center in the origin of the coordinate system in the z domain. On the basis of this equation, we can add that the left-hand side of the s plane is transformed into the interior of this circle, whereas the right-hand side is transformed into the exterior. For the z^{-1} plane

$$z^{-1} = e^{-sT} \tag{1.15}$$

which is also often considered. The relations between the left- and right-hand sides on the s plane and the interior and exterior of the unit circle in the z^{-1} domain are reversed.

The $\mathcal{Z}$, Fourier, and Laplace transforms of functions corresponding to basic signals are shown in Table 1.1 Function $u(t)$ denotes the unit step.

Table 1.1 Transforms of basic signals

$x(t)$	$Z\{u(nT)x(nT)\}$	$\mathcal{F}\{x(t)\}$	$\mathcal{L}\{u(t)x(t)\}$
$\delta(t)$	1	1	1
$u(t)$	$\frac{z}{z-1}$	$\pi\delta(\omega)+\frac{1}{j\omega}$	$\frac{1}{s}$
$sgn(t)$	$\frac{z}{z-1}$	$\frac{2}{j\omega}$	$\frac{1}{s}$
$\Pi_{2k}(t)$	$\frac{z(1-z^{-k})}{z-1}$	$\frac{2\sin k\omega}{\omega}$	$\frac{1-e^{-ks}}{s}$
$e^{j\omega_o t}$	$\frac{z^2-ze^{-j\omega_o T}}{z^2-2z\cos\omega_o T+1}$	$2\pi\delta(\omega-\omega_o)$	$\frac{1}{s-j\omega_o}$
$u(t)e^{-\alpha t}$	$\frac{z}{z-e^{-\alpha T}}$	$\frac{1}{\alpha+j\omega}$	$\frac{1}{s+\alpha}$
$e^{-\alpha\|t\|}$	$\frac{z}{z-e^{-\alpha T}}$	$\frac{2\alpha}{\alpha^2+\omega^2}$	$\frac{1}{s+\alpha}$
$\cos\omega_o t$	$\frac{z^2-z\cos\omega_o T}{z^2-2z\cos\omega_o T+1}$	$\pi\delta(\omega-\omega_o)+\pi\delta(\omega+\omega_o)$	$\frac{s}{s^2+\omega_o^2}$
$u(t)e^{-\alpha t}\cos\beta t$	$\frac{z^2-ze^{-\alpha T}\cos\beta T}{z^2-2ze^{-\alpha T}\cos\beta T+e^{-2\alpha T}}$	$\frac{\alpha+j\omega}{(\alpha+j\omega)^2+\beta^2}$	
$\sin\omega_o t$	$\frac{z\sin\omega_o T}{z^2-2z\cos\omega_o T+1}$	$j\pi\delta(\omega+\omega_o)-j\pi\delta(\omega-\omega_o)$	$\frac{\omega_o}{s^2+\omega_o^2}$
$u(t)e^{-\alpha t}\sin\beta t$	$\frac{ze^{-\alpha T}\sin\beta T}{z^2-2ze^{-\alpha T}\cos\beta T+e^{-\alpha T}}$	$\frac{\beta}{(\alpha+j\omega)^2+\beta^2}$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$

1.2 ALIASING PHENOMENON AND NYQUIST SAMPLING THEOREM

A linear, time-invariant (LTI) system excited by the signal $x(t)$ responds with a continuous-time signal $y(t)$. For the delta excitation $[x(t) = \delta(t)]$ the response is denoted by $h(t)$ and called the pulse response. Any response $y(t)$ of the LTI system can be expressed in the time domain as a convolution:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{1.16}$$

or as a multiplication

$$Y(s) = H(s)X(s), \qquad Y(j\omega) = H(j\omega)X(j\omega) \tag{1.17}$$

in the Laplace and Fourier domains. Similarly, for a discrete-time system we have

$$y_k = x(k) * h_k = \sum_{m=-\infty}^{\infty} x_m h_{k-m} \tag{1.18}$$

in the discrete-time domain, or

$$Y(z) = H(z)X(z), \qquad Y(e^{j\omega T}) = H(e^{j\omega T})X(e^{j\omega T}) \tag{1.19}$$

in the Z and Fourier domains.

