A

abacus

A counting frame that started out, several thousand years ago, as rows of pebbles in the desert sands of the Middle East. The word appears to come from the Hebrew *âbâq* (dust) or the Phoenician *abak* (sand) via the Greek *abax*, which refers to a small tray covered with sand to hold the pebbles steady. The familiar frame-supporting rods or wires, threaded with smoothly running beads, gradually emerged in a variety of places and mathematical forms.

In Europe, there was a strange state of affairs for more than 1,500 years. The Greeks and the Romans, and then the medieval Europeans, calculated on devices with a **place-value system** in which **zero** was represented by an empty line or wire. Yet the written notations didn't have a symbol for zero until it was introduced in Europe in 1202 by **Fibonacci**, via the Arabs and the Hindus.

abacus A special form of the Chinese abacus (c. 1958) consisting of two abaci stacked one on top of the other. *Luis Fernandes*

The Chinese *suan pan* differs from the European abacus in that the board is split into two decks, with two beads on each rod in the upper deck and five beads, representing the digits 0 through 4, on each rod in the bottom. When all five beads on a rod in the lower deck are moved up, they're reset to the original position, and one bead in the top deck is moved down as a carry. When both beads in the upper deck are moved down, they're reset and a bead on the adjacent rod on the left is moved up as a carry. The result of the computation is read off from the beads clustered near the separator beam between the upper and lower decks. In a sense, the abacus works as a 5-2-5-2-5-2 . . . –based number system in which carries and shifts are similar to those in the decimal system. Since each rod represents a digit in a decimal number, the capacity of the abacus is limited only by the number of rods on the abacus. When a user runs out of rods, she simply adds another abacus to the left of the row.

The Japanese *soroban* does away with the dual representations of fives and tens by having only four counters in the lower portion, known as "earth," and only one counter in the upper portion, known as "heaven." The world's largest abacus is in the Science Museum in London and measures 4.7 meters by 2.2 meters.

Abbott, Edwin Abbott (1838–1926)

An English clergyman and author who wrote several theological works and a biography (1885) of Francis Bacon, but is best known for his standard *Shakespearian Grammar* (1870) and the pseudonymously written ***Flatland: A Romance of Many Dimensions*** (by A Square, 1884).[1]

ABC conjecture

A remarkable **conjecture**, first put forward in 1980 by Joseph Oesterle of the University of Paris and David Masser of the Mathematics Institute of the University of Basel in Switzerland, that is now considered one of the most important unsolved problems in **number theory**. If it were proved correct, the proofs of many other famous conjectures and theorems would follow immediately–in some cases in just a few lines. The vastly complex current proof of **Fermat's last theorem**, for example, would reduce to less than a page of mathematical reasoning. The ABC conjecture is disarmingly simple compared to most of the deep questions in number theory and, moreover, turns out to be equivalent to all the main problems that involve **Diophantine equation**s (equations with integer coefficients and integer solutions).

Only a couple of concepts need to be understood to grasp the ABC conjecture. A *square-free number* is an integer that isn't divisible by the square of any number. For example, 15 and 17 are square-free, but 16 (divisible by 4^2) and 18 (divisible by 3^2) are not. The *square-free part* of an integer n, denoted sqp(n), is the largest square-free number that can be formed by multiplying the prime factors of n. For $n = 15$, the prime factors are 5 and 3, and $3 \times 5 = 15$, a square-free number, so that sqp(15) = 15. On the other hand, for $n = 16$, the prime factors are all 2, which means that sqp(16) = 2. In general, if n is square-free, the square-free part of n is just n; otherwise, sqp(n) represents what is left over after all the factors that create a square have been eliminated. In other words, sqp(n) is the product of the distinct **prime number**s that divide n. For example, sqp(9) = sqp(3×3) = 3 and sqp(1,400) = sqp($2 \times 2 \times 2 \times 5 \times 5 \times 7$) = $2 \times 5 \times 7 = 70$.

The ABC conjecture deals with pairs of numbers that have no common factors. Suppose A and B are two such numbers that add to give C. For example, if $A = 3$ and $B = 7$, then $C = 3 + 7 = 10$. Now, consider the square-free part of the product $A \times B \times C$: sqp(ABC) = sqp($3 \times 7 \times 10$) = 210. For most values of A and B, sqp(ABC) > C, as in the prior example. In other words, sqp(ABC)/C > 1. Occasionally, however, this isn't true. For instance, if $A = 1$ and $B = 8$, then $C = 1 + 8 = 9$, sqp(ABC) = sqp($1 \times 8 \times 9$) = sqp($1 \times 2 \times 2 \times 2 \times 3 \times 3$) = $1 \times 2 \times 3 = 6$, and sqp(ABC)/C = $6/9$ = $2/3$. Similarly, if $A = 3$ and $B = 125$, the ratio is $15/64$.

David Masser proved that the ratio sqp(ABC)/C can get arbitrarily small. In other words, given any number greater than zero, no matter how small, it's possible to find integers A and B for which sqp(ABC)/C is smaller than this number. In contrast, the ABC conjecture says that $[\text{sqp}(ABC)]^n/C$ reaches a minimum value if n is any number greater than 1—even a number such as 1.0000000001, which is only barely larger than 1. The tiny change in the expression results in a huge difference in its mathematical behavior. The ABC conjecture in effect translates an infinite number of Diophantine equations (including the equation of Fermat's last theorem) into a single mathematical statement.[144]

Abel, Niels Henrik (1802–1829)

> *The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes.*

A Norwegian mathematician who, independently of his contemporary Évariste **Galois**, pioneered **group** theory and proved that there are no algebraic solutions of the general **quintic** equation. Both Abel and Galois died tragically young–Abel of tuberculosis, Galois in a sword fight.

While a student in Christiania (now Oslo), Abel thought he had discovered how to solve the general quintic algebraically, but soon corrected himself in a famous pamphlet published in 1824. In this early paper, Abel showed the impossibility of solving the general quintic by means of radicals, thus laying to rest a problem that had perplexed mathematicians since the mid-sixteenth century. Abel, chronically poor throughout his life, was granted a small stipend by the Norwegian government that allowed him to go on a mathematical tour of Germany and France. In Berlin he met Leopold Crelle (1780–1856) and in 1826 helped him found the first journal in the world devoted to mathematical research. Its first three volumes contained 22 of Abel's papers, ensuring lasting fame for both Abel and Crelle. Abel revolutionized the important area of elliptic integrals with his theory of **elliptic functions**, contributed to the theory of **infinite series**, and founded the theory of commutative groups, known today as **Abelian groups**. Yet his work was never properly appreciated during his life, and, impoverished and ill, he returned to Norway unable to obtain a teaching position. Two days after his death, a delayed letter was delivered in which Abel was offered a post at the University of Berlin.

Abelian group

A **group** that is **commutative**, that is, in which the result of multiplying one member of the group by another is independent of the order of multiplication. Abelian groups, named after Niels **Abel**, are of central importance in modern mathematics, most notably in **algebraic topology**. Examples of Abelian groups include the **real numbers** (with addition), the nonzero real numbers (with multiplication), and all cyclic groups, such as the **integer**s (with addition).

abracadabra

A word famously used by magicians but which started out as a cabalistic or mystical charm for curing various ailments, including toothache and fever. It was first mentioned in a poem called "Praecepta de Medicina" by the Gnostic physician Quintus Severus Sammonicus in the second century A.D. Sammonicus instructed that the letters be written on parchment in the form of a triangle:

A B R A C A D A B R A
A B R A C A D A B R
A B R A C A D A B
A B R A C A D A
A B R A C A D
A B R A C A
A B R A C
A B R A
A B R
A B
A

This was to be folded into the shape of a cross, worn for nine days suspended from the neck, and, before sunrise, cast behind the patient into a stream running eastward. It was also a popular remedy in the Middle Ages. During the Great Plague, around 1665, large numbers of these amulets were worn as safeguards against infection. The origin of the word itself is uncertain. One theory is that it is based on Abrasax, the name of an Egyptian deity.

PUZZLE

A well-known puzzle, proposed by George Polya (1887–1985), asks how many different ways there are to spell abracadabra in this diamond-shaped arrangement of letters:

A
B B
R R R
A A A A
C C C C C
A A A A A A
D D D D D
A A A A
B B B
R R
A

Solutions begin on page 369.

abscissa

The x-coordinate, or horizontal distance from the y-axis, in a system of **Cartesian coordinates**. Compare with **ordinate**.

absolute

Not limited by exceptions or conditions. The term is used in many different ways in mathematics, physics, philosophy, and everyday speech. Absolute space and absolute time, which, in Newton's universe, form a unique, immutable frame of reference, blend and become deformable in the **space-time** of Einstein. See also **absolute zero**. In some philosophies, the absolute stands behind the reality we see–independent, transcendent, unconditional, and all-encompassing. The American philosopher Josiah Royce (1855–1916) took the absolute to be a spiritual entity whose self-consciousness is imperfectly reflected in the totality of human thought. Mathematics, too, reaches beyond imagination with its absolute **infinity**. See also **absolute value**.

absolute value

The value of a number without regard to its sign. The absolute value, or *modulus,* of a **real number**, r, is the distance of the number from zero measured along the real **number line**, and is denoted $|r|$. Being a distance, it can't be negative; so, for example, $|3| = |-3| = 3$. The same idea applies to the absolute value of a **complex number** $a + ib$, except that, in this case, the complex number is represented by a point on an **Argand diagram**. The absolute value, $|a + ib|$, is the length of the line from the origin to the given point, and is equal to $\sqrt{(a^2 + b^2)}$.

absolute zero

The lowest possible temperature of a substance, equal to 0 Kelvin (K), –273.15°C, or –459.67°F. In classical physics, it is the temperature at which all molecular motion ceases. However, in the "real" world of **quantum mechanics** it isn't possible to stop all motion of the particles making up a substance as this would violate the Heisenberg uncertainty principle. So, at 0 K, particles would still vibrate with a certain small but nonzero energy known as the *zero-point energy.* Temperatures within a few billionths of a degree of absolute zero have been achieved in the laboratory. At such low temperatures, substances have been seen to enter a peculiar state, known as the *Bose-Einstein condensate,* in which their quantum wave functions merge and particles lose their individual identities. Although it is possible to approach ever closer to absolute zero, the third law of thermodynamics asserts that it's impossible to ever attain it. In a deep sense, absolute zero lies at the asymptotic limit of low energy just as the speed of light lies, for particles with mass, at the asymptotic limit of high energy. In both cases, energy of motion (kinetic energy) is the key quantity involved. At the high energy end, as the average speed of the particles of a substance approaches the speed of light, the temperature rises without limit, heading for an unreachable ∞ K.

abstract algebra

To a mathematician, real life is a special case.

–Anonymous

Algebra that is not confined to familiar number systems, such as the **real numbers**, but seeks to solve equations that may involve many other kinds of systems. One of its aims, in fact, is to ask: What other number systems are there? The term *abstract* refers to the perspective taken on the subject, which is very different from that of high school algebra. Rather than looking for the solutions to a particular problem, abstract algebra is interested in such questions as: When does a solution exist? If a solution does exist, is it unique? What general properties does a solution possess? Among the structures it deals with are **group**s, **ring**s, and **field**s. Historically, examples of such structures often arose first in some other field of mathematics, were specified rigorously (axiomatically), and were then studied in their own right in abstract algebra.

Abu'l Wafa (A.D. 940–998)

A Persian mathematician and astronomer who was the first to describe geometrical constructions (see **constructible**) possible only with a straightedge and a fixed compass, later dubbed a "rusty compass," that never alters its radius. He pioneered the use of the tangent function (see **trigonometric function**), apparently discovered the secant and cosecant functions, and compiled tables of sines and tangents at 15′ intervals–work done as part of an investigation into the orbit of the Moon.

abundant number

A number that is smaller than the sum of its **aliquot part**s (proper divisors). Twelve is the smallest abundant number; the sum of its aliquot parts is $1 + 2 + 3 + 4 + 6 = 16$, followed by 18, 20, 24, and 30. A *weird number* is an abundant number that is not *semiperfect;* in other words, n is weird if the sum of its divisors is greater than n, but n is not equal to the sum of any subset of its divisors. The first few weird numbers are 70, 836, 4,030, 5,830, and 7,192. It isn't known if there are any odd weird numbers. A *deficient number* is one that is greater than the sum of its aliquot parts. The first few deficient numbers are 1, 2, 3, 4, 5, 8, and 9. Any divisor of a deficient (or perfect) number is deficient. A number that is not abundant or deficient is known as a **perfect number**.

