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Continuum Mechanics

The science of fluid dynamics describes the motions of liquids and gases and their interaction with solid bodies. There are many ways to further subdivide fluid dynamics into special subjects. The plan of this book is to make the division into compressible and incompressible flows. *Compressible flows* are those where changes in the fluid density are important. A major specialty concerned with compressible flows, *gas dynamics,* deals with high-speed flows where density changes are large and wave phenomena occur frequently. *Incompressible flows,* of either gases or liquids, are flows where density changes in the fluid are not an important part of the physics. The study of incompressible flow includes such subjects as hydraulics, hydrodynamics, aerodynamics, and boundary layer theory. It also contains background information for such special subjects as hydrology, lubrication theory, stratified flows, turbulence, rotating flows, and biological fluid mechanics. Incompressible flow not only occupies the central position in fluid dynamics but is also fundamental to the practical subjects of heat and mass transfer.

Figure 1.1 shows a ship's propeller being tested in a water tunnel. The propeller is rotating, and the water flow is from left to right. A prominent feature of this photograph is the line of vapor that leaves the tip of each blade and spirals downstream. The vapor is not itself important, but it marks a region of very low pressure in the core of a vortex that leaves the tip of each blade. This vortex would exist even if the pressure were not low enough to form water vapor. Behind the propeller one can note a convergence of the vapor lines into a smaller spiral, indicating that the flow behind the propeller is occupying a smaller area and thus must have increased velocity.

An airplane in level flight is shown in Fig. 1.2. A smoke device has been attached to the wingtip so that the core of the vortex formed there is made visible. The vortex trails nearly straight back behind the aircraft. From the sense of the vortex we may surmise that the wing is pushing air down on the inside while air rises outside the tip.

There are obviously some differences in these two situations. The wing moves in a straight path, whereas the ship's propeller blades are rotating. The propeller operates in water, a nearly incompressible liquid, whereas the wing operates in air, a very compressible gas. The densities of these two fluids differ by a factor of 800 : 1. Despite these obvious differences, these two flows are governed by the same laws, and their fluid dynamics are very similar. The purpose of the wing is to lift the airplane; the purpose of the propeller is to produce thrust on the boat. The density of the air as well as that of the water is nearly constant throughout the flow. Both flows have a vortex trailing

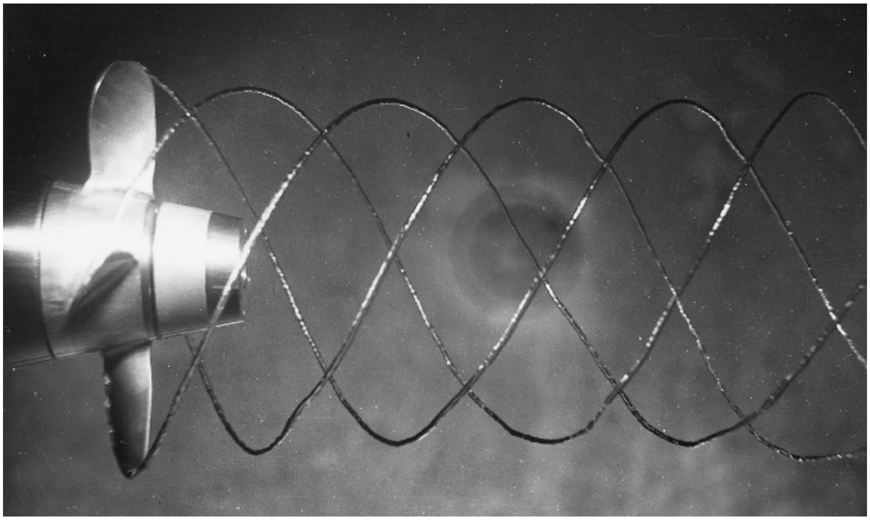

Figure 1.1 Water tunnel test of a ship's propeller. Cavitation vapor marks the tip vortex. Photograph taken at the Garfield Thomas Water Tunnel, Applied Research Laboratory, Pennsylvania State University; supplied with permission by B. R. Parkin.

away from the tip of the surface. This and many other qualitative aspects of these flows are the same. Both are incompressible flows.

In this book we shall learn many characteristics and details of incompressible flows. Equally important, we shall learn when a flow may be considered as incompressible and in exactly what ways the physics of a general flow simplifies for the incompressible case. This chapter is the first step in that direction.

Figure 1.2 Aircraft wingtip vortices. Smoke is introduced at the wingtip to mark the vortex cores. Photograph by W. L. Oberkampf.

