CHAPTER 1

CONVECTION WITH SIMPLIFIED CONSTRAINTS

1.1 INTRODUCTION

Three quarters of a century ago, a paper by Harper and Brown (1922) appeared as an NACA report. It was an elegant piece of work and appears to be the first really significant attempt to provide a mathematical analysis of the interesting interplay between convection and conduction in and upon a single extended surface. Harper and Brown called this a *cooling fin*, which later became known merely as a *fin*. It is most probable that Harper and Brown were the pioneers even though Jakob (1949) pointed out that published mathematical analyses of extended surfaces can be traced all the way back to 1789. At that time, Ingenhouss demonstrated the differences in thermal conductivity of several metals by fabricating rods, coating them with wax, and then observing the melting pattern when the bases of the rods were heated. Jakob also pointed out that Fourier (1822) and Despretz (1822, 1828a,b) published mathematical analyses of the temperature variation of the metal bars or rods.

Although these ancient endeavors may have been quite significant at the time they were written, it appears that the Harper–Brown work should be considered as the forerunner of what has become a burgeoning literature that pertains to a very significant subject area in the general field of heat transfer.

The NACA report of Harper and Brown was inspired by a request from the Engineering Division of the U.S. Army and the U.S. Bureau of Standards in connection with the heat-dissipating features of air-cooled aircraft engines. It is interesting to note that this request came less than halfway through the time period between the first flight of the Wright Brothers at Kitty Hawk and the actual establishment of the U.S. Air Force. The work considered longitudinal fins of rectangular profile and trapezoidal profile (which Harper and Brown called *wedge-shaped fins*) and radial fins of rectangular profile (which Harper and Brown called *circumferential fins*). It also introduced the concept of *fin efficiency*, although the expression employed by Harper and Brown was called the *fin effectiveness*. From this modest, yet masterful beginning, the analysis and evaluation of the performance of individual components of extended surface and arrays of extended surface where individual components are assembled into complicated configurations has become an art.

Harper and Brown (1922) provided thorough analytical solutions for the two-dimensional model for both rectangular and wedge-shaped longitudinal fins and the circumferential fin of uniform thickness. The solutions culminated in expressions for the fin efficiency (called the *effectiveness*) or in correction factors that adjusted the efficiency of the rectangular profile longitudinal fin. They concluded that the use of a one-dimensional model was sufficent and they proposed that the tip heat loss could be accounted for through the use of a corrected fin height, which increases the fin height by a value equal to half of the fin thickness. Lost in the shuffle, however, was the interesting observation that with dx as the differential element of fin height, the differential face surface area of the element is $dx/\cos\kappa$, where κ is the taper angle, which is 90° for rectangular profile straight and circumferential fins as well as for spines of constant cross section.

Schmidt (1926) covered the three profiles considered by Harper and Brown from the standpoint of material economy. He stated that the least material is required for given conditions if the fin temperature gradient (from base to tip) is linear, and he showed how the fin thickness of each type of fin must vary to produce this result. Finding, in general, that the calculated shapes were impractical to manufacture, he proceeded to show the optimum dimensions for longitudinal and radial fins of constant thickness (rectangular profile) and the longitudinal fin of trapezoidal profile. He also considered the longitudinal fin of triangular profile as the case of the longitudinal fin of trapezoidal profile with zero tip thickness.

The case of integral pin fins of different profiles was considered by Bueche and Schau (1936). They determined for conical fins that the heat dissipation was a function of the Biot modulus based on the base radius and aspect ratio of fin height to base radius. They also showed that a weight optimization could be effected.

Murray (1938) considered the problem of the radial fin of uniform thickness (the radial fin of rectangular profile) presenting equations for the temperature gradient and effectiveness under conditions of a symmetrical temperature distribution around the base of the fin. He also proposed that the analysis of extended surface should be based on a set of assumptions that have been known since 1945 as the Murray–Gardner assumptions. These assumptions are deemed to be of considerable importance because their elimination, either one at a time or in combination, provided a series of paths for subsequent investigators to follow.

A stepwise procedure for calculating the temperature gradient and efficiency of fins whose thickness varies in any manner was presented by Hausen (1940). The temperature gradient in conical and cylindrical spines was determined by Focke (1942) who, like Schmidt, showed how the spine thickness must vary to keep the material required to a minimum. He too, found the resulting shape impractical and went on to determine the optimum cylindrical and conical spine dimensions.

Avrami and Little (1942) derived equations for the temperature gradient in thick bar fins and showed under what conditions fins might act as insulators on the base or prime surface. Carrier and Anderson (1944) discussed straight fins of constant thickness, radial fins of constant thickness, and radial fins of constant cross-sectional area, presenting equations for the fin efficiency of each. However, in the latter two cases, the efficiencies are given in the form of an infinite series.

Gardner (1945), in a giant leap forward, derived general equations for the temperature excess profile and fin efficiency for any form of extended surface for which the Murray–Gardner assumptions are applicable and whose thickness varies as some power of the distance measured along an axis normal to the base or prime surface (the fin height). He proposed the profile function (his nomenclature)

$$y = y_b \left(\frac{x}{x_b} \right)^{(1-2n)/(1-n)}$$

for the straight or longitudinal fins,

$$y = y_b \left(\frac{x}{x_b} \right)^{(1-2n)/(2-n)}$$

for spines and

$$y = y_b \left(\frac{x}{x_b} \right)^{-2n/(1-n)}$$

for radial or circumferential (radial) fins.

These equations depend on the assignment of some number to n; for example, the straight fin of rectangular profiles results when $n = 0$ in the first of these. This also serves to show that in Gardner's profile functions, the positive sense of the height coordinate x is in a direction from fin tip to fin base.

With the foregoing profile functions in hand and working with a general differential equation that he derived, Gardner was able to provide solutions for the temperature excess profile in terms of modified Bessel functions. For n equal to zero or an integer,

$$\theta = \theta_b \left(\frac{u}{u_b} \right)^n \frac{I_n(u) + \beta K_n(u)}{I_n(u_b) + \beta K_n(u_b)}$$

and for n equal to a fraction,

$$\theta = \theta_b \left(\frac{u}{u_b} \right)^n \frac{I_n(u) + \beta I_{-n}(u)}{I_n(u_b) + \beta I_{-n}(u_b)}$$

where u depends on the type of fin or spine and where

$$\beta = \frac{I_{n-1}(u_a)}{K_{n-1}(u_a)}$$

if n is equal to zero or an integer and

$$\beta = -\frac{I_{n-1}(u_a)}{I_{1-n}(u_a)}$$

if n is equal to a fraction.

For straight (longitudinal) fins

$$u = 2(1-n)\left(\frac{x}{x_b}\right)^{1/2(1-n)} \sqrt{\frac{h}{k y_b}} x_b$$

for spines

$$u = \frac{2\sqrt{2}\,(2-n)}{3}\left(\frac{x}{x_b}\right)^{3/2(2-n)} \sqrt{\frac{h}{k y_b}} x_b$$

and for radial fins

$$u = (1-n)\left(\frac{x}{x_b}\right)^{1/(1-n)} \sqrt{\frac{h}{k y_b}} x_b$$

The fin efficiency, defined as the ratio of the heat transferred from the fin to the heat that would be transferred by the fin if its thermal conductivity were infinite (if the entire fin were to operate at the base temperature excess), was provided by Gardner for all the fins that he considered. Gardner designated the fin efficiency η, and for n equal to zero or an integer,

$$\eta = \frac{2(1-n)}{u_b[1-(u_a/u_b)^{2(1-n)}]} \frac{I_{n-1}(u_b) - \beta K_{n-1}(u_b)}{I_n(u_b) - \beta K_n(u_b)}$$

and for n equal to a fraction,

$$\eta = \frac{2(1-n)}{u_b[1-(u_a/u_b)^{2(1-n)}]} \frac{I_{n-1}(u_b) - \beta I_{1-n}(u_b)}{I_n(u_b) - \beta I_{-n}(u_b)}$$

Graphs were provided that plotted the efficiency as a function of a parameter that embraced the fin dimensions and thermal properties. Two of the graphs (for straight fins and spines) are reproduced here as Figs. 1.1 and 1.2.

Gardner also pointed out that the terms *fin efficiency* and *fin effectiveness* had not been used consistently in the English literature. He redefined the fin effectiveness as the ratio of the heat transferred through the base of the fin to the heat transferred through the same prime or base surface area if the fin were not present. He also provided a relationship to permit the conversion from fin efficiency to fin effectiveness.

It is felt that the Gardner paper is remarkable for several reasons. First and probably foremost is the fact that he reemphasized the concept of the fin efficiency, thereby creating an itch that literally thousands of equipment designers have been scratching ever since. Moreover, it appears that Gardner was one of the first to demonstrate the use of applied mathematics to yield concepts that engineers could use to build

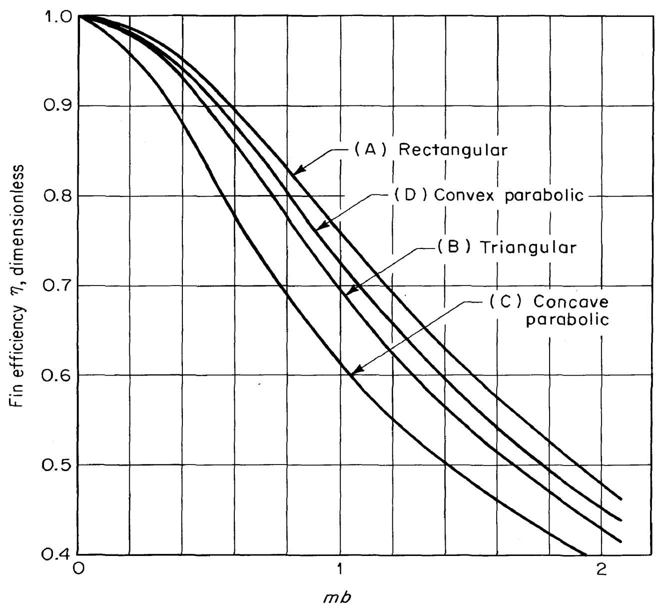

Figure 1.1 Gardner's graph for the efficiency of straight (longitudinal) fins. (Reproduced from *Trans. ASME*, 67, 1945.)

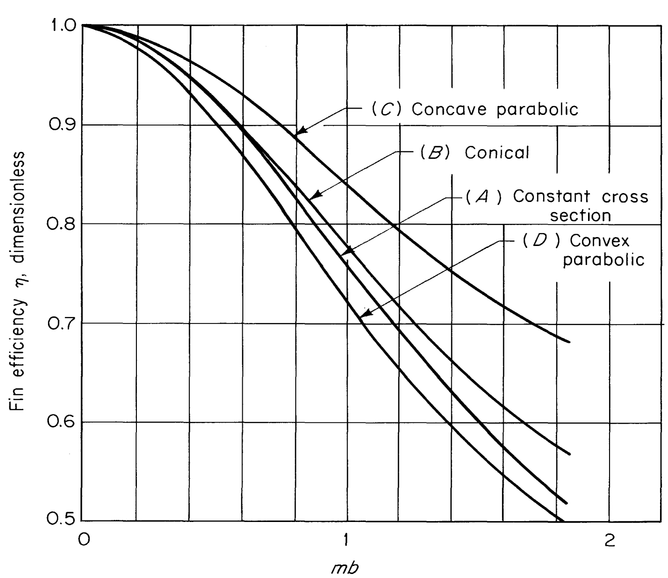

Figure 1.2 Gardner's graph for the efficiency of spines. (Reproduced from *Trans. ASME*, 67, 1945.)

equipment that worked. He may not have been the first to show the modified Bessel functions to the working mechanical engineer, but he certainly provided an intense reexposition of these interesting functions.

One may observe that as 1945, the year of Gardner's paper, drew to a close, the extended surface technology was on a firm foundation. What began with Harper and Brown and what had concluded with the Gardner paper had established useful design equations for the construction of working heat transfer hardware containing finned surfaces. It is also interesting to note that the 1945 volume of *ASME Transactions* contained, in addition to Gardner's pioneering effort, correlations for the heat transfer coefficient between fin and fluid. DeLorenzo and Anderson (1945) provided a correlation for the heat transfer coefficient and friction factor for the longitudinal fin–axial flow exchanger.[1] Jameson (1945) provided a heat transfer correlation, and Gunter and Shaw (1945) presented flow friction data in what were then called *transverse fins* (now *radial fins*).

Attention now turns to a formal study of extended surface heat transfer and begins with fin analyses based on a consideration of all the Murray–Gardner assumptions.

1.2 EXTENDED SURFACE HEAT TRANSFER

A growing number of engineering disciplines are concerned with energy transitions requiring the rapid movement of heat. They produce an expanding demand for high-performance heat transfer components with progressively smaller weights, volumes, costs, or accommodating shapes. *Extended surface heat transfer* is the study of these high-performance heat transfer components with respect to these parameters and of their behavior in a variety of thermal environments. Typical components are found in such diverse applications as air–land–space vehicles and their power sources, in chemical, refrigeration, and cryogenic processes, in electrical and electronic equipment, in convential furnaces and gas turbines, in process heat dissipators and waste heat boilers, and in nuclear-fuel modules.

In the design and construction of various types of heat transfer equipment, simple shapes such as cylinders, bars, and plates are used to implement the flow of heat between a source and a sink. They provide heat-absorbing or heat-rejecting surfaces, and each is known as a *prime surface*. When a prime surface is extended by appendages intimately connected with it, such as the metal tapes and spines on the tubes in Fig. 1.3, the additional surface is known as *extended surface*. In some disciplines, prime surfaces and their extended surfaces are known collectively as extended surfaces to distinguish them from prime surfaces used alone. The latter definition prevails throughout this book. The elements used to extend the prime surfaces are known as *fins*. When the fin elements are conical or cylindrical, they may be referred to as *spines* or *pegs*.

The demands for aircraft, aerospace, gas turbine, air conditioning, and cryogenic auxilliaries have placed particular emphasis on the compactness of the heat exchanger

[1]What Kern (1950) referred to as the *double pipe*.

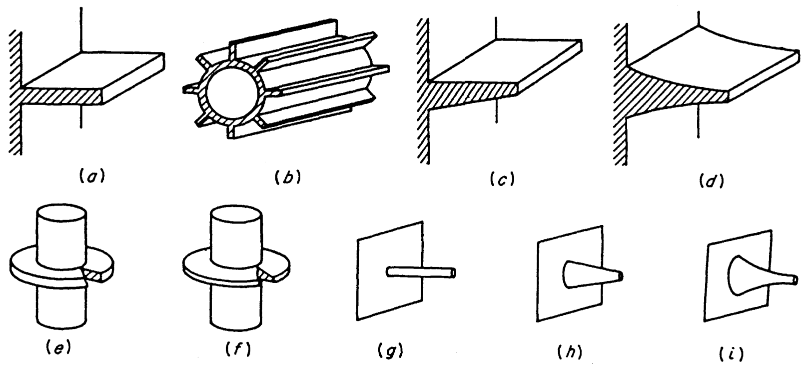

Figure 1.3 Some typical examples of extended surfaces: *(a)* longitudinal fin of rectangular profile; *(b)* cylindrical tube equipped with fins of rectangular profile; *(c)* longitudinal fin of trapezoidal profile; *(d)* longitudinal fin of parabolic profile; *(e)* cylindrical tube equipped with radial fin of rectangular profile; *(f)* cylindrical tube equipped with radial fin of trapezoidal profile; *(g)* cylindrical spine; *(h)* truncated conical spine; *(i)* truncated parabolic spine.

surface, particularly on those surfaces that induce small pressure gradients in the fluids circulated through them. Several are shown in Fig. 1.4. *Compactness* refers to the ratio of heat transfer surface per unit of exchanger volume.

An early definition by Kays and London (1950) established a compact exchanger element as one containing in excess of 245 m^2 per cubic meter of exchanger. Compact exchanger elements have been available with over 4100 m^2 per cubic meter compared with 65 to 130 m^2 per cubic meter for conventional heat exhangers with $\frac{5}{8}$- to 1-in. tubes. Many compact heat exchanger elements consist of prime surface plates or tubes separated by plates, bars, or spines, which also act as fins. As shown in Fig. 1.4*d*, each of the fins may be treated as a single fin with fin height equal to half of the separation plate spacing and with the separation plate acting as the prime surface. Thus, the compact heat exhanger is considered as another form of extended surface.

