1

Lines

One of the very pleasant things about calculus is the fact that we can draw a picture of nearly everything we do. The graph of a function or an equation gives us something concrete to look at and hang our analytic ideas on. The clever device that allows us to geometrize our ideas is of course the Cartesian coordinate system, named after the French mathematician René Descartes. The Cartesian coordinate system not only gives us a pictorial representation of the ideas, but it allows us to use algebraic methods on geometric problems and geometric methods on algebraic problems.

The coordinate system consists of two perpendicular lines, the horizontal one called the ***x*-axis** and the vertical one called the ***y*-axis**. Each point in the plane is identified by a pair of numbers (x, y), where x gives the distance to the right or left of the y-axis and y gives the distance up or down from the x-axis. If $x > 0$, then the point is x units to the right of the y-axis, and if $x < 0$, then the point is $|x|$ units to the left of the y-axis. ($|x|$ denotes the magnitude of the number x, so $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. $|x|$ is called the **absolute value** of x.) Similarly, the point (x, y) is above the x-axis y units if $y > 0$, and below the x-axis if $y < 0$. The numbers x and y are the **coordinates** of the point (x, y). The coordinate axes divide the plane into four **quadrants** (Figure 1.1).

The graph of an equation in x and y is the set of all points whose coordinates satisfy the equation. The graph of an equation is usually a curve in the plane, and here we look at the simplest curves, straight lines.

If (x_1, y_1) and (x_2, y_2) are any two points on a line, then the quantity $m = (y_2 - y_1)/(x_2 - x_1)$ is called the **slope** of the line. You get the same slope, m, no matter what two points you choose. Hence, if (x_1, y_1) is any point on a given line with slope m, and (x, y) is any other point, then

$$\frac{y - y_1}{x - x_1} = m,$$

or

$$y - y_1 = m(x - x_1). \tag{1.1}$$

Equation (1.1) therefore characterizes the line through (x_1, y_1) with slope m. If the line with

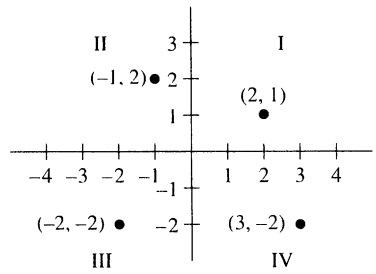

Figure 1.1

slope m goes through the y-axis at $(0, b)$, then the equation is

$$y - b = m(x - 0),$$

or

$$y = mx + b. \tag{1.2}$$

Equation (1.2) is called the **slope-intercept** form, and the number b is called the ***y*-intercept**.

Lines that are not parallel to the y-axis have equations of the form $y = mx + b$. Vertical lines have equations of the form $x = c$. Hence, any line has an equation of the form

$$Ax + By + C = 0, \tag{1.3}$$

where at least one of A and B is nonzero. Conversely, the graph of any equation of the form (1.3) is a line.

EXAMPLE 1.1

Write the equation of the line through $(3, -1)$ and $(1, 2)$. What is the slope, and what is the y-intercept?

Solution

We first find the slope using $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$:

$$m = \frac{2 - (-1)}{1 - 3} = -\frac{3}{2}.$$

Now use either of the given points, say $(3, -1)$, and write the equation (1.1):

$$y - (-1) = \left(-\frac{3}{2}\right)(x - 3),$$

$$y + 1 = -\frac{3}{2}x + \frac{9}{2}.$$

The slope-intercept form is

$$y = -\frac{3}{2}x + \frac{7}{2},$$

so the y-intercept is $\frac{7}{2}$, which is the value of y when $x = 0$.

EXAMPLE 1.2

Write the equation and graph the line through $(3, 1)$ with slope $\frac{1}{2}$. What are the x- and y-intercepts?

Solution

The equation of the line is

$$y - 1 = \frac{1}{2}(x - 3),$$

$$y = \frac{1}{2}x - \frac{1}{2}.$$

The y-intercept is $-\frac{1}{2}$. To find the x-intercept, where the line crosses the x-axis, set $y = 0$ and solve for x:

$$0 = \frac{1}{2}x - \frac{1}{2}; \quad x = 1.$$

The x-intercept is 1. The graph is shown in Figure 1.2.

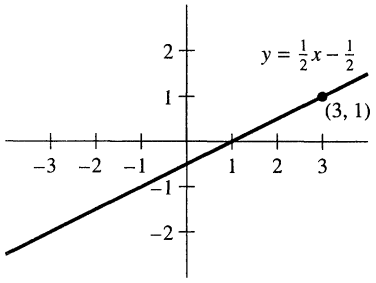

Figure 1.2

Parallel lines have the same slope, but what about perpendicular lines? Let m_1 and m_2 be the slopes of two perpendicular lines. In Figure 1.3 we see that angles BAD and BDC are equal, since their respective sides are perpendicular. Hence, $b/a = c/b$, and $b^2 = ac$. The slopes m_1 and m_2 are given by

$$m_1 = \frac{b}{a} \quad \text{and} \quad m_2 = -\frac{b}{c},$$

so

$$m_1 m_2 = \left(\frac{b}{a}\right)\left(-\frac{b}{c}\right) = -\frac{b^2}{ac} = -1.$$

Lines are perpendicular if and only if their slopes are negative reciprocals of each other.