Achilles and the Tortoise paradox

See **Zeno's paradoxes**.

Ackermann function

One of the most important **function**s in computer science. Its most outstanding property is that it grows astonishingly fast. In fact, it gives rise to **large numbers** so quickly that these numbers, called *Ackermann numbers,* are written in a special way known as **Knuth's up-arrow notation**. The Ackermann function was discovered and studied by Wilhelm Ackermann (1896–1962) in 1928. Ackermann worked as a high school teacher from 1927 to 1961 but was also a student of the great mathematician David **Hilbert** in Göttingen and, from 1953, served as an honorary professor in the university there. Together with Hilbert he published the first modern textbook on mathematical logic. The function he discovered, and that now bears his name, is the simplest example of a well-defined and total function that is also computable but not primitive recursive (PR). "Well-defined and total" means that the function is internally consistent and doesn't break any of the rules laid down to define it. "Computable" means that it can, in principle, be evaluated for all possible input values of its variables. "Primitive recursive" means that it can be computed using only *for loops*–repeated application of a single operation a predetermined number of times. The recursion, or feedback loop, in the Ackermann function overruns the capacity of any *for loop* because the number of loop repetitions isn't known in advance. Instead, this number is itself part of the computation, and grows as the calculation proceeds. The Ackermann function can only be calculated using a *while loop,* which keeps repeating an action until an associated test returns false. Such loops are essential when the programmer doesn't know at the outset how many times the loop will be traversed. (It's now known that everything computable can be programmed using while loops.)

The Ackermann function can be defined as follows:

$$A(0, n) = n + 1 \text{ for } n = 0$$
$$A(m, 0) = A(m - 1, 1) \text{ for } m = 1$$
$$A(m, n) = A(m - 1, A(m, n - 1)) \text{ for } m, n = 1.$$

Two positive integers, m and n, are the input and $A(m, n)$ is the output in the form of another positive integer. The function can be programmed easily in just a few lines of code. The problem isn't the complexity of the function but the awesome rate at which it grows. For example, the innocuous-looking $A(4,2)$ already has 19,729 digits! The use of a powerful large-number shorthand system, such as the up-arrow notation, is indispensable as the following examples show:

$$\begin{aligned} A(1, n) &= 2 + (n + 3) - 3 \\ A(2, n) &= 2 \times (n + 3) - 3 \\ A(3, n) &= 2\uparrow(n + 3) - 3 \\ A(4, n) &= 2\uparrow(2\uparrow(2\uparrow(\ldots\uparrow 2))) - 3 \ (n + 3 \text{ twos}) \\ &= 2\uparrow\uparrow(n + 3) - 3 \\ A(5, n) &= 2\uparrow\uparrow\uparrow(n + 3) - 3, \text{ etc.} \end{aligned}$$

Intuitively, the Ackermann function defines generalizations of multiplication by 2 (iterated additions) and exponentiation with base 2 (iterated multiplications) to iterated exponentiation, iteration of this operation, and so on.[84]

acre

An old unit of **area**, equal to 160 square rods, 4,840 square yards, 43,560 square feet, or 4,046.856 square meters.

acute

From the Latin *acus* for "needle" (which also forms the root for *acid, acupuncture,* and *acumen*). An acute **angle** is less than 90°. An acute **triangle** is one in which all three angles are acute. Compare with **obtuse**.

adjacent

Next to. *Adjacent angles* are next to each other, and thus share one side. *Adjacent sides* of a **polygon** share a vertex.

affine geometry

The study of properties of geometric objects that remain unchanged after parallel projection from one plane to another. During such a projection, first studied by Leonhard **Euler**, each point (x, y) is mapped to a new point $(ax + cy + e, bx + dy + f)$. Circles, angles, and distances are altered by affine transformations and so are of no interest in affine geometry. Affine transformations do, however, preserve collinearity of points: if three points belong to the same straight line, their images (the points that correspond to them) under affine transformations also belong to the same line and, in addition, the middle point remains between the other two points. Similarly, under affine transformations, parallel lines remain parallel; concurrent lines remain concurrent (images of intersecting lines intersect); the ratio of lengths of line segments of a given line remains constant; the ratio of areas of two triangles remains constant; and ellipses, parabolas, and hyperbolas continue to be ellipses, parabolas, and hyperbolas.

age puzzles and tricks

Problems that ask for a person's age or, alternatively, when a person was a certain age, given several roundabout facts. They go back at least 1,500 years to the time of Metrodorus and **Diophantus's riddle**. A number of distinct types of age puzzles sprang up between the sixteenth and early twentieth centuries, in most cases best solved by a little algebra. One form asks: if X is now a years old and Y is now b years old, when will X be c times as old as Y? The single unknown, call it x, can be found from the equation $a + x = c(b + x)$. Another type of problem takes the form: if X is now a times as old as Y and after b years X will be c times as old as Y, how old are X and Y now? In this case the trick is to set up and solve two simultaneous equations: $X = aY$ and $X + b = c(Y + b)$.

PUZZLES

Around 1900, two more variants on the age puzzle became popular. Here is an example of each for the reader to try.

1. Bob is 24. He is twice as old as Alice was when Bob was as old as Alice is now. How old is Alice?
2. The combined ages of Mary and Ann are 44 years. Mary is twice as old as Ann was when Mary was half as old as Ann will be when Ann is three times as old as Mary was when Mary was three times as old as Ann. How old is Ann?"

Solutions begin on page 369.

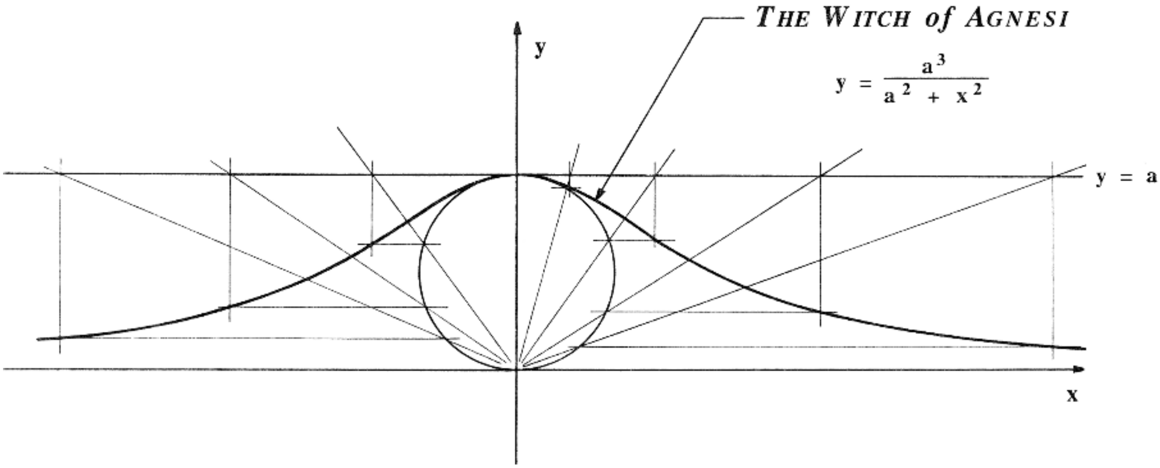
Various mathematical sleights of hand can seem to conjure up a person's age as if by magic. For example, ask a person to multiply the first number of his or her age by 5, add 3, double this figure, add the second number of his or her age to the figure, and tell you the answer. Deduct 6 from this and you will have their age.

Alternatively, ask the person to pick a number, multiply this by 2, add 5, and multiply by 50. If the person has already had a birthday this year and it's the year 2004, she should add 1,754, otherwise she should add 1,753. Each year after 2004 these numbers need to be increased by 1. Finally, the person should subtract the year they were born. The first digits of the answer are the original number, while the last two digits are the person's age.

Here is one more trick. Take your age, multiply it by 7, then multiply again by 1,443. The result is your age repeated three times. (What you have actually done is multiplied by 10,101; if you multiply by 1,010,101, the repetition is fourfold, and so on.)

Agnesi, Maria Gaetana (1718–1799)

An Italian mathematician and scholar whose name is associated with the curve known as the *Witch of Agnesi*. Born in Milan, Maria was one of 24 children of a professor of mathematics at the University of Bologna. A child prodigy, she could speak seven languages, including Latin, Greek, and Hebrew, by the age of 11 and was solving difficult problems in geometry and ballistics by her early teens. Her father encouraged her studies and her appearance at public debates. However, Maria developed a chronic illness, marked by convulsions and headaches, and, from the age of about 20, withdrew socially and devoted herself to mathematics. Her *Instituzioni analitiche ad uso della gioventu italiana,* published in 1748, became a standard teaching manual, and in 1750, she was appointed to the chair of mathematics and natural philosophy at Bologna. Yet she never fulfilled her early promise in terms of making new



Agnesi, Maria Gaetana The Witch of Agnesi curve. *John H. Lienhard*

breakthroughs. After the death of her father in 1752, she moved into theology and, after serving for some years as the directress of the Hospice Trivulzio for Blue Nuns at Milan, joined the sisterhood herself and ended her days in this austere order.

The famous curve that bears her name had been studied earlier, in 1703, by Pierre de **Fermat** and the Italian mathematician Guido Grandi (1671–1742). Maria wrote about it in her teaching manual and referred to it as the *aversiera,* which simply means "to turn." But in translating this, the British mathematician John Colson (1680–1760), the fifth Lucasian professor of mathematics at Cambridge University, confused *aversiera* with *avversiere* which means "witch," or "wife of the devil." And so the name of the curve came down to us as the Witch of Agnesi. To draw it, start with a circle of diameter a, centered at the point $(0, a/2)$ on the y-axis. Choose a point A on the line $y = a$ and connect it to the origin with a line segment. Call the point where the segment crosses the circle B. Let P be the point where the vertical line through A crosses the horizontal line through B. The Witch is the curve traced by P as A moves along the line $y = a$. By a happy coincidence, it does look a bit like a witch's hat! In Cartesian coordinates, its equation is

$$y = a^3/(x^2 + a^2).$$

Ahmes papyrus

See **Rhind papyrus**.

Ahrens, Wilhelm Ernst Martin Georg (1872–1927)

A great German exponent of recreational mathematics whose *Mathematische Unterhaltungen und Spiele*[6] is one of the most scholarly of all books on the subject.

Alcuin (735–804)

A leading intellectual of his time and the probable compiler of *Propositiones ad Acuendos Juvenes* (Problems to sharpen the young), one of the earliest collections of recreational math problems. According to David **Singmaster** and John Hadley: "The text contains 56 problems, including 9 to 11 major types of problem which appear for the first time, 2 major types which appear in the West for the first time and 3 novel variations of known problems. . . . It has recently been realized that the **river-crossing problems** and the crossing-a-desert problem, which appear here for the first time, are probably the earliest known combinatorial problems."

Alcuin was born into a prominent family near the east coast of England. He was sent to York, where he became a pupil and, eventually, in 778, the headmaster, of Archbishop Ecgberht's School. (Ecgberht was the last person to have known the Venerable Bede.) Alcuin built up a superb library and made the school one of the chief centers of learning in Europe. Its reputation became such that, in 781, Alcuin was invited to become master of Charlemagne's Palace School at Aachen and, effectively, minister of education for Charlemagne's empire. He accepted and traveled to Aachen to a meeting of the leading scholars. Subsequently, he was made head of Charlemagne's Palace School and there developed the Carolingian minuscule, a clear, legible script that became the basis of how letters of the present Roman alphabet are written.