1.2.1 Fin Efficiency

It can be shown quite readily that when a fin and its prime surface are exposed to a uniform thermal environment, a unit of fin surface will be less effective than a unit of prime surface. Consider the plate with a longitudinal fin of rectangular cross section shown in Fig. 1.5. Let the inner plate surface remove heat from a source with a uniform heat transfer coefficient and temperature T_1, and let the outer plate and fin surfaces reject it to colder surroundings with a uniform heat transfer coefficient and temperature T_s. The colder surface of the plate is at some intermediate temperature T_p, and the heat from the source leaves the plate because of the temperature potential, $T_p - T_s$. Similarly, the fin surface is at some temperature, T, and the heat leaves the fin because of the temperature potential, $T - T_s$. The heat enters the fin through its base, where it joins the plate and moves continuously through it by conduction. In

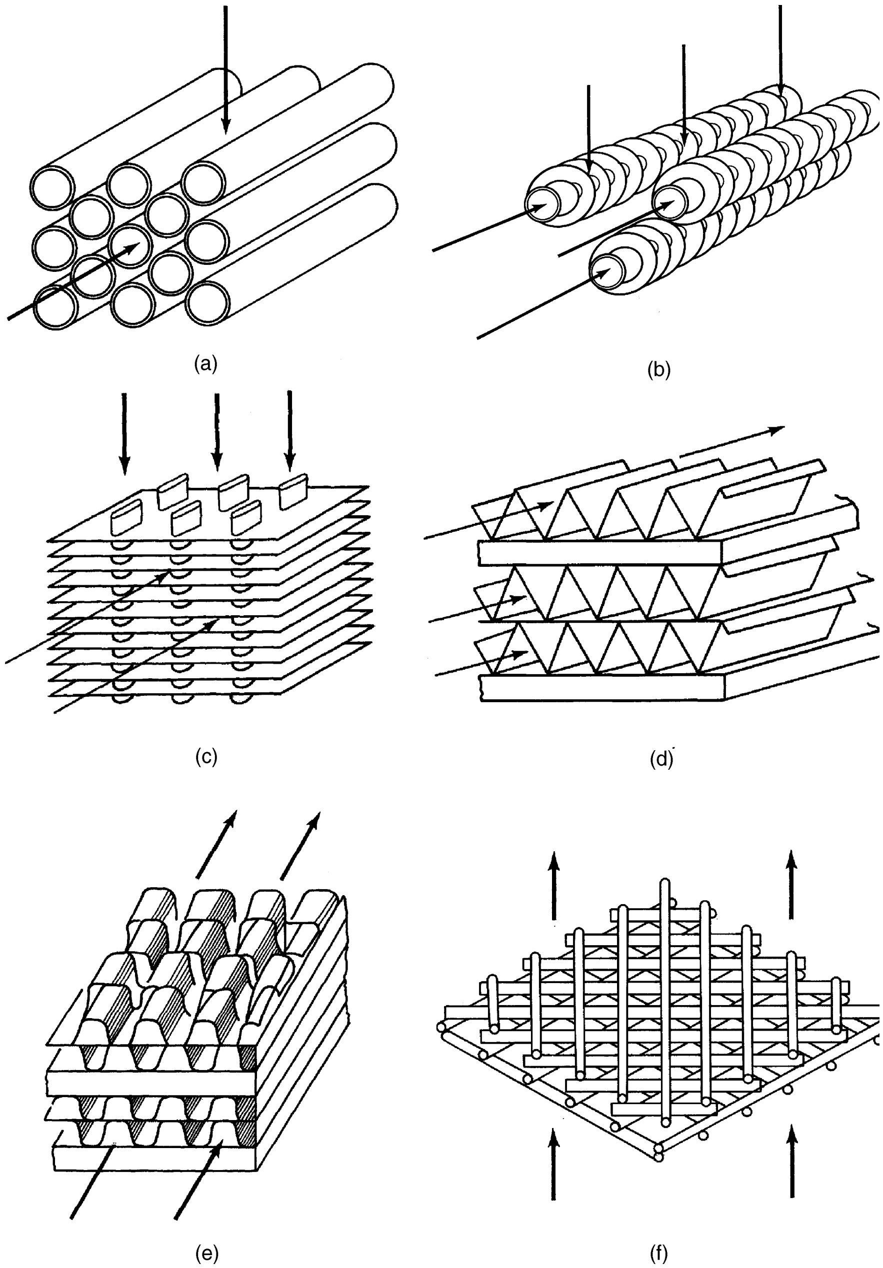

Figure 1.4 Some typical examples of compact heat exchanger surfaces: *(a)* cylindrical tube; *(b)* cylindrical tube with radial fins; *(c)* flat tube with continuous fins; *(d)* plate fin; *(e)* offset plate fin; *(f)* crossed rod matrix. (From W. M. Kays and A. L. London, *Compact Heat Exchangers*, 3rd ed., McGraw-Hill, New York, 1984, by permission.)

most cases, the temperature at the base of the fin will be very nearly the same as T_p. Heat absorbed by the fin through its base can flow toward its tip only if there is a temperature gradient within the fin such that T_p is greater than T. For this condition, because the temperature T varies from the base to the tip of the fin, the temperature potential $T - T_s$ will be smaller than $T_p - T_s$ and a unit of fin surface will be less effective than a unit of plate or prime surface.

This inescapable loss of performance of a unit of fin surface compared to a unit of prime surface is the *inefficiency* of the fin. The *fin efficiency* is defined consistently throughout this book as the ratio of the *actual heat dissipation of a fin to its ideal dissipation if the entire fin were at the same temperature as its base*. Other indexes of perfomance are also employed, such as the *fin effectiveness, weighted fin efficiency, overall passage efficiency, fin resistance*, and *fin input admittance*. Most of these are discussed in later chapters. Fins of given size, shape, and material possess different fin efficiencies, and the efficiency of any fin will vary with its thermal conductivity and the mode of heat transfer with respect to its environment.

1.2.2 Modes of Heat Transfer Involving Fins and Surroundings

The study of extended surface heat transfer in most cases comprises two factors that may conveniently be separated. One factor considers only the movement of the heat within the fin by conduction. The other considers how the fin exchanges heat with the surroundings, which usually involves convection and radiation singly or together. Indeed, a contrived hollow fin with poor thermal conductivity could be exposed to a high temperature on one side such that the movement of the heat within the fin structure could involve internal radiation as well as conduction. In such a case, the analysis would have to include the internal radiation as well. In this chapter convection from the fin faces is the only mode of heat transfer considered.

1.2.3 Limiting Assumptions

Reference has been made to extended surfaces comprising several types of prime surfaces and several types of fins. Much insight on the heat flows, temperature profiles,

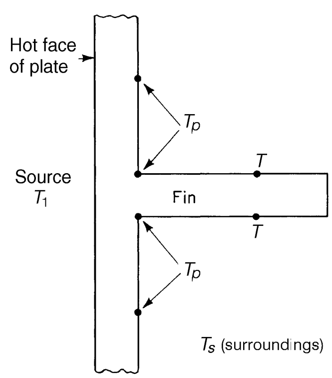

Figure 1.5 Temperature–potential differences in fins.

efficiencies, and optimization of fin parameters can be obtained from analysis of the three fundamental fin geometries shown in Fig. 1.3: longitudinal fins, radial fins, and spines.

Fins of various geometries and thermal conductivities respond differently to identical and uniform heat sources and sinks. Similarly, there are numerous ways in which the temperatures and heat transfer coefficients of sources and sinks may vary. Important to the analysis of fin geometries are the constraints or assumptions that are employed to define and limit the problem and often to simplify its solution. The analysis of the three fundamental fin geometries provided in this chapter employ the assumptions proposed by Murray (1938) and Gardner (1945). These limiting assumptions, which are almost always referred to as the *Murray–Gardner assumptions*, are:

1. The heat flow in the fin and its temperatures remain constant with time.
2. The fin material is homogeneous, its thermal conductivity is the same in all directions, and it remains constant.
3. The convective heat transfer coefficient on the faces of the fin is constant and uniform over the entire surface of the fin.
4. The temperature of the medium surrounding the fin is uniform.
5. The fin thickness is small, compared with its height and length, so that temperature gradients *across* the fin thickness and heat transfer from the edges of the fin may be neglected.
6. The temperature at the base of the fin is uniform.
7. There is no contact resistance where the base of the fin joins the prime surface.
8. There are no heat sources within the fin itself.
9. The heat transferred through the tip of the fin is negligible compared with the heat leaving its lateral surface.
10. Heat transfer to or from the fin is proportional to the temperature excess between the fin and the surrounding medium.

1.3 LONGITUDINAL FINS

1.3.1 Generalized Differential Equation

Gardner (1945) proposed that for the analysis of longitudinal fins, one employ a generalized fin. The differential equation resulting from a heat balance on an element of fin height can be compared termwise with the general form of Bessel's equation as given originally by Douglass in Sherwood and Reed (1938). This method of termwise comparison is demonstrated in Appendix A (Sections A.4.2 and A.6.1 through A.6.5).

Terminology and Coordinate System. Consider the longitudinal fin of arbitrary profile displayed in Fig. 1.6*a* and assume that the fin is dissipating or losing heat to its surroundings. Note that the dimension x pertains to the *height coordinate* which has its origin at the *fin tip* and has a positive orientation from fin tip to *fin base*. The fin profile shown in Fig. 1.6*b* is confined by two curves which are almost always symmetrical, $y = f_2(x)$ and $y = -f_2(x)$, so that the *fin thickness* is $\delta(x) = 2f_2(x)$. The *fin cross section* shown in Fig. 1.6*c* is $A(x) = f_1(x) = 2Lf_2(x)$, where L is the *fin length*, which is directed into the plane of Fig. 1.6. The *edges* of the fin are bounded by the fin profile curves, $\pm f_2(x)$, and the *fin faces* are the *lateral surfaces* of the fin bounded by one of the fin profile curves and the fin length. The fin base is shown by the crosshatched area in Fig. 1.6*c*.

Properties at the fin base, located at $x = b$, are designated by a subscript b. For example, θ_b, q_b, and T_b represent the temperature excess, heat flow, and temperature at the fin base, respectively. Properties at the fin tip are designated by a subscript a (θ_a, q_a, and T_a). In general, the tip is located at $x = a$, but for the longitudinal fins and spines discussed in this chapter, the tip and the origin of the height coordinate is specifically located at $x = a = 0$.

It is customary in extended surface analysis to deal with a *temperature excess*. Let $\theta(x)$, a function of the height coordinate x, be the temperature difference or excess between a point on the fin surface and the surroundings. Thus

$$\theta(x) = T(x) - T_s$$

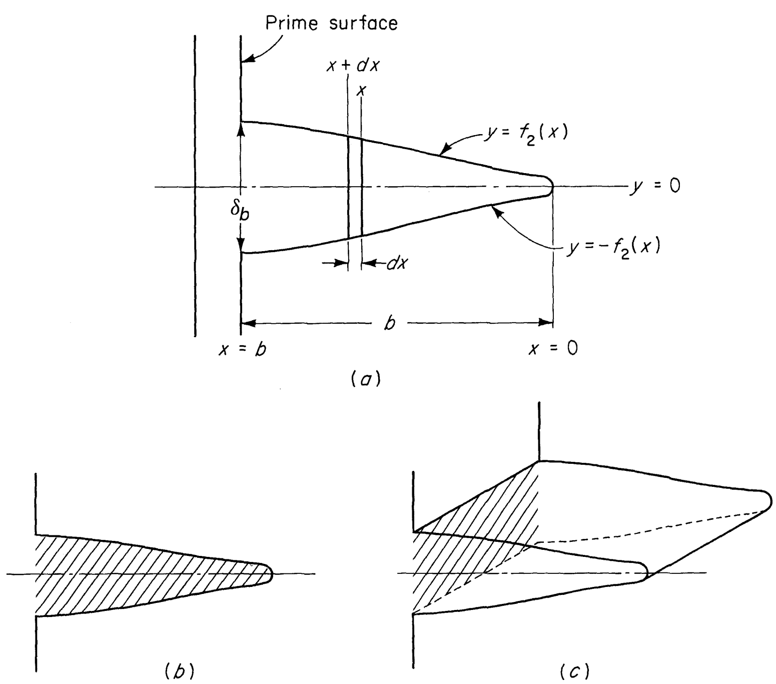

Figure 1.6 Longitudinal fin with arbitrary profile: *(a)* coordinate system; *(b)* fin profile area; *(c)* fin cross-sectional area.

Generalized Differential Equation. The differential equation for the fin temperature profile is formulated from a consideration of the steady-state heat balance over the differential element of height dx. This differential element is bounded by planes parallel to the fin base at x and $x+dx$ and by the confining profile curves, $y = \pm f_2(x)$.

If the fin surface temperature is $T(x)$, so that at dx the temperature is T and k is the fin thermal conductivity, the difference between the heat entering the element by conduction at $x + dx$ and the heat leaving the element by conduction at x is

$$dq = k\frac{d}{dx}\left[f_1(x)\frac{dT}{dx}\right] \tag{1.1}$$

To comply with the assumption of a steady state, the difference in heat conduction into and out of the element dx, as described by eq. (1.1), must be offset by some mode of heat dissipation from the exposed lateral surface of the fin. If the heat is dissipated by convection to the surrounding medium, P is the fin perimeter and h is the convective coefficient,

$$dq = hP(T - T_s)\,dx = 2h[L + f_2(x)](T - T_s)$$

However, by Murray–Gardner assumption 5, which states that the fin thickness must be small in comparison to its height and length, $L >> 2f_2(x)$. Hence

$$dq = 2hL(T - T_s)\,dx \tag{1.2}$$

This also presumes that the element dx on the arbitrary surface described by $f_2(x)$ is equal in height to the element dx on the x-axis.[2] The temperature between a point on the fin and the surroundings at T_s is $\theta = T - T_s$ and because T_s is assumed constant, $d\theta = dT$.

Equations (1.1) and (1.2) may be equated to yield the general differential equation

$$k\frac{d}{dx}\left[f_1(x)\frac{d\theta}{dx}\right] = 2hL\theta\,dx$$

or

$$f_1(x)\frac{d^2\theta}{dx^2} + \frac{df_1(x)}{dx}\frac{d\theta}{dx} - \frac{2h}{k}\theta = 0 \tag{1.3}$$

With $f_1(x) = 2Lf_2(x)$, eq. (1.3) becomes

$$2Lf_2(x)\frac{d^2\theta}{dx^2} + \frac{2Ldf_2(x)}{dx}\frac{d\theta}{dx} - \frac{2h}{k}\theta = 0 \tag{1.4}$$

The profile function $f_2(x)$ for longitudinal fins usually will take the form

$$f_2(x) = \frac{\delta_b}{2}\left(\frac{x}{b}\right)^{(1-2n)/(1-n)} \tag{1.5}$$

[2] Gardner (1945) pointed out that this is generally valid for thin fins and spines because the square of the slope of the fin sides is negligible compared with unity. This is what has come to be called the *length of arc assumption*.

where δ_b is the fin thickness at its base. The particular solution may be obtained by substituting the *boundary conditions* into the general solution to eliminate the arbitrary constants

$$\theta(x = b) = \theta_b \tag{1.6a}$$

and

$$\left.\frac{d\theta}{dx}\right|_{x=0} = 0 \tag{1.6b}$$

The particular solution may also be obtained by substituting the *initial conditions*

$$\theta(x = b) = \theta_b \tag{1.6c}$$

and

$$q(x = b) = q_b = -kA\left.\frac{d\theta}{dx}\right|_{x=b} \tag{1.6d}$$

into the general solution to eliminate the arbitrary constants.

1.3.2 Longitudinal Fin of Rectangular Profile

For the longitudinal fin of rectangular profile displayed with its terminology and coordinate system in Fig. 1.7, the exponent on the general fin profile of eq. (1.5) satisfies the geometry when $n = \frac{1}{2}$. The profile function for this fin then becomes

$$f_2(x) = \frac{\delta_b}{2} = \frac{\delta}{2}$$

because $\delta_b = \delta$ and

$$\frac{df_2(x)}{dx} = 0$$

When these are substituted into eq. (1.4), the governing differential equation becomes

$$\frac{d^2\theta}{dx^2} - \frac{2h}{k\delta}\theta = 0 \tag{1.7}$$

which is an ordinary second-order differential equation with constant coefficients. The general solution is

$$\theta = C_1 e^{mx} + C_2 e^{-mx} \tag{1.8}$$

where m is referred to as the *fin performance factor*,

$$m = \left(\frac{2h}{k\delta}\right)^{1/2}$$

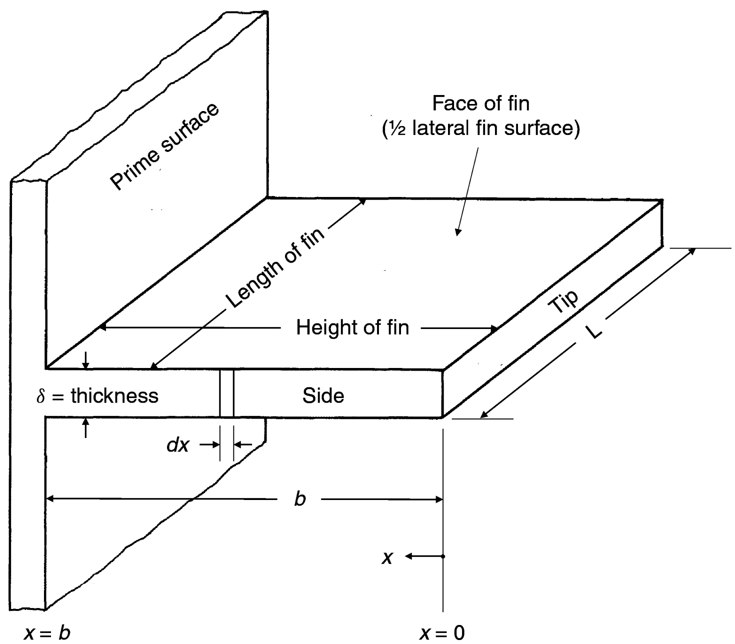

Figure 1.7 Terminology and coordinate system for the longitudinal fin of rectangular profile.