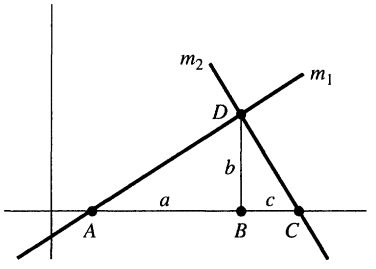

Figure 1.3

EXAMPLE 1.3

Find the line through (3, 2) which is perpendicular to the line $5x + 2y - 7 = 0$, and the line through (3, 2) which is parallel to $5x + 2y - 7 = 0$.

Solution

We write the equation of the given line in slope-intercept form to determine its slope:

$$y = -\frac{5}{2}x + \frac{7}{2}.$$

The given line has slope $-\frac{5}{2}$, so a parallel line has slope $-\frac{5}{2}$ and a perpendicular line has slope $\frac{2}{5}$. Hence, the perpendicular line through (3, 2) is

$$y - 2 = \frac{2}{5}(x - 3),$$

and the parallel line through (3, 2) is

$$(y - 2) = -\frac{5}{2}(x - 3).$$

The angle α a line makes with the x-axis, or any horizontal line, is called the **inclination** of the line. If a line has inclination α, then its slope is $m = \tan\alpha$ (Figure 1.4).

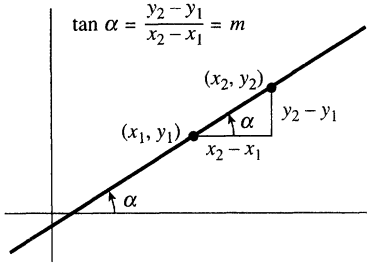

Figure 1.4

EXAMPLE 1.4

(a) If a line has inclination 20°, what is its slope?

(b) If a line has slope 3, what is its inclination?

Solution

Make sure your calculator is set for degrees and read that the slope is $\tan 20° = .36$. To find the inclination of a line with slope 3, read $\tan^{-1} 3 = 71.6°$.

EXAMPLE 1.5

Find the angle between the lines $y = x - 4$ and $y = \frac{1}{2}x - 1$ (Figure 1.5).

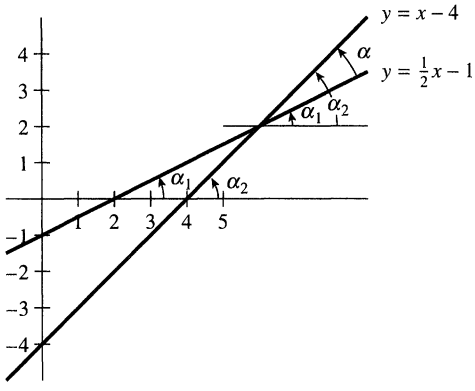

Figure 1.5

Solution

From Figure 1.5 we see that if α_2 is the inclination of $y = x - 4$, and α_1 is the inclination of $y = \frac{1}{2}x - 1$, then the angle α between the lines is $\alpha = \alpha_2 - \alpha_1$. Here

$$\alpha_2 = \tan^{-1} 1 = 45°,$$

$$\alpha_1 = \tan^{-1} \frac{1}{2} = 26.6°,$$

so $\alpha = 18.4°$.

PROBLEMS

1.1 Graph the pairs of points, and then describe the geometric relationship between any two points (a, b) and (b, a); $(1, 0)$ and $(0, 1)$; $(3, 2)$ and $(2, 3)$; $(-1, 2)$ and $(2, -1)$; $(4, -1)$ and $(-1, 4)$.

Graph the lines and determine their slopes.

1.2 $x + 2y = 4$

1.3 $x - y = 2$

1.4 $y = 2x - 3$

1.5 $y = -x - 1$

1.6 $x + 3y = 0$

1.7 $2x + 3y + 6 = 0$

Write the equations of the following lines, and put the equation in the slope-intercept form $y = mx + b$.

1.8 through $(1, 1)$ and $(3, 0)$

1.9 through $(-2, 3)$ and $(4, 1)$

1.10 with y-intercept 7 and slope 2

1.11 with y-intercept -1 and slope -4

1.12 through $(1, 4)$ with slope -3

1.13 through $(2, 3)$ and perpendicular to $x - 2y + 1 = 0$

1.14 through $(1, 2)$ and parallel to $7x - y = 4$

1.15 (i) Show that the lines $x + y - 3 = 0$ and $x - y + 1 = 0$ intersect at $(1, 2)$.

(ii) Show that for any constant k

$$(x - y + 1) + k(x + y - 3) = 0 \qquad (*)$$

is the equation of a line through $(1, 2)$.

(iii) Find k so that the line (*) goes through $(1, 2)$ and $(2, 4)$, and write the equation of the line.

1.16 Find the equation of the line through $(1, 1)$ and the intersection of the lines $x - 2y + 2 = 0$ and $x + y - 4 = 0$. *Hint*: See Problem 1.15. Notice that you don't have to find the intersection of the given lines.

1.17 Find the equation of the line with slope 2 which passes through the intersection of the lines $x + 2y - 5 = 0$ and $3x + y - 7 = 0$.

1.18 Find the equation of the line through the intersection of $x + 3y - 2 = 0$ and $2x + y - 5 = 0$ which is perpendicular to $x + 3y - 2 = 0$.

1.19 Find the angle (i) between the lines $y = 2x$ and $y = 3x$; (ii) between the lines $y = 2x - 5$ and $y = 3x + 7$.

1.20 Find the line through $(3, 4)$ which makes an angle of $25°$ with the x-axis.