Multiplication in the function $\hat{x}(t)$ presented in the second form in (1.3)

$$\hat{x}(t) = [x(t)] \cdot \left[\sum_k \delta(t - kT)\right] \tag{1.20}$$

for LTI systems corresponds to convolution in the frequency domain

$$\begin{aligned}\hat{X}(j\omega) &= X(e^{j\omega T}) \\ &= \frac{1}{2\pi}[X(j\omega)] * \left[\omega_s \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_s)\right] \\ &= \frac{1}{T}\int_{-\infty}^{\infty} X(j\Omega) \sum_{m=-\infty}^{\infty} \delta(\omega - \Omega - m\omega_s)d\Omega \\ &= \frac{1}{T}\sum_{m=-\infty}^{\infty} X[j(\omega - m\omega_s)]\end{aligned} \tag{1.21}$$

It means that the Fourier transform $X(e^{j\omega T})$ of a discrete signal can be obtained as a sum of shifted Fourier transforms $X(j\omega)$ of a continuous-time signal [13]. Each component in this sum is shifted by the integer multiple m of the sampling frequency $\omega_s = 2\pi/T$. It means that the spectrum of the discrete signal can contain high-frequency components of $x(t)$ transposed to low-frequency components. This phenomenon is called aliasing. In order to eliminate aliasing, the signal $x(t)$ is fed to an ideal low-pass filter, called antialiasing filter, with the cutoff frequency $\omega_c \leq \omega_s/2$. In this case, there will be no overlap of frequency components of the signal sampled at the output of this filter. The continuous-time signal can be reconstructed again at the output of the next low-pass filter, called the smoothing filter, excited by a discrete signal. Hence, the signal $x(t)$ which has the Fourier transform $X(j\omega)$ and is sampled at frequency $2\pi/T$, can be reconstructed from its samples if $X(j\omega) = 0$ for all $|\omega| > \pi/T$, (Nyquist sampling theorem). The frequency $\omega_N = \pi/T$ is called the Nyquist frequency. The system for mixed signal processing, containing antialiasing and smoothing filters and presented in Figure 1.1, can be realized as a CMOS circuit on a single chip.

Using the ideal lowpass filter, which has the pulse response

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} \tag{1.22}$$

Figure 1.1 Example of a system for mixed signal processing composed of antialiasing (AF) and smoothing (SF) filters, A/D and D/A converters, and a digital core (DP).

we can obtain the analog signal $x(t)$ at the output of this filter excited by the PUM representation $\hat{x}(t)$ as the convolution $\hat{x}(t) * h(t)$. The reconstruction formula is as follows

$$x(t) = \hat{x}(t) * \left[\frac{\sin(\pi t/T)}{\pi t/T} \right] = \sum_{m=-\infty}^{\infty} x_m \frac{\sin[\pi(t - mT)/T]}{\pi(t - mY)/T} \tag{1.23}$$

1.3 EULER AND BILINEAR TRANSFORMATIONS

LTI systems are described by transfer functions that are rational functions in z and s domains. Discrete-time systems are often designed on the basis of continuous-time systems with the use of the transfer function $H(s)$. However, it is not possible to derive the rational transfer function $H(z)$ from $H(s)$ using the transformation (1.13). Hence, different approximations of relation (1.13) are used. The simplest ones result from the series expansion of exponential functions in the form

$$z = e^{sT} = 1 + \frac{sT}{1!} + \frac{(sT)^2}{2!} + \frac{(sT)^3}{3!} + \ldots \tag{1.24}$$

or

$$z^{-1} = e^{-sT} = 1 + \frac{-sT}{1!} + \frac{(-sT)^2}{2!} + \frac{(-sT)^3}{3!} + \ldots \tag{1.25}$$

and are called the forward and backward Euler transformations:

$$sT = z - 1 \tag{1.26}$$

and

$$sT = 1 - z^{-1} \tag{1.27}$$

respectively.