1.1 CONTINUUM ASSUMPTION

Fluid mechanics, solid mechanics, electrodynamics, and thermodynamics are all examples of physical sciences in which the world is viewed as a continuum. The *continuum assumption* simply means that physical properties are imagined to be distributed throughout space. Every point in space has finite values for such properties as velocity, temperature, stress, and electric field strength. From one point to the next, the properties may change value, and there may even be surfaces where some properties jump discontinuously. For example, the interface between a solid and a fluid is imagined to be a surface where the density jumps from one value to another. On the other hand, the continuum assumption does not allow properties to become infinite or to be undefined at a single isolated point.

Sciences that postulate the existence of a continuum are essentially macroscopic sciences and deal, roughly speaking, with events that may be observed with the unaided eye. Events in the microscopic world of molecules, nuclei, and elementary particles are not governed by continuum laws, nor are they described in terms of continuum ideas. However, there is a connection between the two points of view. Continuum properties may be interpreted as averages of events involving a great number of microscopic particles. The construction of such an interpretation falls into the disciplines of statistical thermodynamics (statistical mechanics) and kinetic theory. From time to time we shall discuss some of the simpler microscopic models that are used for continuum events. This aids in a deeper understanding of continuum properties, but in no way does it make the ideas "truer." The fundamental assumptions of continuum mechanics stand by themselves without reference to the microscopic world.

The continuum concept developed slowly over the course of many years. Leonhard Euler (Swiss mathematician, 1707–1783) is generally credited with giving a firm foundation to the ideas. Previously, scientists had not distinguished clearly between the idea of a point mass and that of a continuum. In his major contributions, Sir Isaac Newton (1642–1727), actually used a primitive form of the point mass as an underlying assumption (he did at times, however, also employ a continuum approach). What we now call *Newton's mechanics* or *classical mechanics* refers to the motion of point masses. In the several centuries following Newton, problems concerning the vibration of strings, the stresses in beams, and the flow of fluids were attacked. In these problems it was necessary to generalize and distinguish point mass properties from continuum properties. The continuum assumption is on a higher level of abstraction and cannot be derived mathematically from the point mass concept. On the other hand, by integration and by introducing notions such as the center of mass and moments of inertia, we can derive laws governing a macroscopic point mass from the continuum laws. Hence, the continuum laws include, as a special case, the laws for a point mass.

1.2 FUNDAMENTAL CONCEPTS, DEFINITIONS, AND LAWS

It is hard to give a precise description of a *fundamental concept* such as mass, energy, or force. They are hazy ideas. We can describe their characteristics, state how they act, and express their relation to other ideas, but when it comes to saying what they are, we must resort to vague generalities. This is not really a disadvantage, because once we work with a fundamental concept for a while and become familiar with its role in physical processes, we have learned the essence of the idea. This is actually all that is required.

Definitions, on the contrary, are very precise. For example, pressure may be defined precisely after we have the ideas of force and area at hand. Once we have made a definition of a certain physical quantity, we may explore its characteristics and deduce its exact relation to other physical quantities. There is no question how pressure is related to force, but there is a certain haziness about what a force is.

The situation is analogous to the task of writing a dictionary. How can we write out the meaning of the first word? By the very nature of a dictionary we must use other words in defining the first word. The dilemma is that those words have not yet been defined. The second word is not much easier than the first. However, after the meanings of a few key words are established, the task becomes much simpler. Word definitions can then be formulated exactly, and subtle distinctions between ideas may be made. As we use the language and see a word in different contexts, we gain a greater appreciation of its essence. At this stage, the problem of which words were the very first to be defined is no longer important. The important thing is the role the word plays in our language and the subtle differences between it and similar words.

Stretching the analogy between a continuum and a dictionary a little bit further, we can draw a correspondence between the molecules of a continuum and the letters of a word. The idea conveyed by the word is essentially independent of our choice of the language and letters to form the word. In the same way, the continuum concepts are essentially independent of the microscopic particles. The microscopic particles are necessary but unimportant.

The mathematical rules by which we predict and explain phenomena in continuum mechanics are called *laws.* Some *restricted laws* apply only to special situations. The equation of state for a perfect gas and Hooke's law of elasticity are examples of this type of law. We shall distinguish laws that apply to all substances by calling them *basic laws*. There are many forms for the basic laws of continuum mechanics, but in the last analysis they may all be related to four laws: the three independent conservation principles for mass, momentum, and energy, plus a fundamental equation of thermodynamics. These suffice when the continuum contains a ‘‘simple substance’’ and gravitational, electrical, magnetic, and chemical effects are excluded. In fluid mechanics, however, we frequently want to include the gravity force. In such cases, a basic law for this force should be added to the list. Problems dealing with electrical, magnetic, and chemical effects would require correspondingly more basic laws.