Application of the boundary conditions of eqs. (1.6*a*) and (1.6*b*) permits evaluation of the arbitrary constants C_1 and C_2. This evaluation yields the point-to-point temperature excess, $\theta(x) = T(x) - T_s$:

$$\theta(x) = \frac{\theta_b \cosh mx}{\cosh mb} \tag{1.9}$$

Because the orientation of the height coordinate x is opposed to the direction of the heat flow in the fin, the heat flow through the base of the fin q_b is obtained from

$$q_b = kA \left. \frac{d\theta}{dx} \right|_{x=b}$$

The fin cross-sectional area is equal to $A = \delta L$, so that in using the derivative of eq. (1.9) evaluated at $x = b$,

$$q_b = \frac{k\delta Lm\theta_b \sinh mb}{\cosh mb}$$

or

$$q_b = k\delta Lm\theta_b \tanh mb \tag{1.10}$$

For the longitudinal fin of rectangular profile, the actual heat flow is given by eq. (1.10). The ideal heat flow is $q_{\text{id}} = hP\theta_b$, where P is the perimeter of the fin,

$P = 2(L + \delta)$. Because $L >> \delta$, the ideal heat flow is $q_{\text{id}} = 2hL\theta_b$, so that the efficiency becomes

$$\eta = \frac{k\delta L m\theta_b \tanh mb}{2hL\theta_b}$$

and by noting that $k\delta/2h = m^2$, the efficiency may be written as

$$\eta = \frac{\tanh mb}{mb} \tag{1.11}$$

Values of η as a function of mb have been plotted from eq. (1.11) in Fig. 1.1, which also displays efficiency values for several other longitudinal fin profiles that will be studied. Observe, however, that Gardner (1945) designates the fin efficiency by ϕ rather than η.

Example 1.1: Longitudinal Fin of Rectangular Profile. A longitudinal fin of rectangular profile is exposed to surroundings at a temperature of 50°C and a heat transfer coefficient of $h = 50.2\ \text{W/m}^2 \cdot \text{K}$. The temperature at the fin base is 90°C and the fin is made from a steel with $k = 33.5\ \text{W/m} \cdot \text{K}$. The fin is 101.6 mm high and 9.525 mm thick. Determine (**a**) the fin efficiency, (**b**) the temperature at the tip of the fin, and (**c**) the dissipation of the fin if it is 250 mm long. (**d–f**) Repeat the foregoing procedure for a quintupled heat transfer coefficient of $h = 251\ \text{W/m}^2 \cdot \text{K}$.

SOLUTION. For the surroundings with $h = 50.2\ \text{W/m}^2 \cdot \text{K}$,

$$\theta_b = 90 - 50 = 40°\text{C} \qquad \delta = 9.525/1000 = 9.525 \times 10^{-3}\ \text{m}$$

$$m = \left(\frac{2h}{k\delta}\right)^{1/2} = \left[\frac{(2)(50.2)}{(33.5)(9.525 \times 10^{-3})}\right]^{1/2} = 17.738\ \text{m}^{-1}$$

$$b = 101.6/1000 = 0.1016\ \text{m} \qquad mb = (17.738)(0.1016) = 1.802$$

(**a**) From eq. (e.11),

$$\eta = \frac{\tanh mb}{mb} = \frac{\tanh 1.802}{1.802} = \frac{0.947}{1.802} = 0.526$$

(**b**) The tip temperature of the fin is determined from eq. (1.9) at $x = 0$, where $\theta(x = 0) = \theta_a$:

$$\theta_a = \frac{\theta_b \cosh mx}{\cosh mb} = \frac{40(\cosh 0)}{\cosh 1.802} = \frac{(40)(1.00)}{3.114} = 12.8°\text{C}$$

so that at $x = 0$, where $T(x = 0) = T_a$,

$$T_a = \theta_a + T_s = 12.8 + 50 = 62.8°\text{C}$$

(**c**) The heat dissipated by the fin is calculated from eq. (1.10):

$$q_b = k\delta L m\theta_b \tanh mb$$

$$= (33.5)(9.525 \times 10^{-3})(0.25)(17.738)(40)(0.947)$$
$$= 53.6 \text{ W}$$

The heat transferred can also be obtained from the ideal heat dissipation and the efficiency determined in part (a). With the surface area of the fin, $S = 2bL$,

$$q_{\text{id}} = 2bLh\theta_b = (2)(0.1016)(0.25)(50.2)(40) = 102.0 \text{ W}$$

the heat dissipated is

$$q_b = \eta q_{\text{id}} = (0.526)(102.0) = 53.6 \text{ W}$$

(**d**) With $h = 251 \text{ W/m}^2 \cdot \text{K}$

$$m = \left(\frac{2h}{k\delta}\right)^{1/2} = \left[\frac{(2)(251)}{(33.5)(9.525 \times 10^{-3})}\right]^{1/2} = 39.664 \text{ m}^{-1}$$
$$mb = (39.664)(0.1016) = 4.030$$

and from eq. (1.11),

$$\eta = \frac{\tanh\ mb}{mb} = \frac{\tanh\ 4.030}{4.030} = \frac{0.999}{4.030} = 0.248$$

(**e**) The tip temperature is determined from eq. (1.9):

$$\theta_a = \frac{\theta_b\ \cosh\ mx}{\cosh\ mb} = \frac{40(\cosh\ 0)}{\cosh\ 4.030} = \frac{40}{28.139} = 1.4°\text{C}$$

so that at $x = 0$, where $T(x = 0) = T_a$,

$$T_a = \theta_a + T_s = 1.4 + 50 = 51.4°\text{C}$$

(**f**) The heat dissipated by the fin is calculated from eq. (1.10) from parts (a), (c), and (d):

$$q_b = k\delta Lm\theta_b \tanh\ mb$$
$$= (33.5)(9.525 \times 10^{-3})(0.25)(39.664)(40)(0.999)$$
$$= 126.5 \text{ W}$$

The heat transferred can also be obtained from the ideal heat dissipation and the efficiency determined in part (d). With

$$q_{\text{id}} = 2bLh\theta_b = (2)(0.1016)(0.25)(251)(40) = 510.0 \text{ W}$$

the heat dissipated is

$$q_b = \eta q_{\text{id}} = (0.248)(510.0) = 126.5 \text{ W}$$

1.3.3 Longitudinal Fin of Triangular Profile

For the longitudinal fin of triangular profile shown in Fig. 1.8, it is noted that the exponent on the general fin profile of eq. (1.5) satisfies the geometry when $n = 0$. The profile function for this fin then becomes

$$f_2(x) = \frac{\delta_b}{2}\frac{x}{b}$$

and

$$\frac{df_2(x)}{dx} = \frac{\delta_b}{2b}$$

When these are substituted into eq. (1.4), the governing differential equation for the temperature excess, $\theta(x) = T(x) - T_s$, becomes

$$x\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} - m^2 b\theta = 0 \tag{1.12}$$

where again, $m = (2h/k\delta)^{1/2}$.

Equation (1.12) is an ordinary second-order differential equation with variable coefficients. As shown in Section A.6.1, its general solution is

$$\theta(x) = C_1 I_0(2m\sqrt{bx}) + C_2 K_0(2m\sqrt{bx}) \tag{1.13}$$

and it can be observed that to have a finite temperature excess at the fin tip where $x = 0$, the arbitrary constant C_2 must equal zero because $K_0(0)$ is unbounded. This leaves

$$\theta(x) = C_1 I_0(2m\sqrt{bx})$$

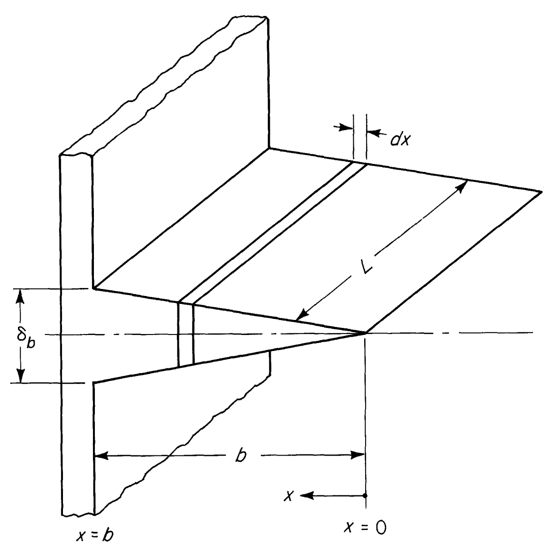

Figure 1.8 Longitudinal fin of triangular profile.

C_1 is evaluated using the boundary condition of eq (1.6a). The particular solution is

$$\theta(x) = \frac{\theta_b I_0(2m\sqrt{bx})}{I_0(2mb)} \tag{1.14}$$

The heat disspated by the fin must equal the heat flow through the base of the fin and is obtained using eq. (1.14), noting that $A = \delta_b L$, writing the Bessel function series expansion for $I_0(2m\sqrt{bx})$, differentiating term by term, and evaluating the derivative at $x = b$. The result is

$$q_b = kA\frac{dT}{dx}\bigg|_{x=b} = \frac{2hL\theta_b I_1(2mb)}{mI_0(2mb)} \tag{1.15}$$

The fin efficiency is the ratio of the actual heat flow given by eq. (1.15) to the ideal heat flow, $q_{\text{id}} = 2hbL\theta_b$:

$$\eta = \frac{2hL\theta_b[I_1(2mb)/mI_0(2mb)]}{2hbL\theta_b} = \frac{I_1(2mb)}{(mb)I_0(2mb)} \tag{1.16}$$

Values of η as a function of mb have been plotted from eq. (1.16) in Fig. 1.1.

1.3.4 Longitudinal Fin of Concave Parabolic Profile

For the longitudinal fin of concave parabolic profile shown in Fig. 1.9, it is noted that the exponent on the general fin profile of eq. (1.5) satisfies the geometry when $n = \infty$. The profile function for this fin then becomes

$$f_2(x) = \frac{\delta_b}{2}\left(\frac{x}{b}\right)^2$$

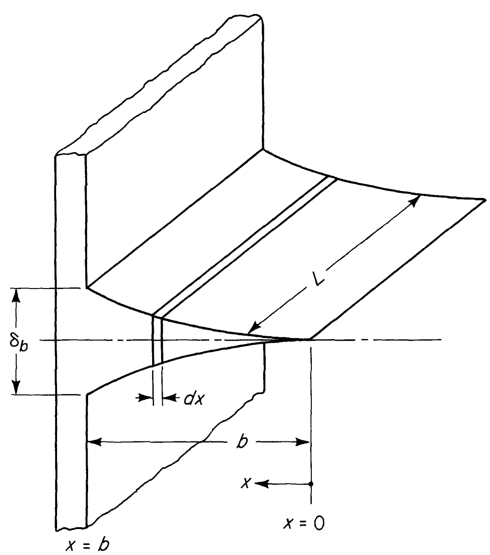

Figure 1.9 Longitudinal fin of concave parabolic profile.

and

$$\frac{df_2(x)}{dx} = \frac{\delta_b}{b}\frac{x}{b}$$

When these are substituted into eq. (1.4), the governing differential equation for the temperature excess, $\theta(x) = T(x) - T_s$, becomes

$$x^2\frac{d^2\theta}{dx^2} + 2x\frac{d\theta}{dx} - m^2b^2\theta = 0 \tag{1.17}$$

where again, $m = (2h/k\delta)^{1/2}$.

Equation (1.17) is an ordinary second-order differential equation with variable coefficients. It is known as an *Euler equation* and its general solution is obtained by making the transformation $x = e^v$ or $v = \ln x$. Then

$$\frac{d\theta}{dx} = \frac{d\theta}{dv}\frac{dv}{dx} = \frac{1}{x}\frac{d\theta}{dv}$$

and

$$\frac{d^2\theta}{dx^2} = \frac{d\,[(1/x)(d\theta/dv)]}{dx} = -\frac{1}{x^2}\frac{d\theta}{dv} + \frac{1}{x}\frac{d(d\theta/dv)}{dx}$$

and after simplification,

$$\frac{d^2\theta}{dx^2} = -\frac{1}{x^2}\frac{d\theta}{dv} + \frac{1}{x^2}\frac{d^2\theta}{dv^2}$$

With these transformations in hand, eq. (1.17) becomes

$$x^2\left(\frac{1}{x^2}\frac{d^2\theta}{dv^2} - \frac{1}{x^2}\frac{d\theta}{dv}\right) + 2x\left(\frac{1}{x}\frac{d\theta}{dv}\right) - m^2b^2\theta = 0$$

Canceling common terms gives an ordinary differential equation with constant coefficients:

$$\frac{d^2\theta}{dv^2} + \frac{d\theta}{dv} - m^2b^2\theta = 0$$

which has as its solution

$$\theta = C_1e^{\alpha v} + C_2e^{\beta v}$$

or in terms of the independent variable x,

$$\theta(x) = C_1x^{\alpha} + C_2x^{\beta} \tag{1.18}$$

where

$$\alpha, \beta = -\tfrac{1}{2} \pm \tfrac{1}{2}(1 + 4m^2b^2)^{1/2}$$

The general solution may be written

$$\theta(x) = C_1 x^{\alpha} + \frac{C_2}{x^{1/\beta}}$$

and it can be observed that at $x = 0$, the temperature excess, $T - T_s$, will be unbounded unless $C_2 = 0$. Therefore,

$$\theta(x) = C_1 x^{\alpha}$$

and from a consideration of the temperature excess at the fin base where $x = b$, the particular solution is obtained as

$$\theta(x) = \theta_b \left(\frac{x}{b}\right)^{\alpha} \tag{1.19}$$

Heat flow through the base of the fin is obtained by differentiating eq. (1.19) and evaluating the derivative at $x = b$. Noting that $A = \delta_b L$, the result is

$$q_b = kA \left.\frac{d\theta}{dx}\right|_{x=b} = \frac{k\delta_b L\theta_b \alpha}{b}$$

or

$$q_b = \frac{k\delta_b L\theta_b}{2b}\left[-1 + \sqrt{1 + (2mb)^2}\right] \tag{1.20}$$

The expression for the fin efficiency results when eq. (1.20) is divided by the ideal heat flow, $q_{\text{id}} = 2hbL\theta_b$:

$$\eta = \frac{k\delta_b L\theta_b \left[-1 + \sqrt{1 + (2mb)^2}\right]}{(2b)(2hbL\theta_b)}$$

This may be simplified by multiplying the numerator and denominator by $-1 - \sqrt{1 + (2mb)^2}$ and noting that as $m^2 = 2h/k\delta_b$,

$$\eta = \frac{-1 + \sqrt{1 + (2mb)^2}}{2(mb)^2} \frac{-1 - \sqrt{1 + (2mb)^2}}{-1 - \sqrt{1 + (2mb)^2}}$$

so that

$$\eta = \frac{2}{1 + \sqrt{1 + (2mb)^2}} \tag{1.21}$$

Values of η as a function of mb have been plotted from eq. (1.21) in Fig. 1.1.

