CHAPTER ONE

ESSENTIAL BACKGROUND

This book is intended for readers who have been exposed to the basic concepts of finite element analysis that are covered in the author's book entitled *Fundamental Finite Element Analysis and Applications: with Mathematica and MATLAB Computations*, (Hoboken, NJ: Wiley, 2005). To minimize references to this *fundamentals book*, some key concepts and equations are summarized in this chapter.

The *finite element method* is a numerical method for obtaining approximate solutions of ordinary and partial differential equations. It is especially powerful when dealing with boundary conditions defined over complex geometries that are common in practical applications. The method can be viewed as a special form of the well-known Galerkin and Rayleigh-Ritz methods. In their classical form it is difficult to apply these methods to two- and three-dimensional problems that involve complicated geometry and boundary conditions. The finite element method overcomes these difficulties by introducing two fundamental concepts.

1. *The solution domain is discretized into elements.* The solution domain is divided into several simpler subdomains called *elements*. Each element has a simple geometry, so appropriate assumed solutions can easily be written for the element. Furthermore, since each element covers only a small portion of the solution domain, a low-degree polynomial can generally be used to describe solution over an element. The differential equations are converted into equivalent integral form called the *weak form*. The weak form can then be evaluated separately over each element and added (assembled) together to represent the entire model.

2. *Coefficients in the assumed solution over an element represent the solution and its appropriate derivatives at the nodes.* In the classical methods the unknown coefficients in the assumed solution do not have physical meaning. They are just mathematical quantities

which when substituted into the assumed solution give an approximate solution. In the finite element method the polynomial coefficients are defined in terms of unknown solutions at the selected points over an element. The locations chosen to define the assumed solution are called *nodes*. Usually, element ends and corners are chosen as nodal locations. The solutions at the nodes are called *nodal degrees of freedom*.

The choice of nodal degrees of freedom is dictated by the order of the derivatives in the essential boundary conditions (EBCs). Since for a second-order problem, the essential boundary conditions do not involve any derivatives, the nodal unknowns for these problems are simply the solution variables. For a fourth-order differential equation, at each node we must choose the solution and its first derivative as degrees of freedom because these are the terms in the essential boundary conditions for this problem. With this choice of nodal degrees of freedom it becomes almost trivial to make the assumed solutions admissible. All that one has to do is to set the corresponding nodal value equal to the value specified by the essential boundary condition.

1.1 STEPS IN A FINITE ELEMENT SOLUTION

The application of the finite element method to a given problem involves the following six steps:

1. Development of element equations
2. Discretization of solution domain into a finite element mesh
3. Assembly of element equations
4. Introduction of boundary conditions
5. Solution for nodal unknowns
6. Computation of solution and related quantities over each element

To review these steps, we consider a simple example of axial deformation of bars.

1.1.1 Two-Node Uniform Bar Element

The simplest element for axial deformation of bars is a two-node line element (Figure 1.1). The element extends from x_1 to x_2 and has a length $L = x_2 - x_1$. The circles represent nodes.

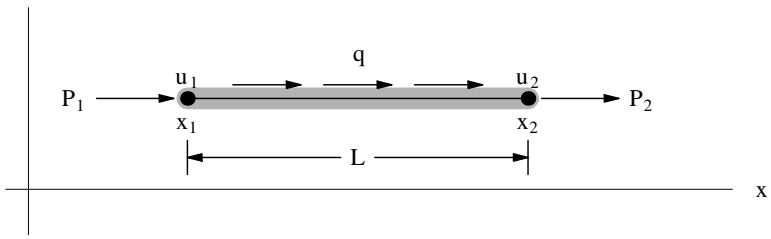


Figure 1.1. Simple two-node element for axial deformation

The unknown solutions at the nodes are indicated by u_1 and u_2. In addition to the distributed load $q(x)$, the element may be subjected to concentrated axial loads P_1 and P_2 applied at the ends of the element. With these assumptions the governing differential equation over an element is

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + q = 0; \quad x_1 < x < x_2$$

Any possible concentrated loads at element ends form natural boundary conditions (NBCs). With the sign convention that tension is positive, the concentrated forces at element ends are related to the derivatives of displacement as

$$-AE\frac{du(x_1)}{dx} = P_1; \qquad AE\frac{du(x_2)}{dx} = P_2$$

The primary unknown is the axial displacement u. Once the displacement is known, axial strain, stress, and force can be computed from the following relationships:

$$\epsilon_x = \frac{du}{dx}; \qquad \sigma_x = E\epsilon_x; \qquad F = A\sigma_x$$

Linear Assumed Solution The assumed solution is a linear interpolation between the nodal unknowns. Thus,

$$u(x) = \begin{pmatrix} \dfrac{x-x_2}{x_1-x_2} & \dfrac{x-x_1}{x_2-x_1} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} N_1 & N_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \boldsymbol{N}^T\boldsymbol{d}$$

$$N_1 = \frac{x-x_2}{x_1-x_2} = -\frac{x-x_2}{L}; \quad N_2 = \frac{x-x_1}{x_2-x_1} = \frac{x-x_1}{L}$$

We will need $u'(x)$ in the later derivation. Differentiating with respect to x, we get

$$u'(x) = \frac{du(x)}{dx} = \begin{pmatrix} \dfrac{1}{x_1-x_2} & \dfrac{1}{x_2-x_1} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} N_1' & N_2' \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv B^T\boldsymbol{d}$$

$$N_1' = \frac{1}{x_1-x_2} = -\frac{1}{L}; \quad N_2' = \frac{1}{x_2-x_1} = \frac{1}{L}$$

Element Equations Using the Galerkin Method The weak form can be written using standard steps of writing the weighted residual, integration by parts, and incorporating the natural boundary conditions. The weighting functions are the same as the interpolation functions N_i. With $u(x)$ as an assumed solution, the residual is

$$e(x) = q + \frac{d}{dx}(AEu')$$

Multiplying by $N_i(x)$ and writing integral over the given limits, the Galerkin weighted residual is

$$\int_{x_1}^{x_2} \left(qN_i + \frac{d}{dx}(AEu')N_i \right) dx = 0$$

Using integration by parts yields

$$AEN_i(x_2)u'(x_2) - AEN_i(x_1)u'(x_1) + \int_{x_1}^{x_2} (qN_i - AEu'N_i')\, dx = 0$$

Given NBCs for the problem, we have

$$-AEu'(x_1) = P_1 \quad \text{and} \quad AEu'(x_2) = P_2$$

Thus, the weak form is

$$P_1N_i(x_1) + P_2N_i(x_2) + \int_{x_1}^{x_2} (qN_i - AEu'N_i')\, dx = 0$$

With the two interpolation functions the two equations for the element are

$$\int_{x_1}^{x_2} (qN_1(x) - AEu'(x)N_1'(x))\, dx + P_1N_1(x_1) + P_2N_1(x_2) = 0$$

$$\int_{x_1}^{x_2} (qN_2(x) - AEu'(x)N_2'(x))\, dx + P_1N_2(x_1) + P_2N_2(x_2) = 0$$

Noting the property of the Lagrange interpolation functions that $N_1(x_1) = 1$, $N_1(x_2) = 0$, $N_2(x_1) = 0$, and $N_2(x_2) = 1$ and writing the two equations together using matrix notation, we have

$$\int_{x_1}^{x_2} \left(\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} q - \begin{pmatrix} N_1' \\ N_2' \end{pmatrix} AEu'(x) \right) dx + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Substituting for $u'(x)$, we have

$$\int_{x_1}^{x_2} \left(\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} q - \begin{pmatrix} N_1' \\ N_2' \end{pmatrix} AE \begin{pmatrix} N_1' & N_2' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) dx + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\int_{x_1}^{x_2} \boldsymbol{N} q\, dx - \int_{x_1}^{x_2} \boldsymbol{B}AE\boldsymbol{B}^T \boldsymbol{d}\, dx + \boldsymbol{r}_p = \boldsymbol{0}$$

where $\boldsymbol{r}_p$ is the 2×1 vector of element nodal loads $(P_1, P_2)^T$ and $\boldsymbol{0}$ is a 2×1 vector of zeros. Rearranging terms and taking nodal degrees of freedom out of the integral sign because they are not functions of x, we have

$$\int_{x_1}^{x_2} \boldsymbol{B}AE\boldsymbol{B}^T\, dx\boldsymbol{d} = \int_{x_1}^{x_2} \boldsymbol{N} q\, dx + \boldsymbol{r}_p \Longrightarrow \boldsymbol{k}\boldsymbol{d} = \boldsymbol{r}_q + \boldsymbol{r}_p$$

where

$$\boldsymbol{k} = \int_{x_1}^{x_2} \boldsymbol{B}AE\boldsymbol{B}^T\,dx; \qquad \boldsymbol{r}_q = \int_{x_1}^{x_2} \boldsymbol{N}q\,dx; \qquad \boldsymbol{r}_p = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

To write explicit equations we must substitute derivatives of the interpolation functions and carry out integrations. For simplicity, it is assumed that AE and q are constant over an element. Thus, we have

$$\boldsymbol{k} = \int_{x_1}^{x_2} \boldsymbol{B}AE\boldsymbol{B}^T\,dx = \begin{pmatrix} \displaystyle\int_{x_1}^{x_2} AE\frac{1}{L^2}\,dx & \displaystyle -\int_{x_1}^{x_2} AE\frac{1}{L^2}\,dx \\ \displaystyle -\int_{x_1}^{x_2} AE\frac{1}{L^2}\,dx & \displaystyle \int_{x_1}^{x_2} AE\frac{1}{L^2}\,dx \end{pmatrix} = \frac{AE}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\boldsymbol{r}_q = \int_{x_1}^{x_2} \boldsymbol{N}q\,dx = \begin{pmatrix} \displaystyle\int_{x_1}^{x_2} -\frac{x-x_2}{L}q\,dx \\ \displaystyle\int_{x_1}^{x_2} \frac{x-x_1}{L}q\,dx \end{pmatrix} = \frac{qL}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We now have equations for the two-node element for the axial deformation problem:

$$\frac{AE}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{qL}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