Before leaving Aachen, Alcuin was responsible for the most prized of the Carolingian codices, now called the

Golden Gospels: a series of illuminated masterpieces written largely in gold, on white or purple vellum. The development of Carolingian minuscule had, indirectly, a major impact on the history of mathematics. Because it was a far more easily readable script than the older unspaced capital, it led to many mathematical works being newly copied into this new style in the ninth century. Most of the works of the ancient Greek mathematicians that have survived did so because of this transcription. Alcuin lived in Aachen from 782 to 790 and again from 793 to 796. In 796, he retired from Charlemagne's Palace School and became abbot of the Abbey of St. Martin at Tours, where he and his monks continued to work with the Carolingian minuscule script.

aleph

The first letter of the Hebrew alphabet, ℵ. It was first used in mathematics by Georg **Cantor** to denote the various orders, or sizes, of **infinity**: $\aleph_0$ (aleph-null), $\aleph_1$ (aleph-one), etc. An earlier (and still used) symbol for infinity, ∞, was introduced in 1655 by John **Wallis** in his *Arithmetica infinitorum* but didn't appear in print until the *Ars conjectandi* by Jakob Bernoulli, published posthumously in 1713 by his nephew Nikolaus Bernoulli (see **Bernoulli Family**).

Alexander's horned sphere A sculpture of a five-level Alexander's horned sphere. *Gideon Weisz, www.gideonweisz.com*

Alexander's horned sphere

In **topology,** an example of what is called a "wild" structure; it is named after the Princeton mathematician James Waddell Alexander (1888–1971) who first described it in the early 1920s. The horned sphere is topologically equivalent to the **simply connected** surface of an ordinary hollow **sphere** but bounds a region that is not simply connected. The horns-within-horns consist of a recursive set–a **fractal**–of interlocking pairs of orthogonal rings (rings set at right angles) of decreasing radius. A rubber band around the base of any horn couldn't be removed from the structure even after infinitely many steps. The horned sphere can be embedded in the plane by reducing the interlock angle between ring pairs from 90° to 0°, then weaving the rings together in an over-under pattern. The sculptor Gideon Weisz has modeled a number of approximations to the structure, one of which is shown in the photograph.

algebra

A major branch of mathematics that, at an elementary level, involves applying the rules of arithmetic to numbers, and to letters that stand for unknown numbers, with the main aim of solving equations. Beyond the algebra learned in high school is the much vaster and more profound subject of **abstract algebra**. The word itself comes from the Arabic *al-jebr,* meaning "the reunion of broken parts." It first appeared in the title of a book, *Al-jebr w'al-mugabalah* (The science of reduction and comparison), by the ninth-century Persian scholar **al-Khowarizmi**–probably the greatest mathematician of his age, and as famous among Arabs as Euclid and Aristotle are to the Western world.

algebraic curve

A curve whose equation involves only *algebraic functions.* These are functions that, in their most general form, can be written as a sum of **polynomial**s in x multiplied by powers of y, equal to zero. Among the simplest examples are straight lines and **conic section**s.

algebraic fallacies

Misuse of **algebra** can have some surprising and absurd results. Here, for example, is a famous "proof" that 1 = 2:

$$\begin{gathered}
\text{Let } a = b. \\
\text{Then } a^2 = ab \\
a^2 + a^2 = a^2 + ab \\
2a^2 = a^2 + ab \\
2a^2 - 2ab = a^2 + ab - 2ab \\
2a^2 - 2ab = a^2 - ab \\
2(a^2 - ab) = 1(a^2 - ab). \\
\text{Dividing both sides by } a^2 - ab \\
2 = 1.
\end{gathered}$$

Where's the mistake? The problem lies with the seemingly innocuous final division. Since $a = b$, dividing by $a^2 - ab$ is the same as dividing by **zero**–the great taboo of mathematics.

Another false argument runs as follows:

$$(n + 1)^2 = n^2 + 2n + 1$$
$$(n + 1)^2 - (2n + 1) = n^2.$$

Subtracting $n(2n + 1)$ from both sides and factorizing gives

$$(n + 1)^2 - (n + 1)(2n + 1) = n^2 - n(2n + 1).$$

Adding $\frac{1}{4}(2n + 1)^2$ to both sides yields

$$(n + 1)^2 - (n + 1)(2n + 1) + \tfrac{1}{4}(2n + 1)^2$$
$$= n^2 - n(2n + 1) + \tfrac{1}{4}(2n + 1)^2.$$

This may be written:

$$[(n + 1) - \tfrac{1}{2}(2n + 1)]^2 = [(n - \tfrac{1}{2}(2n + 1)]^2.$$

Taking square roots of both sides,

$$n + 1 - \tfrac{1}{2}(2n + 1) = n - \tfrac{1}{2}(2n + 1).$$

Therefore,

$$n = n + 1.$$

The problem here is that there are *two* square roots for any positive number, one positive and one negative: the square roots of 4 are 2 and –2, which can be written as ± 2. So the penultimate step should properly read:

$$\pm(n + 1 - \tfrac{1}{2}(2n + 1)) = \pm(n - \tfrac{1}{2}(2n + 1))$$

algebraic geometry

Originally, the geometry of **complex number** solutions to **polynomial** equations. Modern algebraic geometry is also concerned with algebraic varieties, which are a generalization of the solution sets found in the traditional subject, as well as solutions in fields other than complex numbers, for example *finite fields.*

algebraic number

A **real number** that is a **root** of a **polynomial** equation with integer coefficients. For example, any **rational number** a/b, where a and b are nonzero integers, is an algebraic number of degree one, because it is a root of the linear equation $bx - a = 0$. The **square root of two** is an algebraic number of degree two because it is a root of the quadratic equation $x^2 - 2 = 0$. If a real number is not algebraic, then it is a **transcendental number**. Almost all real numbers are transcendental because, whereas the set of algebraic numbers is countably infinite (see **countable set**), the set of transcendental numbers is uncountably infinite.

algebraic number theory

The branch of **number theory** that is studied without using methods such as **infinite series** and **convergence** taken from **analysis.** It contrasts with **analytical number theory**.

algebraic topology

A branch of **topology** that deals with invariants of a topological space that are algebraic structures, often **group**s.

algorithm

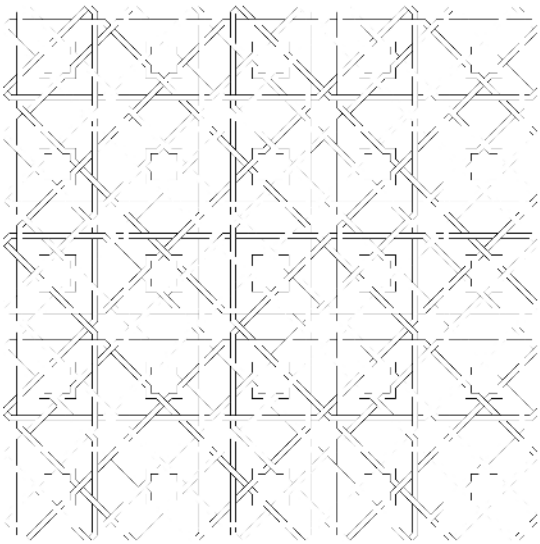
A systematic method for solving a problem. The word comes from the name of the Persian mathematician, **al-Khowarizmi**, and may have been first used by Gottfried **Liebniz** in the late 1600s. It remained little known in Western mathematics, however, until the Russian mathematician Andrei Markov (1903–1987) reintroduced it. The term became especially popular in the areas of math focused on computing and computation.

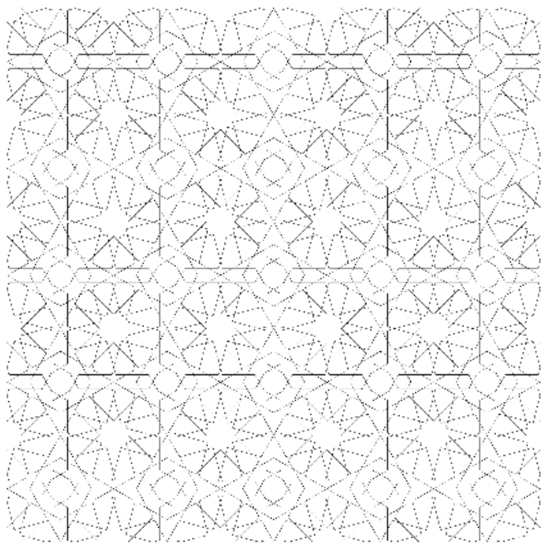
algorithmic complexity

A measure of **complexity** developed by Gregory **Chaitin** and others, based on Claude **Shannon**'s **information theory** and earlier work by the Russian mathematicians Andrei **Kolmogorov** and Ray Solomonoff. Algorithmic complexity quantifies how complex a system is in terms of the shortest computer program, or set of **algorithm**s, needed to completely describe the system. In other words, it is the smallest model of a given system that is necessary and sufficient to capture the essential patterns of that system. Algorithmic complexity has to do with the mixture of repetition and innovation in a complex system. At one extreme, a highly regular system can be described by a very short program or algorithm. For example, the bit string 01010101010101010101 . . . follows from just three commands: print a zero, print a one, and repeat the last two commands indefinitely. The complexity of such a system is very low. At the other extreme, a totally **random** system has a very high algorithmic complexity since the random patterns can't be condensed into a smaller set of algorithms: the program is effectively as large as the system itself. See also **compressible**.

Alhambra

The former palace and citadel of the Moorish kings of Granada, and perhaps the greatest monument to Islamic mathematical art on Earth. Because the Qur'an consid-





Alhambra Computer-generated tilings based on Islamic tile designs such as those found in the Alhambra. *Xah Lee, www.xahlee.org*

ers the depiction of living beings in religious settings blasphemous, Islamic artists created intricate patterns to symbolize the wonders of creation: the repetitive nature of these complex geometric designs suggests the limitless power of God. The sprawling citadel, looming high above the Andalusian plain, boasts a remarkable array of mosaics with tiles arranged in intricate patterns. The Alhambra **tiling**s are *periodic;* in other words, they consist of some basic unit that is repeated in all directions to fill up the available space. All 17 different groups of isometries–the possible ways of repeatedly tiling the plane–are used at the palace. The designs left a deep impression on Maurits **Escher**, who came here in 1936. Subsequently, Escher's art took on a much more mathematical nature, and over the next six years he produced 43 colored drawings of periodic tilings with a wide variety of symmetry types.

aliquot part

Also known as a *proper divisor,* any divisor of a number that isn't equal to the number itself. For instance, the aliquot parts of 12 are 1, 2, 3, 4, and 6. The word comes from the Latin *ali* ("other") and *quot* ("how many"). An *aliquot sequence* is formed by taking the sum of the aliquot parts of a number, adding them to form a new number, then repeating this process on the next number and so on. For example, starting with 20, we get $1 + 2 + 4 + 5 + 10 = 22$, then $1 + 2 + 11 = 14$, then $1 + 2 + 7 = 10$, then $1 + 2 + 5 = 8$, then $1 + 2 + 4 = 7$, then 1, after which the sequence doesn't change. For some numbers, the result loops back immediately to the original number; in such cases the two numbers are called **amicable numbers.** In other cases, where a sequence repeats a pattern after more than one step, the result is known as an *aliquot cycle* or a *sociable chain.* An example of this is the sequence 12496, 14288, 15472, 14536, 14264, . . . The aliquot parts of 14264 add to give 12496, so that the whole cycle begins again. Do all aliquot sequences end either in 1 or in an aliquot cycle (of which amicable numbers are a special case)? In 1888, the Belgian mathematician Eugène Catalan (1814–1894) conjectured that they do, but this remains an open question.

al-Khowarizmi (c. 780–850)

An Arabic mathematician, born in Baghdad, who is widely considered to be the founder of modern day **algebra**. He believed that any math problem, no matter how difficult, could be solved if broken down into a series of smaller steps. The word ***algorithm*** may have derived from his name.

Allais paradox

A **paradox** that stems from questions asked in 1951 by the French economist Maurice Allais (1911–).[8] Which of these would you choose: (A) an 89% chance of receiving an unknown amount and 11% chance of $1 million; or (B) an 89% chance of an unknown amount (the same amount as in A), a 10% chance of $2.5 million, and a 1% chance of nothing? Would your choice be the same if the unknown amount was $1 million? What if the unknown amount was zero?

