1

Introduction

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1.1 ERROR-CONTROLLED ADAPTIVE FINITE-ELEMENT-METHODS

The important advantages of direct variational methods, in particular the Finite Element Method (FEM), against Finite Difference Methods (FDM) are what we call 'logical width and depth, combined with simplicity and robustness', i.e. the same additive methodological concept for a wide class of problems with simple extension to more complicated problems (e.g. from linear to nonlinear theories), and this should be possible with simple algorithmic structures like symmetric and positive definite systems matrices combined with robustness of the solvers and parameter–independent convergence rates in regular cases.

The accuracy of a specific FE analysis thus depends upon the mathematical model and its dimension, the given physical problem, the polynomial degree p of the element-wise trial functions, and the chosen mesh with the local mesh width h. The convergence order of a physically stable and numerically robust primal FE analysis for an elliptic boundary value problem then yields the global *a priori* error estimator in the energy norm of $\|e\|_{E(\Omega)} \stackrel{\text{def}}{=} \|u - u^h\|_{E(\Omega)} \leq C(u, p)h^{\min(r^{-1}, p)}$, where C is mesh independent, r is the regularity of the exact solution, and p is the poly.

It is important to know that a well-posed error controlled adaptive FEM needs careful information about:

- the existence and uniqueness of the analytical solution of a problem;
- the existence in the approximation space of the FEM;
- robustness properties, i.e. mesh-independence of the unified interpolation and stability constant C in the above *a priori* error estimate.

If these criteria are fulfilled, the adaptive analysis–controlled by proper *a posteriori* error estimates–yields the *a priori* convergence rate, and can even be improved to exponential convergence in the case of quasi-optimal hp-adaptivity for which general error measures are not yet available because a measure of regularity–which is not local–has also to be considered.

Today, two types of error estimators are well known for linear elliptic problems, namely residual and averaging estimators. Global estimators can be extended to local or goal-oriented estimators by the product of the error of the original or primal problem and

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the global error (pollution error) of the corresponding dual problem (according to Betti's reciprocal theorem for linear elliptic problems), which can be extended to nonlinear problems by stepwise linearizations where the estimator of the dual problem is always linear. It should be clear that an estimator has to have a bound by definition, usually at least an upper bound in the case of a primal FEM for elliptic well-posed boundary value problems. Lower bounds depend upon the interpolation properties, and are harder to get than upper bounds.

Relative error estimators can usually be computed explicitly at the element level, where residual estimators need the traction jumps to neighboured elements. Absolute or quantitative error estimators need to solve local implicit problems with higher test spaces with respect to the previous analysis (due to the crucial *Galerkin*-orthogonality of the discretization error and the test space used). In the case of residual estimates, local *Dirichlet* or *Neumann* problems have to be solved for enhanced test functions, and in the case of averaging estimates, improved local quantities are computed by testing with higher polynomials in the form of monomes, also discussed as Superconvergent Patch Recovery (SPR) by *Zienkiewicz* and *Zhu*.

Experience has shown that the additional effort for computing absolute error estimates yields sharper bounds and improved effectivity indices, such that the adaptive process becomes more efficient, which especially holds for weakly elliptic problems (like strong anisotropy) or thin-walled structures where the isosurfaces of the strain energy are flat ellipsoids. But it is still an open problem to prove that a certain 'good' error estimator with a sharp bound is also a 'good' indicator for optimal efficiency of the whole adaptive computational process, and vice versa.

The error analysis of geometrically nonlinear problems can be treated by tangential error estimators, using the implicit function theorem. Severe problems arise in the closed neighbourhood of singularity points of the solution path, like bifurcation and turning points. In the case of time-dependent problems like elastoplasticity (where the time in the flow rule is only a load parameter) or viscoelasticity, the usual operator split with the FEM in space and the FDM in time (e.g. with backward *Euler* integration) does not yield a fully coupled error analysis in space and time. There is particularly the problem of the accumulation error in time, for which a proper estimate is not possible. To control this, a crude method is to repeat the whole calculation with smaller time steps (which is not efficient), or to use a type of pilgrim step procedure (for adaptive time stepping), controlled by the estimated error in time, i.e. reducing the time increment some steps before, and noting the accumulation error. Therefore, only a *Galerkin* method in space and time (continuous or discontinuous in time) yields the possibility of a proper error analysis. However, in realizing that, an additional dimension of the problem (i.e. large algebraic equation systems) has to be solved. This can be overcome by using parallel processing, and distributing the spatial unknowns to the available processors for each time slab, with a certain number of finite time steps in between.

A similar important problem is the stable and efficient time integration of dynamic deformation processes in fulfilling all conservation principles of mechanics.

Error estimation of mixed FEMs is also covered in this book. Two-field elliptic problems were mathematically investigated for special problems with related test and trial spaces, such as the *Brezzi–Douglas–Marini* spaces for elastic plates in bending.

In summary, there is strong need for further research in the whole area of error-controlled adaptive FEMs.

1.2 MISSING FEATURES AND PROPERTIES OF TODAY'S GENERAL PURPOSE FE PROGRAMS FOR STRUCTURAL MECHANICS

1. **Reliability**

   – absolute quantitative global and local error estimation (with bounds!);
   – variable h-, hp- and hpd-Ansatz-concepts;
   – flexible remeshing of 2D- and 3D-complex structures for error controlled adaptive FEMs; *a priori* mesh refinement by graded meshes is available in general;
   – checking numerical instability phenomena (like locking and softening (spurious modes));
   – integrated error controlled model adaptivity in disturbed subdomains, and for changes of material and system behavior;
   – sensitivity analysis and structural optimization with error control.

2. **Efficiency**

   – combination of different loads with adaptive FEMs;
   – efficiency control of adaptive FEMs by effectivity indices;
   – providing direct and interactive multi-level solvers for unstructured meshes of multi-dimensional large problems, including parallel computing;
   – efficient solvers for large scale critical and post-critical static analysis;
   – flexible adaptive post-critical quasistatic and dynamic analysis.

3. **Robustness**

   – *Galerkin* space-time integration of parabolic and hyperbolic problems;
   – p- and hp-discretizations for primal and mixed methods avoiding numerical instabilities;
   – numerical stability and condition number checks of direct and iterative algorithms.

4. **Insight and safety aspects in general**

   – control of equilibrium in arbitrary sections defined by the user;
   – vector plots for resulting tractions in arbitrary defined curved sections of 3D-systems;
   – knowledge-based recommendations and warnings concerning:
     - mathematical model, e.g. 5 or 7 parameter shell theory or 3D-theory,
     - type of inelastic deformations,
     - numerical method,

  - ansatz-functions (h-, p-, hp-version), p-order,
  - solvers;

- transition from continuous damage to discrete failure;
- quasistatic/dynamic ultimate load and lifetime analysis;
- shake down and low cycle fatigue analysis, i.e. consideration of load spaces;
- failure progress of total ductile/brittle structures.

This list should give some input into discussions about goals and strategies of further software development.

Of course, alternative methods and techniques have entered computational mechanics, mainly for special tasks, like the mesh-free method, Salerkins space-time elements and new types of mixed elements. But the core of FEM as a weighted residual concept and related numerical methods with a big variety of related software implementations will be very probably predominant for many decades.

The main issue of today is the coupling with general CAD-programs on all data- and program-interfaces, which is one of the conditions for quasi-automatic error-controlled adaptivity.

Another important task is the coupling of error-controlled mathematical modeling – also as multi-scale modeling with the error-controlled adaptivity of the approximation error.