Element Equations Using the Rayleigh-Ritz Method The same equations can also be derived using the Rayleigh-Ritz method. The potential energy function for the element is as follows:

$$\Pi = \text{strain energy } (U) - \text{work done by applied forces } (W)$$

For axial deformations the strain energy is

$$U = \frac{1}{2}\int_{x_1}^{x_2} AE(u'(x))^2\,dx$$

The square of the first derivative of the assumed solution must be written carefully so that the nodal unknowns can be taken out of the integral sign. To achieve this goal, we proceed as follows:

$$(u'(x))^2 = (u'(x))^T u'(x) = (\boldsymbol{B}^T\boldsymbol{d})^T\boldsymbol{B}^T\boldsymbol{d} = \boldsymbol{d}^T\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{d}$$

where, we have used the rule for the transpose of product of two matrices, $(\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T\boldsymbol{A}^T$. The strain energy term can now be evaluated as follows:

$$U = \frac{1}{2}\int_{x_1}^{x_2} AE\boldsymbol{d}^T\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{d}\,dx = \frac{1}{2}\boldsymbol{d}^T\int_{x_1}^{x_2} AE\boldsymbol{B}\boldsymbol{B}^T\,dx\boldsymbol{d} \equiv \frac{1}{2}\boldsymbol{d}^T\boldsymbol{k}\boldsymbol{d}$$

where

$$\boldsymbol{k} = \int_{x_1}^{x_2} AE\boldsymbol{B}\boldsymbol{B}^T\, dx = \frac{AE}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The work done by the distributed load can be evaluated as follows:

$$W_q = \int_{x_1}^{x_2} qu\, dx = \int_{x_1}^{x_2} q\boldsymbol{N}^T\boldsymbol{d}\, dx = \boldsymbol{r}_q^T\boldsymbol{d} \equiv \boldsymbol{d}^T\boldsymbol{r}_q$$

where

$$\boldsymbol{r}_q^T = \int_{x_1}^{x_2} q\boldsymbol{N}^T\, dx \Longrightarrow \boldsymbol{r}_q = \int_{x_1}^{x_2} \boldsymbol{N}q\, dx = \frac{qL}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The work done by the concentrated loads applied at element ends is

$$W_p = P_1u_1 - P_2u_2 = \begin{pmatrix} P_1 & P_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \boldsymbol{r}_p^T\boldsymbol{d} \equiv \boldsymbol{d}^T\boldsymbol{r}_p$$

The potential energy can now be written as

$$\Pi = U - W = \frac{1}{2}\boldsymbol{d}^T\boldsymbol{k}\boldsymbol{d} - \boldsymbol{r}_q^T\boldsymbol{d} - \boldsymbol{r}_p^T\boldsymbol{d} = \boldsymbol{d}^T\left(\frac{1}{2}\boldsymbol{k}\boldsymbol{d} - \boldsymbol{r}_q - \boldsymbol{r}_p\right)$$

The necessary conditions for minimum of the potential energy give

$$\frac{\partial\Pi}{\partial\boldsymbol{d}} = \left(\frac{1}{2}\boldsymbol{k}\boldsymbol{d} - \boldsymbol{r}_q - \boldsymbol{r}_p\right) + \frac{1}{2}\boldsymbol{k}\boldsymbol{d} = \boldsymbol{0}$$

Rearranging terms, we get the same element equations as those obtained using the Galerkin method:

$$\boldsymbol{k}\boldsymbol{d} = \boldsymbol{r}_q + \boldsymbol{r}_p \Longrightarrow \frac{AE}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{qL}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

The following example illustrates the remaining five steps in the finite element solution process.

Example 1.1 Rotating Bar Determine axial stress distribution in a bar, shown in the Figure 1.2, that is rotating at 500 rpm. The problem can be treated as one-dimensional with the following governing differential equation:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + \rho Ax\omega^2 = 0; \qquad 0 < x < L$$

$$u(0) = 0; \qquad EA\frac{du(L)}{dx} = 0$$

where x is a coordinate along the axis of the bar, $u(x)$ is the axial displacement, L is the length of the bar, E is Young's modulus, A is the cross-sectional area, ρ is the mass density, and ω is the angular velocity in rad/s. The axial stress is $\sigma_x = E\, du/dx$. An exact analytical

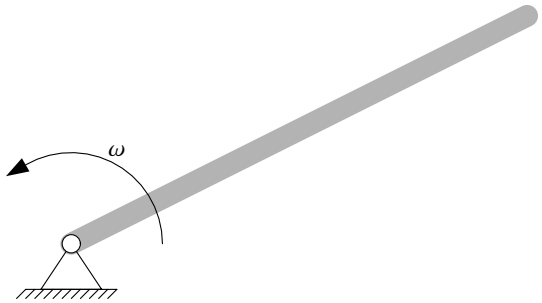


Figure 1.2. Rotating bar

solution of the problem is

$$\sigma_{x,\text{exact}} = \frac{\rho\omega^2}{2}(L^2 - x^2)$$

Compare finite element solutions with one, two, and three elements. Use the following numerical data: $L = 80$ cm, $E = 200$ GPa, $A = 250$ mm^2, and $\rho = 7850$ kg/m^3. For this problem, $q(x) = \rho A x \omega^2$. Since q is not constant, the $\boldsymbol{r}_q$ vector needs to be computed.

$$\boldsymbol{r}_q = \int_{x_1}^{x_2} \boldsymbol{N} q \, dx = \begin{pmatrix} \displaystyle\int_{x_1}^{x_2} -\frac{x - x_2}{L}\rho A x \omega^2 \, dx \\ \displaystyle\int_{x_1}^{x_2} \frac{x - x_1}{L}\rho A x \omega^2 \, dx \end{pmatrix} = \begin{pmatrix} \dfrac{L^2 \rho A \omega^2}{6} \\ \dfrac{L^2 \rho A \omega^2}{3} \end{pmatrix}$$

$$\text{Element equations: } \begin{pmatrix} \dfrac{EA}{L} & -\dfrac{EA}{L} \\ -\dfrac{EA}{L} & \dfrac{EA}{L} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \dfrac{L^2 \rho A \omega^2}{6} \\ \dfrac{L^2 \rho A \omega^2}{3} \end{pmatrix}$$

Using $N \cdot m$ units the solution is as follows.

One-Element Solution

$$\text{Element nodes: } \left\{x_1 \to 0, x_2 \to \tfrac{4}{5}\right\}$$

$$\begin{pmatrix} 62500000 & -62500000 \\ -62500000 & 62500000 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 573.899 \\ 1147.8 \end{pmatrix}$$

Essential boundary conditions:

dof	Value
u_1	0

Incorporating EBC, the final system of equations is

$$(62500000)(u_2) = (1147.8)$$

Solution for nodal unknowns:

dof	x	Solution
u_1	0	0
u_2	$\frac{4}{5}$	0.0000183648

Solution over elements:

$$\text{Interpolation functions: } \boldsymbol{N}^T = \left\{1 - \frac{5x}{4}, \frac{5x}{4}\right\}$$

$$\text{Nodal values: } \boldsymbol{d}^T = \{0, 0.0000183648\}$$

$$\text{Solution: } u(x) = \boldsymbol{N}^T\boldsymbol{d} = 0.000022956x$$

Solution summary:

	Range	Solution
1	$0 \le x \le \frac{4}{5}$	$0.000022956x$

Two-Element Solution

$$\text{Nodal locations: } \left\{0, \tfrac{2}{5}, \tfrac{4}{5}\right\}$$

Element 1:

$$\text{Element nodes: } \left\{x_1 \to 0, x_2 \to \tfrac{2}{5}\right\}$$

$$\begin{pmatrix} 125000000 & -125000000 \\ -125000000 & 125000000 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 143.475 \\ 286.95 \end{pmatrix}$$

Element 2:

$$\text{Element nodes: } \left\{x_2 \to \tfrac{2}{5}, x_3 \to \tfrac{4}{5}\right\}$$

$$\begin{pmatrix} 125000000 & -125000000 \\ -125000000 & 125000000 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 573.899 \\ 717.374 \end{pmatrix}$$

Global equations before boundary conditions:

$$\begin{pmatrix} 125000000 & -125000000 & 0 \\ -125000000 & 250000000 & -125000000 \\ 0 & -125000000 & 125000000 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 143.475 \\ 860.849 \\ 717.374 \end{pmatrix}$$

Essential boundary conditions:

dof	Value
u_1	0

Incorporating EBCs, the final system of equations is

$$\begin{pmatrix} 250000000 & -125000000 \\ -125000000 & 125000000 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 860.849 \\ 717.374 \end{pmatrix}$$

Solution for nodal unknowns:

dof	x	Solution
u_1	0	0
u_2	$\frac{2}{5}$	0.0000126258
u_3	$\frac{4}{5}$	0.0000183648

Solution over elements:

Element 1:

$$\text{Nodes: } \left\{x_1 \to 0, x_2 \to \frac{2}{5}\right\}$$

$$\text{Interpolation functions: } \boldsymbol{N}^T = \left\{1 - \frac{5x}{2}, \frac{5x}{2}\right\}$$

$$\text{Nodal values: } \boldsymbol{d}^T = \{0, 0.0000126258\}$$

$$\text{Solution: } u(x) = \boldsymbol{N}^T\boldsymbol{d} = 0.0000315645x$$

Element 2:

$$\text{Nodes: } \left\{x_1 \to \tfrac{2}{5}, x_2 \to \tfrac{4}{5}\right\}$$

$$\text{Interpolation functions: } \boldsymbol{N}^T = \left\{2 - \frac{5x}{2}, \frac{5x}{2} - 1\right\}$$

$$\text{Nodal values: } \boldsymbol{d}^T = \{0.0000126258, 0.0000183648\}$$

$$\text{Solution: } u(x) = \boldsymbol{N}^T\boldsymbol{d} = 0.0000143475x + 6.88679 \times 10^{-6}$$

Solution summary:

	Range	Solution
1	$0 \le x \le \frac{2}{5}$	$0.0000315645x$
2	$\frac{2}{5} \le x \le \frac{4}{5}$	$0.0000143475x + 6.88679 \times 10^{-6}$

Three-Element Solution

$$\text{Nodal locations: } \left\{0, \tfrac{4}{15}, \tfrac{8}{15}, \tfrac{4}{5}\right\}$$

Element 1:

$$\text{Element nodes: } \left\{x_1 \to 0, x_2 \to \tfrac{4}{15}\right\}$$

$$\begin{pmatrix} 187500000 & -187500000 \\ -187500000 & 187500000 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 63.7666 \\ 127.533 \end{pmatrix}$$

