CHAPTER 1

MATERIAL MECHANICS

1.1 INTRODUCTION

Any rheological approach relies on a description of the motion within the frame of continuum mechanics, and on the assumption of some mechanical behavior intrinsic to the material under study. In practice, in order to determine this behavior, one uses viscometric flows that, because of their simplicity, provide straightforward relations between stress components and flow history under certain conditions. The theory of viscometric flows, which provides a complete theoretical frame for analyzing such data in rheological terms, has nevertheless been developed within the frame of a specific class of materials, *simple fluids*, which have a vanishing memory.

Although the validity of these assumptions (continuum medium, simple fluid) is clear for simple liquids, such as water, alcohol, and oil, it is not obvious for more complex materials such as pastes or granular materials. Indeed, we often observe these materials to be strongly heterogeneous at our scale of observation, and thus may question the validity of the continuum assumption. Moreover these materials may have a dual behavior, behaving like solids under some conditions and liquids under other conditions. For example, powders or foams remain stationary like a solid under the action of gravity alone, but begin to flow like a liquid under vibration (powders) or squeezing (foams). Various pasty materials also exhibit time-dependent properties; their viscosity may continuously increase when left at rest but, during flow, may decrease in time. Thus, pastes and granular materials cannot be considered a priori as simple fluids with vanishing memory.

Rheometry of Pastes, Suspensions, and Granular Materials: Applications in Industry and Environment
By Philippe Coussot

There is thus a need to review the basic tools of (continuum) fluid mechanics and rheometry in a more general frame including the specificities of pastes and granular materials. In this context we first review the definitions and principles of continuum mechanics and their physical origin, focusing on the validity of the continuum assumption for pastes and granular materials (Section 1.2). Then we examine the role and structure of the constitutive equation, and show that pastes and granular materials belong to a class intermediate between solids and fluids (Section 1.3). Usual viscometric flows and the corresponding methods of analysis are reviewed in Section 1.4 taking into account as far as possible the specific behavior pattern of these materials.

1.2 CONTINUUM MECHANICS

1.2.1 Definition of a Material

A *material* can be defined as the ensemble of elements of matter contained in a connex volume. The fact that we identify this ensemble as a single material means that in any two parts of this volume we would find similar ensembles of elements of matter. Nevertheless, for any given material, this definition would probably fail if viewed on the scale of some basic components of this matter (atoms, molecules, bubbles, solid particles, etc.), since on this scale the material clearly varies from one (very small) part to another. This implies that our evaluation of the material in a given part must account for a certain number of basic components over which we can proceed to some average of a physical property q (see Figure 1.1). It follows that the minimum scale (l) on which one can reasonably "observe" the system and effectively consider it as a material

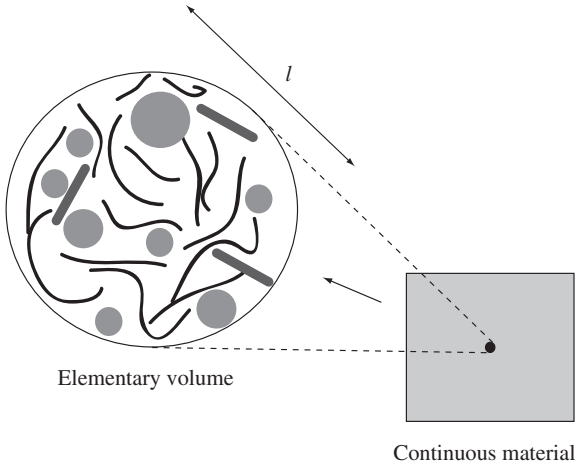

Figure 1.1 Aspect of an elementary volume (left) of a material, continuous on our scale of observation (right), but including different types of matter.

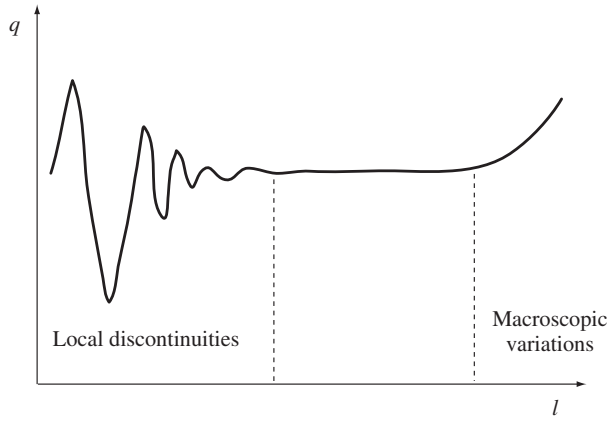

Figure 1.2 Variation of a physical property (q) averaged over a given volume of material as a function of the extent of this volume (l).

(i.e., the minimum volume of a given part of the system) corresponds to the point beyond which the result of this (q) average no longer varies when this scale further increases (see Figure 1.2). These volumes will be referred to as *elementary parts* (or *volumes*) of the material, while the basic components identified above will be referred to as the *elements*. However, it is worth noting that, when further increasing the scale of observation, macroscopic variations resulting from thermal and/or mechanical constraints imposed to the material begin to play a role (see Figure 1.2). These variations are precisely the phenomena that continuum mechanics or thermodynamics intends to describe with the help of mathematical concepts. The appropriate scale of elementary parts is thus situated between the range of rapid variations of q due to matter discontinuity and the range of macroscopic variations. When these two ranges recover, it is not possible to consider the system under the continuum assumption, as the flow is in a "discrete" regime.