Another transformation, not so simple as the Euler transformations, but with very interesting properties, is the bilinear transformation

$$\frac{sT}{2} = \frac{z-1}{z+1} = \frac{1-z^{-1}}{1+z^{-1}} \tag{1.28}$$

or

$$z^{-1} = \frac{1 - sT/2}{1 + sT/2} \tag{1.29}$$

which can be obtained from the series representation of the ln function

$$sT = \ln(z) = 2\left[\frac{z-1}{z+1} + \frac{(z-1)^3}{3(z+1)^3} + \frac{(z-1)^5}{5(z+1)^5} + \ldots\right] \tag{1.30}$$

We see from relations (1.26) and (1.27) that the imaginary axis $s = j\omega$ in the s domain corresponds to the line tangent to the unit circle at the point (0, 1) on the z and z^{-1} planes, respectively. The left-hand side of the s plane corresponds to half-plane on the left-hand side of this tangent in the z domain and on the right-hand side in the z^{-1} domain. Let us note that the exact transformation (1.13) transforms the left-hand side of the s plane into the interior or exterior of the unit circle in the z and z^{-1} domains, respectively. For $s = j\omega$ the bilinear relation (1.29) yields $|z| = 1$ and, like the exact transformation (1.13), transforms the imaginary axis in the analog domain into the unit circle in the discrete domain. These relations between analog s and discrete z domains are shown in Figure 1.2.

The Euler and bilinear transformations impose scaling of frequencies ω_a and ω_d in analog and discrete domains. In the case of bilinear transformation, introducing into (1.28) the frequencies $s = j\omega_a$ and $z = e^{j\omega_d T}$, we obtain

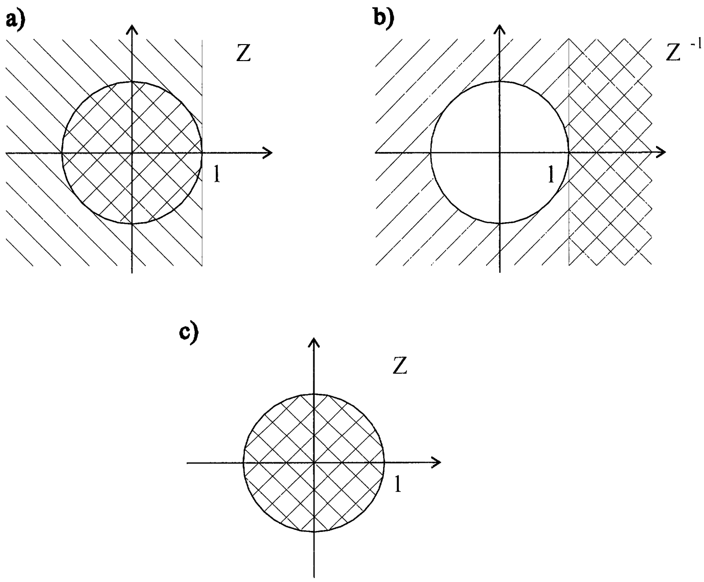

Figure 1.2 Transformations between analog and discrete domains for forward Euler (a), backward Euler (b), and bilinear (c) transformations.

$$\frac{\omega_a T}{2} = \tan\frac{\omega_d T}{2} \tag{1.31}$$

Let us note that this relation compresses the whole frequency axis in the analog domain into the frequency range limited by the Nyquist frequency $\omega_N = \pi/T$. This property makes discrete filters obtained on the basis of prototype analog filters more selective. On the other hand, the design process of discrete filters requires the analog filter to change its frequencies according to the relation (1.31), in order to obtain the desired frequencies in the counterpart discrete filter. This stage of the design process is called prewarping.

1.4 TWO-DIMENSIONAL DISCRETE COSINE TRANSFORM}

The Fourier transform presented in the previous sections can also be used for two-dimensional (2-D) processing. However, the optimum transform for image compression is the Karhunen–Loeve transformation (KLT) [31], because it packs the greatest amount of energy in the smallest number of elements in the frequency domain of a 2-D signal and minimizes the total entropy of the signal sequence. Unfortunately, the basis functions of KLTs are image-dependent, which is the most important implementation-related deficiency. It is observed that the two-dimensional (2-D) discrete cosine transform (DCT) has the output close to the output produced by the KLT [3], and uses image-independent basis functions. Hence, DCT-based image coding is applied in all video compression standards. In these standards, the image is divided into 8 × 8 blocks in the spatial domain and DCT transforms them into 8 × 8 blocks in the 2-D frequency domain. Such block size is convenient with respect to computational complexity. Larger sizes do not offer significantly better compression.