Most people don't like risk and so prefer the better chance of winning $1 million in option A. This choice is firm when the unknown amount is $1 million, but seems to waver as the amount falls to nothing. In the latter case, the risk-averse person favors B because there isn't much difference between 10% and 11%, but there's a big difference between $1 million and $2.5 million. Thus the choice between A and B depends on the unknown amount, even though it is the same unknown amount independent of the choice. This flies in the face of the so-called *independence axiom,* that rational choice between two alternatives should depend only on how those two alternatives differ. Yet, if the amounts involved in the problem are reduced to tens of dollars instead of millions of dollars, people's behavior tends to fall back in line with the axioms of rational choice. In this case, people tend to choose option B regardless of the unknown amount. Perhaps when presented with such huge numbers, people begin to calculate qualitatively. For example, if the unknown amount is $1 million the options are essentially (A) a fortune guaranteed or (B) a fortune almost guaranteed with a small chance of a bigger fortune and a tiny chance of nothing. Choice A is then rational. However, if the unknown amount is nothing, the options are (A) a small chance of a fortune ($1 million) and a large chance of nothing, and (B) a small chance of a larger fortune ($2.5 million) and a large chance of nothing. In this case, the choice of B is rational. Thus, the Allais paradox stems from our limited ability to calculate rationally with such unusual quantities.

almost perfect number

A description sometimes applied to the powers of 2 because the **aliquot part**s (proper divisors) of 2^n sum to $2^n - 1$. So a power of 2 is a deficient number (one that is less than the sum of its proper divisors), but only just. It isn't known whether there is an odd number n whose divisors (excluding itself) sum to $n - 1$.

alphamagic square

A form of **magic square**, introduced by Lee **Sallows**,[278–280] in which the number of letters in the word for each number, in whatever language is being used, gives rise to another magic square. In English, for example, the alphamagic square:

5 (five)	22 (twenty-two)	18 (eighteen)
28 (twenty-eight)	15 (fifteen)	2 (two)
12 (twelve)	8 (eight)	25 (twenty-five)

generates the square:

4	9	8
11	7	3
6	5	10

A surprisingly large number of 3 × 3 alphamagic squares exist—in English and in other languages. French allows just one 3 × 3 alphamagic square involving numbers up to 200, but a further 255 squares if the size of the entries is increased to 300. For entries less than 100, none occurs in Danish or in Latin, but there are 6 in Dutch, 13 in Finnish, and an incredible 221 in German. Yet to be determined is whether a 3 × 3 square exists from which a magic square can be derived that, in turn, yields a third magic square—a magic triplet. Also unknown is the number of 4 × 4 and 5 × 5 language-dependent alphamagic squares. Here, for example, is a four-by-four English alphamagic square:

26	37	48	59
49	58	27	36
57	46	39	28
38	29	56	47

alphametic

A type of **cryptarithm** in which a set of words is written down in the form of a long addition sum or some other mathematical problem. The object is to replace the letters of the alphabet with decimal digits to make a valid arithmetic sum. The word *alphametic* was coined in 1955 by James **Hunter**. However, the first modern alphametic, published by Henry **Dudeney** in the July 1924 issue of *Strand Magazine,* was "Send more money," or, setting it out in the form of a long addition:

$$\begin{array}{r} \text{SEND} \\ \underline{\text{MORE}} \\ \text{MONEY} \end{array}$$

and has the (unique) solution:

$$\begin{array}{r} 9567 \\ \underline{1085} \\ 10652 \end{array}$$

PUZZLES

The reader is invited to try to solve the following elegant examples:

1. Earth, air, fire, water: nature. (Herman Nijon)
2. Saturn, Uranus, Neptune, Pluto: planets. (Peter J. Martin)
3. Martin Gardner retires. (H. Everett Moore)

Solutions begin on page 369.

Two rules are obeyed by every alphametic. First, the mapping of letters to numbers is one-to-one; that is, the same letter always stands for the same digit, and the same digit is always represented by the same letter. Second, the digit zero isn't allowed as the left-most digit in any of the numbers being added or in their sum. The best alphametics are reckoned to be those with only one correct answer.

Altekruse puzzle

A symmetrical 12-piece **burr puzzle** for which a patent was granted to William Altekruse in 1890. The Altekruse family is of Austrian-German origin and, curiously, the name means "old cross" in German, which has led some authors to incorrectly assume that it was a pseudonym. William Altekruse came to the United States as a young man in 1844 with his three brothers to escape being drafted into the German army. The Altekruse puzzle has an unusual mechanical action in the first step of disassembly by which two halves move in opposition to each other, unlike the more familiar burr types that have a key piece or pieces. Depending on how it is assembled, this action can take place along one, two, or all three axes independently but not simultaneously.

alternate

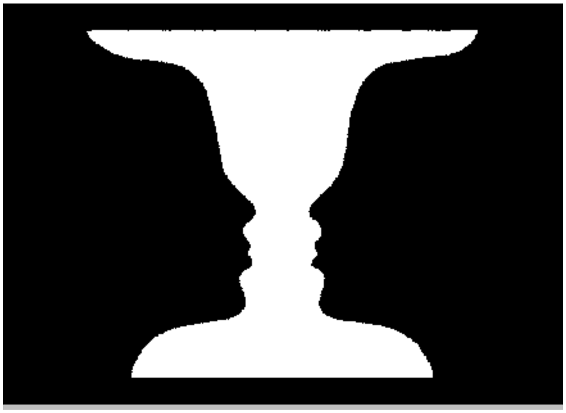
A mathematical term with several different meanings: (1) *Alternate angles* are angles on opposite sides and opposite ends of a line that cuts two parallel lines. (2) A well-known theorem called the *alternate segment theorem* involves the segment on the opposite side of a given chord of a circle. (3) An *alternate hypothesis* in statistics is the alternative offered to the **null hypothesis**. (4) To alternate is to cycle backward and forward between two different values, for example, 0, 1, 0, 1, 0, 1, ...

altitude

A perpendicular line segment from one **vertex** of a figure or solid to an edge or face opposite to that vertex. Also the length of such a line segment.

ambiguous figure

An **optical illusion** in which the subject or the perspective of a picture or shape may suddenly switch in the mind of the observer to another, equally valid possibility. Often the ambiguity stems from the fact that the figure and ground can be reversed. An example of this is the vase/profile illusion, made famous by the Danish psychologist Edgar John Rubin (1886–1951) in 1915, though earlier versions of the same illusion can be found in many eighteenth-century French prints depicting a variety of vases, usually in a naturalistic setting, and profiles of particular people. The same effect can be created in three dimensions with a suitably shaped solid vase. In some ambiguous figures, the features of a person or of an animal can suddenly be seen as different features of another individual. Classic examples include the old woman–young woman illusion and the duck–rabbit illusion. **Upside-down picture**s involve a special case of dual-purpose features in which the reversal is accomplished not mentally, by suddenly "seeing" the alternative, but physically, by turning the picture 180°. Ambiguity can also occur, particularly in some geometric drawings, when there is confusion as to which are the front and the back faces of a figure, as in the **Necker cube**, the **Thiery figure**, and **Schröder's reversible staircase**.



ambiguous figure The Rubin vase illusion: one moment a vase, the next two people face to face.

ambiguous connectivity

See **impossible figure**.

Ames room

The famous distorted room illusion, named after the American ophthalmologist Adelbert Ames Jr. (1880–1955), who first constructed such a room in 1946 based on a concept by the German physicist Hermann Helmholtz in the late nineteenth century. The Ames room looks cubic when seen with one eye through a specially positioned peephole; however, the room's true shape is trapezoidal. The floor, ceiling, some walls, and the far windows are trapezoidal surfaces; the floor appears level but is actually at an incline (one of the far corners being

Ames room Misleading geometry makes these identical twins appear totally different in size. *Technische Universitat, Dresden*

much lower than the other); and the walls are slanted outward, though they seem perpendicular to the floor. This shape makes it look as if people or objects grow or shrink as they move from one corner of the room to another. See also **distortion illusion**.[142, 178]

amicable numbers

A pair of numbers, also known as *friendly numbers,* each of whose **aliquot part**s add to give the other number. (An aliquot part is any divisor that doesn't include the number itself.) The smallest amicable numbers are 220 (aliquot parts 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110, with a sum of 284) and 284 (aliquot parts 1, 2, 4, 71, and 142, with a sum of 220). This pair was known to the ancient Greeks, and the Arabs found several more. In 1636, Pierre de **Fermat** rediscovered the amicable pair 17,296 and 18,416; two years later René **Descartes** rediscovered a third pair, 9,363,584 and 9,437,056. In the eighteenth century, Leonhard **Euler** drew up a list of more than 60. Then, in 1866, B. Nicolò Paganini (not the violinist), a 16-year-old Italian, startled the mathematical world by announcing that the numbers 1,184 and 1,210 were amicable. This second-lowest pair of all had been completely overlooked! Today, the tally of known amicable numbers has grown to about 2.5 million. No amicable pair is known in which one of the two numbers is a square. An unusually high proportion of the numbers in amicable pairs ends in either 0 or 5. A *happy amicable pair* is an amicable pair in which both numbers are **happy number**s; an example is 10,572,550 and 10,854,650. See also **Harshad number**.

amplitude

Size or magnitude. The origin of the word is the same Indo-European *ple* root that gives us *plus* and *complement.* The more immediate Latin source is *amplus* for "wide." Today, **amplitude** is used to describe, among other things, the distance a **periodic** function varies from its central value, and the magnitude of a **complex number**.

anagram

The rearrangement of the letters of a word or phrase into another word or phrase, using all the letters only once. The best anagrams are meaningful and relate in some way to the original subject; for example, "stone age" and "stage one." There are also many remarkable examples of long anagrams. " 'That's one small step for a man; one giant leap for mankind.' Neil Armstrong" becomes "An 'Eagle' lands on Earth's Moon, making a first small permanent footprint."

PUZZLES

The reader is invited to untangle the following anagrams that give clues to famous people:

1. A famous German waltz god.
2. Aha! Ions made volts!
3. I'll make a wise phrase.

Solutions begin on page 369.

An *antonymous anagram,* or *antigram,* has a meaning opposite to that of the subject text; for example, "within earshot" and "I won't hear this." *Transposed couplets,* or *pairagrams,* are single word anagrams that, when placed together, create a short meaningful phrase, such as "best bets" and "lovely volley." A rare *transposed triplet,* or *trianagram,* is "discounter introduces reductions." See also **pangram**.

anallagmatic curve

A curve that is invariant under inversion (see **inverse**). Examples include the **cardioid**, **Cassinian ovals**, **limaçon of Pascal**, **strophoid**, and **Maclaurin trisectrix**.

analysis

A major branch of mathematics that has to do with *approximating* certain mathematical objects, such as numbers or **function**s, in terms of other objects that are easier to understand or to handle. A simple example of analysis is the calculation of the first few decimal places of **pi** by writing it as the **limit** of an **infinite series**. The origins of analysis go back to the seventeenth century, when people such as Isaac **Newton** began investigating how to approximate locally–in the neighborhood of a point–the behavior of quantities that vary continuously. This led to an intense study of limits, which form the basis of understanding infinite series, differentiation, and integration.