1.3.5 Longitudinal Fin of Convex Parabolic Profile

For the longitudinal fin of convex parabolic profile shown in Fig. 1.10, it is noted that the exponent on the general fin profile of eq. (1.5) satisfies the geometry when $n = \frac{1}{3}$. The profile function for this fin then becomes

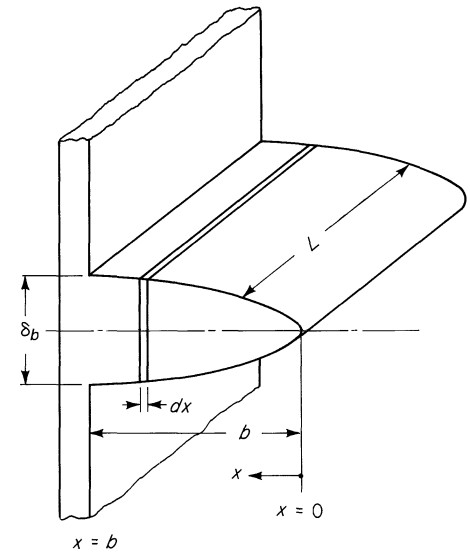

Figure 1.10 Longitudinal fin of convex parabolic profile.

$$f_2(x) = \frac{\delta_b}{2}\left(\frac{x}{b}\right)^{1/2}$$

and

$$\frac{df_2(x)}{dx} = \frac{\delta_b}{4\sqrt{bx}}$$

When these are substituted into eq. (1.4), the governing differential equation for the temperature excess, $\theta(x) = T(x) - T_s$, becomes

$$\sqrt{x}\frac{d^2\theta}{dx^2} + \frac{1}{2\sqrt{x}}\frac{d\theta}{dx} - m^2\sqrt{b}\,\theta = 0 \tag{1.22}$$

where again, $m = (2h/k\delta)^{1/2}$.

A termwise comparison (see Section A.6.2) with the general Bessel equation leads to the general solution for the temperature excess, $\theta(x) = T(x) - T_s$:

$$\theta(x) = x^{1/4}\left[C_1 I_{1/3}\left(\tfrac{4}{3}mb^{1/4}x^{3/4}\right) + C_2 I_{-1/3}\left(\tfrac{4}{3}mb^{1/4}x^{3/4}\right)\right] \tag{1.23}$$

Evaluation of the arbitrary constants in eq. (1.23) requires careful consideration of the infinite series expansions of the two Bessel functions.

Define a transformed variable u:

$$u \equiv \tfrac{4}{3}mb^{1/4}x^{3/4}$$

so that eq. (1.23) may be rewritten as

$$\theta(u) = \Omega u^{1/3}[C_1 I_{1/3}(u) + C_2 I_{-1/3}(u)] \tag{1.24}$$

where

$$\Omega \equiv \left(\frac{3}{4mb^{1/4}}\right)^{1/3}$$

and where the boundary conditions of eqs. (1.6*a*) and (1.6*b*) in terms of the transformed variable u are

$$\theta\left(u = u_b = \tfrac{4}{3}mb\right) = \theta_b \tag{1.25a}$$

and

$$\left.\frac{d\theta}{du}\right|_{u=0} = 0 \tag{1.25b}$$

Use of the boundary condition of eq. (1.25*b*) requires multiplication of each of the terms of the infinite series expansion for $I_{1/3}(u)$ and $I_{-1/3}(u)$ by $u^{1/3}$ followed by a term-by-term differentiation. When this procedure is performed, the term involving

$$\frac{d}{du}\left[u^{1/3}I_{1/3}(u)\right]$$

becomes unbounded at $x = 0$. This requires that $C_1 = 0$. Then application of the boundary condition of eq. (1.25*a*) yields a value for C_2 such that the particular solution of eq. (1.24) in terms of u becomes

$$\theta(u) = \frac{\Omega u^{1/3}\theta_b I_{-1/3}(u)}{\Omega u_b^{1/3} I_{-1/3}(u_b)} = \theta_b\left(\frac{u}{u_b}\right)^{1/3} \frac{I_{-1/3}(u)}{I_{-1/3}(u_b)}$$

and in terms of x,

$$\theta(x) = \theta_b\left(\frac{x}{b}\right)^{1/4} \frac{I_{-1/3}\left(\frac{4}{3}mb^{1/4}x^{3/4}\right)}{I_{-1/3}\left(\frac{4}{3}mb\right)} \tag{1.26}$$

The heat flow through the base of the fin is obtained by differentiating eq. (1.26) term by term and evaluating the derivative at $x = b$. Again noting that $A = \delta bL$,

$$q_b = k\delta_b Lm\theta_b \frac{I_{2/3}\left(\frac{4}{3}mb\right)}{I_{-1/3}\left(\frac{4}{3}mb\right)} \tag{1.27}$$

The fin efficiency can be obtained by taking the ratio of eq. (1.27) to the ideal heat flow, $q_{\text{id}} = 2hbL\theta_b$:

$$\eta = \frac{q_b}{2hbL\theta_b} = \frac{mk\delta_b\theta_b I_{2/3}\left(\frac{4}{3}mb\right)}{2hb\theta_b I_{-1/3}\left(\frac{4}{3}mb\right)}$$

or

$$\eta = \frac{1}{mb} \frac{I_{2/3}\left(\frac{4}{3}mb\right)}{I_{-1/3}\left(\frac{4}{3}mb\right)} \tag{1.28}$$

Values of η as a function of mb have been plotted from eq. (1.28) in Fig. 1.1.

Example 1.2: Comparison of Longitudinal Fins of Different Profiles. Longitudinal fins of different profiles are exposed to surroundings at a temperature of 20°C and a heat transfer coefficient of $h = 40\ \text{W/m}^2 \cdot \text{K}$. In all cases, the temperature at the fin base is 90°C and the fins are made from a steel with $k = 30\ \text{W/m} \cdot \text{K}$. All fins are 10 cm high with bases 0.80 cm thick. Compare the fin efficiencies, the dissipation of the fin per unit length, and the tip temperatures if the profiles are (**a**) rectangular, (**b**) triangular, (**c**) concave parabolic, and (**d**) convex parabolic.

SOLUTION. For all fins

$$\theta_b = 90 - 20 = 70°\text{C} \qquad \delta = 0.80/1000 = 0.008\ \text{m}$$

$$m = \left(\frac{2h}{k\delta}\right)^{1/2} = \left[\frac{(2)(40)}{(30)(0.008)}\right]^{1/2} = 18.257\ \text{m}^{-1}$$

$$b = 10/100 = 0.100\ \text{m} \qquad mb = (18.257)(0.100) = 1.8257$$

(**a**) For the rectangular profile, by eq. (1.11),

$$\eta = \frac{\tanh\ mb}{mb} = \frac{\tanh\ 1.8257}{1.8257} = \frac{0.949}{1.8257} = 0.520$$

by eq. (1.10),

$$\begin{aligned} q_b &= k\delta_b L m\theta_b \tanh\ mb \\ &= (30)(0.008)(1.00)(18.257)(70)(0.949) \\ &= 291.1\ \text{W} \end{aligned}$$

and by eq. (1.9),

$$\theta_a = \frac{\theta_b\ \cosh\ mx}{\cosh\ mb} = \frac{70\ \cosh\ 0}{\cosh\ 1.8257} = \frac{70}{3.184} = 22.0°\text{C}$$

so that

$$T_a = \theta_a + 20 = 22 + 20 = 44°\text{C}$$

(**b**) For the triangular profile,[3]

$$2mb = (2)(1.8257) = 3.6514$$

[3]Numerical values for all of the modified Bessel functions in this example can be obtained from tables or computer codes.

By eq. (1.16),

$$\eta = \frac{I_1(2mb)}{mbI_0(2mb)} = \frac{I_1(3.6514)}{1.8257I_0(3.6514)} = \frac{7.1133}{(1.8257)(8.3327)} = 0.468$$

by eq. (1.15),

$$\begin{aligned} q_b &= \frac{2hL\theta_b I_1(2mb)}{mI_0(2mb)} \\ &= \frac{(2)(40)(1.00)(70)(7.1133)}{(18.257)(8.3327)} \\ &= 261.8 \text{ W} \end{aligned}$$

and by eq. (1.14),

$$\theta_a = \frac{\theta_b I_0(2m\sqrt{bx})}{I_0(2mb)} = \frac{70I_0(0)}{8.3327} = \frac{(70)(1.00)}{8.3327} = 8.4°\text{C}$$

so that

$$T_a = \theta_a + 20 = 8.4 + 20 = 28.4°\text{C}$$

(**c**) For the concave parabolic profile, by eq. (1.21),

$$\eta = \frac{2}{1 + \sqrt{1 + (2mb)^2}} = \frac{2}{1 + \sqrt{1 + (3.6514)^2}} = 0.418$$

and by eq. (1.20),

$$\begin{aligned} q_b &= \frac{k\delta_b L\theta_b}{2b}\left[-1 + \sqrt{1 + (2mb)^2}\right] \\ &= \frac{(30)((0.008)(1.00)(70)}{(2)(0.100)}\left[-1 + \sqrt{1 + (3.6514)^2}\right] \\ &= 234.0 \text{ W} \end{aligned}$$

Equation (1.19) shows that at $x = 0$, $\theta(x = 0) = 0$ and the tip temperature will approximate the temperature of the surroundings. Thus,

$$T_a \approx 20°\text{C}$$

(**d**) For the convex parabolic profile, by eq. (1.28),

$$\eta = \frac{1}{mb}\frac{I_{2/3}\left(\frac{4}{3}mb\right)}{I_{-1/3}\left(\frac{4}{3}mb\right)}$$

Here

$$\tfrac{4}{3}mb = 2.4343$$

and

$$\eta = \frac{1}{1.826} \frac{I_{2/3}(2.434)}{I_{-1/3}(2.434)} = \frac{2.7419}{(1.826)(3.0512)} = 0.492$$

and by eq. (1.27),

$$\begin{aligned} q_b &= k\delta_b L m \theta_b \frac{I_{2/3}\left(\frac{4}{3}mb\right)}{I_{-1/3}\left(\frac{4}{3}mb\right)} \\ &= (30)(0.008)(1.00)(70)(18.257)\left(\frac{2.7419}{3.0512}\right) \\ &= 275.6 \text{ W} \end{aligned}$$

Here, too, the temperature at the tip of the fin will approximate the temperature of the surroundings, This can be deduced from eq. (1.26), where the presence of the $(x/b)^{1/4}$ term causes θ_b to reduce to zero at $x = 0$. Thus $T_a \approx 20°\text{C}$.

Following is a summary of the performance of the four fins for the conditions imposed:

Fin Profile	η	q_b (W)	T_a (°C)
Rectangular	0.520	291.1	44.0
Triangular	0.468	261.8	28.4
Concave parabolic	0.418	234.0	≈ 20
Convex parabolic	0.492	275.6	≈ 20

1.3.6 Longitudinal Fin of Least Material

A discussion of the longitudinal fin that yields minimum weight is presented in Chapter 3.

1.4 RADIAL FINS

1.4.1 Generalized Differential Equation

Consider the radial fin of arbitrary profile shown in Fig. 1.11. A generalized differential equation can be developed for any radial fin of arbitrary profile function by a procedure similar to that used for the longitudinal fin whose profile is confined by two symmetrical curves that are functions of *the radial coordinate* r, $y = f_2(r)$ and $y = -f_2(r)$. Observe that the fin height is in the direction of the radial coordinate and that the origin of this coordinate is taken at the point where $r = 0$.

The difference in heat conducted into the differential element at r and that leaving the element a $r + dr$ in terms of the temperature excess $\theta = T - T_s$ is

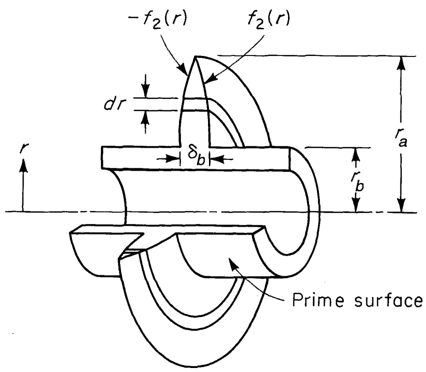

Figure 1.11 Radial fin of arbitrary profile.

$$dq = k\frac{d}{dr}\left[(2\pi r)2f_2(r)\frac{d\theta}{dr}\right]dr$$

This is the equation for a time-invariant steady-state system, and in accordance with an energy balance, it can be equated to the heat leaving the element dr by convection:

$$dq = 2h(2\pi r\, dr)\theta$$

The heat balance is

$$4\pi k\frac{d}{dr}\left[rf_2(r)\frac{d\theta}{dr}\right]dr = 4\pi h\theta r\, dr$$

or

$$k\left[f_2(r)\frac{d^2\theta}{dr^2} + f_2(r)\frac{d\theta}{dr} + r\frac{df_2(r)}{dr}\frac{d\theta}{dr}\right] = h\theta r$$

which, upon rearrangement, yields the generalized differential equation

$$f_2(r)\frac{d^2\theta}{dr^2} + \frac{f_2(r)}{r}\frac{d\theta}{dr} + \frac{df_2(r)}{dr}\frac{d\theta}{dr} - \frac{h}{k}\theta = 0 \tag{1.29}$$

1.4.2 Radial Fin of Rectangular Profile

For the radial fin of rectangular profile shown in Fig. 1.12, the profile function is

$$f_2(r) = \frac{\delta}{2}$$

and its derivative

$$\frac{df_2(r)}{dr} = 0$$

With these substituted, eq. (1.29) becomes

$$r^2\frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - m^2r^2\theta \tag{1.30}$$

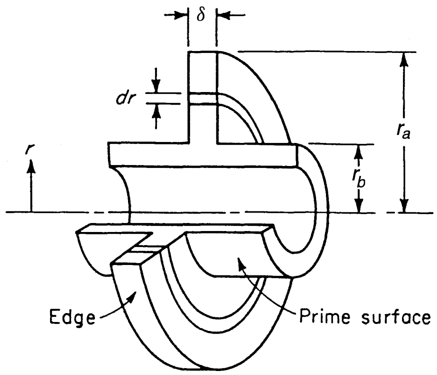

Figure 1.12 Radial fin of rectangular profile.

where $m = (2h/k\delta)^{1/2}$. Equation (1.30) is Bessel's modified equation, which has a solution in terms of the modified Bessel functions:

$$\theta(r) = C_1 I_0(mr) + C_2 K_0(mr) \tag{1.31}$$

The arbitrary constants are evaluated using the boundary conditions

$$\theta(r = r_b) = \theta_b \tag{1.32a}$$

and

$$\left.\frac{d\theta}{dr}\right|_{r=r_a} = 0 \tag{1.32b}$$

When these boundary conditions are used with eq. (1.31), two simultaneous equations in C_1 and C_2 result

$$\begin{aligned} \theta_b &= C_1 I_0(mr_b) + C_2 K_0(mr_b) \\ 0 &= C_1 I_1(mr_a) - C_2 K_1(mr_a) \end{aligned}$$

When C_1 and C_2 are evaluated and inserted into eq. (1.31), the equation for the temperature excess becomes

$$\theta(r) = \frac{\theta_b[K_1(mr_a)I_0(mr) + I_1(mr_a)K_0(mr)]}{I_0(mr_b)K_1(mr_a) + I_1(mr_a)K_0(mr_b)} \tag{1.33}$$

Note that when $r = r_b$, eq. (1.33) reduces to $\theta = \theta_b$, as it should.

The heat flow through the base is determined[4] from the general relationship

$$q_b = -2\pi r_b k\delta \left.\frac{d\theta}{dr}\right|_{r=r_b}$$

[4]Here the minus sign must be used because the temperature gradient decreases with increasing height coordinate r.