2-D DCT can be expressed as

$$y_{kl} = \frac{c(k)c(l)}{4}\sum_{i=0}^{7}\sum_{j=0}^{7} x_{ij}\cos\frac{(2i+1)k\pi}{16}\cos\frac{(2j+1)l\pi}{16} \tag{1.32}$$

where $k, l = 0, 1, \ldots, 7$ and

$$c(k) = \begin{cases} \frac{1}{\sqrt{2}}, & k = 0 \\ 1, & k \neq 0 \end{cases} \tag{1.33}$$

Assuming that the matrices Y and X are composed of elements y_{ij} and x_{ij}, $i, j = 0, 1, \ldots, 7$, respectively, the relation (1.32) can be also written in the matrix form as

$$Y = CXC^t \tag{1.34}$$

where the matrix of coefficients C is as follows:

$$C = \frac{1}{2}\begin{bmatrix} d & d & d & d & d & d & d & d \\ a & c & e & g & -g & -e & -c & -a \\ b & f & -f & -b & -b & -f & f & b \\ c & -g & -a & -e & e & a & g & -c \\ d & -d & -d & d & d & -d & -d & d \\ e & -a & g & c & -c & -g & a & -e \\ f & -b & b & -f & -f & b & -b & f \\ g & -e & c & -a & a & -c & e & -g \end{bmatrix} \tag{1.35}$$

$a = \cos(\pi/16)$, $b = \cos(2\pi/16)$, $c = \cos(3\pi/16)$, $d = \cos(4\pi/16)$, $e = \cos(5\pi/16)$, $f = \cos(6\pi/16)$, $g = \cos(7\pi/16)$.

The main property of 2-D DCT, with respect to implementation, is separability. On the basis of the matrix equation (1.34), written in the form

$$Y = Z'C', \qquad Z = X'C' \tag{1.36}$$

we can realize 2-D DCT with two 1-D ones. The matrix X denotes one input 8×8 block, and its transposition X' in the relation $Z = X'C'$ means that it is read out column by column. The matrix Z, containing intermediate results, is obtained with the use of 1-D DCT, and is saved in a memory array. Transposition of this matrix in the first equation in (1.36) means that the elements of Z obtained successively for the current block are memorized in row cells and for the previous block are read out from column cells of the memory array. The intermediate results are processed in the same way as the input matrix X, giving the output signal matrix Y. The implementation of a 2-D DCT processor will be presented in the second part of this book.

The matrix relations (1.36) can be expressed as

$$y_n = c(n)\sqrt{\frac{2}{N}}\sum_{k=0}^{N-1} x(k)\cos\frac{(2k+1)n\pi}{2N} \tag{1.37}$$

describing a 1-D DCT in an explicit form, where $n = 0, 1, \ldots, N-1$. Equation (1.37) can be used to show the relationship between DCT and DFT given by (1.9), [47]. On the basis of $x(k)$, a $2N$-point sequence ξ_k can be obtained as

$$\xi_k = \begin{cases} x(k), & 0 \le k \le N-1 \\ x_{2N-k-1}, & N \le k \le 2N-1 \end{cases} \tag{1.38}$$

Let us note that the second half of ξ_k for $k = N, \ldots, 2N-1$ is a mirror image of the first half of ξ_k for $k = 0, \ldots, N-1$. The $2N$-point DFT of ξ_k is, from the definition (1.9), given by

$$\begin{aligned} X_{2\mathrm{N}}(n) &= \sum_{k=0}^{2N-1} \xi_k e^{-j2\pi nk/(2N)} \\ &= \sum_{k=0}^{N-1} x(k)e^{-j2\pi nk/(2N)} + \sum_{k=N}^{2N-1} x_{2N-k-1}e^{-j2\pi nk/(2N)} \end{aligned} \tag{1.39}$$

for $n = 0, \ldots, 2N-1$. The first summation on the right-hand side of the above equation can be written as

$$\sum_{k=0}^{N-1} x(k)e^{-j2\pi nk/(2N)} = e^{jn\pi/(2N)} \sum_{k=0}^{N-1} x(k)e^{-j(2k+1)n\pi/(2N)} \tag{1.40}$$