Element 2:

$$\text{Element nodes: } \left\{x_2 \to \tfrac{4}{15}, x_3 \to \tfrac{8}{15}\right\}$$

$$\begin{pmatrix} 187500000 & -187500000 \\ -187500000 & 187500000 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 255.066 \\ 318.833 \end{pmatrix}$$

Element 3:

$$\text{Element nodes: } \left\{x_3 \to \tfrac{8}{15}, x_4 \to \tfrac{4}{5}\right\}$$

$$\begin{pmatrix} 187500000 & -187500000 \\ -187500000 & 187500000 \end{pmatrix}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 446.366 \\ 510.133 \end{pmatrix}$$

Global equations before boundary conditions:

$$\begin{pmatrix} 187500000 & -187500000 & 0 & 0 \\ -187500000 & 375000000 & -187500000 & 0 \\ 0 & -187500000 & 375000000 & -187500000 \\ 0 & 0 & -187500000 & 187500000 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 63.7666 \\ 382.599 \\ 765.199 \\ 510.133 \end{pmatrix}$$

Essential boundary conditions:

dof	Value
u_1	0

Incorporating EBC, the final system of equations is

$$\begin{pmatrix} 375000000 & -187500000 & 0 \\ -187500000 & 375000000 & -187500000 \\ 0 & -187500000 & 187500000 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 382.599 \\ 765.199 \\ 510.133 \end{pmatrix}$$

Solution for nodal unknowns:

dof	x	Solution
u_1	0	0
u_2	$\frac{4}{15}$	8.8423×10^{-6}
u_3	$\frac{8}{15}$	0.0000156441
u_4	$\frac{4}{5}$	0.0000183648

Solution over elements:

Element 1:

$$\text{Nodes: } \left\{x_1 \to 0, x_2 \to \tfrac{4}{15}\right\}$$

$$\text{Interpolation functions: } \boldsymbol{N}^T = \left\{1 - \frac{15x}{4}, \frac{15x}{4}\right\}$$

$$\text{Nodal values: } \boldsymbol{d}^T = \{0, 8.8423 \times 10^{-6}\}$$

$$\text{Solution: } u(x) = \boldsymbol{N}^T\boldsymbol{d} = 0.0000331586x$$

Element 2:

$$\text{Nodes: } \left\{x_1 \to \tfrac{4}{15}, x_2 \to \tfrac{8}{15}\right\}$$

$$\text{Interpolation functions: } \boldsymbol{N}^T = \left\{2 - \frac{15x}{4}, \frac{15x}{4} - 1\right\}$$

$$\text{Nodal values: } \boldsymbol{d}^T = \{8.8423 \times 10^{-6}, 0.0000156441\}$$

$$\text{Solution: } u(x) = \boldsymbol{N}^T\boldsymbol{d} = 0.0000255066x + 2.04053 \times 10^{-6}$$

Element 3:

$$\text{Nodes: } \left\{x_1 \to \tfrac{8}{15}, x_2 \to \tfrac{4}{5}\right\}$$

$$\text{Interpolation functions: } \boldsymbol{N}^T = \left\{3 - \frac{15x}{4}, \frac{15x}{4} - 2\right\}$$

$$\text{Nodal values: } \boldsymbol{d}^T = \{0.0000156441, 0.0000183648\}$$

$$\text{Solution: } u(x) = \boldsymbol{N}^T\boldsymbol{d} = 0.0000102027x + 0.0000102027$$

Solution summary:

	Range	Solution
1	$0 \le x \le \frac{4}{15}$	$0.0000331586x$
2	$\frac{4}{15} \le x \le \frac{8}{15}$	$0.0000255066x + 2.04053 \times 10^{-6}$
3	$\frac{8}{15} \le x \le \frac{4}{5}$	$0.0000102027x + 0.0000102027$

The exact solution and the axial stresses obtained from the three solutions are plotted in Figure 1.3. Because of the linear assumed solution over an element, the axial stress is constant and hence the finite element solution shows a stepped pattern. The solution is getting better as the number of elements is increased. From the three-element solution we notice that the average stress from the two adjacent elements is very close to the exact solution.

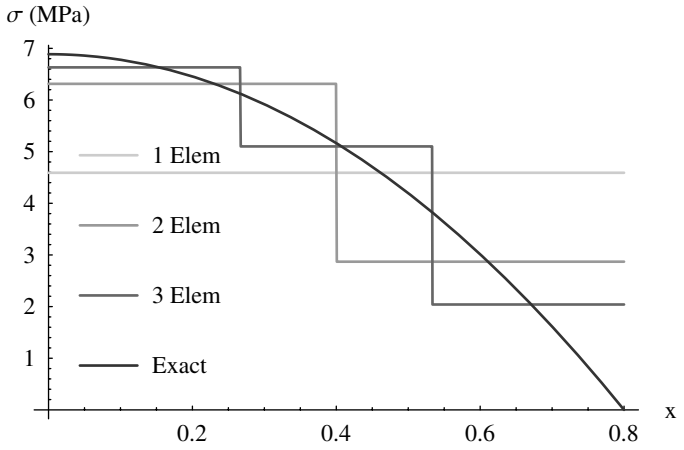


Figure 1.3. Stress distribution from finite element and exact solutions

1.2 INTERPOLATION FUNCTIONS

A two-node line element for a second-order problem is shown in Figure 1.4. The element extends from x_1 to x_2. For a second-order problem the nodal degrees of freedom are the unknown solutions at the nodes and are indicated by u_1 and u_2.

Figure 1.4. Simple two-node element for second-order problems

Since there are two nodes on the element, a linear solution can be written over the element by starting from a linear polynomial with two coefficients and then evaluating these coefficients in terms of the nodal unknowns:

$$u(x) = a_0 + a_1 x$$

$$\text{At } x = x_1: \quad u(x_1) = u_1 \Longrightarrow u_1 = a_0 + a_1 x_1$$

$$\text{At } x = x_2: \quad u(x_2) = u_2 \Longrightarrow u_2 = a_0 + a_1 x_2$$

Solving the two equations for a_0 and a_1, we get

$$a_0 = -\frac{u_1 x_2 - u_2 x_1}{x_1 - x_2}; \qquad a_1 = -\frac{u_2 - u_1}{x_1 - x_2}$$

Thus, linear solution over elements in terms of nodal degrees of freedom is

$$u(x) = -\frac{u_1 x_2 - u_2 x_1}{x_1 - x_2} - \frac{u_2 - u_1}{x_1 - x_2} x$$

Collecting together terms involving u_1 and u_2, we can write this solution as

$$u(x) = \frac{x - x_2}{x_1 - x_2} u_1 + \frac{x - x_1}{x_2 - x_1} u_2$$

This is the finite element form of a linear solution. It clearly is equivalent to the linear polynomial. However, unlike the polynomial coefficients a_0 and a_1, which do not have any physical meaning, coefficients u_1 and u_2 are the solutions at the two nodes of the element. This is a key feature of the finite element form of the assumed solution and has the following three major advantages.

1. To make the solution admissible in the polynomial form, we must use the essential boundary conditions to set up equations and then solve them to find values for one or more parameters. However, in the finite element form, making the solution admissible is trivial. For example, if the essential boundary condition is $u(x_2) = 5$, all, we have to do is to set $u_2 = 5$ and we are done.

2. The finite element form is suitable for direct assembly of element equations. Assume that we are using two elements to model a solution domain, as shown in Figure 1.5. The node 2 is common between the two elements. We can get a piecewise linear solution for the entire domain just by making sure that at the common node the two elements use the same nodal value. This simple observation makes it possible to perform computations independently for each element and then simply to add the contributions from each element. During this assembly process the nodes common between different elements are assigned the same nodal degree of freedom.

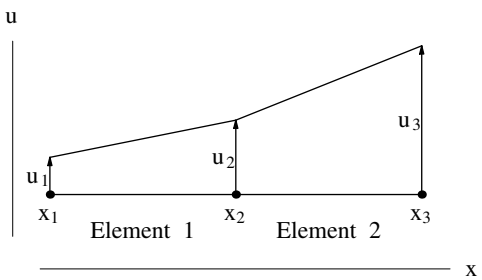


Figure 1.5. Simple two-element model

3. To add more terms to the assumed solution, we now have two options. We can use higher-order polynomials to derive higher-order finite element solutions by following exactly the same procedure as that used for deriving the linear solution. Alternatively, we can simply use a large number of linear elements. By using a sufficiently large number of

simple elements, we can get reasonably accurate solutions to even complicated differential equations.

1.2.1 Lagrange Interpolation for Second-Order Problems

An interpolation function is a smooth function that passes through all data points in a given data set. Thus, treating the nodal locations and unknown solutions at these locations as data points, the problem of writing a finite element solution for second-order differential equations boils down to choosing an appropriate interpolation technique.

A variety of interpolation methods are available in the literature. One simple well-known technique is the *Lagrange interpolation*. If, we have n data points $\{x_i, u_i\}$, $i = 1, 2, \ldots, n$ [where $u(x_i) \equiv u_i$], the Lagrange interpolation formula for passing a polynomial of degree $n - 1$ through these points can be written as

$$u(x) = \sum_{i=1}^{n} L_i(x)u_i \equiv \begin{pmatrix} L_1 & L_2 & L_3 & \cdots \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{pmatrix} \equiv \boldsymbol{N}^T\boldsymbol{d}$$

where $L_i(x)$ are the Lagrange interpolation functions given by the formula

$$L_i(x) \equiv N_i = \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \equiv \frac{x - x_1}{x_i - x_1} \times \frac{x - x_2}{x_i - x_2} \times \cdots \times \frac{x - x_{i-1}}{x_i - x_{i-1}} \times \frac{x - x_{i+1}}{x_i - x_{i+1}} \times \cdots \times \frac{x - x_n}{x_i - x_n}$$

The symbol Π indicates product of terms. For writing the ith interpolation function, the product includes all terms except the ith. Thus, there are $n - 1$ terms in the product, and each interpolation function is of degree $n - 1$, where n is the number of data points.