Newton's second law is familiar to all students from their earliest course in physics:

$$F = Ma = M \frac{d^2x}{dt^2}$$

This law relates the ideas of force, mass, and acceleration. It should not be considered as a definition of force. It is our responsibility to identify and formulate all the different types of forces. In this law we usually consider distance, time, mass, and force to be fundamental concepts, and acceleration to be a defined quantity. Newton's law tells us that these quantities cannot take on independent values but must always maintain a certain relationship.

Which concepts are taken to be fundamental and which are defined is a matter of tradition and convenience. For example, we usually take length and time as fundamental and consider velocity to be defined by the time derivative of the position. On the other hand, we might take velocity and time as fundamental concepts and then consider distance to be defined by the integral

$$x = \int_0^t v \, dt$$

This would be unusual and awkward; however, it is conceptually as valid as defining velocity from the ideas of distance and time.

In this book we do not emphasize the philosophical aspects and the logical construction of continuum mechanics. This task belongs to a branch of mathematics called *rational mechanics.* Our efforts will fall short of its standards of rigor. Our purpose is to understand the physics and to quantify (if possible) practical situations in fluid mechanics. We do not intend to sacrifice accuracy, but we cannot afford the luxury of a highly philosophical approach.

1.3 SPACE AND TIME

The natural independent variables of continuum mechanics are three-dimensional space and time. We assume all the concepts and results of Euclidean geometry: length, area, parallel lines, and so on. Euclidean space is the setting for the progress of events as time proceeds independently. With these assumptions about the nature of time and space, we have ruled out relativistic effects and thereby limited the scope of our subject.

To measure space and other physical quantities, it is necessary to introduce a coordinate system. This brings up the question of how a quantity such as energy might depend on the coordinate system in which it is calculated. One of the major facts of physics is the existence of special coordinate systems called *inertial frames*. The laws of physics have exactly the same mathematical form when quantities are measured from any inertial coordinate system. The magnitude of the momentum or the magnitude of the energy will be different when measured in different coordinates; however, the physical laws deal only with changes in these quantities. Furthermore, the laws have a structure such that the same change will be observed from any inertial system. All inertial coordinate systems are related by *Galilean transformations,* in which one coordinate system is in uniform translational motion with respect to the other. Furthermore, any coordinate system that is in uniform translational motion with respect to an inertial system is also an inertial system. We sometimes say that a coordinate system that is fixed with respect to the "distant stars" is an *inertial coordinate system.* Of course, we cannot be too precise about this concept, or we run into relativity. The laboratory is not an inertial coordinate system because of Earth's rotation and acceleration. Nevertheless, many events occur in such a short time that Earth's rotation may be neglected and laboratory coordinates may be taken as an inertial system.

As mentioned above, all the facts of Euclidean geometry are assumed to apply to space, while time is a parameterlike independent variable that proceeds in a forward direction. At any instant in time we may define a *control volume,* or control region, as any closed region in space. It is our invention. The boundary is called a *control surface,* and we prescribe its motion in any manner we choose. The purpose of a control region is to focus our attention on physical events at the boundary and within the region. The ideas of control surface and control volume are generalizations of the *Euler cut* that were refined and promoted in the engineering literature by Prandtl. *Control surface* is a literal translational of the German *kontrollflache*. In German, ‘‘control’’ has the meaning of accounting; hence a ‘‘control surface’’ is a place where one must keep track of physical events (Vincenti, 1982).

It will be useful to define four types of regions that depend on how the surface of the region moves with time (Fig. 1.3). A *fixed region* (FR) is one where the control surface does not move at all but is fixed in space. We might imagine a fixed region as enclosing a compressor as shown in Fig. 1.3. The region surface cuts through the inlet and outlet pipes, and fluid flows across these surfaces into or out of the region. At another place the control surface must cut through the shaft that drives the compressor. Here we imagine that the control surface is stationary even though the material that composes the shaft is moving tangentially to the surface. When we use a fixed region, we must allow material to either cross the surface or slide along it.

The second type of region is called a *material region* (MR), because the surface moves with the local velocity of the material. Consider a bubble of gas that is rising through a liquid. As the bubble rises, it expands in size and the gas inside exhibits circulatory motion. A material region that just encloses the gas has a local velocity composed of three parts: the rising velocity of the bubble, the expansion velocity of the bubble, and the gas velocity at the interface due to the internal circulation (a sliding velocity tangent to the surface). If we omit the velocity of the internal circulation, the region will no longer strictly fit the definition of a material region. The surface will still always enclose the same material, but the surface will not have the local material velocity.