whereas the second one can be written as

$$\begin{aligned} \sum_{k=N}^{2N-1} x_{2N-k-1}e^{-j2\pi nk/(2N)} &= \sum_{k=N-1}^{0} x(k)e^{-j2\pi n(2N-k-1)/(2N)} \\ &= e^{jn\pi(1-4N)/(2N)} \sum_{k=N-1}^{0} x(k)e^{j(2k+1)n\pi/(2N)} \\ &= e^{jn\pi/(2N)} \sum_{k=0}^{N-1} x(k)e^{j(2k+1)n\pi/(2N)} \end{aligned} \tag{1.41}$$

Introducing the results from (1.40) and (1.41) into (1.39), we obtain

$$X_{2N}(n) = 2e^{jn\pi/(2N)} \sum_{k=0}^{N-1} x(k) \cos\frac{(2k+1)n\pi}{2N} \tag{1.42}$$

for $n = 0, \ldots, 2N-1$. Hence, the DCT transform y_n in (1.37) can be obtained from the $2N$-point DFT using the equation

$$y_n = \frac{c(n)}{\sqrt{2N}} e^{-jn\pi/(2N)} X_{2N}(n) \tag{1.43}$$

for $n = 0, \ldots, N-1$.

1.5 TRANSFER FUNCTIONS OF A 2-D MULTIPORT NETWORK

Transfer functions H of LTI systems are often described in analog (s) or discrete (z) complex domains, as can be seen in relations (1.17) and (1.19). In this section, we will consider relations between transfer functions of a system described in different complex domains. The formulae that will be presented refer to two-dimensional systems. The corresponding relationships for one-dimensional systems can be easily obtained as a special case of formulae introduced for 2-D systems.

Transfer functions H^{mn}, $m = 1, \ldots, M$, $n = 1, \ldots, N$, of a 2-D LTI network are rational functions of two complex variables. H^{mn} is an element of the matrix H that describes a linear 2-D multiport network shown in Figure 1.3. N denotes the number of inputs, whereas M denotes the number of outputs. The elements of the input and output vector signals x and y are also functions of the complex variables. Each variable belongs to the s or z domain. Hence, there are four equivalent representa-

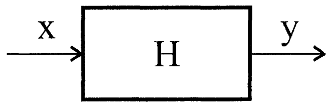

Figure 1.3 Symbol of a linear 2-D multiport network.

tions of the transfer functions H^{mn}, $m = 1, \ldots, M$, $n = 1, \ldots, N$: discrete, ($H^{mn} \leftarrow Z_1A^{mn}Z_2'/Z_1BZ_2'$), analog ($H^{mn} \leftarrow S_1P^{mn}S_2'/S_1QS_2'$), and hybrid ($H^{mn} \leftarrow S_1P_h^{mn}Z_2'/S_1Q_hZ_2'$) and ($H^{mn} \leftarrow Z_1A_h^{mn}S_2'/Z_1B_hS_2'$), as is shown in Figure 1.4, [21]. Let us note that each representation can have different numerators. However, they all have a common denominator.

The polynomials in the numerators and the denominator are written in the matrix form. The elements of matrices A^{mn}, B, P^{mn}, and Q are equal to coefficients of respective polynomials, and Z_i, S_i are vectors composed of complex variables z_i, s_i:

$$Z_i = [z_i^{-k} \cdots z_i^{-1}\ 1] \qquad S_i = [s_i^k \cdots s_i\ 1] \qquad i = 1, 2 \tag{1.44}$$

where the elements of the vectors Z_i, and S_i, $i = 1, 2$, are ordered in descending powers of variables z_i^{-1}, s_i, respectively. The sign $'$ denotes a transposed matrix or vec-