Using this formula, a linear interpolation ($n = 2$) can be written as follows:

$$u(x) = N_1(x)u_1 + N_2(x)u_2$$

$$N_1(x) = \frac{x - x_2}{x_1 - x_2}; \qquad N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

These are the same shape functions as those derived in the section. A quadratic polynomial ($n = 3$) can be written as follows:

$$u(x) = N_1(x)u_1 + N_2(x)u_2 + N_3(x)u_3$$

$$N_1(x) = \frac{x - x_2}{x_1 - x_2} \times \frac{x - x_3}{x_1 - x_3}; \qquad N_2(x) = \frac{x - x_1}{x_2 - x_1} \times \frac{x - x_3}{x_2 - x_3}; \qquad N_3(x) = \frac{x - x_1}{x_3 - x_1} \times \frac{x - x_2}{x_3 - x_2}$$

1.2.2 Hermite Interpolation for Fourth-Order Problems

Essential boundary conditions for a fourth-order differential equation consist of a solution and its first derivative. To satisfy these essential boundary conditions at the ends of an element, we must choose the solution and its first derivative as nodal degrees of freedom.

Figure 1.6. Two-node element for fourth-order problems

Thus, the simplest two-node line element for a fourth-order problem has a total of four degrees of freedom, as shown in Figure 1.6. The element extends from x_1 to x_2 and has a length $L = x_2 - x_1$. For simplicity, the left end of the element is assumed at $x = 0$ and the right end is at $x = L$.

The Lagrange interpolation formula cannot be used to interpolate when, we have the solution and its first derivative specified at each data point. We can generate two independent interpolations between $\{(x_1, u_1), (x_2, u_2)\}$ and $\{(x_1, u_1'), (x_2, u_2')\}$ if we like. However, then there will be no connection between the solution and its first derivative. The appropriate method to interpolate between data values and their first derivatives at each point is known as *Hermite interpolation*.

General Formula for Hermite Interpolation Given function values u_i and their first derivatives u_i' at n data points, $x_1, x_2, \ldots, x_n$, a polynomial of degree $2n - 1$ that fits all data can be written:

$$u(x) = \sum_{i=1}^{n} P_i(x)u_i + \sum_{i=1}^{n} Q_i(x)u_i' \equiv \begin{pmatrix} P_1 & Q_1 & \cdots & P_n & Q_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_1' \\ \vdots \\ u_n \\ u_n' \end{pmatrix} = \boldsymbol{N}^T\boldsymbol{d}$$

where $P_i(x)$ and $Q_i(x)$ are polynomials of degree $2n - 1$. These polynomials are expressed in terms of Lagrange interpolation functions $L_i(x)$ as follows:

$$P_i(x) = (1 - 2L_i'(x_i)(x - x_i))L_i^2(x)$$
$$Q_i(x) = (x - x_i)L_i^2(x)$$

The functions P_i interpolate between the data values and Q_i between the derivative values. Each interpolation function is of degree $2n - 1$, where n is the number of data points. Note that functions $P_i(x)$ are 1 at the ith data point and zero at all other data points. The first derivatives of P_i, $i = 1, \ldots,$ are zero at all data points. On the other hand, the functions $Q_i(x)$ are zero at all data points and their first derivatives are 1 at the ith point and zero everywhere else. That is,

$$P_i(x_j) = \delta_{ij}; \qquad P_i'(x_j) = 0$$
$$Q_i(x_j) = 0; \qquad Q_i'(x_j) = \delta_{ij}$$

where δ_{ij} is the Dirac delta function, which is equal to 1 when $i = j$ and equal to 0 otherwise. As an example, we use this formula to write shape functions for the element in Figure 1.6.

Data point locations: $\{0, L\}$

Lagrange interpolation functions for these points:

$$\left\{L_1 = -\frac{x-L}{L}, L_2 = \frac{x}{L}\right\}$$

Derivatives of Lagrange interpolation functions:

$$\left\{L_1' = -\frac{1}{L}, L_2' = \frac{1}{L}\right\}$$

Derivatives at the given points:

$$\left\{L_1'(0) = -\frac{1}{L}, L_2'(L) = \frac{1}{L}\right\}$$

Using these, the Hermite interpolations for data values are

$$\left\{P_1 = \frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1, P_2 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right\}$$

and those for derivative values are

$$\left\{Q_1 = \frac{x^3}{L^2} - \frac{2x^2}{L} + x, Q_2 = \frac{x^3}{L^2} - \frac{x^2}{L}\right\}$$

Thus, the assumed solution for the element is

$$u(x) = \left(\frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1 \quad \frac{x^3}{L^2} - \frac{2x^2}{L} + x \quad \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \quad \frac{x^3}{L^2} - \frac{x^2}{L}\right)\begin{pmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{pmatrix}$$

1.2.3 Lagrange Interpolation for Rectangular Elements

We can use products of appropriate Lagrange interpolation formulas in the x and y directions to write interpolation functions for rectangular elements. The procedure is justified in detail for a four-node element followed by several examples of product Lagrange interpolation functions for higher-order rectangular elements.

Four-Node Rectangular Element For a four-node rectangular element with dimensions $2a \times 2b$ and origin at the center (x_c, y_c) as shown in Figure 1.7, we note the following:

1. Along side 1–2, $t = -b$ is constant and therefore the interpolation functions must be functions only of s. Thus,

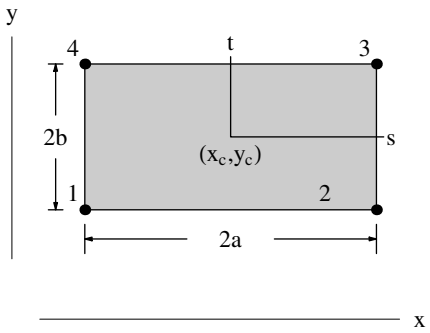


Figure 1.7. Four-node rectangular element

$$u(s,-b) = \begin{pmatrix} n_1(s) & n_2(s) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv \begin{pmatrix} -\dfrac{s-a}{2a} & \dfrac{a+s}{2a} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where the interpolation functions n_1 and n_2 are written using the one-dimensional Lagrange interpolation formula.

2. Similarly, along side 4–3, $t = b$ and therefore the interpolation functions are

$$u(s,b) = \begin{pmatrix} n_4(s) & n_3(s) \end{pmatrix} \begin{pmatrix} u_4 \\ u_3 \end{pmatrix} \equiv \begin{pmatrix} -\dfrac{s-a}{2a} & \dfrac{a+s}{2a} \end{pmatrix} \begin{pmatrix} u_4 \\ u_3 \end{pmatrix}$$

Expanding to include all four degrees of freedom, the interpolations for $u(s,-b)$ and $u(s,b)$ can be written

$$u(s,-b) = \begin{pmatrix} -\dfrac{s-a}{2a} & \dfrac{a+s}{2a} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

$$u(s,b) = \begin{pmatrix} 0 & 0 & \dfrac{a+s}{2a} & -\dfrac{s-a}{2a} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

3. In the t direction we can consider linear interpolation between $u(s,-b)$ and $u(s,b)$. Using the Lagrange interpolation formula in the t direction, we have

$$u(s,t) = \begin{pmatrix} -\dfrac{t-b}{2b} & \dfrac{b+t}{2b} \end{pmatrix} \begin{pmatrix} u(s,-b) \\ u(s,b) \end{pmatrix}$$

Thus, the interpolation for $u(s,t)$ can be written

$$u(s,t) = \begin{pmatrix} -\dfrac{t-b}{2b} & \dfrac{b+t}{2b} \end{pmatrix} \begin{pmatrix} -\dfrac{s-a}{2a} & \dfrac{a+s}{2a} & 0 & 0 \\ 0 & 0 & \dfrac{a+s}{2a} & -\dfrac{s-a}{2a} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

Multiplying the two matrices, we get the following interpolation functions:

$$u(s,t) = \begin{pmatrix} \dfrac{(a-s)(b-t)}{4ab} & \dfrac{(a+s)(b-t)}{4ab} & \dfrac{(a+s)(b+t)}{4ab} & \dfrac{(a-s)(b+t)}{4ab} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

Each interpolation function can be seen as simply a product of the linear Lagrange interpolation functions in the s and t directions.

Nine-Node Rectangular Element To use quadratic interpolation in both the s and t directions, we need three nodes along each line. Thus, we have a nine-node rectangular element $2a \times 2b$ as shown in Figure 1.8. Along the bottom, middle, and top in the s direction we can define quadratic interpolation functions. Now in the t direction we can also use quadratic interpolation. Thus, interpolation functions for the element are as follows:

$$\begin{pmatrix} \dfrac{s(s-a)t(t-b)}{4a^2b^2} \\ -\dfrac{(s-a)(a+s)t(t-b)}{2a^2b^2} \\ \dfrac{s(a+s)t(t-b)}{4a^2b^2} \\ -\dfrac{s(a+s)(t-b)(b+t)}{2a^2b^2} \\ \dfrac{s(a+s)t(b+t)}{4a^2b^2} \\ -\dfrac{(s-a)(a+s)t(b+t)}{2a^2b^2} \\ \dfrac{s(s-a)t(b+t)}{4a^2b^2} \\ -\dfrac{s(s-a)(t-b)(b+t)}{2a^2b^2} \\ \dfrac{(s-a)(a+s)(t-b)(b+t)}{a^2b^2} \end{pmatrix}$$

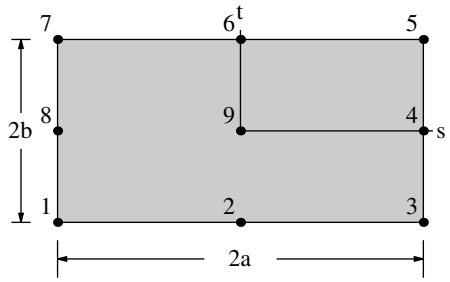


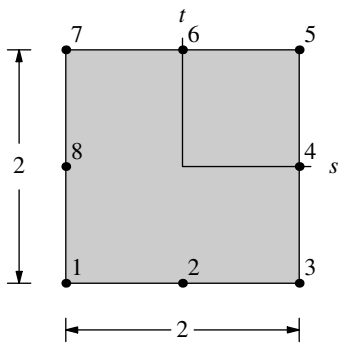
Figure 1.8. Nine-node rectangular element

Serendipity Shape Functions As, we have just seen, with quadratic Lagrange interpolations in both directions, we get a nine-node element. To reduce the required input data, it is generally preferred to have an element with no interior nodes. That is, an eight-node element with quadratic solution along all four sides is more popular than a nine-node element. Furthermore, with the product Lagrange interpolations, it is not possible to have elements that have, say, a quadratic solution along one side and a linear solution in the same direction but on the opposite side of the element. These types of elements are useful in the transition region when switching from a higher-order element to a lower element in the same model. (See Chapter 7 of the fundamentals book for an example.)