The third type of region is one where the surface velocity is the same at each location, but varies with time $w_i = W_i(t)$. For example, consider a region surrounding a

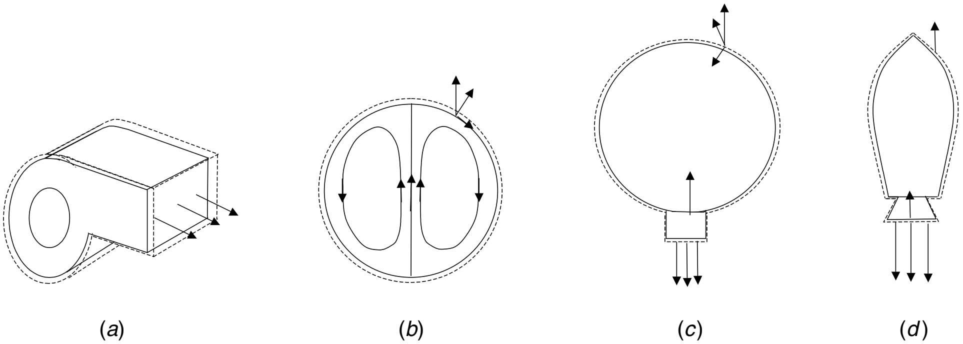
(*a*) (*b*) (*c*) (*d*)

Figure 1.3 Control regions: (*a*) fixed region around a centrifugal blower; (*b*) material region around a rising bubble; (*c*) arbitrary region around a moving and collapsing balloon; (*d*) constant volume region around a rocket.

rocket. Material is ejected from the rocket nozzle and the region moves; however, the volume of region is constant. This is called a *volume region* (VR).

Any control region that does not fall into the first three categories is called an *arbitrary region* (AR). An example of an arbitrary region is given by a toy balloon that has been turned loose to move freely through the air. Choose the surface of the region to coincide with the balloon everywhere except at the mouth, where air is escaping. At this point the surface cuts across the plane of the exit and the air crosses the surface of the region. Such a region is very useful for an analysis; however, it must be classed as an arbitrary region.

In the examples above, the regions have been of finite size and have obviously been chosen in order to perform an engineering analysis. Control regions are also very useful for conceptual and theoretical purposes. When they are used for these purposes, one often considers a sequence of regions that become smaller and smaller. An example of this type of reasoning is presented in Section 1.4.

1.4 DENSITY, VELOCITY, AND INTERNAL ENERGY

Density is the mass per unit volume of a substance and is one of our fundamental concepts. We consider that the continuum has a density at every point in space. The following thought experiment is a popular way to illustrate the concept. Consider a specific point in space, and choose a fixed control region that encloses the point. Imagine that we freeze the molecules and then count the number of them within the region. With this information we form the ratio of the mass of the material to the volume of the region, that is, the average density of the control region. Let L be a measure of the size of the control region: L might be the distance across the central point to a certain position on the control surface. The experiment is then repeated with a smaller but geometrically similar control region. Each time the results are plotted as in Fig. 1.4. A logarithmic scale for L is used because L ranges over many orders of magnitude. When L is very large, say a mile, the measurement represents an average that might have little to do with the local fluid density. As L becomes small, the experiment produces a consistent number for M/V even as L ranges over several orders of magnitude. This number is the density at point P. Finally, the control region becomes so small that L approaches the distance between molecules. With only a few molecules within the volume, the ratio M/V jumps as the control region shrinks past a molecule. To continue the process produces even more scatter in M/V.

If we begin the process again with a different-shaped control region, we find a different curve for very large values of L, but as the length becomes a millimeter or so, the same plateau in M/V may occur. If so, it will be valid to take a continuum viewpoint and define a density at point P. Mathematically, the definition is expressed by

$$\rho = \lim_{L \to 0} \frac{\Sigma\, m_i}{V} \tag{1.4.1}$$

where the summation occurs over all particles within the region. The limit process $L \to 0$ is understood to go toward zero but never to reach a molecular scale.