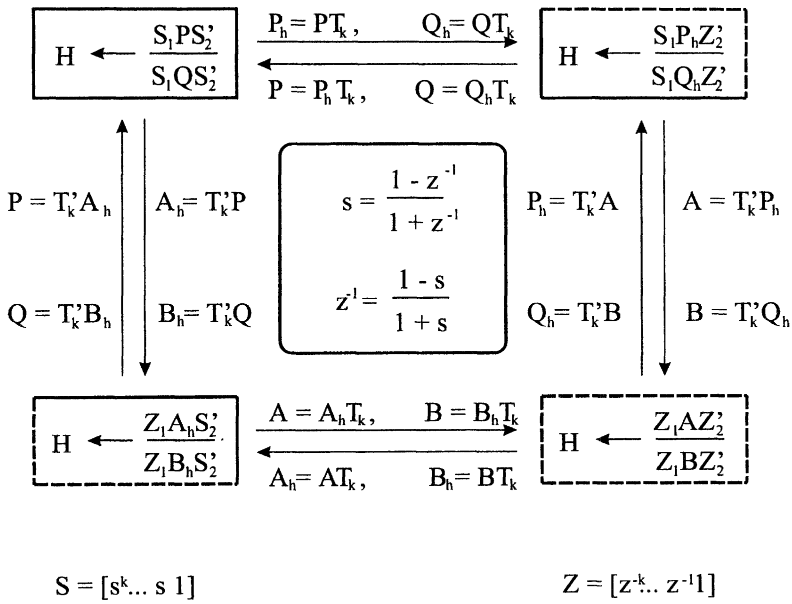

Figure 1.4 Representations of 2-D network transfer functions.

tor. Let us note that the transfer function in the discrete domain is usually written in the form

$$H^{mn} \leftarrow \frac{\widetilde{Z}_1 \hat{A}^{mn} \widetilde{Z}_2'}{\widetilde{Z}_1 \hat{B} \widetilde{Z}_2'} \tag{1.45}$$

where the elements of the vector $\widetilde{Z}_i$ are ordered in ascending powers of the variable z_i^{-1}

$$\widetilde{Z}_i = [1 z_i^{-1} \cdots z_i^{-k}] \tag{1.46}$$

and where $\hat{A}^{mn}$, $\hat{B}$ denotes the matrices A^{mn}, B transposed with respect to both diagonals. The description given by (1.45) is called the standard form of a transfer function.

The vectors S_1, S_2, Z_1, and Z_2 are used for describing polynomials in the numerators and denominators of the transfer functions. It does not mean that polynomials are of the same order with respect to the given variable because some rows or columns in the matrices A^{mn}, B, P^{mn}, and Q may be composed of zero elements. For example, the denominator of the transfer function of a nonrecursive filter is equal to 1 in the digital domain. One can describe this filter by the matrix B in the form

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{1.47}$$

We assume that the discrete and analog variables are bilinearly transformed

$$s_i = \frac{1 - z_i^{-1}}{1 + z_i^{-1}}$$

$$z_i^{-1} = \frac{1 - s_i}{1 + s_i}, \qquad i = 1, 2 \tag{1.48}$$

The above relationships are obtained from (1.28) and (1.29) where, for the sake of simplicity, we will assume that the sampling periods in both dimensions $i = 1, 2$ are $T = 2$. Under these assumptions, we can obtain all transfer function representations, multiplying matrices A^{mn}, B, P^{mn}, and Q by the transformation matrix T_k. The matrix T_k can be generated in a recurrent manner:

$$T_0 = [1],\ T_1 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},\ T_2 = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix}, \dots, T_k \tag{1.49}$$

The procedure for construction of these matrices is as follows. The $(i-1)$th row of the matrix T_{j-1} and the ith row of the matrix T_j, $i = 2, \cdots, j+1, j = 1, \cdots, k$, always form two neighboring rows of a Pascal triangle with $T_0 = [1]$. For example, the first row of T_0, the second row of T_1, the third row of T_2, and the fourth row of T_3, etc. form a Pascal triangle. Similarly, the first row of T_1, the second of T_2, the third row of T_3, etc., or the first row of T_2 and the second row of T_3, etc. also form Pascal triangles. As far as the first row of each matrix is concerned, the lth element of the first row of the jth matrix is the lth element of the last row of the same matrix multiplied by $(-1)^{j+l+1}$.