The result, after differentiating eq. (1.33) and evaluating at $r = r_b$, is

$$q_b = 2\pi r_b \delta k m \theta_b \frac{I_1(mr_a)K_1(mr_b) - K_1(mr_a)I_1(mr_b)}{I_0(mr_b)K_1(mr_a) + I_1(mr_a)K_0(mr_b)} \tag{1.34}$$

The ideal heat flow

$$q_{\text{id}} = 2\pi(r_a^2 - r_b^2)h\theta_b$$

can be used with eq. (1.34) to determine the fin efficiency:

$$\eta = \frac{q_b}{q_{\text{id}}} = \frac{2\pi r_b \delta k m \theta_b}{2\pi(r_a^2 - r_b^2)h\theta_b} \frac{I_1(mr_a)K_1(mr_b) - K_1(mr_a)I_1(mr_b)}{I_0(mr_b)K_1(mr_a) + I_1(mr_a)K_0(mr_b)}$$

and by noting that $m^2 = 2h/k\delta$, an alternative form is obtained:

$$\eta = \frac{2r_b}{m(r_a^2 - r_b^2)} \frac{I_1(mr_a)K_1(mr_b) - K_1(mr_a)I_1(mr_b)}{I_0(mr_b)K_1(mr_a) + I_1(mr_a)K_0(mr_b)} \tag{1.35}$$

The fin efficiency expressed by eq. (1.35) does not lend itself to comparison with the efficiencies of fins of other radial profiles. However, eq. (1.35) can be adjusted by expressing the efficiency in terms of the radius ratio,

$$\rho \equiv \frac{r_b}{r_a} \tag{1.36}$$

and a parameter ϕ defined by

$$\phi \equiv (r_a - r_b)^{3/2} \left(\frac{2h}{kA_p}\right)^{1/2} \tag{1.37}$$

where A_p is the profile area of the fin:

$$A_p = \delta(r_a - r_b)$$

The arguments of the Bessel functions in eq. (1.35) can be expressed in terms of the profile area:

$$mr_a = \left(\frac{2h}{k\delta}\right)^{1/2} r_a = r_a \left[\frac{2h(r_a - r_b)}{kA_p}\right]^{1/2}$$

or

$$mr_a = r_a(r_a - r_b)^{1/2} \left(\frac{2h}{kA_p}\right)^{1/2} \tag{1.38}$$

In a similar fashion,

$$mr_b = r_b(r_a - r_b)^{1/2} \left(\frac{2h}{kA_p}\right)^{1/2} \tag{1.39}$$

The fin height is the difference between the fin radii, $b = r_a - r_b$. If both numerators and denominators of eqs. (1.38) and (1.39) are multiplied by $b = r_a - r_b$, the result is

$$mr_a = \frac{r_a(r_a - r_b)^{3/2}(2h/kA_p)^{1/2}}{r_a - r_b} = \frac{r_a\phi}{r_a - r_b} \tag{1.40}$$

and an identical procedure gives

$$mr_b = \frac{r_b\phi}{r_a - r_b} \tag{1.41}$$

Now define two additional radius functions,

$$R_a \equiv \frac{1}{1 - r_b/r_a} = \frac{1}{1 - \rho}$$

and

$$R_b \equiv \rho R_a = \frac{\rho}{1 - \rho}$$

Substituting these into eqs. (1.40) and (1.41) yields

$$mr_a = \frac{\phi}{1 - \rho} = \phi R_a$$

and

$$mr_b = \frac{\rho\phi}{1 - \rho} = \phi R_b$$

Finally, the portion of eq. (1.35) preceding the final term may be expressed in terms of ϕ and ρ so that the efficiency of the radial fin of rectangular profile becomes

$$\eta = \frac{2\rho}{\phi(1 + \rho)} \frac{I_1(\phi R_a)K_1(\phi R_b) - I_1(\phi R_b)K_1(\phi R_a)}{I_0(\phi R_b)K_1(\phi R_a) + I_1(\phi R_a)K_0(\phi R_b)} \tag{1.42}$$

Gardner's (1945) plot of the efficiencies of radial fins of rectangular profile is provided in Fig. 1.13. The curves are for values of x_e/x_b (Gardner's nomenclature), which is the reciprocal of ρ.

Example 1.3: Radial Fin of Rectangular Profile. A radial fin of rectangular profile is exposed to surroundings at a temperature of 35°C and a heat transfer coefficient of $h = 40\ \text{W/m}^2 \cdot \text{K}$. The temperature at the fin base is 110°C and the fin is made from a steel with $k = 40\ \text{W/m} \cdot \text{K}$ with outer and inner diameters of 25 and 10 cm. The fin thickness is 0.25 cm. Determine (**a**) the fin efficiency, (**b**) the temperature at the tip of the fin, and (**c**) the dissipation of the fin.

SOLUTION. For this particular radial fin,

$$\theta_b = 110 - 35 = 75°\text{C} \qquad \delta = 0.0025\ \text{m}$$

$$r_a = 0.125\ \text{m} \qquad r_b = 0.050\ \text{m}$$

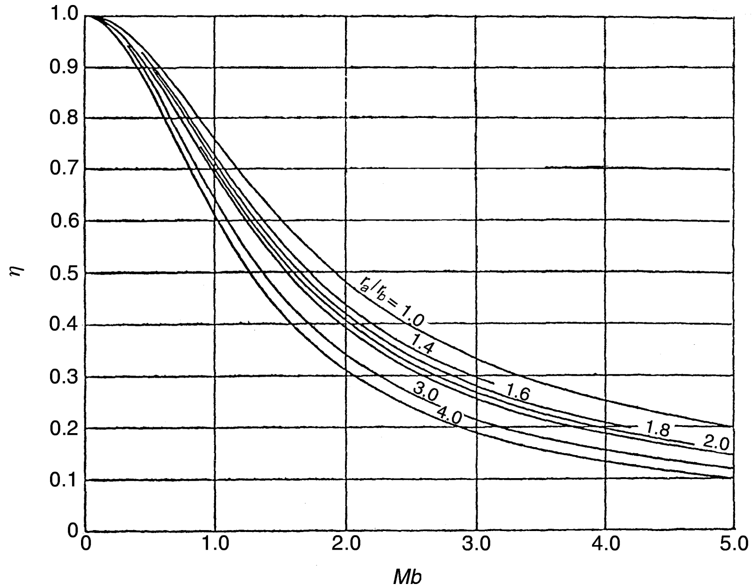

Figure 1.13 Gardner's graph for the efficiency of radial fins. (Reproduced from *Trans. ASME*, 67, 1945.)

$$\rho = \frac{r_a}{r_b} = \frac{0.050}{0.125} = 0.400$$

$$\begin{aligned} m &= \left(\frac{2h}{k\delta}\right)^{1/2} \\ &= \left[\frac{(2)(40)}{(40)(0.0025)}\right]^{1/2} \\ &= 28.284 \text{ m}^{-1} \\ A_p &= \delta(r_a - r_b) \\ &= (0.0025)(0.125 - 0.050) \\ &= 1.875 \times 10^{-4} \text{ m}^2 \end{aligned}$$

and

$$\begin{aligned} \phi &= (r_a - r_b)^{3/2}\left(\frac{2h}{kA_p}\right)^{1/2} \\ &= (0.125 - 0.050)^{3/2}\left[\frac{(2)(40)}{(40)(1.875 \times 10^{-4})}\right]^{1/2} \\ &= (0.0205)(103.28) \\ &= 2.121 \end{aligned}$$

Then

$$\phi R_a = \frac{\phi}{1-\rho} = \frac{2.121}{0.600} = 3.536$$

and

$$\phi R_b = \frac{\rho\phi}{1-\rho} = \frac{(0.400)(2.121)}{0.600} = 1.414$$

(**a**) For the fin efficiency, either eqs. (1.35) or (1.42) applies. Both involve the evaluation of six Bessel functions involving mr_a or mr_b as arguments. Tables of software provide

$$\begin{aligned}
I_0(\phi R_b) &= I_0(mr_b) &= 1.5661\\
I_1(\phi R_b) &= I_1(mr_b) &= 0.8992\\
I_1(\phi R_a) &= I_1(mr_a) &= 6.4081\\
K_0(\phi R_b) &= K_0(mr_b) &= 0.2387\\
K_1(\phi R_b) &= K_1(mr_b) &= 0.3136\\
K_1(\phi R_a) &= K_1(mr_a) &= 0.0213
\end{aligned}$$

The final term in eq. (1.35) or (1.42) is evaluated as

$$\frac{(6.4081)(0.3136) - (0.8992)(0.0213)}{(1.5661)(0.0213) + (6.4081)(0.2387)} = 1.2733$$

so that by eq. (1.42),

$$\eta = \frac{2\rho}{\phi(1+\rho)}(1.2733) = \frac{(2)(0.400)}{(2.121)(1.400)}(1.2733) = 0.343$$

(**b**) For the tip temperature, use eq. (1.33) with $r = r_a = 0.125$ m. This requires the evaluation of two more modified Bessel functions. Tables or software provide

$$\begin{aligned}
I_0(\phi R_a) &= I_0(mr_a) &= 7.5897\\
K_0(\phi R_a) &= K_0(mr_a) &= 0.0189
\end{aligned}$$

and the combination of modified Bessel functions in the bracketed term of eq. (1.33) becomes 0.1804.

$$\theta(r = r_a) = \theta_a = 0.1804\theta_b = 0.1804(75) = 13.5°\text{C}$$

and

$$T_a = \theta_a + T_s = 13.5 + 35 = 48.5°\text{C}$$

(**c**) The heat dissipation can be obtained from eq. (1.34) or the efficiency. With the fin surface

$$S = 2\pi(r_a^2 - r_b^2) = 2\pi(0.0131) = 8.247 \times 10^{-2}\ \text{m}^2$$

the heat dissipation is

$$q_b = hS\eta\theta_b = 40(8.247 \times 10^{-2})(0.343)(75) = 84.9 \text{ W}$$

1.4.3 Radial Fin of Hyperbolic Profile

For the radial fin of hyperbolic profile shown in Fig. 1.14, the profile function is

$$f_2(r) = \frac{C}{r}$$

where C is a constant. The derivative of the profile function is

$$\frac{df_2(r)}{dr} = -\frac{C}{r^2}$$

and the cross section will be

$$f_1(r) = (2\pi r)2f_2(r) = 4\pi r\frac{C}{r} = 4\pi C$$

which shows that the cross section of this fin is a constant. Moreover, $f_2(r_b) = \delta_b/2$ and the constant becomes $C = \delta_b r_b/2$. Substitution of the profile function into eq. (1.29) gives, upon rearrangement, the governing differential equation for the temperature excess

$$\frac{d^2\theta}{dr^2} - \frac{m^2}{r_b}r\theta = 0 \tag{1.43}$$

where again $m^2 = 2h/k\delta_b$.

With

$$M^2 \equiv \frac{m^2}{r_b}$$

the general solution can be obtained as

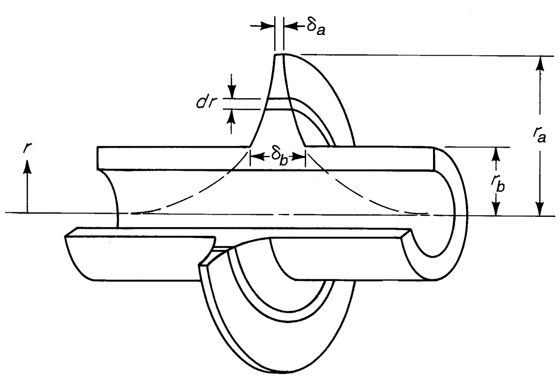

Figure 1.14 Radial fin of hyperbolic profile.

$$\theta(r) = r^{1/2}\left[C_1 I_{1/3}\left(\tfrac{2}{3}Mr^{3/2}\right) + C_2 I_{-1/3}\left(\tfrac{2}{3}Mr^{3/2}\right)\right] \tag{1.44}$$

where the arbitrary constants C_1 and C_2 are evaluated from the boundary conditions defined by eqs. (1.32) and where the details for obtaining this general solution may be found in Section A.6.3.

As in the case of the longitudinal fin of convex parabolic profile, it is helpful to make a transformation of variables. Let

$$u \equiv \tfrac{2}{3}Mr^{3/2}$$

so that

$$\frac{du}{dr} = Mr^{1/2}$$

and

$$r^{1/2} = \left(\frac{3u}{2M}\right)^{1/3} = \Omega u^{1/3}$$

where

$$\Omega = \left(\frac{3}{2M}\right)^{1/3}$$

The use of the foregoing permits the representation

$$\theta(r) = \Omega u^{1/3}[C_1 I_{1/3}(u) + C_2 I_{-1/3}(u)] \tag{1.45}$$

Equation (1.45) is identical in form with the general solution for the longitudinal fin of convex parabolic profile. However, the transformed boundary conditions in this case become

$$\theta\left(u = u_b = \tfrac{2}{3}Mr_b^{3/2}\right) = \theta_b \tag{1.46a}$$

and

$$\theta'\left(u = u_a = \frac{2}{3}Mr_a^{3/2}\right) = 0 \tag{1.46b}$$

where the prime is used to indicate the derivative with respect to u.

The use of the second boundary condition requires term-by-term differentiation of the product terms involving $u^{1/3}$ and the modified Bessel functions. In using eq. (1.46b), one obtains in terms of r:

$$0 = r_a^{1/2}\left[C_1 I_{-2/3}\left(\tfrac{2}{3}Mr_a^{3/2}\right) + C_2 I_{2/3}\left(\tfrac{2}{3}Mr_a^{3/2}\right)\right]$$

When this is coupled with eq. (1.32a), eq. (1.44) gives

$$\theta_b = r_b^{1/2}\left[C_1 I_{1/3}\left(\tfrac{2}{3}Mr_b^{3/2}\right) + C_2 I_{-1/3}\left(\tfrac{2}{3}Mr_b^{3/2}\right)\right]$$

These equations permit evaluation of the arbitrary constants C_1 and C_2. The particular solution for the temperature excess in the radial fin of hyperbolic profile, incorporating the solutions for C_1 and C_2, is

$$\theta(r) = \theta_b \Lambda \left(\frac{r}{r_b}\right)^{1/2} \tag{1.47}$$

where

$$\Lambda = \frac{I_{2/3}\left(\frac{2}{3}Mr_a^{3/2}\right) I_{1/3}\left(\frac{2}{3}Mr^{3/2}\right) - I_{-2/3}\left(\frac{2}{3}Mr_a^{3/2}\right) I_{-1/3}\left(\frac{2}{3}Mr^{3/2}\right)}{I_{2/3}\left(\frac{2}{3}Mr_a^{3/2}\right) I_{1/3}\left(\frac{2}{3}Mr_b^{3/2}\right) - I_{-2/3}\left(\frac{2}{3}Mr_a^{3/2}\right) I_{-1/3}\left(\frac{2}{3}Mr_b^{3/2}\right)}$$

Equation (1.47) is seen to reduce to θ_b when $r = r_b$, as it should.

The heat flow through the fin base is determined from

$$q_b = -2\pi r_b \delta_b k \frac{d\theta}{dr}$$

using the derivative of eq. (1.47) evaluated at $r = r_b$. The differentiation involves two product terms:

$$I_{1/3}\left(\tfrac{2}{3}Mr^{3/2}\right)$$

and

$$I_{-1/3}\left(\tfrac{2}{3}Mr^{3/2}\right)$$

and is most readily accomplished using the transformed variable u and the condition at the base given by eq. (1.46*a*). Termwise differentiation and subsequent use of eq. (1.46*a*) lead to the expression for the fin heat dissipation, which is the heat flow through the base:

$$q_b = 2\pi k \delta_b \theta_b M \psi r_b^{3/2} \tag{1.48}$$

where

$$\psi = \frac{I_{2/3}(u_a) I_{-2/3}(u_b) - I_{-2/3}(u_a) I_{2/3}(u_b)}{I_{-2/3}(u_a) I_{-1/3}(u_b) - I_{2/3}(u_a) I_{1/3}(u_b)}$$

The ideal heat flow is

$$q_{\text{id}} = 2\pi (r_a^2 - r_b^2) h \theta_b$$

Hence, the efficiency will be given by the ratio of eq. (1.48) to this ideal heat flow:

$$\eta = \frac{2\pi k \delta_b \theta_b M r_b^{3/2} \psi}{2\pi (r_a^2 - r_b^2) h \theta_b} = \frac{2 r_b \psi}{m (r_a^2 - r_b^2)} \tag{1.49}$$

The fin efficiency given by eq. (1.49) does not lend itself to comparison with the efficiencies of other radial profiles but can be adjusted by expressing the efficiency in terms of the radius ratio ρ. To do this, it is necessary to evaluate the fin profile area,

$$A_p = \int_{r_b}^{r_a} 2f_2(r)\,dr = \int_{r_b}^{r_a} 2\left(\frac{\delta_b r_b}{2r}\right) dr = \delta_b r_b \ \ln\ \frac{r_a}{r_b} = \delta_b r_b \ \ln\ \frac{1}{\rho}$$

This profile area may be used in the efficiency relationship expressed by eq. (1.49). The base thickness δ_b is evaluated in terms of the profile area:

$$\delta_b = \frac{A_p}{r_b\ \ln(1/\rho)}$$

so that

$$m = \left(\frac{2h}{k\delta_b}\right)^{1/2} = \left[\frac{2hr_b\ \ln(1/\rho)}{kA_p}\right]^{1/2}$$

and

$$M = \frac{m}{r_b^{1/2}} = \left[\frac{2h\ \ln(1/\rho)}{kA_p}\right]^{1/2}$$

The terms exclusive of ψ in eq. (1.49) can be represented by

$$\frac{2r_b}{m(r_a^2 - r_b^2)} = \frac{2r_b^{1/2}}{(2h/kA_p)^{1/2}\,[\ln(1/\rho)]^{1/2}\,(r_a - r_b)(r_a + r_b)}$$
$$= \frac{2r_b^{1/2}(r_a - r_b)^{1/2}}{\phi\,[\ln(1/\rho)]^{1/2}\,(r_a + r_b)}$$