Modern analysis is subdivided into several areas: real analysis (the study of derivatives and integrals of real-valued functions); functional analysis (the study of spaces of functions); **harmonic analysis** (the study of **Fourier series** and their abstractions); **complex analysis** (the study of functions from the complex plane to the complex plane that are complex differentiable); and **nonstandard analysis** (the study of **hyperreal numbers** and their functions, which leads to a rigorous treatment of infinitesimals and of infinitely large numbers).

analytical geometry

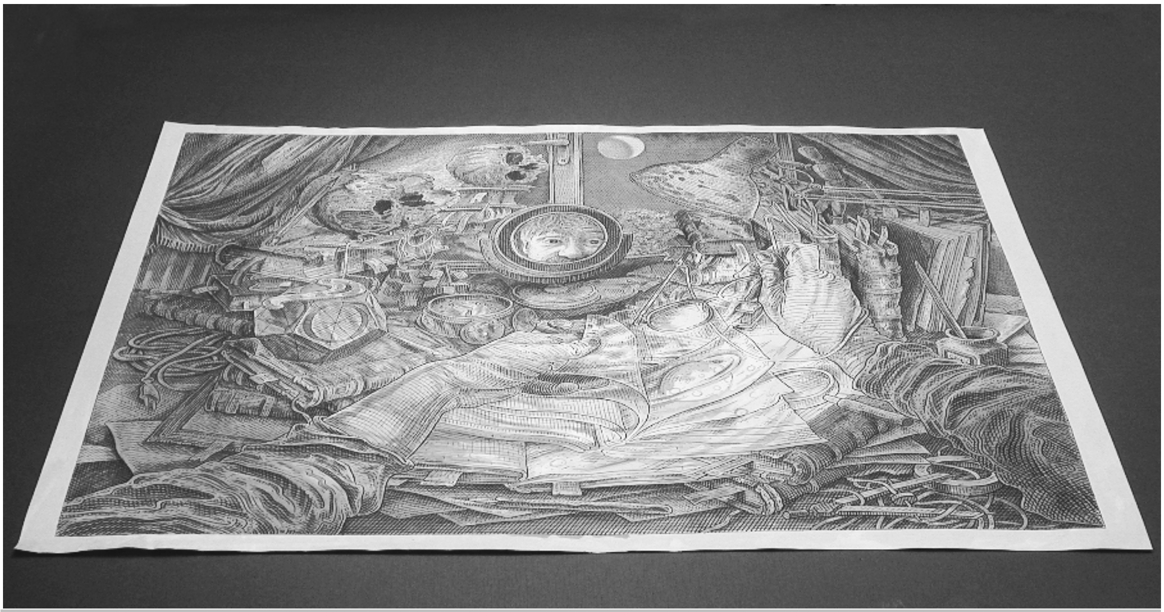
Also known as *coordinate geometry* or *Cartesian geometry,* the type of geometry that describes points, lines, and shapes in terms of **coordinates**, and that uses **algebra** to prove things about these objects by considering their coordinates. René **Descartes** laid down the foundations for analytical geometry in 1637 in his *Discourse on the Method of Rightly Conducting the Reason in the Search for Truth in the Sciences,* commonly referred to as *Discourse on Method.* This work provided the basis for **calculus**, which was introduced later by Isaac **Newton** and Gottfried **Leibniz**.

analytical number theory

The branch of **number theory** that uses methods taken from **analysis**, especially **complex analysis**. It contrasts with **algebraic number theory**.

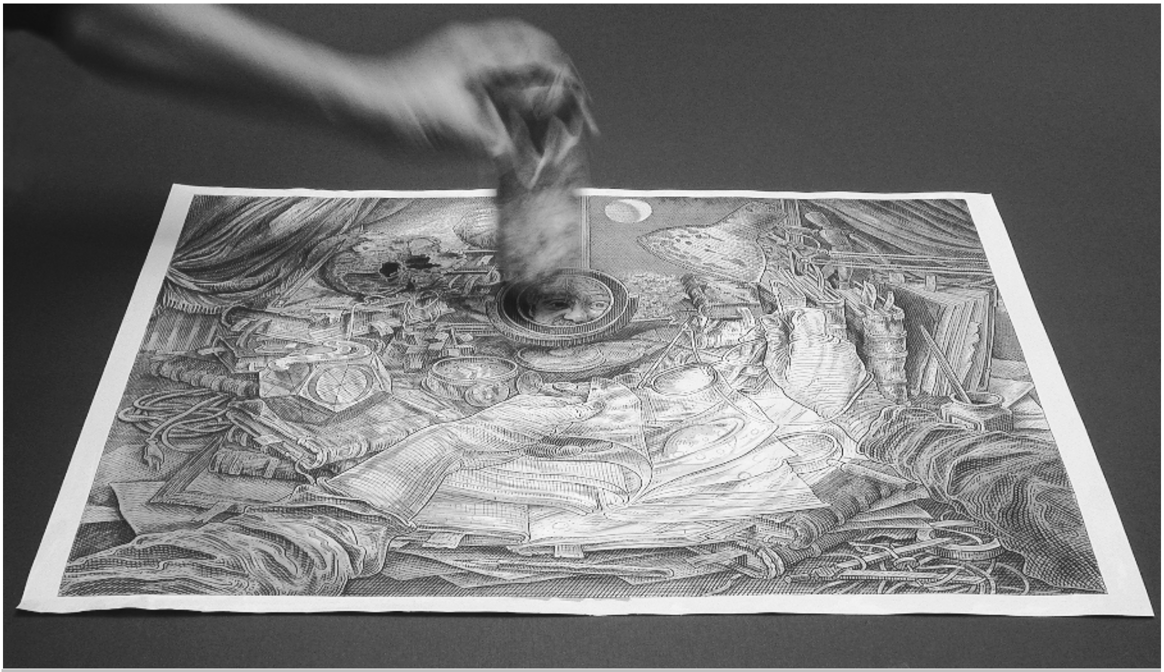
anamorphosis

The process of distorting the perspective of an image to such an extent that its normal appearance can only be restored by the observer completely changing the way he looks at the image. In *catoptric anamorphosis,* a curved mirror, usually of cylindrical or conical shape, is used to restore an anamorphic picture to its undistorted form. In other kinds of anamorphism, the observer has to change her viewing position–for example, by looking at the picture almost along its surface. Some anamorphic art adds deception by concealing the distorted image in an



anamorphosis "Self-portrait with Albert" is a clever example of anamorphic art by the Hungarian artist Istvan Orosz. The artist's hands over his desk and a small round mirror in which the artist's face is reflected can be seen in the etching. *Istvan Orosz*

(continued)



anamorphosis *(continued)* A cylindrical mirror is placed over the circle. *Istvan Orosz*

The mirror reveals a previously unsuspected aspect of the picture. The distorting effect of the curved mirror is to undistort a face hidden amid the shapes on the desk: the face of Albert Einstein. Orosz created this etching for an exhibition in Princeton, where the great scientist lived. *Istvan Orosz* otherwise normal looking picture. At one time, artists who had the mathematical knowledge to create anamorphic pictures kept their calculations and grids well-guarded secrets. Now it is relatively easy to create such images by computer.

angle

My geometry teacher was sometimes acute, and sometimes obtuse, but he was always right.

–Anonymous

The opening between two lines or two planes that meet; the word comes from the Latin *angulus* for "sharp bend." Angles are measured in **degree**s. A right angle has 90°, an **acute** angle less than 90°, and an **obtuse** angle has between 90° and 180°. If an angle exceeds the straight angle of 180°, it is said to be convex. *Complementary angles* add to 90°, and *supplementary angles* make a total of 180°.

angle bisection

See **bisecting an angle**.

angle trisection

See **trisecting an angle**.

animals' mathematical ability

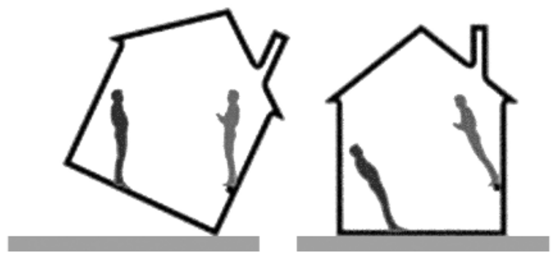
Many different species, including rats, parrots, pigeons, raccoons, and chimpanzees, are capable of doing simple calculations. Tests on dogs have shown that they have a basic grasp of cardinality–the number of things on offer. If they're shown a pile of treats and then shown the pile again after it has been concealed and the number of treats changed slightly, they will react differently than if there's been no change. However, not all purported animal math talents stand the test of time. At the turn of the century, a horse named Clever Hans wowed audiences with his counting skills. His trainer would pose a problem, and the horse would tap out the answer. In the end, though, it was found that Hans couldn't really add or subtract but was instead responding to subtle, unintended clues from its trainer, who would visibly relax when the horse reached the correct number.

annulus

The region between the smaller and the larger of two circles that share a common center.

antigravity houses and hills

The House of Mystery in the Oregon Vortex, Gold Hill, Oregon, built during the Great Depression in the 1930s, can claim to be the first "antigravity house." It spawned many imitators around the United States and in other parts of the world. Such buildings give rise to some spectacular visual effects, which seem bewildering until the underlying cause is revealed. Of course, the visitor guides are not forthcoming about what is really going on and make fantastic claims about magnetic or gravitational anomalies, UFOs, or other weird and wonderful phenomena. The fact is that all the stunning effects stem from clever construction and concealment that make an incline seem like a horizontal in the mind of the visitor. All antigravity houses are built on hills, with a typical incline of about 25°. But unlike a normal house on the side of a hill, an antigravity house is built so that its walls are perpendicular to the (inclined) ground. In addition, the area around the house is surrounded by a tall fence that prevents the visitor from establishing a true horizontal. Thus compelled to fall back on experience, the visitor assumes that the floor of the house is horizontal and that the walls are vertical with respect to Earth's gravity. All the stunning visual shenanigans follow from this.



antigravity houses and hills Visitors to a "house of mystery," believing that the floor of the house is horizontal, may be astonished by the apparent gravity-defying effects (right). All these effects are easily understood, however, when it is realized that the entire house tilts at the same angle as a hill on which it is built.

In addition to man-made antigravity illusions, there are also a number of remarkable natural locations around the world where gravity seems to be out of kilter. One example is the "Electric Brae," known locally as Croy Brae, in Ayrshire, Scotland. This runs the quarter-mile from the bend overlooking the Croy railway viaduct in the west (86 meters above sea level) to the wooded Craigencroy Glen (92 meters above sea level) to the east. While there is actually a slope of 1 in 86 (a rise of 1 meter for every 86 meters horizontally) upward from the bend at the Glen, the configuration of the land on either side of the road creates the illusion that the slope runs the other way. The author is among countless folk who have parked their cars with the brakes off on this stretch of road and been amazed to see it roll apparently uphill. See also **distortion illusion**.

antimagic square

An $n \times n$ square arrangement of the numbers 1 to n^2 such that the totals of the n rows and n columns and two long diagonals form a sequence of $(2n + 2)$ consecutive integers. There are no antimagic squares of size 2×2 and 3×3 but plenty of them for larger sizes. Here is a 4×4 example:

1	13	3	12
15	9	4	10
7	2	16	8
14	6	11	5

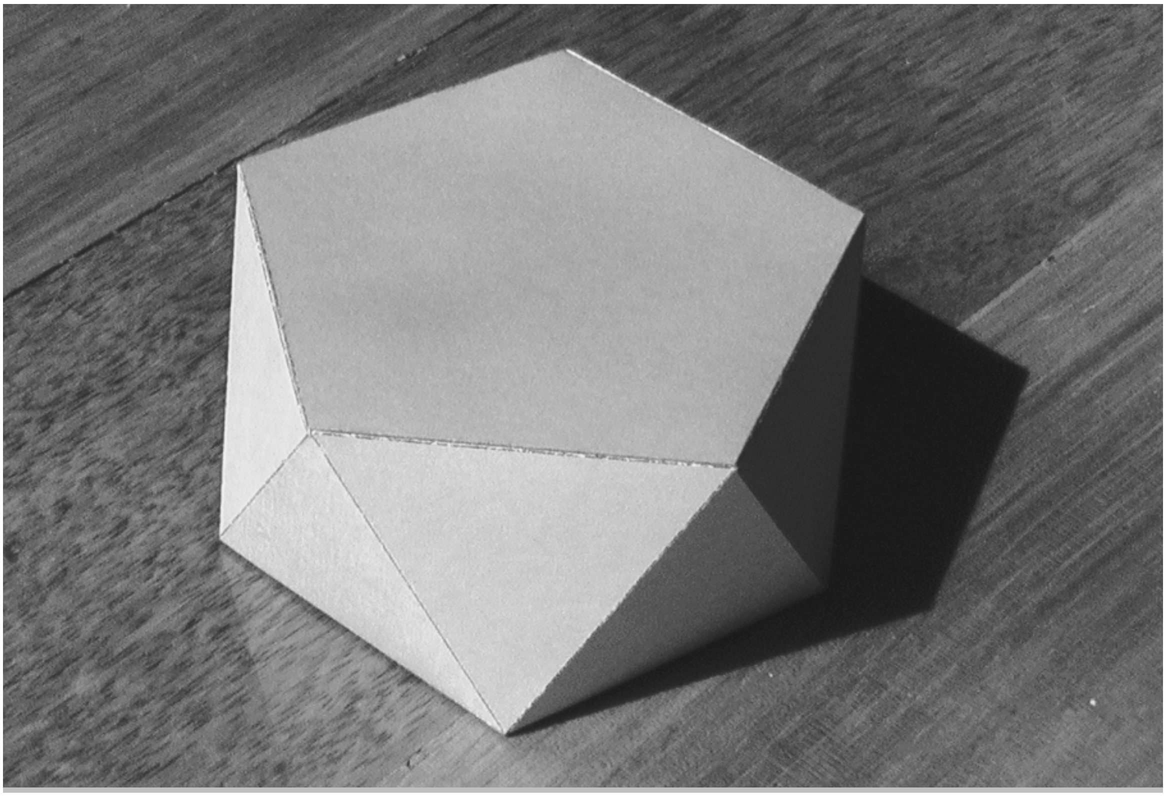
See also **magic square**.