Using the bilinear transformation we can write

$$\begin{bmatrix} s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix} = \frac{1}{(1+z^{-1})^{n-1}} \begin{bmatrix} (1-z^{-1})^{n-1} \\ \vdots \\ (1+z^{-1})^{n-2}(1-z^{-1}) \\ (1+z^{-1})^{n-1} \end{bmatrix} \tag{1.50}$$

and

$$\begin{bmatrix} s^{n} \\ s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix} = \frac{1}{(1+z^{-1})^{n}} \begin{bmatrix} (1-z^{-1})^{n} \\ (1+z^{-1})(1-z^{-1})^{n-1} \\ \vdots \\ (1+z^{-1})(1+z^{-1})^{n-2}(1-z^{-1}) \\ (1+z^{-1})(1+z^{-1})^{n-1} \end{bmatrix} \tag{1.51}$$

The comparison of (1.50) and (1.51), for $n = 1$ and $n = k$, yields

$$S' = T_k Z' \tag{1.52}$$

where the scaling factor $1/(1+z^{-1})^k$, which does not affect the transfer function H^{mn}, has been dropped. We see that both s to z^{-1} and z^{-1} to s transformations in (1.48) have the same form. Hence, similarly to (1.52), we can write

$$Z' = T_k S' \tag{1.53}$$

and we see that, instead of the inverse matrix T_k^{-1}, the matrix T_k can be used for the inverse z to s transformation. We find that

$$T_k T_k = 2^k U \tag{1.54}$$

where U is a unit matrix. Hence, the normalization factor of matrix T_k is $1/\sqrt{2^k}$. The transposition of (1.52) and (1.53) gives

$$S = ZT_k' \tag{1.55}$$

and

$$Z = ST_k' \tag{1.56}$$

which completes the proof of relations

$$\begin{aligned}
&P_h^{mn} = P^{mn}T_k,\ P^{mn} = P_h^{mn}T_k,\ Q_h = QT_k,\ Q = Q_hT_k\\
&A_h^{mn} = A^{mn}T_k,\ A^{mn} = A_h^{mn}T_k,\ B_h = BT_k,\ B = B_hT_k\\
&P^{mn} = T_k'A_h^{mn},\ A_h^{mn} = T_k'P^{mn},\ Q = T_k'B_h,\ B_h = T_k'Q\\
&A^{mn} = T_k'P_h^{mn},\ P_h^{mn} = T_k'A^{mn},\ B = T_k'Q_h,\ Q_h = T_k'B
\end{aligned} \tag{1.57}$$

shown in the scheme in Figure 1.4.

1.6 PROBLEMS

1. On the basis of (1.6) prove that the Fourier series (1.4) in the complex form is equivalent to the Fourier series in the trigonometric form:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi}{T_p}nt\right) + b_n \sin\left(\frac{2\pi}{T_p}nt\right)\right] \tag{1.58}$$

2. On the basis of the definition (1.10), calculate the Laplace transform $X(s)$ of the function $x(t) = e^{\lambda t}$.
3. Calculate the Laplace transforms shown in Table 1.1 of the functions $e^{-\alpha t}$, $\sin \omega_o t$, $\cos \omega_o t$, $e^{-\alpha t}\sin\beta t$, and $e^{-\alpha t}\cos \beta t$.
   Hint: Use the result from the previous example, introducing $\lambda = -\alpha$, $\lambda = j\omega_o$, or $\lambda = -\alpha + j\beta$.
4. Calculate the Fourier series defined by (1.4), (1.5) of the periodic function

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{1.59}$$

5. Prove the relation

$$\mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \delta(t - kT)\right\} = \omega_s \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_s) \tag{1.60}$$

used in (1.21), which means that the Fourier transform of a sequence of impulses is also a sequence of impulses.

Hint: Use the inverse Fourier transform given in (1.7) and the result of the previous problem.

6. Prove the relation (1.31).
7. Choose appropriate relations from (1.57) and calculate $H_1'(z_1, s_2)$, $H_1''(s_1, z_2)$, and $H_2(z_1, z_2)$ for the transfer function

$$H(s_1, s_2) = \frac{[s_1 \; 1]P[s_2^2 \; s_2 \; 1]'}{[s_1 \; 1]Q[s_2^2 \; s_2 \; 1]'} \tag{1.61}$$

where

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \tag{1.62}$$

Note that the 2-D transfer function (1.61) is obtained from the transfer function of the first-order high-pass filter:

$$H_o(s) = \frac{s}{s+1} \tag{1.63}$$

after the substitution $s = s_1 + s_2^2$.