As in the case of the radial fin of rectangular profile,

$$\phi = (r_a - r_b)^{3/2}\left(\frac{2h}{kA_p}\right)^{1/2}$$

and a further simplification through the use of the radius ratio ρ provides

$$\frac{2r_b^{1/2}(r_a - r_b)^{1/2}}{\phi\,[\ln(1/\rho)]^{1/2}\,r_a\,(1 + r_b/r_a)} = \frac{2r_b^{1/2}r_a^{1/2}\,(1 - r_b/r_a)^{1/2}}{\phi\,[\ln(1/\rho)]^{1/2}\,r_a\,(1 - r_b/r_a)}$$
$$= \frac{1}{\phi}\left[\frac{4\rho(1-\rho)}{(1+\rho)^2\ \ln(1/\rho)}\right]^{1/2}$$

Now define

$$R_b \equiv \tfrac{2}{3}Mr_b^{3/2}$$

and

$$R_a \equiv \tfrac{2}{3}Mr_a^{3/2}$$

These may be expressed in terms of ϕ and ρ:

$$\begin{aligned} R_b &= \frac{2}{3} M r_b^{3/2} \\ &= \frac{2}{3}\left(\frac{2h}{kA_p}\right)^{1/2}\left(\ln\frac{1}{\rho}\right)^{1/2} r_b^{3/2} \\ &= \frac{2}{3}\phi\left(\ln\frac{1}{\rho}\right)^{1/2}\left(\frac{\rho}{1-\rho}\right)^{3/2} \end{aligned}$$

and

$$\begin{aligned} R_a &= \frac{2}{3} M r_a^{3/2} \\ &= \frac{2}{3}\left(\frac{2h}{kA_p}\right)^{1/2}\left(\ln\frac{1}{\rho}\right)^{1/2} r_a^{3/2} \\ &= \frac{2}{3}\phi\left(\ln\frac{1}{\rho}\right)^{1/2}\left(\frac{1}{1-\rho}\right)^{3/2} \end{aligned}$$

The fin efficiency of the radial fin of hyperbolic profile with all the foregoing terms substituted into eq. (1.49) becomes

$$\eta = \frac{1}{\phi}\zeta\left[\frac{4\rho(1-\rho)}{(1+\rho)^2(\ln 1/\rho)}\right]^{1/2} \tag{1.50}$$

where

$$\zeta = \frac{I_{2/3}(R_a)I_{-2/3}(R_b) - I_{-2/3}(R_a)I_{2/3}(R_b)}{I_{-2/3}(R_a)I_{-1/3}(R_b) - I_{2/3}(R_a)I_{1/3}(R_b)}$$

an expression that is a function only of ϕ and ρ.

The efficiencies of radial fins with hyperbolic profile are plotted in Fig. 1.15 for values of the radius ratio, $\rho = 0.8$ and 0.4. These efficiencies may be compared with the radial fins of rectangular profile, which are also displayed. By inspection, the hyperbolic profile fins yield a higher efficiency because equal profile areas and fin heights provide a greater proportion of both profile and cross-sectional areas near the fin base.

1.4.4 Radial Fin of Triangular Profile

An exact solution for the temperature excess of the radial fin of triangular profile was provided by Smith and Sucec (1969), who followed the lead provided by Bert (1963). The configuration is displayed in Fig. 1.16, where it can be observed that the profile function and its derivative are

$$f_2(r) = \frac{\delta_b}{2b}(r_a - r)$$

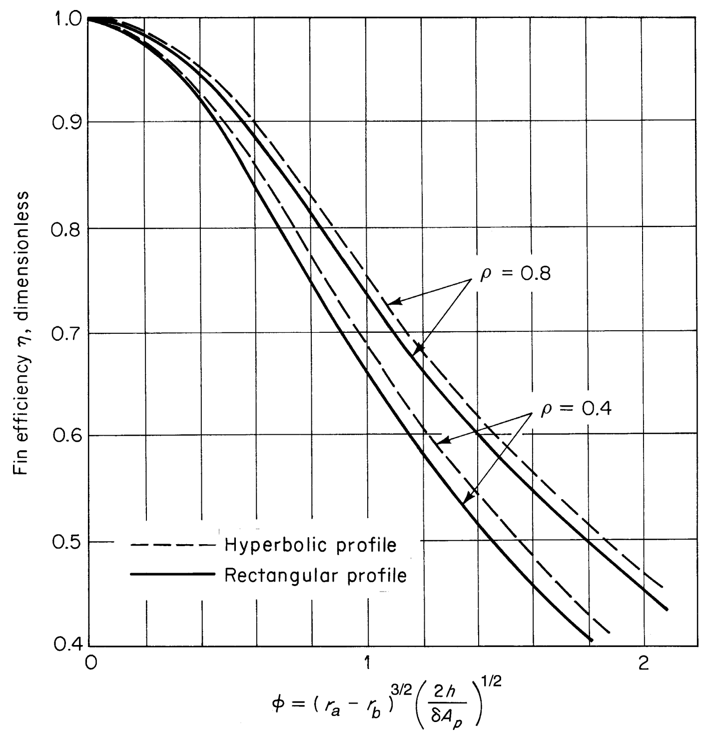

Figure 1.15 Comparison of fin efficiences: radial fins of rectangular and hyperbolic profile.

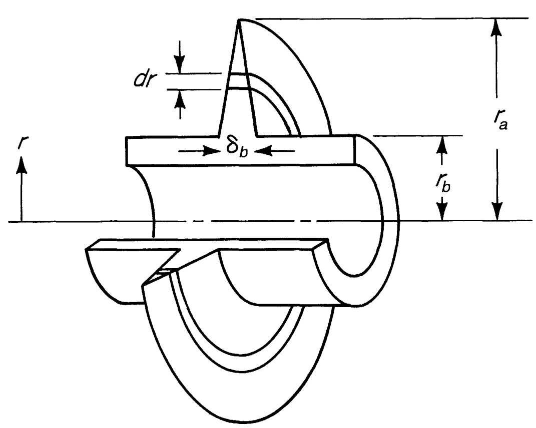

Figure 1.16 Coordinate system for radial fin of triangular profile.

and

$$\frac{df_2(r)}{dr} = -\frac{\delta_b}{2r}$$

and where the fin height is

$$b = r_a - r_b$$

With these substituted into eq. (1.29), the governing differential equation for the temperature excess results:

$$r(r_a - r)\frac{d^2\theta}{dr^2} - (r_a - 2r)\frac{d\theta}{dr} - bm^2 r\theta = 0$$

where, here too, $m = (2h/k\delta_b)^{1/2}$.

If a transformation is made,

$$v \equiv r_a - r$$

so that

$$dv = -dr$$

the differential equation for temperature excess can be transformed to

$$v(r_a - v)\frac{d^2\theta}{dv^2} - (r_a - 2v)\frac{d\theta}{dv} - bm^2(r_a - v)\theta = 0 \qquad (1.51)$$

and this equation can be solved by the method of Frobenius.

Assume that

$$\theta = v^p(a_0 + a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4 + \cdots)$$

so that

$$\begin{aligned}\frac{d\theta}{dv} &= pv^{p-1}(a_0 + a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4 + \cdots)\\ &\quad + v^p(a_1 + 2a_2 v + 3a_3 v^2 + 4a_4 v^3 + \cdots)\end{aligned}$$

and

$$\begin{aligned}\frac{d^2\theta}{dv^2} &= p(p-1)v^{p-2}(a_0 + a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4 + \cdots)\\ &\quad + 2pv^{p-1}(a_1 + 2a_2 v + 3a_3 v^2 + 4a_4 v^3 + \cdots)\\ &\quad + v^p(2a_2 + 6a_3 v + 12a_4 v^2 + \cdots)\end{aligned}$$

The procedure to be followed here is identical to the one employed in Section A.4.3. When the assumed values of θ and its derivatives are substituted into eq. (1.51), a series involving v to various powers of p results:

$$A_1 v^{p-1} + A_2 v^p + A_3 v^{p+1} + A_4 v^{p+2} + \cdots = 0$$

where A_1, A_2, and A_3 are

$$A_1 = p^2 r_a a_0$$

$$A_2 = [(p^2 + 2p + 1)r_a]a_1 - (p^2 + p + bm^2 r_a)a_0$$

$$A_3 = [(p^2 + 4p + 4)r_a]a_2 - (p^2 + 3p + 2 + bm^2 r_a)a_1 + bm_2 a_0$$

The indicial equation derives from the sum of the entries in the column headed by the lowest power of v. Because eq. (1.51) demands that the coefficients A_k be identically equal to zero, it is observed from A_1 that

$$p^2 r_a a_0 = 0$$

and because r_a is a physical dimension that cannot equal zero, a trivial solution would result if $a_0 = 0$. Thus the only alternative is that $p = 0$. The theory than says that the solution for θ will be

$$\theta = C_1 \sum_{k=0}^{\infty} a_k v^k + C_2 \ln v \left(\sum_{k=0}^{\infty} a_k v^k + \sum_{k=0}^{\infty} b_k v^k \right)$$

But to keep θ finite at $r = r_a$, where $v = r_a - r_a = 0$, C_2 must be zero, and after application of the boundary condition of eq. (1.32a), the particular solution will be

$$\theta = C_1 \sum_{k=0}^{\infty} a_k v^k \tag{1.52}$$

The coefficients, $a_0, a_1, a_2, a_3, \ldots$ are all related by a recurrence relationship, and it is determined by looking at the coefficients, $A_1, A_2, A_3, \ldots$ with $p = 0$. For A_1 with $p = 0$,

$$r_a a_1 = bm^2 r_a a_0$$

or

$$a_1 = bm^2 a_o \tag{1.53}$$

It can be shown after a rather laborious procedure that for $k \geq 2$,

$$a_k = \frac{[k(k-1) + br_a m^2]a_{k-1} - bm^2 a_{k-2}}{k^2 r_a} \tag{1.54}$$

with

$$a_o = \frac{\theta_b}{1 + (mb)^2 + \sum_{k=2}^{\infty} (a_k/a_0) b^k} \tag{1.55}$$

Use of eqs. (1.54) and (1.55) in eq. (1.52) allows a computation of the heat flow and the fin efficiency. A plot of the efficiency is shown in Fig. 1.17.

1.4.5 Radial Fin of Least Material

A discussion of the longitudinal fin that yields minimum weight is presented in Chapter 3.

1.5 SPINES

1.5.1 Generalized Differential Equation

Gardner (1945) also proposed a profile function for spines:

$$f_2(x) = \frac{\delta_b}{2}\left(\frac{x}{b}\right)^{(1-2n)/(2-n)} \tag{1.56}$$

With the appropriate value of n, eq. (1.56) may be used for the development of a generalized differential equation for spines.

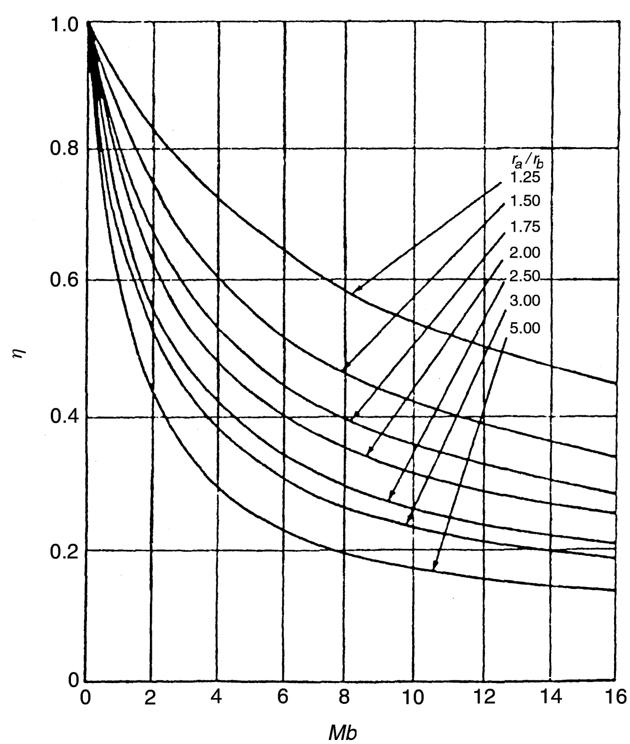

Figure 1.17 Efficiency of radial fin of triangular profile. (Reproduced from *Trans. ASME*, 91, 1969.)

Figure 1.18 shows a spine of arbitrary profile. It can be seen that the spine cross-sectional normal to the flow of heat, the confining profile, and the perimeter of the spine are all perfectly arbitrary functions of the distance x from the tip of the spine. A differential equation for the temperature excess, $\theta(x) = T(x) - T_s$, may be written in a manner similar to that for the longitudinal and radial fins by considering the heat conduction into and out of the element dx through the cross section $f_1(x)$:

$$dq = k\frac{d}{dx}\left[f_1(x)\frac{d\theta}{dx}\right]dx$$

This difference in heat flow into and out of the element must equal that dissipated by the surface of the spine. If dissipation occurs solely by convection and h is the convection coefficient,

$$dq = hf_3(x)\theta\ dx$$

where $f_3(x)$ defines a perimeter function $P(x)$ which depends on the distance x from the origin of the coordinate system. The energy balance involving conduction and convection with regard to the element dx yields

$$k\frac{d}{dx}\left[f_1(x)\frac{d\theta}{dx}\right] = hf_3(x)\theta$$

and upon rearrangement, the generalized differential equation becomes

$$f_1(x)\frac{d^2\theta}{dx^2} + \frac{df_1(x)}{dx}\frac{d\theta}{dx} - \frac{k}{k}f_3(x)\theta = 0 \tag{1.57}$$

The relationships between $f_1(x)$ and $f_2(x)$ and $f_3(x)$ and $f_2(x)$ are

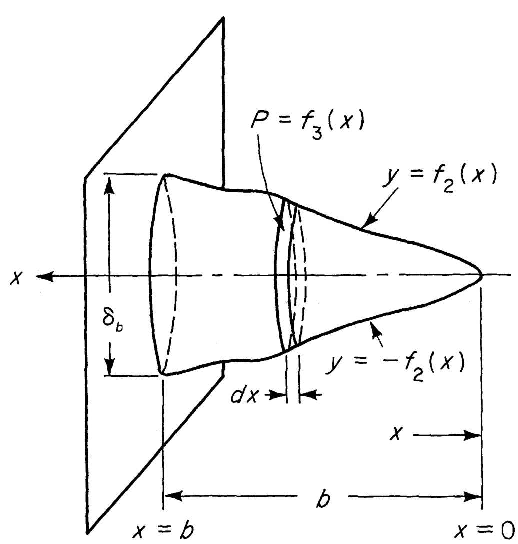

Figure 1.18 Spine of arbitrary profile.

$$f_1(x) = \pi[f_2(x)]^2 \quad \text{and} \quad f_3(x) = 2\pi f_2(x)$$

Thus eq. (1.57) may also be written as

$$[f_2(x)]^2 \frac{d^2\theta}{dx^2} + \frac{d}{dx}[f_2(x)]^2 \frac{d\theta}{dx} - \frac{2k}{k} f_2(x)\theta = 0 \tag{1.58}$$

Equation (1.58) is a second-order differential equation with variable coefficients except where the spine cross section normal to the heat flow is constant. It may be solved via termwise comparison with the general Bessel equation and is identical to the procedure used for the generalized longitudinal fin. Moreover, a comparison of Figs 1.6 and 1.18 indicates that the boundary conditions are the same for both cases and the general solution of eq. (1.58) will have two arbitrary constants evaluated using the boundary conditions of eqs. (1.6):

$$\theta(x = b) = \theta_b \tag{1.6a}$$

and

$$\left.\frac{d\theta}{dx}\right|_{x=0} = 0 \tag{1.6b}$$

1.5.2 Spines of Constant Cross Section

Cylindrical Spine. For the cylindrical spine shown in Fig. 1.19, the profile function of eq. (1.56) matches the configuration when $n = \frac{1}{2}$. Actually, δ_b in eq. (1.56) may be replaced by the spine diameter d. Thus

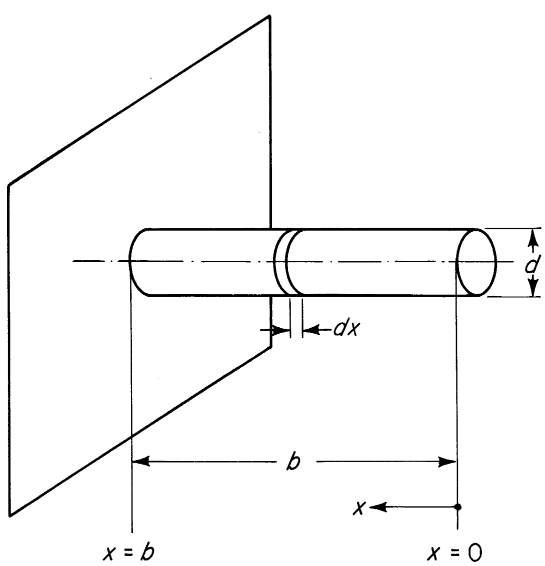

Figure 1.19 Cylindrical spine.

$$f_2(x) = \frac{d}{2}$$

$$[f_2(x)]^2 = \frac{d^2}{4}$$

and

$$\frac{d}{dx}[f_2(x)]^2 = 0$$

These values may be substituted into eq. (1.58) to obtain

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{1.59}$$

where

$$m = \left(\frac{4h}{kd}\right)^{1/2}$$

Equation (1.59) is identical in form to eq. (1.7). The general solution, the boundary conditions, the particular solution, the heat flow through the fin base, and the fin efficiency all have the same form as those for the longitudinal fin of rectangular profile. The exceptions involve the use of the spine diameter instead of the fin thickness and the performance factor $m = (4h/kd)^{1/2}$ instead of $m = (2h/k\delta_b)^{1/2}$. These comparisons yield for the temperature excess,

$$\theta(x) = \theta_b \frac{\cosh\, mx}{\cosh\, mb} \tag{1.60}$$

for the heat flow through the spine base and the heat dissipation,

$$q_b = \frac{\pi}{4} kd^2 m\theta_b \tanh\, mb \tag{1.61}$$

and for the fin efficiency,

$$\eta = \frac{\tanh\, mb}{mb} \tag{1.62}$$

Values of η using eq. (1.62) have been plotted in Fig. 1.2.