The following interpolation functions, known as *serendipity shape functions*, have been derived for square elements with any number of side nodes from four to eight. The functions are written for 2×2 square elements with the origin at the center. These elements act as *master* or *parent elements* for arbitrary quadrilaterals using the mapping concept reviewed in a later section.

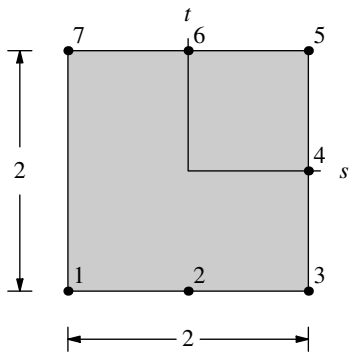
1. Interpolation functions for an eight-node element:

$$N = \begin{pmatrix} -\frac{1}{4}(-1+s)(-1+t)(1+s+t) \\ \frac{1}{2}(-1+s^2)(-1+t) \\ \frac{1}{4}(-1+t)(1-s^2+t+st) \\ -\frac{1}{2}(1+s)(-1+t^2) \\ \frac{1}{4}(1+s)(1+t)(-1+s+t) \\ -\frac{1}{2}(-1+s^2)(1+t) \\ \frac{1}{4}(-1+s)(1+s-t)(1+t) \\ \frac{1}{2}(-1+s)(-1+t^2) \end{pmatrix}$$



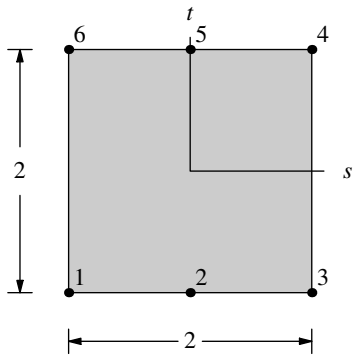
2. Interpolation functions for a seven-node element:

$$N = \begin{pmatrix} -\frac{1}{4}(-1+s)\,s\,(-1+t) \\ \frac{1}{2}(-1+s^2)(-1+t) \\ \frac{1}{4}(-1+t)(1-s^2+t+st) \\ -\frac{1}{2}(1+s)(-1+t^2) \\ \frac{1}{4}(1+s)(1+t)(-1+s+t) \\ -\frac{1}{2}(-1+s^2)(1+t) \\ \frac{1}{4}(-1+s)\,s\,(1+t) \end{pmatrix}$$



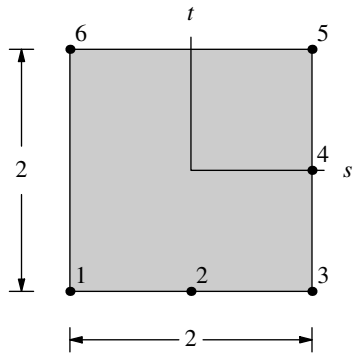
3. Interpolation functions for a six-node element (opposite sides quadratic):

$$\boldsymbol{N} = \begin{pmatrix} \frac{1}{4}(-1+s)\,s\,(1-t) \\ -\frac{1}{2}(-1+s)(1+s)(1-t) \\ \frac{1}{4}\,\mathrm{s}\,(1+s)(1-t) \\ \frac{1}{4}\,\mathrm{s}\,(1+s)(1+t) \\ -\frac{1}{2}(-1+s)(1+s)(1+t) \\ \frac{1}{4}(-1+s)\,s\,(1+t) \end{pmatrix}$$



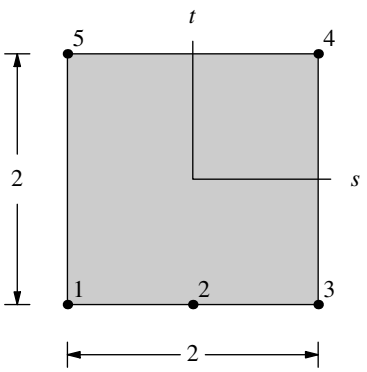
4. Interpolation functions for a six-node element (adjacent sides quadratic):

$$\boldsymbol{N} = \begin{pmatrix} -\frac{1}{4}(-1+s)\,s\,(-1+t) \\ \frac{1}{2}(-1+s^2)(-1+t) \\ \frac{1}{4}(-1+t)(1-s^2+t+s\,t) \\ -\frac{1}{2}(1+s)(-1+t^2) \\ \frac{1}{4}(1+s)\,t\,(1+t) \\ -\frac{1}{4}(-1+s)(1+t) \end{pmatrix}$$



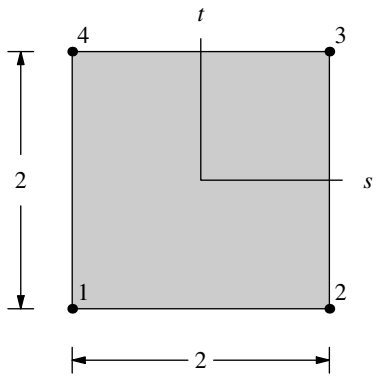
5. Interpolation functions for a five-node element:

$$\boldsymbol{N} = \begin{pmatrix} -\frac{1}{4}(-1+s)\,s\,(-1+t) \\ \frac{1}{2}(-1+s^2)(-1+t) \\ -\frac{1}{4}\,\mathrm{s}\,(1+s)(-1+t) \\ \frac{1}{4}(1+s)(1+t) \\ -\frac{1}{4}(-1+s)(1+t) \end{pmatrix}$$



6. Interpolation functions for a four-node element:

$$N = \begin{pmatrix} \frac{1}{4}(-1+s)(-1+t) \\ -\frac{1}{4}(1+s)(-1+t) \\ \frac{1}{4}(1+s)(1+t) \\ -\frac{1}{4}(-1+s)(1+t) \end{pmatrix}$$



1.2.4 Triangular Elements

A typical three-node triangular element is shown in Figure 1.9. The nodal coordinates are (x_1, y_1), (x_2, y_2), and (x_3, y_3). For each side, n defines its normal and c a coordinate that runs along that side. A complete linear polynomial in two dimensions has three terms, and thus a finite element solution for the element is based on the following polynomial:

$$u(x, y) = c_0 + c_1 x + c_2 y = \begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$$

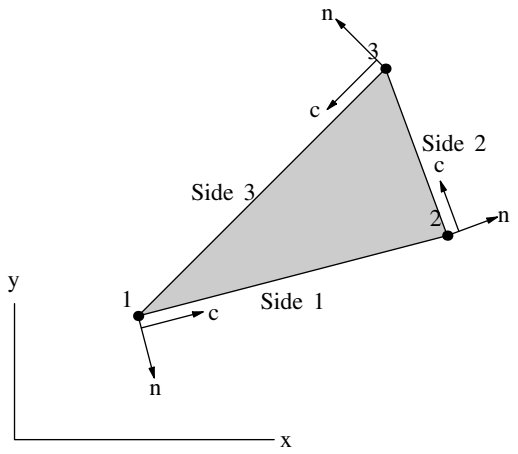


Figure 1.9. Three-node triangular element

The coefficients c_0, c_1, and c_2 can be expressed in terms of nodal degrees of freedom by evaluating the polynomial at the nodes as follows:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$$

Inverting the 3×3 matrix, we get

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \frac{1}{2A} \begin{pmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

where A is the area of the triangle and, we have introduced the notation

$$\begin{aligned} f_1 &= x_2y_3 - x_3y_2; \qquad f_2 = x_3y_1 - x_1y_3; \qquad f_3 = x_1y_2 - x_2y_1 \\ b_1 &= y_2 - y_3; \qquad b_2 = y_3 - y_1; \qquad b_3 = y_1 - y_2 \\ c_1 &= x_3 - x_2; \qquad c_2 = x_1 - x_3; \qquad c_3 = x_2 - x_1 \\ A &= \tfrac{1}{2}(-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3) \equiv \tfrac{1}{2}(f_1 + f_2 + f_3) \end{aligned}$$

Substituting the coefficients into the polynomial, we have

$$u(x, y) = \begin{pmatrix} 1 & x & y \end{pmatrix} \frac{1}{2A} \begin{pmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Carrying out multiplication, we get the finite element assumed solution over the element:

$$u(x, y) = \begin{pmatrix} N_1 & N_2 & N_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \boldsymbol{N}^T \boldsymbol{d}$$

$$\begin{aligned} N_1 &= \frac{1}{2A}(x_3(y - y_2) + x(y_2 - y_3) + x_2(-y + y_3)) \equiv \frac{1}{2A}(xb_1 + yc_1 + f_1) \\ N_2 &= \frac{1}{2A}(x_3(-y + y_1) + x_1(y - y_3) + x(-y_1 + y_3)) \equiv \frac{1}{2A}(xb_2 + yc_2 + f_2) \\ N_3 &= \frac{1}{2A}(x_2(y - y_1) + x(y_1 - y_2) + x_1(-y + y_2)) \equiv \frac{1}{2A}(xb_3 + yc_3 + f_3) \end{aligned}$$

Note that each of the interpolation functions N_i is 1 at the ith node and 0 at every other node.

It is also possible to form triangles by *collapsing* one side of a quadrilateral defined by using the mapping concept (see Chapter 6 of the fundamentals book). This procedure is convenient from a programming point of view and is the one implemented in most commercial computer programs.

1.3 INTEGRATION BY PARTS

The *integration-by-parts formula* is used to rewrite an integral of a product of a derivative of a function, say $f(x)$, and another function, say $g(x)$, as follows:

$$\int_{x_0}^{x_l} \left[\frac{d}{dx}(f(x))\right] g(x)\,dx = f(x_l)g(x_l) - f(x_0)g(x_0) - \int_{x_0}^{x_l} \left[\frac{d}{dx}(g(x))\right] f(x)\,dx$$

Note that the integrand must involve the product of a function and the derivative of another function. The application of the formula produces two terms that are evaluated at the ends of integration domain and another integral in which the derivative shifts from one function to the other.