antiprism

A **semi-regular polyhedron** constructed from two *n-sided* **polygon**s and $2n$ triangles. An antiprism is like a **prism** in that it contains two copies of any chosen regular polygon, but is unlike a prism in that one of the copies is given a slight twist relative to the other. The polygons are connected by a band of triangles pointing alternately up and down. At each vertex, three triangles and one of the chosen polygons meet. By spacing the two polygons at the proper distance, all the triangles become equilateral. Antiprisms are named square antiprisms, pentagonal antiprisms, and so on. The simplest, the triangular antiprism, is better known as the **octahedron**.

aperiodic tiling

A **tiling** made from the same basic elements or tiles that can cover an arbitrarily large surface without ever exactly repeating itself. For a long time it was thought that whenever tiles could be used to make an aperiodic tiling, those same tiles could also be fitted together in a different way to make a **periodic tiling**. Then, in the 1960s, mathematicians began finding sets of tiles that were uniquely aperiodic. In 1966, Robert Berger produced the first set of 20,426 aperiodic tiles, and soon lowered this number to 104. Over the next few years, other mathematicians reduced the number still further.



antiprism A pentagonal antiprism. *Robert Webb, www.software3d.com; created using Webb's Stella program*

In 1971, Raphael Robinson found a set of six aperiodic tiles based on notched squares; then, in 1974, Roger **Penrose** found a set of two colored aperiodic tiles (see **Penrose tilings**). The coloring can be dispensed with if the pieces are notched. There is a set of three convex (meaning no notches) aperiodic tiles, but it isn't known if there is a set of two such tiles or even a single tile (see **Einstein problem**). In three dimensions, Robert Ammann found two aperiodic polyhedra, and Ludwig Danzer found four aperiodic tetrahedra.

Apéry's constant

The number defined by the formula $\zeta/(3) = \sum_{n=1}^{\infty} 1/n^3$, where ζ is the **Riemann zeta function**: It has the value 1.202056 . . . and gives the odds (1 in 1.202056 . . .) of any three positive integers, picked at random, having no common divisor. In 1979, the French mathematician Roger Apéry (1916–1994) stunned the mathematical world with a proof that this number is irrational.[11] Whether it is a **transcendental number** remains an open question.

apex

The **vertex** of a cone or a pyramid.

apocalypse number

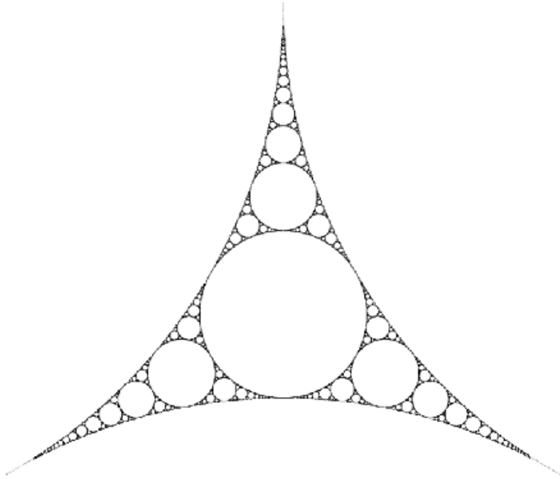
See **beast number**.

Apollonius of Perga (c. 255–170 B.C.)

A highly influential Greek mathematician (born in a region of what is now Turkey), known as the "Great Geometer," whose eight-part work *On Conics* introduced such terms as *ellipse, parabola,* and *hyperbola.* **Euclid** and others had written earlier about the basic properties of **conic sections** but Apollonius added many new results, particularly related to **normals** and **tangents** to the various conic curves. One of the most famous questions he raised is known as the **Apollonius problem**. He also wrote widely on other subjects including science, medicine, and philosophy. In *On the Burning Mirror* he showed that parallel rays of light are not brought to a focus by a spherical mirror (as had been previously thought), and he discussed the focal properties of a parabolic mirror. A few decades after his death, Emperor Hadrian collected Apollonius's works and ensured their publication throughout his realm.

Apollonius problem

A problem first recorded in *Tangencies,* written around 200 B.C. by **Apollonius of Perga**. Given three objects in the plane, each of which may be a circle *C,* a point *P* (a degenerate circle), or a line *L* (part of a circle with infinite radius), find another circle that is tangent to (just touches) each of the three. There are ten cases: *PPP, PPL, PLL, LLL, PPC, PLC, LLC, LCC, PCC, CCC.* The two easiest involve three points or three straight lines and were first solved by **Euclid**. Solutions to the eight other cases, with the exception of the three-circle problem, appeared in *Tangencies;* however, this work was lost. The most difficult case, to find a tangent circle to any three other circles, was first solved by the French mathematician François Viète (1540–1603) and involves the simultaneous solution of three **quadratic** equations, although, in principle, a solution could be found using just a compass and a straightedge. Any of the eight circles that is a solution to the general three-circle problem is called an *Apollonius circle.* If the three circles are mutually tangent then the eight solutions collapse to just two, which are known as **Soddy circles**. A **fractal** is produced by starting with three mutually tangent circles and creating a fourth–the inner Soddy circle–that is nested between the original three. The process is repeated to yield three more circles nested between sets of three of these, and then repeated again indefinitely. The points that are never inside a circle form a fractal set called the *Apollonian gasket,* which has a fractional dimension of about 1.30568.



Apollonius problem The Apollonian gasket.

apothem

Also known as a *short radius,* the perpendicular distance from the center of a **regular polygon** to one of its sides. It is the same as the radius of a circle inscribed in the polygon.

apotome

One of Euclid's categories of **irrational number**s. An apotome has the form $\sqrt{(\sqrt{A} - \sqrt{B})}$. The corresponding number with a "+" sign is called a *binomial* in Euclid's scheme.

applied mathematics

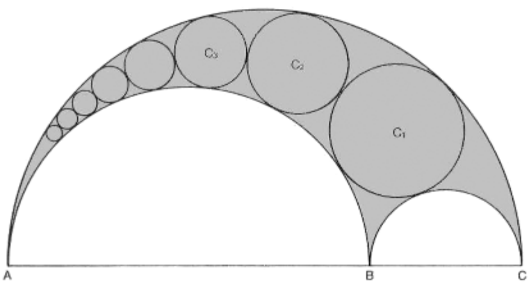
Mathematics for the sake of its use to science or society.

Arabic numeral

A **numeral** written with an Arabic digit alone: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, or in combination: 10, 11, 12, . . . 594, ...

arbelos

A figure bounded by three semicircles *AB, BC,* and *AC,* where *ABC* is a straight line. Archimedes (about 250 B.C.) called it an *arbelos*–the Greek word for a knife of the same shape used by shoemakers to cut and trim leather–and wrote about it in his *Liber assumptorum* (Book of lemmas). Among its properties are that the sum of the two smaller arc lengths is equal to the larger; the area of the arbelos is $\pi/4$ times the product of the two smaller diameters (*AB* and *BC*); and the area of the arbelos is equal to the area of a circle whose diameter is the length of a perpendicular segment drawn from the tangent point *B* of the two smaller semicircles to the point *D,* where it meets the larger semicircle. The circles inscribed on each half of *BD* of the arbelos (called *Archimedes's circles*) each have a diameter of $(AB)(BC)/(AC)$. Furthermore, the smallest circumcircle of these two circles has an area equal to that of the arbelos. **Pappus of Alexandria** wrote on the relations of the chain of circles, C_1, C_2, C_3, . . . (called a *Pappus chain* or an *arbelos train*) that are mutually tangent to the two largest semicircles and to each other. The centers of these circles lie on an ellipse and the diameter of the *n*th circle is $(1/n)$ times the base of the perpendicular distance to the base of the semicircle.



arbelos A Pappus chain of circles, C_1, C_2, C_3, . . . , inside an arbelos (shaded region).

arc

Any part of a curved line or part of the circumference of a circle; the word comes from the Latin *arcus* for a bow, which also gives rise to *arch. Arc length* is the distance along part of a curve.

arch

A strong, curved structure, traditionally made from wedge-shaped elements, that may take many different forms and that provides both an opening and a support for overlying material. Two common forms are the semicircular arch, first used by the Romans, and the pointed Gothic arch. The semicircular arch is the weaker of the two because it supports all the weight on the top and tends to flatten at its midpoint. It also requires massive supporting walls since all the stress on the arch acts purely downward. The pointed arch, by contrast, directs stresses both vertically and horizontally, so that the walls can be thinner, though buttressing may be required to prevent the walls from collapsing sideways. See also **Vesica Piscis**.

Archimedean dual

See **Catalan solid**.

Archimedean solid

A convex **semi-regular polyhedron**; a solid made from regular polygonal sides of two or more types that meet in a uniform pattern around each corner. (A regular polyhedron, or **Platonic solid**, has only one type of polygonal side.) There are 13 Archimedean solids (see table "Archimedian Solids"). Although they are named after their discoverer, the first surviving record of them is in the fifth book of the *Mathematical Collection* of **Pappus of Alexandria**. The **dual**s of the Archimedean solids (made by replacing each face with a vertex, and each vertex with a face) are commonly known as **Catalan solid**s. Apart from the Platonic and Archimedean solids, the only other convex uniform polyhedra with regular faces are **prism**s and **antiprism**s. This was shown by Johannes **Kepler**, who also gave the names generally used for the Archimedean solids. See also **Johnson solid**.