Example 1.4: Cylindrical Spine. A cylindrical rod is used as a spine. Its diameter is 0.875 cm and its height is 8 cm. It is fabricated of a steel with $k = 32$ W/m · K and is exposed to surroundings at a temperature of 30°C via a heat transfer coefficient of $h = 50$ W/m^2 · K. The temperature at the fin base of the spine is 85°. Determine (**a**) the fin efficiency, (**b**) the tip temperature, and (**c**) the heat dissipation. (**d–f**) Repeat the foregoing procedure for a thermal conductivity of $k = 200$ W/m · K.

SOLUTION. For the surroundings, with $h = 50$ W/m^2 · K,

$$\theta_b = 85 - 30 = 55°\text{C} \qquad d = 8.75/1000 = 8.75 \times 10^{-3}\text{ m}$$

$$m = \left(\frac{4h}{kd}\right)^{1/2} = \left[\frac{(4)(50)}{(32)(8.75 \times 10^{-3})}\right]^{1/2} = 26.726\text{ m}^{-1}$$

$$b = 8/100 = 0.08\text{ m} \qquad mb = (26.726)(0.08) = 2.138$$

(**a**) From eq. (1.62),

$$\eta = \frac{\tanh\, mb}{mb} = \frac{\tanh\, 2.138}{2.138} = \frac{0.973}{2.138} = 0.455$$

(**b**) The tip temperature of the fin is determined from eq. (1.60) at $x = 0$, where $\theta(x = 0) = \theta_a$:

$$\theta_a = \frac{\theta_b\, \cosh\, mx}{\cosh\, mb} = \frac{55(\cosh\, 0)}{\cosh\, 2.138} = \frac{(55)(1.00)}{4.301} = 12.8°\text{C}$$

so that at $x = 0$, where $T(x = 0) = T_a$,

$$T_a = \theta_a + T_s = 12.8 + 30 = 42.8°\text{C}$$

(**c**) The heat dissipated by the fin is calculated from eq. (1.61):

$$\begin{aligned} q_b &= \frac{\pi}{4} kd^2 m\theta_b\, \tanh\, mb \\ &= \frac{\pi}{4}(32)(8.75 \times 10^{-3})^2(26.726)(55)(0.973) \\ &= 2.75\text{ W} \end{aligned}$$

The heat transferred can also be obtained from the ideal heat dissipation and the efficiency determined in part (a). With the surface area of the fin,

$$S = \pi db = \pi(8.75 \times 10^{-3})(0.08) = 2.199 \times 10^{-3}\text{ m}^2$$

$$q_{\text{id}} = hS\theta_b = 50(2.199 \times 10^{-3})(55) = 6.045\text{ W}$$

the heat dissipated is

$$q_b = \eta q_{\text{id}} = (0.455)(6.045) = 2.75\text{ W}$$

(**d**) With $k = 200\text{ W/m}\cdot\text{K}$,

$$m = \left(\frac{4h}{kd}\right)^{1/2} = \left[\frac{(4)(50)}{(200)(8.75 \times 10^{-3})}\right]^{1/2} = 10.690\text{ m}^{-1}$$

$$mb = (10.690)(0.08) = 0.855$$

and from eq. (1.11),

$$\eta = \frac{\tanh\, mb}{mb} = \frac{\tanh\, 0.855}{0.855} = \frac{0.694}{0.855} = 0.811$$

(**e**) The tip temperature is determined from eq. (1.9):

$$\theta_a = \frac{\theta_b \cosh mx}{\cosh mb} = \frac{55(\cosh 0)}{\cosh 0.855} = \frac{55}{1.389} = 39.6°\text{C}$$

so that at $x = 0$, where $T(x = 0) = T_a$,

$$T_a = \theta_a + T_s = 39.6 + 30 = 69.6°\text{C}$$

(**f**) The heat transferred can also be obtained from the ideal heat dissipation and the efficiency. With

$$S = \pi d b = \pi(8.75 \times 10^{-3})(0.008) = 2.199 \times 10^{-3}\ \text{m}^2$$

and

$$q_{\text{id}} = hS\theta_b = (50)(2.199 \times 10^{-3})(55) = 6.045\ \text{W}$$

the heat dissipated is

$$q_b = \eta q_{\text{id}} = (0.811)(6.045) = 4.91\ \text{W}$$

Rectangular Spine. For the rectangular spine shown in Fig. 1.20, the profile function of eq. (1.56) also matches the configuration when $n = \frac{1}{2}$. Here, however, the sides are designated by δ_1 and δ_2. With

$$f_1(x) = \delta_1\delta_2$$
$$f_3(x) = 2(\delta_1 + \delta_2)$$

and

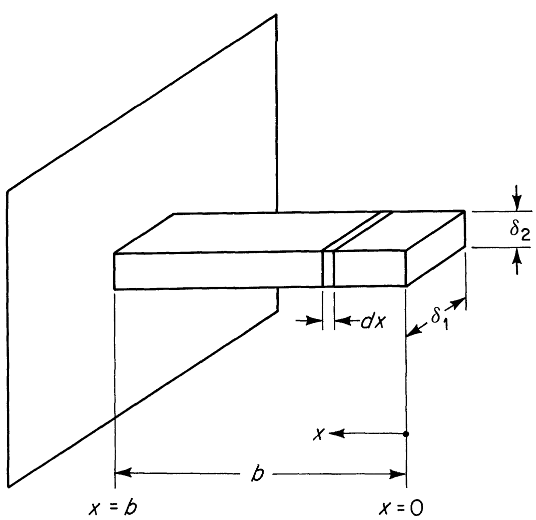

Figure 1.20 Rectangular spine.

$$\frac{df_1(x)}{dx} = 0$$

substituted into eq. (1.57), the differential equation for temperature excess is obtained:

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{1.63}$$

Here m has a special form involving the perimeter of the spine:

$$m = \left(\frac{hP}{kA}\right)^{1/2} = \left[\frac{2h(\delta_1 + \delta_2)}{k\delta_1\delta_2}\right]^{1/2}$$

which is a general expression for m for spines and all values given previously for spines of constant cross section are specific values of this expression. For the special case of the square cross section, $\delta = \delta_1 = \delta^2$, and eq. (1.63) applies with $m = (4h/k\delta)^{1/2}$.

Equation (1.63) is identical with eq. (1.59). Hence the temperature excess and the efficiency will be given by eqs. (1.60) and (1.62), respectively, as long as the proper value of m is employed. The heat flow through the base is

$$q_b = k\delta_1\delta_2 m\theta_b \tanh mb \tag{1.64}$$

Elliptical Spine. The cross section for the elliptical spine is shown in Fig. 1.21, where it may be noted that the semimajor and semiminor axes are designated, respectively, by δ_1 and δ_2. Here

$$f_1(x) = \pi\delta_1\delta_2$$

$$\frac{df_1(x)}{dx} = 0$$

and

$$f_3(x) = \pi(\delta_1 + \delta_2)\left(1 + \frac{p^2}{4} + \frac{p^4}{64} + \frac{p^6}{256} + \cdots\right)$$

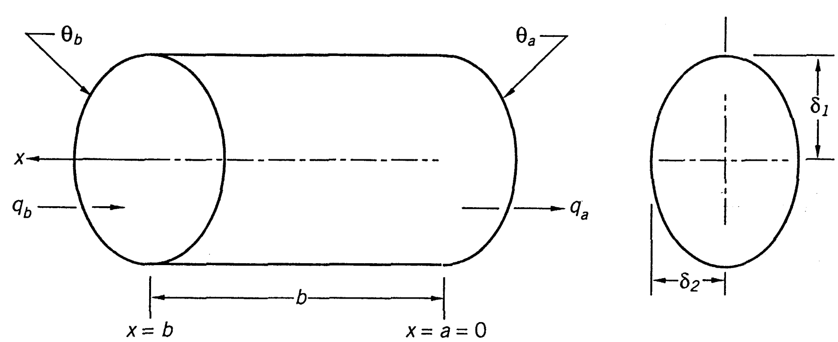

Figure 1.21 Cross section for the elliptical spine.

where

$$p = \frac{\delta_1 - \delta_2}{\delta_1 + \delta_2}$$

Here, too, the differential equation for the temperature excess is given by

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{1.59}$$

but with

$$m = \left(\frac{hP}{kA}\right)^{1/2} = \left[\frac{h(\delta_1 + \delta_2)}{k(\delta_1\delta_2)}\left(1 + \frac{p^2}{4} + \frac{p^4}{64} + \frac{p^6}{256} + \cdots\right)\right]^{1/2}$$

The temperature excess and the fin efficiency are given by eqs. (1.60) and (1.62), respectively, but the heat dissipated is given by

$$q_b = \left[hk\pi^2\delta_1\delta_2(\delta_1\delta_2)\left(1 + \frac{p^2}{4} + \frac{p^4}{64} + \frac{p^6}{256} + \cdots\right)^{1/2}\theta_b \tanh mb\right] \tag{1.65}$$

1.5.3 Conical Spine

For the conical spine shown in Fig. 1.22, the profile function is defined by eq. (1.56) with $n = -1$. Hence

$$f_2(x) = \frac{\delta_b}{2}\frac{x}{b}$$

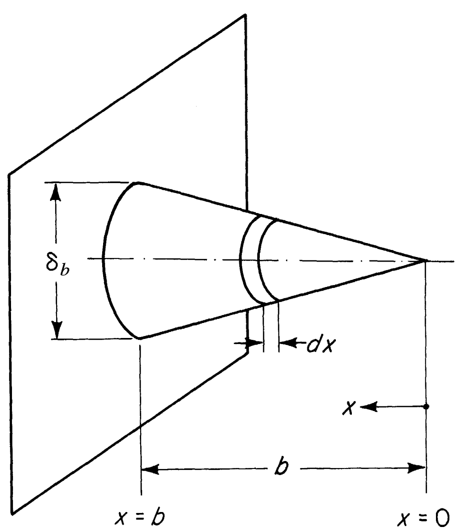

Figure 1.22 Conical spine.

and

$$\frac{df_2(x)}{dx} = \frac{\delta_b}{2b}$$

Use of these in eq. (1.58) gives the differential equation for the temperature excess, $\theta(x) = T(x) - T_s$:

$$x^2 \frac{d^2\theta}{dx^2} + 2x\frac{d\theta}{dx} - M^2 x\theta = 0 \tag{1.66}$$

where

$$M = (2m^2 b)^{1/2}$$

and $m = (2h/k\delta_b)^{1/2}$.

The general solution of eq. (1.66) is shown in Section A.6.5 to be

$$\theta(x) = x^{-1/2}[C_1 I_1(2M\sqrt{x}) + C_2 K_1(2M\sqrt{x})] \tag{1.67}$$

where C_1 and C_2 are arbitrary constants to be evaluated from the boundary conditions of eqs. (1.6). However, it can be noted immediately that in order to have a finite temperature excess at $x = 0, C_2$ must equal zero because $K_1(2M\sqrt{x})/\sqrt{x}$ is unbounded[5] at $x = 0$. Hence, only C_1 need be evaluated. This evaluation is carried out at $x = b$, and when the result is substituted into eq. (1.67), the particular solution for the temperature excess is obtained:

$$\theta(x) = \theta_b \left(\frac{b}{x}\right)^{1/2} \frac{I_1(2M\sqrt{x})}{I_1(2M\sqrt{b})} \tag{1.68}$$

which reduces, as it should, to θ_b at $x = b$.

The heat flow through the base can be obtained via differentiation of eq. (1.68), evaluation at $x = b$, and substitution into

$$q_b = kA\frac{d\theta}{dx}\bigg|_{x=b}$$

As in many previous cases, differentiation of eq. (1.68) is best accomplished by using a transformation of variable

$$u \equiv 2M\sqrt{x}$$

so that

$$\frac{d\theta}{dx} = \frac{du}{dx}\frac{d\theta}{du} = \frac{2M^2}{u}\frac{d\theta}{du}$$

and with $d\theta/du$ evaluated at

[5]Multiplication of each term of the infinite series for $I_1(2M\sqrt{x})$ by $1/\sqrt{x}$ will show that no term in $I_1(2M\sqrt{x})/\sqrt{x}$ is unbounded at $x = 0$.

$$u_b = 2M\sqrt{b}$$

the heat flow through the base in terms of u is

$$q_b = \frac{\pi k \delta_b^2 \theta_b M^3 \sqrt{b}}{u_b^2} \frac{I_2(u_b)}{I_1(u_b)}$$

In terms of x it is

$$q_b = \frac{\pi k \delta_b^2 \theta_b M}{4\sqrt{b}} \frac{I_2(2M\sqrt{b})}{I_1(2M\sqrt{b})} \tag{1.69}$$

The surface area of the conical spine is the integral of the perimeter function evaluated between the limits $x = 0$ and $x = b$:

$$S = \int_{x=0}^{x=b} f_3(x)\, dx = \int_{x=0}^{x=b} \pi \frac{\delta_b}{b} x\, dx = \frac{\pi}{2} \delta_b b$$

The ideal heat flow is obtained from this surface with operation at the base temperature excess:

$$q_{\text{id}} = h\left(\frac{\pi}{2}\delta_b b\right)\theta_b$$

Then the efficiency with $M = m\sqrt{2b}$ is

$$\eta = \frac{q_b}{q_{\text{id}}} = \frac{\pi k \delta_b \theta_b M I_2(2M\sqrt{b})}{(\pi/2)\, h \delta_b b \theta_b I_1(2M\sqrt{b})}$$

or

$$\eta = \frac{q_b}{q_{\text{id}}} = \frac{\sqrt{2} I_2(2\sqrt{2}\, mb)}{(mb) I_1(2\sqrt{2}\, mb)} \tag{1.70}$$

Values of η as a function of mb employing eq. (1.70) have been plotted in Fig. 1.2.[6]

1.5.4 Spine of Concave Parabolic Profile

The coordinate system for the spine of concave parabolic profile is shown in Fig. 1.23. It requires that the exponent on the profile function of eq. (1.56) be derived from $n = \infty$. The profile function for this spine then becomes

$$f_2(x) = \frac{\delta_b}{2}\left(\frac{x}{b}\right)^2$$

and

[6] A typographical error appears in the original Gardner (1945) paper. It shows a difference by the $\sqrt{2}$ in the modified Bessel function argument. Correspondence with Gardner confirmed eq. (1.70), and Gardner provided an errata in 1976.