To derive the weak form for two- and three-dimensional boundary value problems, we need a formula equivalent to integration by parts for one-dimensional problems. This is accomplished by using the well-known Gauss and Green-Gauss theorems from vector calculus. These theorems are presented here in the context of a two-dimensional problem. Their extension to the three dimensions is obvious.

1.3.1 Gauss's Divergence Theorem

Consider a vector of functions $\boldsymbol{F} = (F_1(x, y), F_2(x, y))^T$. The divergence of $\boldsymbol{F}$ is defined as

$$\text{div}\,\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

According to Gauss's divergence theorem, the area integral of $\text{div}\,\boldsymbol{F}$ is equal to the line integral of its dot product with the unit outer normal $\boldsymbol{n}$. That is,

$$\iint_A \text{div}\,\boldsymbol{F}\,dA = \int_c \boldsymbol{F}^T\boldsymbol{n}\,dc$$

The $dA = dx\,dy$ is the differential area. All terms in the boundary integral must be expressed in terms of the boundary coordinate c. The dc in the boundary integral is the length of a differential segment on the boundary.

$$dc = \sqrt{dx^2 + dy^2}$$

Written explicitly, we have

$$\iint_A \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA = \int_c (F_1\,n_x + F_2 n_y)\,dc$$

where n_x and n_y are the components of the unit normal to the boundary.

1.3.2 Green-Gauss Theorem

Given another function $g(x, y)$, the integral of the divergence of the product $g\boldsymbol{F}$ can be written as

$$\iint_A \operatorname{div}(g\boldsymbol{F})\, dA = \int_c g\boldsymbol{F}^T\boldsymbol{n}\, dc$$

Using the product rule, the divergence of the product $g\boldsymbol{F}$ can be written

$$\operatorname{div}(g\boldsymbol{F}) = \frac{\partial(gF_1)}{\partial x} + \frac{\partial(gF_2)}{\partial y} = g\frac{\partial F_1}{\partial x} + F_1\frac{\partial g}{\partial x} + g\frac{\partial F_2}{\partial y} + F_2\frac{\partial g}{\partial y}$$

Rearranging terms yields

$$\operatorname{div}(g\boldsymbol{F}) = g\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + \begin{pmatrix}\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}\begin{pmatrix}F_1\\F_2\end{pmatrix}$$

Noting that the row vector in the second term is the transpose of the gradient vector of g, we have

$$\nabla g \equiv \begin{pmatrix}\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}^T$$

and

$$\operatorname{div}(g\boldsymbol{F}) = g(\operatorname{div}\boldsymbol{F}) + (\nabla g)^T\boldsymbol{F}$$

Thus, the integral of $\operatorname{div}(g\boldsymbol{F})$ can be expressed as follows:

$$\iint_A g(\operatorname{div}\boldsymbol{F})\, dA + \iint_A (\nabla g)^T\boldsymbol{F}\, dA = \int_c g\boldsymbol{F}^T\boldsymbol{n}\, dc$$

Rearranging terms, we get the following relationship, commonly known as the Green-Gauss theorem:

$$\iint_A g(\operatorname{div}\boldsymbol{F})\, dA = \int_c g\boldsymbol{F}^T\boldsymbol{n}\, dc - \iint_A (\nabla g)^T\boldsymbol{F}\, dA$$

1.3.3 Green-Gauss Theorem as Integration by Parts in Two Dimensions

The Green-Gauss theorem can be written in a form very similar to the usual one-dimensional integration by parts by taking the vector of functions with $F_1 = f$ and $F_2 = 0$ [i.e., $\boldsymbol{F} = (f(x, y), 0)^T$]. The divergence of this function is

$$\operatorname{div}\boldsymbol{F} = \frac{\partial f}{\partial x}$$

and thus applying the Green-Gauss theorem, we have

$$\iint_A g\left(\frac{\partial f}{\partial x}\right) dA = \int_c g f n_x \, dc - \iint_A \frac{\partial g}{\partial x} f \, dA$$

Similarly, by taking the vector of functions with $F_1 = 0$ and $F_2 = f$, we can see that

$$\iint_A g\left(\frac{\partial f}{\partial y}\right) dA = \int_c g f n_y \, dc - \iint_A \frac{\partial g}{\partial y} f \, dA$$

Note that if the derivative on function $f(x, y)$ is with respect to x, the boundary integral involves n_x, and vice versa. Similar to the one-dimensional case, this form of the Green-Gauss theorem gives an area integral in which the derivative is switched from one function to the other. We also get a boundary integral involving the product of the functions and a component of the unit normal. These formulas, and their obvious extensions to three dimensions, are used in subsequent chapters to obtain weak forms for various boundary value problems.

1.4 NUMERICAL INTEGRATION USING GAUSS QUADRATURE

The integrands in typical finite elements are quite complicated and it generally is not possible to evaluate the required integrals in a closed form. We must use numerical integration in these situations. Simple numerical integration techniques such as the trapezoidal rule and Simpson's formula do not work that well in finite element computations. The Gauss quadrature, also called the Gauss-Legendre quadrature, is more suitable for finite element applications and has become the standard tool. It is reviewed briefly in this section.

1.4.1 Gauss Quadrature for One-Dimensional Integrals

In the Gauss quadrature, the basic derivation assumes that the integral of a function $f(s)$ is to be evaluated over an interval $-1 \le s \le 1$. The main idea is to represent the integral in the form

$$I = \int_{-1}^{1} f(s) \, ds \approx w_1 f(s_1) + w_2 f(s_2) + \cdots + w_n f(s_n)$$

The n points $(s_1, s_2, \ldots)$ are known as *Gauss points* and the corresponding coefficients $(w_1, w_2, \ldots)$ are known as *weights*. The locations of Gauss points and the appropriate weights are determined by requiring that the equation above give exact integrals for constant, linear, quadratic, and so on, terms in a polynomial.

Gauss point locations and weights for formulas up to six points are listed in Table 1.1. The table also indicates the order of the polynomial that is integrated exactly by the corresponding formula. Note the given locations and weights must be used only when evaluating integrals from -1 to 1. For different integration limits, an appropriate change of variables must be introduced prior to using these values.

TABLE 1.1. Gauss Point Locations and Weights for One-Dimensional Integrals

Point	Locations	Weights	Order
1	0	2	1
2	−0.57735 02691 89626	1	3
	0.57735 02691 89626	1	
3	−0.77459 66692 41483	0.55555 55555 55556	
	0	0.88888 88888 88889	
	0.77459 66692 41483	0.55555 55555 55556	
4	−0.86113 63115 94053	0.34785 48451 37454	7
	−0.33998 10435 84856	0.65214 51548 62546	
	0.33998 10435 84856	0.65214 51548 62546	
	0.86113 63115 94053	0.34785 48451 37454	
5	−0.90617 98459 38664	0.23692 68850 56189	9
	−0.53846 93101 05683	0.47862 86704 99366	
	0	0.56888 88888 88889	
	0.53846 93101 05683	0.47862 86704 99366	
	0.90617 98459 38664	0.23692 68850 56189	
6	−0.93246 95142 03152	0.17132 44923 7917	11
	−0.66120 93864 66265	0.36076 15730 48139	
	−0.23861 91860 83197	0.46791 39345 72691	
	0.23861 91860 83197	0.46791 39345 72691	
	0.66120 93864 66265	0.36076 15730 48139	
	0.93246 95142 03152	0.17132 44923 7917	

1.4.2 Gauss Quadrature for Area Integrals

The one-dimensional Gauss quadrature formulas extend easily to two-dimensional integrals as long as the integration region is a 2×2 square with the origin at the center, as shown in Figure 1.10. Thus, we consider evaluating the following integral:

$$I = \int_{-1}^{1} \int_{-1}^{1} f(s, t)\, ds\, dt$$

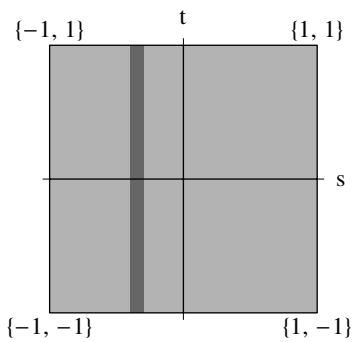


Figure 1.10. A 2×2 square area for two-dimensional Gauss quadrature

Considering a vertical strip, shown in dark shading in the figure, the integral in the t direction for a given s can be evaluated using an n-point one-dimensional Gauss quadrature formula:

$$I \approx \int_{-1}^{1} \left(\sum_{j=1}^{n} w_j f(s, t_j) \right) ds$$

The resulting s integrals can be evaluated using m points in the s direction, giving

$$I \approx \sum_{i=1}^{m} w_i \left(\sum_{j=1}^{n} w_j f(s_i, t_j) \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_i w_j f(s_i, t_j)$$

TABLE 1.2. Gauss Point Locations and Weights for Two-Dimensional Integrals

Quadrature	Points	s_i	t_j	$w_i \times w_j$
1×1	1	0	0	4
2×2	1	−0.57735	−0.57735	1
	2	−0.57735	0.57735	1
	3	0.57735	−0.57735	1
	4	0.57735	0.57735	1
3×3	1	−0.774597	−0.774597	0.308642
	2	−0.774597	0	0.493827
	3	−0.774597	0.774597	0.308642
	4	0	−0.774597	0.493827
	5	0	0	0.790123
	6	0	0.774597	0.493827
	7	0.774597	−0.774597	0.308642
	8	0.774597	0	0.493827
	9	0.774597	0.774597	0.308642
4×4	1	−0.861136	−0.861136	0.121003
	2	−0.861136	−0.339981	0.226852
	3	−0.861136	0.339981	0.226852
	4	−0.861136	0.861136	0.121003
	5	−0.339981	−0.861136	0.226852
	6	−0.339981	−0.339981	0.425293
	7	−0.339981	0.339981	0.425293
	8	−0.339981	0.861136	0.226852
	9	0.339981	−0.861136	0.226852
	10	0.339981	−0.339981	0.425293
	11	0.339981	0.339981	0.425293
	12	0.339981	0.861136	0.226852
	13	0.861136	−0.861136	0.121003
	14	0.861136	−0.339981	0.226852
	15	0.861136	0.339981	0.226852
	16	0.861136	0.861136	0.121003

Note that the total number of points is $m \times n$. Usually the same number of points is used in both directions and we therefore get $1 \times 1 = 1$, $2 \times 2 = 4$, $3 \times 3 = 9, \ldots$ point formulas. Table 1.2 shows the locations and weights of some of these formulas. To save space, only six significant figures are shown in the table. Actual numerical computations should use more precision.