Archimedean spiral

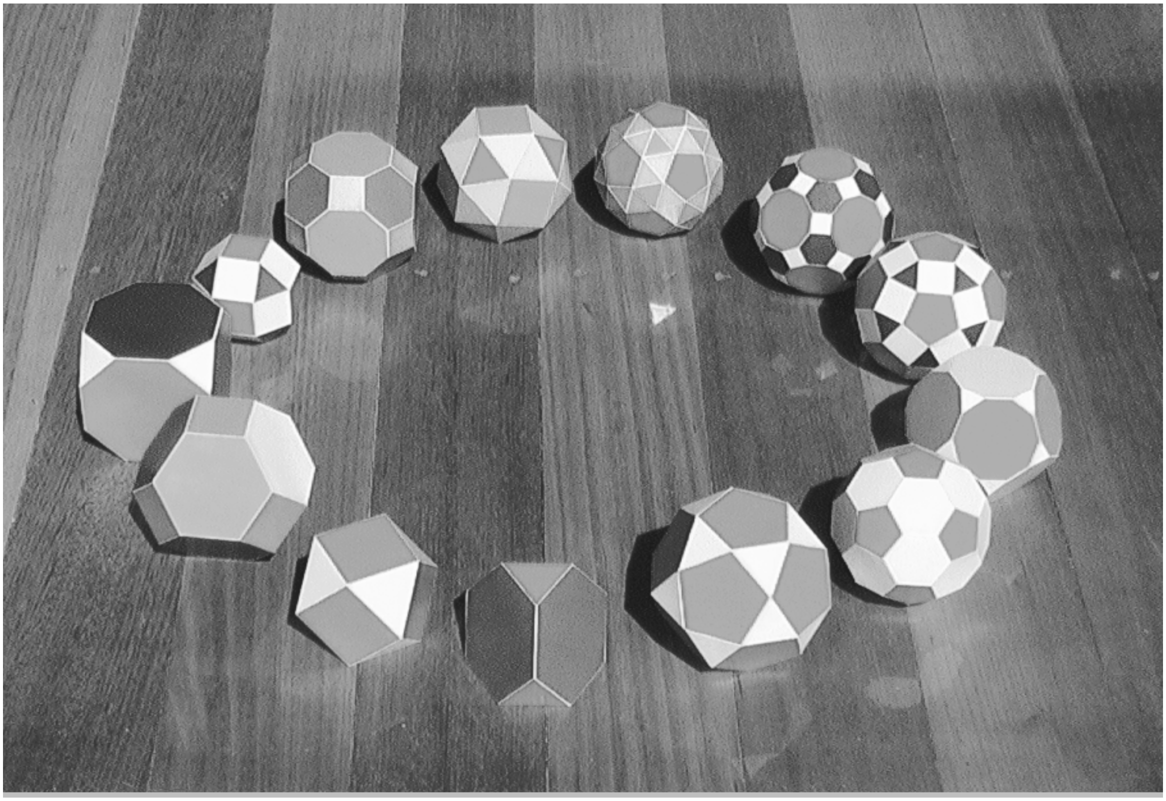
A **spiral**, like that of the groove in a phonograph record, in which the distance between adjacent coils, measured radially out from the center, is constant. **Archimedes** was the first to study it and it was the main subject of his treatise *On Spirals*. The Archimedean spiral has a very simple equation in **polar coordinates** (r, θ):

$$r = a + b\theta$$

where *a* and *b* can be any real numbers. Changing the parameter *a* turns the spiral, while *b* controls the distance

Archimedean Solids

Name	Number of Vertices	Faces	Edges
Truncated tetrahedron	8 = 4 + 4	12	18
Truncated cube	14 = 8 + 6	24	36
Truncated octahedron	14 = 6 + 8	24	36
Truncated dodecahedron	32 = 20 + 12	60	90
Truncated icosahedron	32 = 12 + 20	60	90
Cuboctahedron	14 = 8 + 6	12	24
Icosidodecahedron	32 = 20 + 12	30	60
Snub dodecahedron	92 = 80 + 12	60	150
Rhombicuboctahedron	26 = 8 + 18	24	48
Great rhombicosidodecahedron	62 = 30 + 20 + 12	120	180
Rhombicosidodecahedron	62 = 20 + 30 + 12	60	120
Great rhombicuboctahedron	26 = 8 + 12 + 6	48	72
Snub cube	38 = 32 + 6	24	60



Archimedean solid The complete set of Archimedean solids, starting far left and going clockwise: truncated cube, small rhombicuboctahedron, great rhombicuboctahedron, snub cube, snub dodecahedron, great rhombicosidodecahedron, small rhombicosidodecahedron, truncated dodecahedron, truncated icosahedron (soccer ball), icosidodecahedron, truncated tetrahedron, cuboctahedron, and truncated octahedron. *Robert Webb, www.software3d.com; created using Webb's Stella program* between the arms. The Archimedean spiral is distinguished from the **logarithmic spiral** by the fact that successive arms have a fixed distance (equal to $2\pi b$ if θ is measured in radians), whereas in a logarithmic spiral these distances form a **geometric sequence**. Note that the Archimedean spiral has two possible arms that coil in opposite directions, one for $\theta > 0$ and the other for $\theta < 0$. Many examples of spirals in the man-made world, such as a watch spring or the end of a rolled carpet, are either Archimedean spirals or another curve that is very much like it, the **circle involute**.

Archimedean tessellation

Also known as a *semiregular tessellation,* a **tiling** that uses only regular **polygon**s arranged so that two or more different polygons are around each vertex and each vertex involves the same pattern of polygons. There are eight such tessellations, two involving triangles and squares, two involving triangles and hexagons, and one each involving squares and octagons; triangles and dodecagons; squares, hexagons, and dodecagons; and triangles, squares, and hexagons.

Archimedes of Syracuse (c. 287–212 B.C.)

One of the greatest mathematicians and scientists of all time. He became a popular figure because of his involvement in the defense of Syracuse against the Roman siege in the first and second Punic Wars when his war machines helped keep the Romans at bay. He also devised a scheme to move a full-size ship, complete with crew and cargo, by pulling a single rope, and invented the irrigation device known as the Archimedean screw. According to one of many legends about him, he is said to have discovered the principle of buoyancy while taking a bath and then ran into the street naked shouting "eureka" ("I found it!").

In his book *The Sand-Reckoner,* he described a positional **number system** and used it to write the equivalent of numbers up to 8×10^{64}—the number of grains of sand he thought it would take to fill the universe. He devised a rule-of-thumb method to do private calculations that closely resembles integral calculus (2,000 years before its "discovery"), but then switched to geometric proof for his results. He demonstrated that the ratio of a circle's perimeter to its diameter is the same as the ratio of the circle's area to the square of the radius. Although he didn't call this ratio "**pi**," he showed how to work it out to arbitrary accuracy and gave an approximation of it as "exceeding 3 in less than $1/7$ but more than $10/71$."

Archimedes was the first, and possibly the only, Greek mathematician to introduce mechanical curves (those traced by a moving point) as legitimate objects of study, and he used the **Archimedean spiral** to square the circle. He proved that the area and volume of the **sphere** are in the same ratio to the area and volume of a circumscribed straight cylinder, a result that pleased him so much that he made it his epitaph. Archimedes is probably also the first mathematical physicist on record, and the best before Galileo and Isaac **Newton**. He invented the field of statics, enunciated the law of the lever, the law of equilibrium of fluids, and the law of buoyancy, and was the first to identify the concept of center of gravity. He is also, perhaps erroneously, credited with the invention of a square **dissection** puzzle known as the **loculus of Archimedes**. Many of his original works were lost when the library at Alexandria burned down and they survive only in Latin or Arabic translations. Plutarch wrote of him: "Being perpetually charmed by his familiar siren, that is, by his geometry, he neglected to eat and drink and took no care of his person; that he was often carried by force to the baths, and when there he would trace geometrical figures in the ashes of the fire, and with his finger draws lines upon his body when it was anointed with oil, being in a state of great ecstasy and divinely possessed by his science."

Archimedes's cattle problem

A fiendishly hard problem involving very **large numbers** that **Archimedes** presented in a 44-line letter to **Eratosthenes,** the chief librarian at Alexandria. It ran as follows:

> If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colors, one milk white, another a glossy black, a third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all of the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls,

went to pasture together. Now the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each color, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

But come, understand also all these conditions regarding the cattle of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colors in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

The answer to the first part of the problem–the smallest solution for the total number of cattle–turns out to be 50,389,082. But when the extra two constraints in the second part are factored in, the solution is vastly larger. The approximate answer of 7.76×10^{202544} was found in 1880 by A. Amthor, having reduced the problem to a form called a **Pell equation**.[9] His calculations were continued by an ad hoc group called the Hillsboro Mathematical Club, of Hillsboro, Illinois, between 1889 and 1893. The club's three members (Edmund Fish, George Richards, and A. H. Bell) calculated the first 31 digits and the last 12 digits of the smallest total number of cattle to be

77602714064868182695302328332**09 . . .** 719455081800

though the two digits in bold should be 13.[31] In 1931, a correspondent to the *New York Times* wrote: "Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete [exact] number [of cattle], it is obvious that the world will never have a complete solution." But *obvious* and *never* are words designed to make fools of prognosticators. Enter the computer. In 1965, with the help of an IBM 7040, H. C. Williams, R. A. German, and C. R. Zarnke reported a complete solution to the cattle problem, though it was 1981 before all 202,545 digits were published, by Harry Nelson, who used a Cray-1 supercomputer to generate the answer, which begins: $7.760271406486818269530232833213 \ldots \times 10^{202544}$.[341]

Archimedes's square
See **loculus of Archimedes**.

area
A measure of surface extension in two-dimensional space. *Area* is the Latin word for a vacant piece of level ground and still carries this common meaning. The French shortened form *are* denotes a square of land with a side length of 10 meters, that is, an area of 100 square meters. A *hectare* is a hundred are.

area codes
North American telephone area codes seem to have been chosen at random. But there was a method to their selection. In the mid-1950s when direct dialing of long-distance calls first became possible, it made sense to assign area codes that took the shortest time to dial to the larger cities. Almost all calls were from rotary dials. Area codes such as 212, 213, 312, and 313 took very little time for the dial to return to its starting position compared, for example, to numbers such as 809, 908, 709. The quickest-to-dial area codes were assigned to the places expected to receive the most direct-dialed calls. New York City got 212, Chicago 312, Los Angeles 213, and Washington, D.C., 202, which is a little longer to dial than 212, but much shorter than others. In order of decreasing size and estimated amount of telephone traffic, the numbers grew larger: San Francisco got 415, Miami 305, and so on. At the other end of the spectrum came places like Hawaii (the last state annexed in 1959) with 808, Puerto Rico with 809, and Newfoundland with 709. The original plan (still in use until about 1993) was that area codes had a certain construction to the numbers: the first digit is 2 through 9, the second digit is 0 or 1, and the third digit is 1 through 9. Three-digit numbers with two zeros are special codes, that is, 700, 800, or 900. Three-digit numbers with two ones are for special local codes such as 411 for local directory assistance, 611 for repairs, and so forth.

Argand diagram
A way of representing **complex number**s as points on a coordinate plane, also known as the *Argand plane* or the *complex plane,* using the x-axis as the real axis and the y-axis as the imaginary axis. It is named for the French amateur mathematician Jean Robert Argand

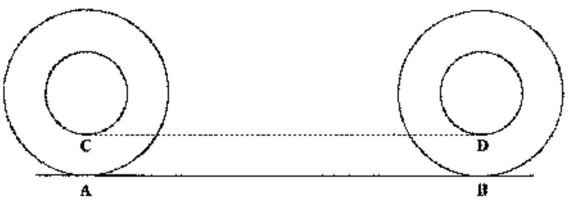
(1768–1822) who described it in a paper in 1806.[14] John **Wallis** suggested a similar method 120 years earlier and Casper **Wessel** extensively developed it. But Wessel's paper was published in Danish and wasn't circulated in the languages more common to mathematics at that time. In fact, it wasn't until 1895 that his paper came to the attention of the mathematical community–long after the name "Argand diagram" had stuck.

argument

(1) The input for a **function**. (2) The angle between ZO, where Z is the point representing a complex number on an **Argand diagram** and O is the origin, and the real axis. (3) A mathematical proof, possibly an informal one.

Aristotle's wheel

A **paradox** mentioned in the ancient Greek text *Mechanica,* whose author is unknown but is suspected by some to have been Aristotle. The paradox concerns two concentric circles on a wheel, as shown in the diagram. A **one-to-one** correspondence exists between points on the larger circle and those on the smaller circle. Therefore, the wheel should travel the same distance regardless of whether it is rolled from left to right on the top straight line or on the bottom one. This seems to imply that the two circumferences of the different-sized circles are equal, which is impossible. How can this apparent contradiction be resolved? The key lies in the (false) assumption that a one-to-one correspondence of points means that two curves must have the same length. In fact, the cardinalities of points in a line segment of any length (or even an infinitely long line or an infinitely large n-dimensional Euclidean space) are all the same. See also **infinity**.



Aristotle's wheel The outer circle turns once when going from A to B, as does the inner circle when going from C to D. Yet AB is the same length as CD. How can this be, since the circles are a different size?

arithmetic

A branch of mathematics concerned with doing calculations with numbers using addition, subtraction, multiplication, and division.

arithmetic mean

The sum of n given numbers divided by n. See also **geometric mean** and **harmonic mean**.

arithmetic sequence

Also known as an *arithmetic progression,* a finite **sequence** of at least three numbers, or an infinite sequence, whose terms differ by a constant, known as the *common difference.* For example, starting with 1 and using a common difference of 4 we can get the finite arithmetic sequence: 1, 5, 9, 13, 17, 21, and also the infinite sequence 1, 5, 9, 13, 17, 21, 25, 29, . . . , $4n + 1$, ... In general, the terms of an arithmetic sequence with the first term a_0 and common difference d, have the form $a_n = dn + a_0$ ($n = 1, 2, 3, \ldots$). Does every increasing sequence of integers have to contain an arithmetic progression? Surprisingly, the answer is no. To construct a counterexample, start with 0. Then for the next term in the sequence, take the *smallest possible integer* that doesn't cause an arithmetic progression to form in the sequence constructed thus far. (There must be such an integer because there are infinitely many integers beyond the last term, and only finitely many possible progressions that the new term could complete.) This gives the nonarithmetic sequence 0, 1, 3, 4, 9, 10, 12, 13, 27, 28, ...