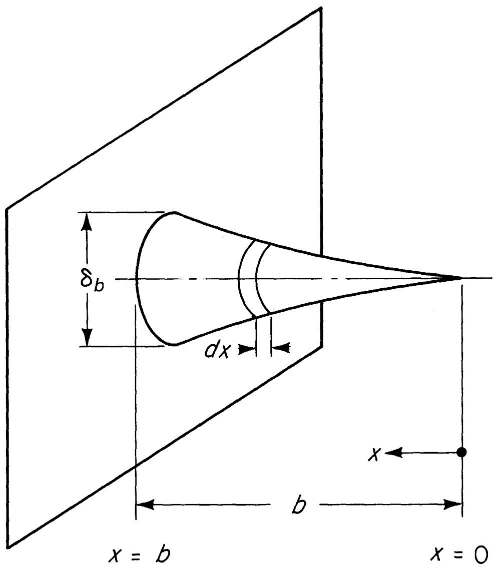

Figure 1.23 Spine of concave parabolic profile.

$$\frac{df_2(x)}{dx} = \frac{\delta_b x}{b^2}$$

Use of these in eq. (1.58) gives the governing differential equation for the temperature excess, $\theta(x) = T(x) - T_s$:

$$x^4 \frac{d^2\theta}{dx^2} + 4x^3 \frac{d\theta}{dx} - M^2 x^2 \theta = 0 \tag{1.71}$$

where $M = \sqrt{2}\, mb$ and $m = (2h/k\delta)^{1/2}$.

Equation (1.71) is recognized as an *Euler equation*, and by a procedure similar to that used for the longitudinal fin of concave parabolic profile, a particular solution is obtained:

$$\theta(x) = \theta_b \left(\frac{x}{b}\right)^{\alpha} \tag{1.72}$$

where

$$\alpha = -\tfrac{3}{2} + \tfrac{1}{2}\sqrt{(9 + 4M^2)^{1/2}}$$

The heat flow through the base of the fin is obtained by evaluating

$$q_b = k \left(\frac{\pi}{4}\delta_b^2\right) \left.\frac{d\theta}{dx}\right|_{x=b}$$

and using eq. (1.72):

$$q_b = \frac{\pi k \delta_b^2 \theta_b [-3 + (9 + 4M^2)^{1/2}]}{8b} \tag{1.73}$$

The spine surface area is

$$S = \int_{x=0}^{x=b} f_3(x)\,dx = \int_{x=0}^{x=b} \pi\delta_b \left(\frac{x}{b}\right)^2 dx = \frac{1}{3}\pi\delta_b b$$

which can be used to obtain the ideal heat dissipation,

$$q_{\text{id}} = \tfrac{1}{3}h\pi\delta_b b\theta_b$$

The efficiency is

$$\eta = \frac{q_b}{q_{\text{id}}} = \frac{3\pi k\delta_b^2\alpha\theta_b}{4\pi h\delta_b b^2\theta_b} = \frac{3k\delta_b\alpha}{4hb^2}$$

which may be adjusted to give

$$\eta = \frac{3}{2m^2b^2}\left[-\frac{3}{2} + \frac{1}{2}\sqrt{(9+4M^2)^{1/2}}\right]$$

or after suitable algebraic readjustment, the final and simplest form for the longitudinal fin of concave parabolic profile:

$$\eta = \frac{2}{1 + \left(1 + \frac{8}{9}m^2b^2\right)^{1/2}} \tag{1.74}$$

Values of η as a function of mb have been plotted from eq. (1.74) in Fig. 1.2.

1.5.5 Spine of Convex Parabolic Profile

The coordinate system for the spine of convex parabolic profile shown in Fig. 1.24. It requires that the exponent on the general fin profile of eq. (1.56) be obtained when $n = 0$. The profile function for this spine then becomes

$$f_2(x) = \frac{\delta_b}{2}\left(\frac{x}{b}\right)^{1/2}$$

and

$$\frac{df_2(x)}{dx} = \frac{\delta_b}{4}\left(\frac{1}{bx}\right)^{1/2}$$

When these are substituted into eq. (1.58), the governing differential equation for the temperature excess, $\theta(x) = T(x) - T_s$, becomes

$$x\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} - M^2\sqrt{x}\,\theta = 0 \tag{1.75}$$

where

$$M = \left(2m^2b^{1/2}\right)^{1/2}$$

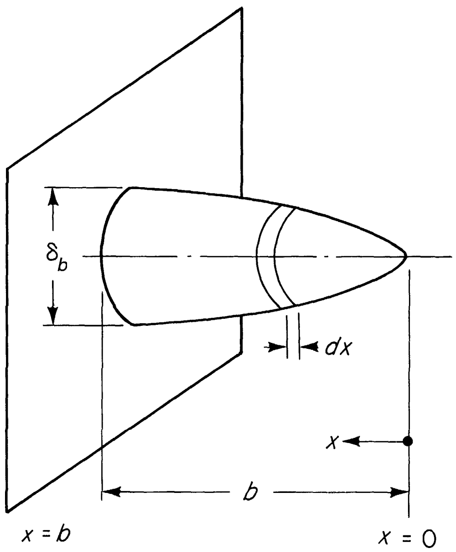

Figure 1.24 Spine of convex parabolic profile.

and $m = (2h/k\delta)^{1/2}$.

As indicated in Section A.6.4, the general solution of eq. (1.75) is

$$\theta(x) = C_1 I_0\left(\tfrac{4}{3}Mx^{3/4}\right) + C_2 K_0\left(\tfrac{4}{3}Mx^{3/4}\right) \tag{1.76}$$

and it may be noted that because $K_0(0)$ is unbounded, in order to maintain a finite temperature excess at $x = 0$, C_2 must be zero. Thus eq. (1.76) reduces to

$$\theta(x) = C_1 I_0\left(\tfrac{4}{3}Mx^{3/4}\right)$$

Then C_1 may be evaluated by using the boundary condition of eq. (1.6*a*), and this leads to the particular solution

$$\theta(x) = \frac{\theta_b I_0\left(\frac{4}{3}\sqrt{2}\,mb^{1/4}x^{3/4}\right)}{I_0\left(\frac{4}{3}\sqrt{2}\,mb\right)} \tag{1.77}$$

The heat flow through the base of the fin is obtained by making the transformation

$$u \equiv \tfrac{4}{3}\sqrt{2}\,mb^{1/4}x^{3/4}$$

Then, because at $x = b$

$$u(x = b) = u_b = \tfrac{4}{3}\sqrt{2}\,mb$$

after transformation of eq. (1.76),

$$q_b = kA\frac{d\theta}{du}\bigg|_{u=u_b} = \frac{\pi}{4}k\delta_b^2\theta_b\left(\frac{16m^4b}{3u}\right)^{1/3}\frac{d}{du}\left[\frac{I_0(u)}{I_0(u_b)}\right]_{u=u_b}$$

The heat flow through the base in terms of the transformed variable u is

$$q_b = \frac{\sqrt{2}}{4} k\pi \delta_b^2 \theta_b m \frac{I_1(u_b)}{I_0(u_b)}$$

and in terms of the specified spine dimensions,

$$q_b = \frac{\sqrt{2}}{4} k\pi \delta_b^2 \theta_b m \frac{I_1\left(\frac{4}{3}\sqrt{2}\, mb\right)}{I_0\left(\frac{4}{3}\sqrt{2}\, mb\right)} \tag{1.78}$$

Once more, consider the surface area of the spine:

$$S = \int_{x=0}^{x=b} f_3(x)\, dx = \int_{x=0}^{x=b} \pi \delta_b \left(\frac{x}{b}\right)^{1/2} dx = \frac{2}{3}\pi \delta_b b$$

which can be used to obtain the ideal heat dissipation,

$$q_{\text{id}} = hS\theta_b = \tfrac{2}{3} h\pi \delta_b b\theta_b$$

The spine efficiency then becomes

$$\eta = \frac{q_b}{q_{\text{id}}} = \frac{\left(\sqrt{2}/4\right) k\pi \delta_b^2 m\theta_b I_1\left(\frac{4}{3}\sqrt{2}\, mb\right)}{\left(\frac{2}{3}\right)\pi \delta_b hb\theta_b I_0\left(\frac{4}{3}\sqrt{2}mb\right)}$$

or

$$\eta = \frac{3\sqrt{2}}{4} \frac{I_1\left(\frac{4}{3}\sqrt{2}mb\right)}{(mb) I_0\left(\frac{4}{3}\sqrt{2}mb\right)} \tag{1.79}$$

Values of η as a function of mb have been plotted from eq. (1.79)[7] in Fig. 1.2.

Example 1.5: Spines of Different Profiles. Spines of cylindrical, conical, concave parabolic and convex parabolic, profiles are exposed to surroundings at a temperature of 25°C via a heat transfer coefficient of $h = 40\ \text{W/m}^2 \cdot \text{K}$. In all cases, the spine base temperature is 100°, the spine thermal conductivity is, $k = 100\ \text{W/m} \cdot \text{K}$, the spine base diameter is 0.92 cm, and the spine height is 10 cm. Compare the fin efficiencies and heat dissipations of the four spines.

SOLUTION. For all the spines

$$\theta_b = 100 - 25 = 75°\text{C}$$

$$\delta_b = 0.0092\ \text{m} \quad \text{and} \quad b = 0.10\ \text{m}$$

[7]Equation (1.79) is at variance with the original Gardner (1945) reference. Correspondence with Gardner confirmed eq. (1.78) and Gardner published an errata in 1976.

(**a**) For the cylindrical spine with $\delta_b = d$,

$$m = \left(\frac{4h}{kd}\right)^{1/2} = \left[\frac{(4)(40)}{(100)(0.0092)}\right]^{1/2} = 13.188 \text{ m}^{-1}$$

and

$$mb = (13.188)(0.100) = 1.319$$

By eq. (1.62),

$$\eta = \frac{\tanh\, mb}{mb} = \frac{\tanh\, 1.319}{1.319} = \frac{0.866}{1.319} = 0.657$$

and by eq. (1.61),

$$\begin{aligned} q_b &= \frac{\pi}{4} kd^2 m\theta_b \tanh\, mb \\ &= \frac{\pi}{4}(100)(0.0092)^2(13.188)(75)(0.866) \\ &= 5.70 \text{ W} \end{aligned}$$

(**b**) For the conical spine with $\delta_b = 0.0092$ m,

$$m = \left(\frac{2h}{k\delta_b}\right)^{1/2} = \left[\frac{(2)(40)}{(100)(0.0092)}\right]^{1/2} = 9.325 \text{ m}^{-1}$$

and

$$mb = (9.325)(0.100) = 0.9325$$

By eq. (1.70),

$$\frac{\sqrt{2}\, I_2(2\sqrt{2}\, mb)}{(mb) I_1(2\sqrt{2}\, mb)}$$

It is interesting to note that some software and mathematical tables may not include values for $I_2(x)$. The user may have to resort to a computation using the infinite series or recognize that the recurrence relationships may help. Employment of entry 24 in Table A.1.1 gives

$$I_{n+1}(x) = I_{n-1} - \frac{2n}{x} I_n(x)$$

so that with $n = 1$ and

$$x = 2\sqrt{2}\, mb = 2\sqrt{2}\,(0.9325) = 2.638$$

one finds that

$$I_0(2.638) = 3.6585$$

$$I_1(2.638) = 2.8505$$

and then

$$I_2(2.638) = 1.4969$$

This makes the efficiency

$$\frac{\sqrt{2}\, I_2(2\sqrt{2}\, mb)}{(mb) I_1(2\sqrt{2}\, mb)} = \frac{\sqrt{2}\,(1.4969)}{(0.933)(2.8505)} = 0.796$$

The heat dissipation is calculated from

$$q_b = \frac{\pi k \delta_b^2 \theta_b M}{4\sqrt{b}} \frac{I_2(2M\sqrt{b})}{I_1(2M\sqrt{b})} \tag{1.69}$$

where

$$M = m\sqrt{2b} = 9.325\sqrt{(2)(0.100)} = 4.170$$

and where

$$2M\sqrt{b} = 2\sqrt{2}\, mb = 2\sqrt{2}\,(0.9325) = 2.638$$

Thus

$$\begin{aligned} q_b &= \frac{\pi k \delta_b^2 \theta_b M}{4\sqrt{b}} \frac{I_2(2M\sqrt{b})}{I_1(2M\sqrt{b})} \\ &= \frac{\pi(100)(0.0092)^2(75)(4.170)}{4\sqrt{0.100}} \left(\frac{1.4969}{2.8505}\right) \\ &= 3.45\ \text{W} \end{aligned}$$

Observe that the conical spine operates at a higher efficiency than the cylindrical spine ($0.796 > 0.657$), but it dissipates substantially less heat ($3.45\ \text{W} < 5.70\ \text{W}$).

(**c**) For the concave parabolic spine with $\delta_b = 0.0092$ m, $mb = 0.9325$, and by eq. (1.74),

$$\begin{aligned} \eta &= \frac{2}{1 + \left(1 + \frac{8}{9} m^2 b^2\right)^{1/2}} \\ &= \frac{2}{1 + \left[1 + \frac{8}{9}(0.9325)^2\right]^{1/2}} \\ &= 0.858 \end{aligned}$$

By eq. (1.73),

$$q_b = \frac{\pi k \delta_b^2 \theta_b [-3 + (9 + 4M^2)^{1/2}]}{8b} \tag{1.73}$$

In this case,

$$M = \sqrt{2}\, mb = \sqrt{2}\,(0.9325) = 1.319$$

and

$$\begin{aligned} q_b &= \frac{\pi(100)(0.0092)^2(75)\{-3 + [9 + 4(1.319)^2]^{1/2}\}}{(8)(0.10)} \\ &= 2.48 \text{ W} \end{aligned}$$

(**d**) For the convex parabolic spine with $\delta_b = 0.0092$ m, $mb = 0.9325$ and by eq. (1.79),

$$\eta = \frac{3\sqrt{2}}{4} \frac{I_1\left(\frac{4}{3}\sqrt{2}\, mb\right)}{(mb) I_0\left(\frac{4}{3}\sqrt{2}\, mb\right)}$$

With

$$\tfrac{4}{3}\sqrt{2}\, mb = \tfrac{4}{3}\sqrt{2}\,(0.9325) = 1.758$$

then

$$I_0(1.758) = 1.9354$$

$$I_1(1.758) = 1.2652$$

and the efficiency is computed as

$$\eta = \frac{3\sqrt{2}}{4} \frac{1.2652}{(0.9325)(1.9354)} = 0.744$$

By eq. (1.78),

$$\begin{aligned} q_b &= \frac{\sqrt{2}}{4} k\pi \delta_b^2 \theta_b m \frac{I_1\left(\frac{4}{3}\sqrt{2}\, mb\right)}{I_0\left(\frac{4}{3}\sqrt{2}\, mb\right)} \\ &= \frac{\sqrt{2}}{4}(100)\pi(0.0092)^2(75)(9.325)\left(\frac{1.2652}{1.9354}\right) \\ &= 4.30 \text{ W} \end{aligned}$$

The efficiencies and heat dissipations of all four spines in the surroundings imposed are summarized as follows:

Spine Profile	η	q_b (W)
Cylindrical	0.657	5.70
Conical	0.796	3.45
Concave parabolic	0.858	2.48
Convex parabolic	0.744	4.30

1.6 NOMENCLATURE

Roman Letter Symbols

A cross-sectional or profile area, m^2; combination of coefficients in Frobenius analysis
a coefficient in Frobenius analysis, dimensions vary
b fin height, m
C arbitrary constant, dimensionless
d diameter, m; derivative or differential, dimensions vary
f function, dimensions vary
h heat transfer coefficient, $W/m^2 \cdot K$
I modified Bessel function of the first kind
K modified Bessel function of the second kind
k thermal conductivity, $W/m \cdot K$
L fin length, m
M modified fin performance parameter, dimensions vary
m fin performance parameter, m^{-1}
P fin perimeter, m
p ratio of elliptical spine diameters, dimensionless
q heat flow, W
R radius function, dimensionless
r radial coordinate, m; radius, m
S surface area, m^2
T temperature, K
u transformed variable, dimensionless
v transformed variable, dimensionless; variable in Frobenius expansion, dimensionless
x height coordinate, m
y fin thickness function, m

Greek Letter Symbols

β ratio of Bessel functions, dimensionless
Δ change in, dimensionless
δ fin thickness, m; side of rectangular spine, m; diameter of elliptical spine, m
ζ combination of Bessel functions, dimensionless
η fin efficiency, dimensionless

θ temperature excess, K
Λ combination of Bessel functions, dimensionless
ρ radius ratio, dimensionless
ϕ combination of terms, dimensionless
ψ combination of Bessel functions, dimensionless
Ω combination of terms, dimensionless

Roman Letter Subscripts

a tip of fin
b base of fin
id ideal
n order of Bessel function
p profile area
s surroundings

Roman Letter Superscripts

k exponent in Frobenius solution
n indicates type of profile
p exponent in Frobenius solution

Greek Letter Superscipts

α exponent in Euler equation general solution, dimensionless
β exponent in Euler equation general solution, dimensionless