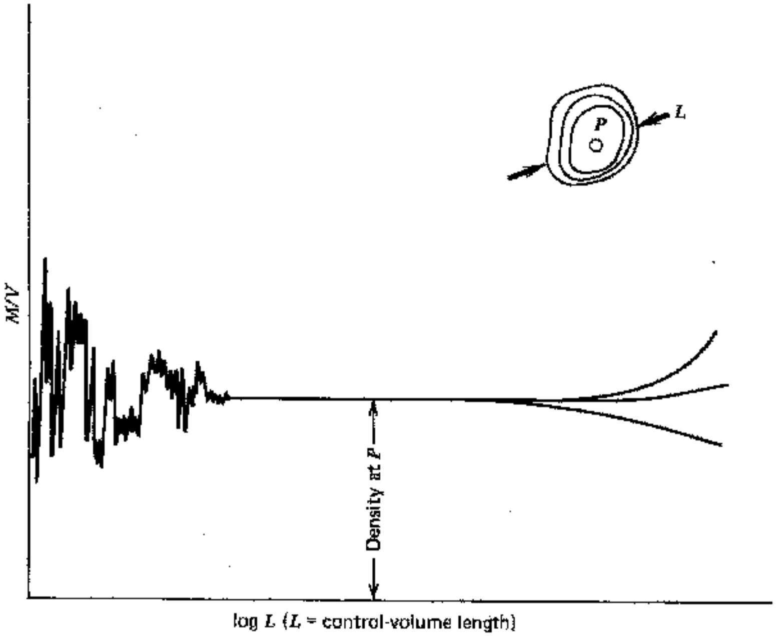

Figure 1.4 Thought experiment to define density.

In a flow where the number of molecules changes rapidly over a distance comparable to intermolecular distances, the continuum assumption will be suspect. To illustrate this, consider the problem of computing the internal structure of a shock wave. The thickness of a shock wave is only a few times the *mean free path* (the average distance a molecule travels before colliding with another molecule). Over this distance the density may increase by a factor of 2. Can the density profile be computed using continuum assumptions? This problem is a borderline case, and it turns out that the continuum calculation gives reasonable answers. In ordinary engineering situations, density gradients occur over distances on the order of centimeters, and the continuum assumption is unquestionably valid.

We can gain a better insight into the continuum assumption by reviewing some of the molecular properties of air. Air at atmospheric conditions contains 3×10^{19} molecules in 1 cm^3. Numbers like this are hard to comprehend. How long would it take to count the molecules in 1 mm^3 of air? Suppose that a superfast electronic counter can count at the rate of 1 million molecules per second. A simple calculation shows that for a cubic millimeter of air we would have to let the counter run for

$$3 \times 10^{10}\ \text{s} = 8.3 \times 10^{6}\ \text{h} = 3.5 \times 10^{5}\ \text{days} = 1000\ \text{yr}$$

A cubic millimeter was chosen for this example because the time to count for a cubic centimeter would also be hard to comprehend.

A few other facts about air at standard conditions are worth noting. The mean free path is about $8 \times 10^{-8}\ \text{m} \approx 0.1\ \mu\text{m}$, and this is about 25 times the distance between molecules (3×10^{-9} m). In other words, a molecule passes about 25 molecules before it collides with another molecule. The number of molecules in a cube that is one mean free path on each side is 15,000, still a large number. It can be predicted by kinetic theory that the density of this volume will fluctuate in time by only 0.8% root mean square (rms). If we reduce the side of our volume to 0.1 mean free path, we now have only 15 molecules and the density fluctuation will be 25%. These numbers show that the mean free path also offers a convenient dividing line between the continuum and microscopic worlds. Another interesting fact about simple gases (as standard conditions) is that the distance between molecules is about 10 times the size of a simple molecule. (The nucleus of an atom is about 1/100,000 of the size of the atom.)

In liquids, the size required for the continuum hypothesis to be valid is somewhat smaller than for gases; however, the mean free path concept is not valid for liquids. The distances between molecules and the sizes of the molecules are roughly the same in liquids, so a smaller volume is required for a reasonable formulation of the density.

Velocity is another fundamental continuum concept that is based on the volume-limiting process. There are actually two ways to define fluid velocity: the molar-averaged velocity and the mass-averaged velocity. They may have different values if the fluid is a chemical mixture. The *mass-averaged velocity* is formed by the vector sum of all particle velocities with the mass used as a weighing factor:

$$\mathbf{v} = \lim_{L\to 0} \frac{\Sigma\, m_i \mathbf{v}_i}{\Sigma\, m_i} \tag{1.4.2}$$

The mass-averaged velocity is natural for problems of fluid flow where the momentum equation is important. The product $\rho\mathbf{v}$ gives the momentum per unit volume averaged over all particles. If the fluid is a chemical mixture, the average motion of one chemical species may not be in the direction of $\mathbf{v}$. We define the *molar-averaged velocity* of chemical species k by summing only over molecules of that species:

$$\mathbf{V}^{(k)} = \lim_{L\to 0} \frac{\Sigma\, \mathbf{v}_i^{(k)}}{n^{(k)}}$$

In this expression $n^{(k)}$ is the number of molecules of species k within the volume. The molar-averaged velocity of the entire mixture is the vector sum over all molecules divided by the total number of molecules:

$$\mathbf{V} = \lim_{L\to 0} \frac{\Sigma\, \mathbf{v}_i}{n}$$

Only if the fluid has a uniform chemical composition are the two velocities equal, $\mathbf{V} = \mathbf{v}$. In situations where there is mass diffusion or there are chemical reactions, it is sometimes more convenient to employ a molar-averaged velocity. Since we deal only with fluids of uniform composition in this book, the mass-averaged velocity will always be used.