1.4.3 Gauss Quadrature for Volume Integrals

Extension of one-dimensional Gauss quadrature to three-dimensional integrals follows the same line of reasoning as for the two-dimensional case. The integration region now is a $2 \times 2 \times 2$ cube with the origin at the center. Thus, we consider evaluating the following integral:

$$I = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(r, s, t)\, dr\, ds\, dt$$

Taking m points in the r direction, n points in the s direction, and p points in the t direction, the $m \times n \times p$ Gauss quadrature for volume integration is

$$I \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} w_i w_j w_k f(r_i, s_j, t_k)$$

Usually, the same number of points are used in each direction. Table 1.3 shows the locations and weights of some of these formulas.

1.5 MAPPED ELEMENTS

The mapping concept makes finite element computations possible for arbitrary-shaped elements. For mapping two-dimensional areas, the master area is a 2×2 square in the s, t coordinates. The interpolation functions are written by substituting numerical values of the nodal coordinates of the master area into the expressions given for rectangular elements. Mapping between x and y coordinates is obtained by treating the coordinates of the actual element as the data for Lagrange interpolation:

$$x(s, t) = N_1 x_1 + N_2 x_2 + \cdots + N_n x_n$$
$$y(s, t) = N_1 y_1 + N_2 y_2 + \cdots + N_n y_n$$

The interpolation functions for the master element are the serendipity shape functions presented earlier. For an element whose four sides are arcs of a circle (defined by three nodes on each side), we use the functions for an eight-node element. For an element whose four sides are straight lines (defined by two nodes on each side), we use the functions for a four-node element. For elements with some sides arcs of circles and others straight lines, we use the appropriate five- to seven-node serendipity shape functions.

TABLE 1.3. Gauss Point Locations and Weights for Three-Dimensional Integrals

Quadrature	Points	r_i	s_j	t_k	$w_i \times w_j \times w_k$
$1 \times 1 \times 2$	1	0	0	0	8
$2 \times 2 \times 2$	1	−0.57735	−0.57735	−0.57735	1
	2	−0.57735	−0.57735	−0.57735	1
	3	−0.57735	0.57735	−0.57735	1
	4	−0.57735	0.57735	0.57735	1
	5	0.57735	−0.57735	−0.57735	1
	6	0.57735	−0.57735	0.57735	1
	7	0.57738	0.55735	−0.57735	1
	8	0.57735	0.57735	0.57735	1
3×3	1	−0.774597	−0.774597	−0.774597	0.171468
	2	−0.774597	−0.774597	0	0.274348
	3	−0.774597	−0.774597	0.774597	0.171468
	4	−0.774597	0	−0.774597	0.274348
	5	−0.774597	0	0	0.438957
	6	−0.774597	0	0.774597	0.274348
	7	−0.774597	0.774597	−0.774597	0.171468
	8	−0.774597	0.774597	0	0.274348
	9	−0.774597	0.774597	0.774597	0.171468
	10	0	−0.774597	−0.774597	0.274348
	11	0	−0.774597	0	0.438957
	12	0	−0.774597	0.774597	0.274348
	13	0	0	−0.774597	0.438957
	14	0	0	0	0.702332
	15	0	0	0.774597	0.438957
	16	0	0.774597	−0.774597	0.274348
	17	0	0.774597	0	0.438957
	18	0	0.774597	0.774597	0.274348
	19	0.774597	−0.774597	−0.774597	0.171468
	20	0.774597	−0.774597	0	0.274348
	21	0.774597	−0.774597	0.774597	0.171468
	22	0.774597	0	−0.774597	0.274348
	23	0.774597	0	0	0.438957
	24	0.774597	0	0.774597	0.274348
	25	0.774597	0.774597	−0.774597	0.171468
	26	0.774597	0.774597	0	0.274348
	27	0.774597	0.774597	0.774597	0.171468

1.5.1 Restrictions on Mapping of Areas

Mapping must be one-to-one to be useful in finite element computations. To develop an appropriate criterion to verify this, we use the chain rule and write total derivatives of the mapping as

$$dx = \frac{\partial x}{\partial s}\,ds + \frac{\partial x}{\partial t}\,dt; \qquad dy = \frac{\partial y}{\partial s}\,ds + \frac{\partial y}{\partial t}\,dt$$

Writing the two differentials in a matrix form, we have

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} ds \\ dt \end{pmatrix}$$

For given ds and dt values, this equation determines corresponding values of dx and dy. For mapping to be one-to-one, we should be able get unique values of ds and dt for any given values of dx and dy. That is, we should be able to get an inverse relationship:

$$\begin{pmatrix} ds \\ dt \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{pmatrix}^{-1} \begin{pmatrix} dx \\ dy \end{pmatrix} \equiv \frac{1}{\det \boldsymbol{J}} \begin{pmatrix} \dfrac{\partial y}{\partial t} & -\dfrac{\partial x}{\partial t} \\ -\dfrac{\partial y}{\partial s} & \dfrac{\partial x}{\partial s} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

where $\boldsymbol{J}$ is the 2×2 Jacobian matrix and $\det \boldsymbol{J}$ is its determinant and is simply called the *Jacobian*:

$$\boldsymbol{J} = \begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{pmatrix} \equiv \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}; \qquad \det \boldsymbol{J} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$$

It is evident that $\det \boldsymbol{J}$ must not be zero anywhere over $-1 \le s, t \le 1$ for the mapping to be invertible. This requirement is satisfied as long as $\det \boldsymbol{J}$ is either positive or negative over the entire master element. By adopting a convention of defining areas by moving in a counterclockwise direction around the boundaries, $\det \boldsymbol{J}$ should always be positive. Thus, for the mapping to be valid, we have the following criterion:

$$\text{mapping is valid if } \det \boldsymbol{J} > 0 \text{ for all } -1 \le s, t \le 1$$

1.5.2 Derivatives of the Assumed Solution

The interpolation functions are written over the master element in terms of s, t. However, the element equations need x and y derivatives of these interpolation functions. Using the mapping $x(s, t)$ and $y(s, t)$ and employing the chain rule of differentiation, the derivatives of the ith interpolation can be written as follows:

$$\frac{\partial N_i}{\partial s} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial N_i}{\partial t} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial t}$$

Writing the two equations together in matrix form, we have

$$\begin{pmatrix} \dfrac{\partial N_i}{\partial s} \\ \dfrac{\partial N_i}{\partial t} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{pmatrix}$$

Here we notice that the 2×2 matrix is the transpose of the Jacobian matrix. The x and y derivatives of the ith interpolation function can thus be computed from its s and t derivatives and the inverse of the $\boldsymbol{J}^T$ matrix as follows:

$$\begin{pmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{pmatrix} = \boldsymbol{J}^{-T} \begin{pmatrix} \dfrac{\partial N_i}{\partial s} \\ \dfrac{\partial N_i}{\partial t} \end{pmatrix} = \frac{1}{\det \boldsymbol{J}} \begin{pmatrix} \dfrac{\partial y}{\partial t} & -\dfrac{\partial y}{\partial s} \\ -\dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} \end{pmatrix} \begin{pmatrix} \dfrac{\partial N_i}{\partial s} \\ \dfrac{\partial N_i}{\partial t} \end{pmatrix} \equiv \frac{1}{\det \boldsymbol{J}} \begin{pmatrix} J_{22} & -J_{21} \\ -J_{12} & J_{11} \end{pmatrix} \begin{pmatrix} \dfrac{\partial N_i}{\partial s} \\ \dfrac{\partial N_i}{\partial t} \end{pmatrix}$$

Separating the x and y derivatives, we have

$$\frac{\partial N_i}{\partial x} = \frac{1}{\det \boldsymbol{J}} \left(J_{22} \frac{\partial N_i}{\partial s} - J_{21} \frac{\partial N_i}{\partial t} \right)$$

$$\frac{\partial N_i}{\partial y} = \frac{1}{\det \boldsymbol{J}} \left(-J_{12} \frac{\partial N_i}{\partial s} + J_{11} \frac{\partial N_i}{\partial t} \right)$$

1.5.3 Evaluation of Area Integrals

In developing element equations, we need to perform integrations of the following form over the area of the element:

$$\boldsymbol{k} = \iint_A \boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^T \, dA$$

where typically $\boldsymbol{B}$ involves derivatives of interpolation functions and $\boldsymbol{C}$ is a matrix involving material properties. The arbitrary element area in the x, y coordinate is already mapped into a 2×2 square, and therefore with the change of variables the integral can be written as follows:

$$\boldsymbol{k} = \int_{-1}^{1} \int_{-1}^{1} \boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^T \det \boldsymbol{J} \, ds \, dt$$

Using m points in the s direction and n points in the t direction, we have an $m \times n$ Gauss quadrature formula. The element matrices are then evaluated numerically as

$$\boldsymbol{k} = \int_{-1}^{1} \int_{-1}^{1} \boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^T \det \boldsymbol{J} \, ds \, dt = \sum_{i=1}^{m} \sum_{j=1}^{n} w_i w_j \boldsymbol{B}(s_i, t_j) \boldsymbol{C}(s_i, t_j) \boldsymbol{B}^T(s_i, t_j) \det \boldsymbol{J}(s_i, t_j)$$

where (s_i, t_j) are the Gauss points and w_i and w_j are the corresponding weights.

The number of integration points used in evaluating these matrices depends on the order of terms present in the integrands. The $\boldsymbol{B}$ matrix not only involves known derivatives of $\boldsymbol{N}$ with respect to s and t but also terms from the Jacobian matrix $\boldsymbol{J}$ and its determinant. Thus,

the complexity of the terms in the integrand depends on the mapping. If the actual elements are rectangles or squares, $\boldsymbol{J}$ involves constant terms and it is fairly easy to determine the number of points necessary to evaluate these integrals exactly. However, for a general quadrilateral, $\boldsymbol{J}$ is not constant, and thus the integrands in these matrices are not simple polynomials. Therefore, in general, integration over mapped elements is approximate and does introduce another source of error in the finite element results. As long as the elements are not too distorted, the following integration rules give reasonable results.