If the terms of an arithmetic sequence are added together the result is an *arithmetic series,* $a_0 + (a_0 + d) + \ldots + (a_0 + (n-1)d)$, the sum of which is given by:

$$S_n = n/2\ (2a_0 + (n-1)d) = n/2\ (a_0 + a_n).$$

See also **geometric sequence**.

around the world game

See **Icosian game**.

array

A set of numbers presented in a particular pattern, usually a grid. Matrices (see **matrix**) and **vector**s are examples of arrays.

Arrow paradox

The oldest and best-known **paradox** related to voting. The American economist Kenneth Arrow (1921–) showed that it is impossible to devise a perfect democratic voting system. In his book *Social Choice and Individual Values,*[16] Arrow identified five conditions that are universally regarded as essential for any system in which social decisions are based on individual voting preferences. The Arrow paradox is that these five conditions are logically inconsistent: under certain conditions, at least one of the essential conditions will be violated.

arrowhead
See **dart**.

artificial intelligence (AI)
The subject of "making a machine behave in ways that would be called intelligent if a human were so behaving," according John McCarthy, who coined the term in 1955. How can we tell if a computer has acquired AI at a human level? One way would be to apply the **Turing test**, though not everyone agrees that this test is foolproof (see **Chinese room**). Certainly, AI has not developed at nearly the rate many of its pioneers expected back in the 1950s and 1960s. Meanwhile, progress in fields such as **neural network**s and **fuzzy logic** continues to be made, and most computer scientists have no doubt that it is only a matter of time before computers are outperforming their biological masters in a wide variety of tasks beyond those that call for mere number-crunching ability.

artificial life
A lifelike pattern that may emerge from a **cellular automaton** and appear organic in the way it moves, grows, changes shape, reproduces, aggregates, and dies. Artificial life was pioneered by the computer scientist Chris Langton, and has been researched extensively at the Santa Fe Institute. It is being used to model various **complex system**s such as ecosystems, the economy, societies and cultures, and the immune system. The study of artificial life, though controversial, promises insights into natural processes that lead to the buildup of structure in self-organizing (see **self-organization**) systems.

associative
Three numbers, x, y, and z, are said to be *associative under addition* if

$$x + (y + z) = (x + y) + z,$$

and to be *associative under multiplication* if

$$x \times (y \times z) = (x \times y) \times z.$$

In general, three elements a, b, and c of a set S are associative under the binary operation (an operation that works on two elements at a time) $*$ if

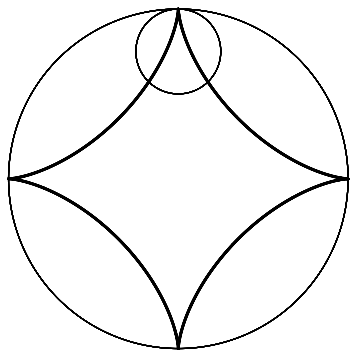
$$a * (b * c) = (a * b) * c.$$

The word incorporates the Greek root *soci,* from which we also get *social,* and may have been first used in the modern mathematical sense by William **Hamilton** around 1850. Compare with **distributive** and **commutative**.

astroid
A **hypocycloid**—the path of a point on a circle rolling inside another circle—for which the radius of the inner circle is four times smaller than that of the larger circle; this ratio results in the astroid having four cusps. The astroid was first studied by the Danish astronomer Ole Römer in 1674, in his search for better shapes for gear teeth, and later by Johann Bernoulli (1691) (see **Bernoulli family**), Gottfried **Leibniz** (1715), and Jean **d'Alembert** (1748). Its modern name comes from the Greek *aster* for "star" and was introduced in a book by Karl Ludwig von Littrow published in Vienna in 1836; before this, the curve had a variety of names, including tetracuspid (still used), cubocycloid, and paracycle. The astroid has the Cartesian equation

$$x^{2/3} + y^{2/3} = r^{2/3}$$

where r is the radius of the fixed outer circle, and $r/4$ is the radius of the rolling circle. Its area is $3\pi r^2/8$, or $3/2$ times that of the rolling circle, and its length is $6r$. The astroid is a sextic curve and also a special form of a **Lamé curve**. It has a remarkable relationship with the quadrifolium (see **rose curve**): the radial, pedal, and orthoptic of the astroid are the quadrifolium, while the cata**caustic** of the quadrifolium is the astroid. The



astroid As a small circle rolls around the inside of a larger one with exactly four times its circumference, a point on the rim of the small circle traces out an astroid. *© Jan Wassenaar, www.2dcurves.com* astroid is also the catacaustic of the **deltoid** and the **evolute** of the ellipse.

asymptote

A curve that gets closer and closer to a fixed straight line without ever actually touching it. Imagine facing along the direction of a great wall that is just a meter to your left. Every second, you walk forward a meter and at the same time move sideways slightly so that you halve the distance between you and the wall. The path you follow is an asymptote. The word comes from the Greek roots *a* (not), *sum* (together), and *piptein* (to fall), so that it literally means "not falling together" and was originally used in a wider sense to describe any two curves that don't intersect. **Proclus** writes about both asymptotic lines and symptotic lines (those that do cross). Nowadays, "symptotic" is almost never heard, and "asymptote" is used mainly to denote lines that serve as a limiting barrier for some curve as one of its parameters approaches plus or minus infinity. The "~" symbol is often used to show that one **function** is asymptotic to another. For example, $f(x) \sim g(x)$ indicates that the ratio of the functions $f(x)$ to $g(x)$ approaches 1 as x tends to infinity. Asymptotes are not always parallel to the x- and y-axes, as shown by the graph of $x + 1/x$, which is asymptotic to both the y-axis and the diagonal line $y = x$.

Atiyah, Michael Francis (1929–)

An English mathematician who has contributed to many topics in mathematics, notably dealing with the relationships between geometry and **analysis**. In **topology**, he developed *K-theory*. He proved the *index theorem* on the number of solutions of elliptic differential equations, linking **differential geometry**, topology, and analysis–a theorem that has been usefully applied to quantum theory. Atiyah was influential in initiating work on gauge theories and applications to nonlinear differential equations, and in making links between topology and quantum field theory. Theories of superspace and supergravity, and string theory, were all developed using ideas introduced by him.

Atomium, the

A giant steel monument in Heysel Park, Brussels, Belgium, consisting of 9 spheres that represent the body-centered cubic structure of an iron crystal magnified 150 billion times. Designed by the architect André Waterkeyn and built for the 1958 World's Fair, the 103-meter-high Atomium was originally meant to stand for only 6 months. It may be the world's largest cube. Each of its spheres have a diameter of 18 meters and are connected by escalators. Three of the upper spheres have no vertical support, and so for safety reasons are not open to the public. However, the top sphere offers a panoramic view of Brussels through its windows, and the lower spheres contain various exhibitions.

attractor

A trajectory, or set of points in **phase space**, toward which nearby orbits converge, and which is stable. Specific types of attractor include **fixed-point attractor**, **periodic attractor**, and **chaotic attractor**.

Aubel's theorem

Given a quadrilateral and a square drawn on each side of it, the two lines connecting the centers of the squares on opposite sides are perpendicular and of equal length.

autogram

See **self-enumerating sentence**.

automorphic number

Also known as an *automorph,* a number n whose square ends in n. For instance 5 is automorphic, because $5^2 = 25$, which ends in 5. A number n is called *trimorphic* if n^3 ends in n. For example $49^3 = 117{,}649$, is trimorphic. Not all trimorphic numbers are automorphic. A number n is called *tri-automorphic* if $3n^2$ ends in n; for example 6,667 is tri-automorphic because $3 \times 667^2 = 133{,}346{,}667$ ends in 7.

automorphism

An **isomorphism** from a **set** onto itself. An *automorphism group* of a **group** G is the group formed by the automorphisms of G (bijections from G to itself that preserve the multiplication). Similarly, one can consider the automorphism groups of other structures such as **ring**s and **graph**s, by looking at bijections that preserve their mathematical structure.

Avagadro constant

One of the best known examples of a **large number** in science. It is named after the Italian physicist Amedio Avagadro (1776–1856) and is defined as the number of carbon atoms in 12 grams of pure carbon, or, more generally as the number of atoms of n grams in an element with atomic weight n. It has the value $6.02214199 \times 10^{23}$.

average

A vague term that usually refers to the **arithmetic mean** but can also signify the **median**, **mode**, **geometric mean**, or weighted mean. The word stems from a commercial practice of the shipping age. The root *aver* means to declare, and the shippers of goods would declare the value of their goods. When the goods were sold, a deduction was made from each person's share, based on their declared value, for a portion of the loss or "average."

axiom

A statement that is considered to be true without need of **proof**. The term *axiom* comes from the Greek *axios* meaning "worthy" and was used by many Greek philosophers and mathematicians, including **Aristotle**. Curiously, **Euclid**, whose axioms are best known of all, seems to have favored a more general phrase meaning "common notion."

axiom of choice

An **axiom** in **set theory** that is one of the most controversial axioms in mathematics; it was formulated in 1904 by the German mathematician Ernst Zermelo (1871–1953) and, at first, seems obvious and trivial. Imagine there are many–possibly an unlimited number of–boxes in front of you, each of which has at least one thing in it. The axiom of choice (AC) says simply that you can always choose one item out of each box. More formally, if $\mathbf{S}$ is a collection of nonempty sets, then there exists a set that has exactly one element in common with every set S of $\mathbf{S}$. Put another way, there exists a function f with the property that, for each set S in the collection, $f(S)$ is a member of S. Bertrand **Russell** summed it up neatly: "To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed." His point is that the two socks in a pair are identical in appearance, so, to pick one of them, we have to make an arbitrary choice. For shoes, we can use an explicit rule, such as "always choose the left shoe." Russell specifically mentions *infinitely many* pairs, because if the number is finite then AC is superfluous: we can pick one member of each pair using the definition of "nonempty" and then repeat the operation finitely many times using the rules of formal logic.

AC lies at the heart of a number of important mathematical arguments and results. For example, it is equivalent to the *well-ordering principle,* to the statement that for any two **cardinal number**s m and n, then $m < n$ or $m = n$ or $m > n$, and to *Tychonoff's theorem* (the product of any collection of compact spaces in topology is compact). Other results hinge upon it, such as the assertion that every infinite set has a denumerable subset. Yet AC was strongly attacked when it was first suggested, and still makes some mathematicians uneasy. The central issue is what it means to *choose* something from the sets in question and what it means for the choosing function to *exist.* This problem is brought into sharp focus when $\mathbf{S}$ happens to be the collection of all nonempty subsets of the **real number**s. No one has ever found a suitable choosing function for this collection, and there are good reasons to suspect that no one ever will. AC just mandates that there *is* such function. Because AC conjures up sets without offering workable procedures, it is said to be *nonconstructive,* as are any theorems whose proofs involve AC. Another reason that some mathematicians aren't greatly enamored with AC is that it implies the existence of some bizarre counterintuitive objects, the most famous and notorious example of which is the **Banach-Tarski paradox**. The main reason for accepting AC, as the majority of mathematicians do (albeit often reluctantly), is that it is useful. However, as a result of work by Kurt **Gödel** and, later, by Paul Cohen, it has been proven to be independent of the remaining axioms of set theory. Thus there are no contradictions in choosing to reject it; among the alternatives are to adopt a contradictory axiom or to use a completely different framework for mathematics, such as **category theory**.

axis

A line with respect to which a curve or figure is drawn, measured, rotated, and so forth. The word comes from the Greek root *aks* for a point of turning or rotation and seems to have first been used in English by Thomas Digges around 1570 in reference to the rotational axis of a right circular cone.