The term *fluid particle* has at least two meanings in common usage. The first is a moving-point concept. Here we envision a point that moves with the local fluid velocity at each place in space. A line traced through the flow field by this method is called a *particle path*. We say that the point that traces the path is a fluid particle, or *material point*. For some purposes—for instance, to talk about the expansion of a fluid—it is necessary to consider a small chunk of the fluid. This second meaning for the term *fluid* *particle* is made precise by considering a small MR and allowing the size of the region to tend to zero. Which of the two meanings is intended is usually obvious from the context. Note that because of molecular diffusion, a fluid particle does not always consist of the same molecules. As a particle moves through the flow, it gains and loses molecules because of random molecular motions.

The third fundamental concept that we cover in this section is *internal energy*. The particle velocity defined above is the average velocity of the molecule, the velocity we observe from our macroscopic world. As far as the macroscopic world is concerned, the kinetic energy of this bulk motion is

$$\text{bulk-motion kinetic energy per unit mass} = \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} \tag{1.4.3}$$

However, this will not account completely for all the energy of the molecular translational motions. The true total kinetic energy sums the molecular velocities:

$$\text{total kinetic energy of translation} = \lim_{L\to 0}\frac{\Sigma\, m_i\frac{1}{2}\mathbf{v}_i\cdot\mathbf{v}_i}{\Sigma\, m_i} \tag{1.4.4}$$

The difference between Eqs. 1.4.4 and 1.4.3, the energy that is hidden from direct macroscopic observation, is the thermodynamic internal energy due to random translational motion. We can formulate an expression for this internal energy by introducing the random molecular velocities (denoted by a prime). To do this we subtract from each molecular velocity $\mathbf{v}_i$ the average fluid velocity $\mathbf{v}$:

$$\mathbf{v}_i' = \mathbf{v}_i - \mathbf{v}$$

In terms of $\mathbf{v}_i'$ the translational internal energy is expressed as

$$\text{internal energy from random translation velocities} = \frac{\Sigma\, m_i\frac{1}{2}\mathbf{v}_i'\cdot\mathbf{v}_i'}{\Sigma\, m_i} \tag{1.4.5}$$

Thus, the total molecular kinetic energy is split into two parts: a macroscopic part, which is observable as bulk motion, and a microscopic part, which is part of the internal energy. There are many other forms of microscopic energy that are hidden from our continuum world: molecular vibration, rotation, potential energies of molecular configurations, potentials of molecules close to each other, and so on. All of these forms of microscopic energy are accounted for in the thermodynamic internal energy.

The three properties discussed above—density, velocity, and internal energy—are basic and can be defined even when thermodynamic equilibrium does not exist.

1.5 INTERFACE BETWEEN PHASES

The interface between two phases offers some special difficulties in continuum mechanics. The most obvious problem is that the thickness of the interface is small compared to intermolecular distances. Consider for a moment a gas in contact with a liquid (Fig. 1.5). In the liquid the molecules are closely packed and exert strong attractive forces on each other. For a molecule that is deep within the liquid, these forces come from all

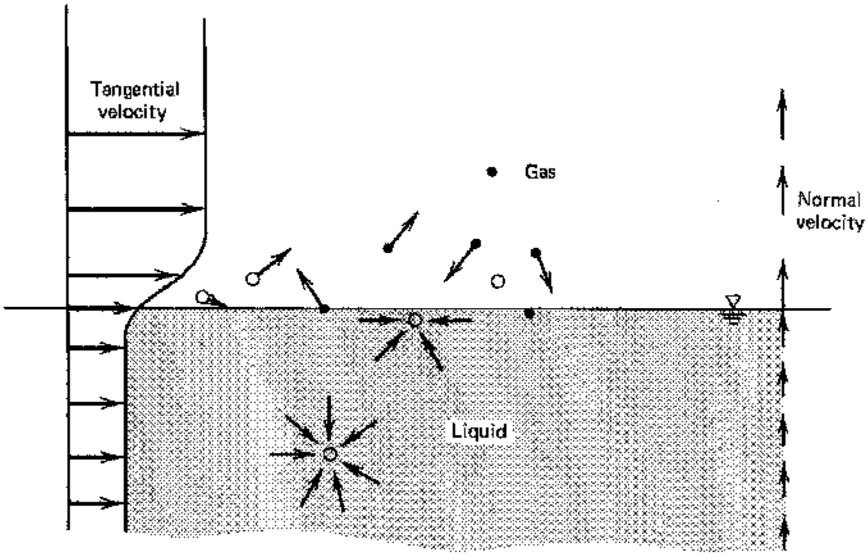

Figure 1.5 Liquid–gas interface. The tangential velocity is continuous, but the normal velocity may have a discontinuity.

directions. As we approach the surface the situation changes, because the neighboring liquid molecules are only on one side. The other side is occupied by a gas. Gas molecules are constantly bombarding the surface, becoming mingled with liquid, and sometimes being absorbed. If we idealize the interface as a surface with zero thickness, we must in general assign to it physical properties; it is a two-dimensional world. Each physical property then has a two-dimensional analogue in the interface; corresponding to density, for example, we have the mass per unit area (the absorbed mass). Energy per unit volume has a surface analogue in the energy per unit area. This includes not only the energy associated with the motions of interface molecules, but also the energy of the special configuration of molecules at the interface.