1. *Four-node quadrilaterals.* Use 2×2 integration (four points).
2. *Eight-node quadrilaterals.* Use 3×3 integration (nine points).

1.5.4 Evaluation of Boundary Integrals

Elements involving a boundary over which a nonzero natural boundary condition is specified require computation of integrals of the form

$$\boldsymbol{r}_q = \int_c q\boldsymbol{N}_c \, dc$$

where c is a coordinate along the boundary and q is a given function expressed in terms of the boundary coordinate. In general, the element boundary is an arbitrary curve in an x, y coordinate system. Once mapped, each boundary curve is either a horizontal or a vertical line representing a side of the master square element. The boundary coordinate c for each side is mapped to a coordinate $-1 \le a \le 1$ for each side that is shown in Figure 1.11. The mapping for each side, $x(a)$ and $y(a)$, can be obtained from the element mapping as follows:

For side 1: $a = s$; $\quad t = -1$

For side 2: $s = 1$; $\quad a = t$

For side 3: $a = -s$; $\quad t = 1$

For side 4: $s = -1$; $\quad a = -t$

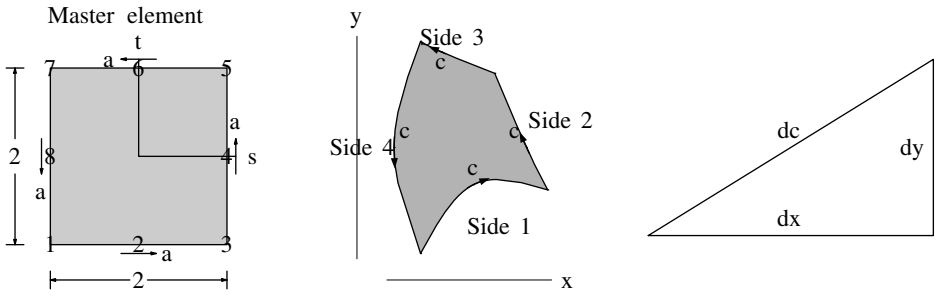


Figure 1.11. Boundary coordinates on the master and actual elements

To write interpolation functions $\boldsymbol{N}_c$ along a side, we substitute the appropriate s, t values in the interpolation functions $\boldsymbol{N}(s, t)$. The only task remaining is to relate the differential length dc along an element side to the differential line segment da along the side of the master element. This can be done easily by noting that

$$dc = \sqrt{dx^2 + dy^2}$$

Dividing both sides by da, we have

$$\frac{dc}{da} = \sqrt{\left(\frac{dx}{da}\right)^2 + \left(\frac{dy}{da}\right)^2} \Longrightarrow dc = J_c\, da$$

where J_c is the Jacobian of a side:

$$J_c = \sqrt{\left(\frac{dx}{da}\right)^2 + \left(\frac{dy}{da}\right)^2}$$

The boundary integrals along the side of the element can now be written

$$\boldsymbol{r}_q = \int_{-1}^{1} q\boldsymbol{N}_c J_c\, da$$

This integral can easily be evaluated using one-dimensional Gauss quadrature formulas:

$$\boldsymbol{r}_q = \int_{-1}^{1} q\boldsymbol{N}_c J_c\, da = \sum_{i=1}^{n} w_i q(a_i)\boldsymbol{N}_c(a_i) J_c(a_i)$$

where the a_i are the one-dimensional Gauss points and the w_i are the corresponding weights.

◆ See *Mathematica*/MATLAB Implementation 1.1 on the book Web site.

PROBLEMS

1.1 An engineering analysis problem is formulated in terms of the following second-order boundary value problem:

$$-ux^4 - u' + u'' = x; \qquad 0 < x < 1$$
$$u(0) = 4 \quad \text{and} \quad u'(1) = 1$$

Derive a suitable weak form for use with the Galerkin method. Indicate clearly how the boundary conditions will be handled.

1.2 An engineering analysis problem is formulated in terms of the following ordinary differential equation:

$$\frac{d^2u}{dx^2} - x\frac{du}{dx} = u; \qquad 0 < x < 1$$
$$u(0) = \frac{du(0)}{dx} - 2; \qquad \frac{du(1)}{dx} = 1$$

Obtain a suitable weak form for the problem. What is the order of the differential equation? Is the boundary condition at $x = 0$ a natural or an essential boundary condition? Is the boundary condition at $x = 1$ a natural or an essential boundary condition?

1.3 Steady-state heat flow through long, hollow circular cylinders can be described by the following ordinary differential equation:

$$\frac{d}{dr}\left(kA\frac{dT(r)}{dr}\right) + AQ = 0; \qquad r_i < r < r_o$$
$$T(r_i) = T_i; \qquad T(r_o) = T_0$$

where r is the radial coordinate, $T(r)$ is the temperature, k is the thermal conductivity, Q is the heat generation per unit area, $A = 2\pi rL$ the surface area, L is the length of the cylinder, r_i is the inner radius, and r_o is the outer radius. The boundary conditions specify the temperature on the inside and outside of the cylinder respectively. Derive finite element equations for a typical two-node linear element for the problem with nodes at r_1 and r_2. Assume that k and Q are constant over the element. Note that A is a function of r and is not constant over the element.

1.4 Consider the solution of the following second-order boundary value problem using two-node linear elements:

$$\frac{d^2u}{dx^2} = \frac{du}{dx}; \qquad 0 < x < 100$$
$$u(0) = 50; \qquad u(100) = 10$$

(a) Show that the following is an appropriate weak form for a typical linear element with nodes at arbitrary locations x_1 and x_2:

$$\int_{x_1}^{x_2} (u'(w_i + w_i'))\, dx = 0$$

where the $w_i(x)$ are suitable weighting functions.

(b) Using the weak form given in part (a) and the assumed solution written in terms of the interpolation functions

$$u(x) = \begin{pmatrix} N_1 & N_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}; \qquad u'(x) = \begin{pmatrix} N_1' & N_2' \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

show that the element equations for a two-node linear element for this problem are as follows:

$$\begin{pmatrix} \displaystyle\int_{x_1}^{x_2} (N_1 + N_1')N_1'\,dx & \displaystyle\int_{x_1}^{x_2} (N_1 + N_1')N_2'\,dx \\ \displaystyle\int_{x_1}^{x_2} (N_2 + N_2')N_1'\,dx & \displaystyle\int_{x_1}^{x_2} (N_2 + N_2')N_2'\,dx \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(c) Carrying out integrations, the element equations in part (b) can be expressed as

$$\frac{1}{2L} \begin{pmatrix} -L+2 & L-2 \\ -L-2 & L+2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $L = x_2 - x_1$, the element length. Using three of these elements, with nodes located at 0, 60, 90, and 100, determine an approximate solution of the problem.

1.5 Consider the following boundary value problem:

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) = \frac{2}{x^2}; \qquad 1 < x < 2$$

$$u(1) = 2 \quad \text{and} \quad \frac{du}{dx}(2) = -\frac{1}{4}$$

Compare the solution and its first derivative obtained by using the following two models:

(a) Use two equal-length linear finite elements.

(b) Use one quadratic finite element.

1.6 A four-node quadrilateral element is shown in Figure 1.12.

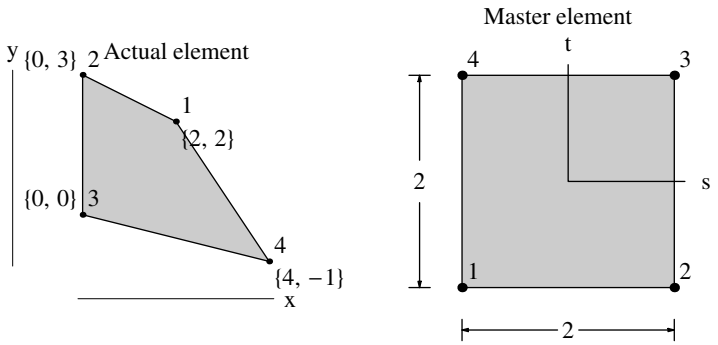


Figure 1.12.

(a) Develop an appropriate mapping to map the actual element into the 2×2 master element shown in the figure.

(b) Compute $\partial N_3 / \partial y$ where $N_3(s, t)$ is the third interpolation function for the master element. Give the value of the derivative at node 3 of the element.

(c) Compute $\iint_A N_3 \, dA$, where $N_3(s, t)$ is the third interpolation function for the master element and A is the area of the actual element. Use numerical integration with a 1×1 Gauss quadrature formula. No credit will be given if any other integration method is used, even if the integral is correct.

(d) Compute $\int_c N_{3c} \, dc$, where c is line 3–4 of the element and N_{3c} is the third interpolation function for the master element expressed in a coordinate that runs along this line. Use numerical integration with a two-point Gauss quadrature formula. No credit will be given if any other integration method is used, even if the integral is correct.

1.7 A four-node quadrilateral element is shown in Figure 1.13.

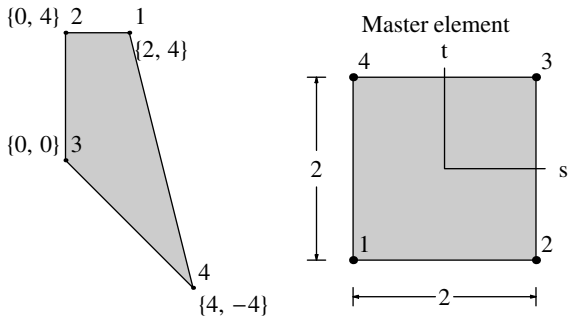


Figure 1.13.

(a) Develop an appropriate mapping to map the actual element into the 2×2 master element shown in the figure. Verify that the mapping is good.

(b) Compute $\partial N_2 / \partial x$, where $N_2(s, t)$ is the second interpolation function for the master element. Give the value of the derivative at the element centroid ($s = t = 0$).

(c) Compute $\iint_A N_4^2 \, dA$, where $N_4(s, t)$ is the fourth interpolation function for the master element and A is the area of the actual element. Use numerical integration with a 2×1 Gauss quadrature formula. No credit will be given if any other integration method is used, even if the answer is correct.