The two-dimensional interface world is much more complicated than our world. The geometry is non-Euclidean. Conservation laws are complicated because mass, momentum, and energy may change through interactions with the three-dimensional world. Deviations from theory are usual, because a few foreign molecules contaminating the surface can have a great influence. We shall not go into the thermodynamics and fluid mechanics of interfaces; the interested reader should consult Edwards et al. (1991).

Interfacial physics and chemistry are subjects in themselves. To make progress in our main interest, fluid mechanics, we shall have to assume a very simple model of the interface. In a great many practical applications, this model will suffice. We assume that an interface is a surface of zero thickness, which contains no mass, momentum, or energy. Across the interface the density is allowed to jump discontinuously. On the other hand, the temperature and tangential velocity are assumed to be continuous. This assumption is justified because molecules from both sides are constantly colliding and equilibrating within the surface layer. These ideas are illustrated in Fig. 1.5, where a gas flows over a liquid. Molecules leaving the surface and moving back into either fluid have the same tangential velocity. In other words, the velocity of fluid within the interface has only one value. This assumption, called the *no-slip condition,* is not an obvious fact. Indeed, it was once the subject of a long debate [see Goldstein (1965, p. 676) for a brief history]. The debate concerned surface tension and the fact that some liquids are attracted to certain solids whereas others are not. It turns out that wettability is not important and that the no-slip condition applies in general to all substances.

The velocity perpendicular to the interface is discontinuous whenever mass is transferred across the surface. This situation is illustrated by considering a vaporizing liquid. There is a continuous flow of vapor away from the surface with a mass flux $\rho v|_{\text{vap}}$. This must be balanced by an equal flux into the surface from the liquid side of $\rho v|_{\text{liq}}$. Since the two densities are quite different, the velocities must also be different. The discontinuity in normal velocity and the continuity of tangential velocity apply even if the surface itself is in motion.

1.6 CONCLUSIONS

In this first chapter we have attempted to define the scope and nature of fluid mechanics. The three fundamental continuum concepts of density, velocity, and energy were introduced. We shall introduce many more concepts as they are needed in later chapters. In all of our work we shall limit ourselves to exclude magnetic, electrical, and chemical effects. The fluids in the problems that we study will always be assumed to be homogeneous, simple, compressible substances. Even with all of these restrictions, there will be plenty of material to cover.

Perhaps the most fundamental restriction in our subject is the continuum assumption. The characteristic size of the flow must be a continuum scale length. There is a famous physical phenomenon called *Brownian motion,* which illustrates this restriction very nicely. The botanist Robert Brown, while observing life-forms in a water droplet by means of a microscope, noticed that some pollen particles in the water had a jittery motion. The motion was actually a random vibration where the velocity was abruptly changing direction at a high frequency. It gave the particles a fuzzy appearance. The pollen particles were a few micrometers in size, maybe 100 times the intermolecular spacing in water. Later, the reason for this random meandering of the particles was correctly ascribed to unequal and fluctuating molecular forces. The particle was not large enough that molecular bombardment on one side was always exactly counterbalanced on the other side.

Calculations of the motion were finally made by Einstein and Smoluchowski. They used an ad hoc mixture of molecular and continuum ideas. The random driving force was taken from molecular concepts, and a continuum viscous retarding force was assumed. Situations of this type, in the gray area between continuum mechanics and kinetic theory, have grown into what is now called *colloidal science*. It marks a boundary of continuum fluid mechanics where body sizes become comparable with molecular sizes (see Fig. 1.6).

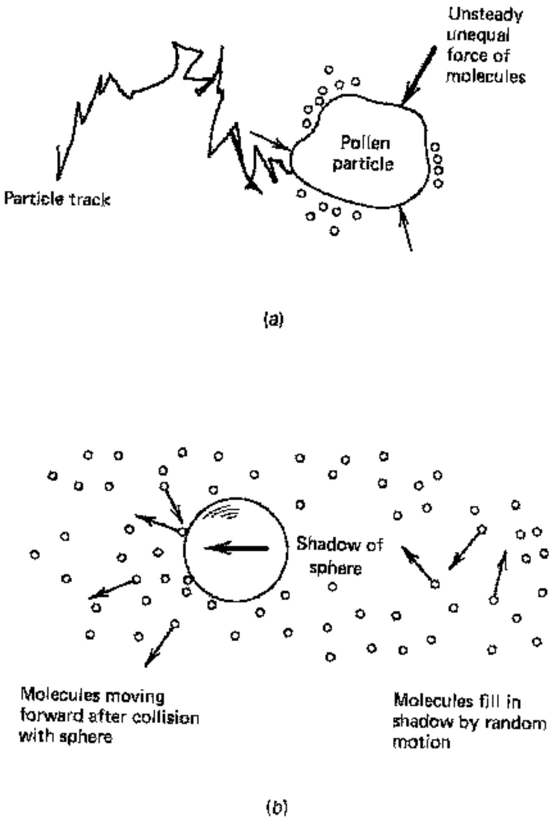

Figure 1.6 Failures of the continuum assumption: (*a*) body size compares with molecular dimensions (very small particle in a liquid); (*b*) body size compares with distance between molecules (sphere moving through a rarefied gas).

Another boundary for the continuum assumption occurs for finite-size bodies in gas flow. As the density is reduced and vacuum conditions approached, either at high altitudes or in vacuum systems, the distance between molecules may become several centimeters. Now the body size may be comparable to the mean free path. Consider a sphere shooting through a rarefied gas. Molecules that collide with the front of the sphere are sent forward several sphere diameters before they interact with other molecules and influence the gas motion. Behind the sphere there is a partial vacuum swept out by its motion. Several diameters back, the random molecular velocities fill this region in once more. This flow field is much different than the one we would find if the mean free path were very small compared to the diameter. The extension of fluid mechanics into this region is called *rarefied-gas dynamics*.

These illustrations show two ways in which the continuum assumption may fail: The characteristic length in the flow (the body diameter) may be so small that it compares with the molecular dimensions, or the mean free path of the fluid may be comparable with the characteristic length of the body.

PROBLEMS

1.1 (B) Consider an unsteady one-dimensional flow where the density and velocity depend on x and t. A Galilean transformation into a new set of variables x', t' is given by the equations $x = x' + Vt'$, $t = t'$, where V is a constant velocity. For the moment, let $f = f(x, t)$ stand for a function that we wish to express in the x'–t' coordinate system. By careful use of the chain rules of calculus, find expressions for $\partial f/\partial t'$ and $\partial f/\partial x'$. Next, consider the *substantial derivatives* of ρ and v, which are

$$\frac{\partial \rho'}{\partial t'} + v' \frac{\partial \rho'}{\partial x'}, \qquad \frac{\partial v'}{\partial t'} + v' \frac{\partial v'}{\partial x'}$$

Show that the substantial derivatives above have exactly the same mathematical form when transformed into the x–t coordinate system (note that $\rho' = \rho$ and $v' = v - V$).

1.2 (A) A droplet of liquid is moving through a gas. It evaporates uniformly, does not deform, and has no internal circulation. A control region coinciding with the liquid is what type of region?

1.3 (A) A droplet of liquid is moving through a gas. It does not evaporate or deform, but it does have an internal (and surface) circulation. Describe the velocity of a material region whose surface encloses the droplet.

1.4 (B) A material region was defined as one where the surface velocity $\mathbf{w}$ is everywhere equal to the fluid velocity $\mathbf{v}$. Such a region always contains the same fluid. Can you define $\mathbf{w}$ in a less restrictive way and still have a region that always contains the same material?

1.5 (C) Prove that the average of the random molecular velocities $\mathbf{v}_i'$ (see Eqs. 1.4.2 and 1.4.5) is zero, that is,

$$\lim_{L \to 0} \sum m_i \mathbf{v}_i' = 0$$

1.6 (B) Using Problem 1.5, prove that the total kinetic energy (per unit mass) of translational molecular motion may be split into two parts as follows:

$$\sum m_i \frac{1}{2} \mathbf{v}_i \cdot \mathbf{v}_i = \sum m_i \frac{1}{2} \mathbf{v}_i' \cdot \mathbf{v}_i' + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \sum m_i$$

1.7 (C) We have found that the continuum assumption cannot be applied to events with a characteristic length that is on the order of molecular scales. Is there a characteristic time interval for which the continuum assumption is not valid?

1.8 (B) The momentum $\mathbf{p}_j$ of a molecule is equal to the product $m_i\mathbf{v}_i$. From the definitions of ρ and $\mathbf{v}$, show that the product $\rho\mathbf{v}$ is the total momentum of all molecules per unit volume.