1.2.2 Continuum Assumption

The material can thus be considered as *continuous* when the averages of physical properties (density, force, velocity, temperature, etc.) over the elementary parts of the material slowly vary from one such part to one of its neighboring parts. More precisely, the continuum assumption implies that the physical variables, which assume the values of property averages, may be described by continuous and continuously derivable functions of space and time. This will make it possible to describe the material evolutions from a set of equations relating an ensemble of variables and their changes in time and space. Such an approach opens the

way to a quantitative description of the material's properties and, in particular, its mechanical behavior.

Now let us examine the conditions under which the continuum assumption is valid. In practice, anticipating the definitions of some variables in Section 1.2.3, it appears that the continuities of density and velocity are the two main factors to consider. The density continuity is the easiest to predict and control because it relies on geometric considerations; we can define a minimum scale beyond which the continuum assumption is valid for the density from the direct observation of the distribution of material components in space. A practical means consists in computing the ratio of some characteristic flow length to the dimensions of the largest elements. The description of material properties with the continuum assumption becomes more precise as this ratio increases. It is difficult to propose a general rule, but in practice, as a general rule of thumb, a ratio of the order of 30 should ensure a good approximation of reality using the continuum approach.[1]

The problem is less simple for the velocity continuity, because the velocity field results from an interplay between the material behavior, the boundary conditions, and the thermomechanical constraints. Flow instabilities or discontinuities may result from this coupling, the effects of which can seldom be predicted a priori. Typically, slow flows of pastes or granular materials give rise to shear localization or wall slip (see Chapter 3). In that case the flow is composed of at least two regions, one in which the material flows and the other in which it remains rigid. At a sufficiently large scale of observation (L_1), the apparent thickness of the region of flow is so small that the velocity field may seem to exhibit a discontinuity (see Figure 1.3). Generally, this velocity field in fact appears as continuous on a sufficiently small scale, but the continuum assumption is not valid at this scale if the geometric continuity (see above) fails when using the thickness of the flowing region as the characteristic flow length, that is, when the ratio of this thickness to the typical particle size (L) remains sufficiently large. Otherwise, the sheared region does not correspond to a flow of the homogeneous material and may be regarded as an interface through which the velocity is discontinuous.

Finally, when localization occurs, because the thickness of the sheared region and thus the effective continuity of the variables depend on several material and flow characteristics in possible interplay, there is no general criterion that may

[1] This result may be obtained from the following rough approach. Let us consider a suspension of solid particles (of elementary volume V) in a gas (solid volume fraction ϕ); measuring the density of the material over a volume of material Ω takes into account $\phi\Omega/V$ particles, a number that may typically fluctuate from one volume to another in the suspension. The error with respect to the density is thus of the order of 10% for $\phi\Omega/V \approx 10$; for $\phi = 35\%$, we get the ratio for the dimensions of an elementary volume of material to the dimensions of the particle $b/a \approx (\Omega/V) \approx 3$; now, for a relevant mathematical description of the flow characteristics, there must exists at least $\sim$10 such elementary volumes along a typical length, which leads to a ratio of the characteristic flow length to the dimensions of the largest elements of 30.

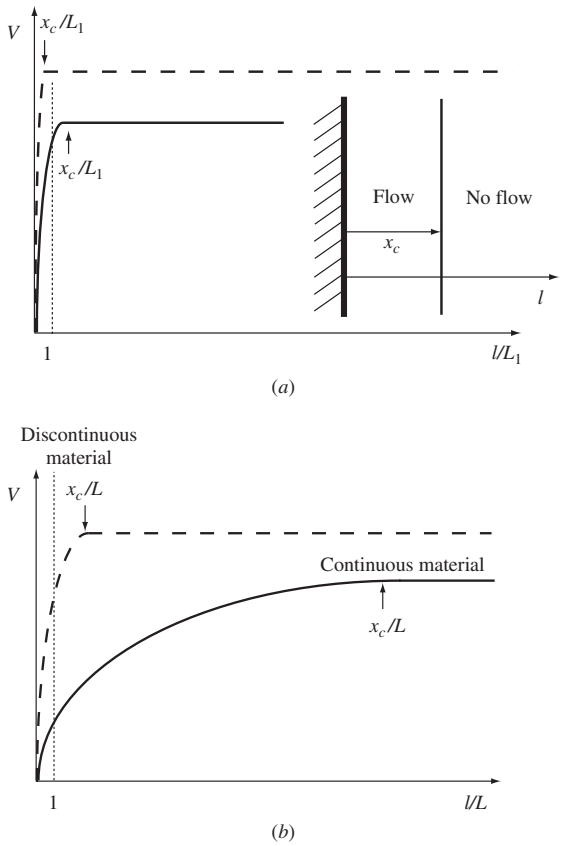

Figure 1.3 Possible aspects of the velocity field at a sufficiently large scale of observation (L_1) (a) and on an observation scale covering only the basic components (L) (b) in a material for which shear localization occurs, that is, where flow occurs only in a region of thickness x_c from the solid–fluid interface. Both velocity fields appear to be discontinuous at the scale L_1, but the first one (unbroken line) is continuous with respect to density, while the second one (dashed line) is discontinuous on the smaller scale L.

ensure a priori the validity of the continuum assumption for a given material. In practice, the validity of the continuum assumption must be checked a posteriori. In this book we will assume that the continuum assumption is satisfied and, in specific cases, discuss the origin of possible discontinuities in terms of flow or material characteristics.

1.2.3 Main Variables

Density and Concentration

The first physical property that can be identified directly derives from the definition of the material; this is the spatial distribution of matter within the volume.

In order to neglect the local specificities of matter, it is logical to use a generic property, specifically, the *mass*, which we will compute for a given volume of material by adding the mass values for all the individual elements of matter contained in that material. Under the continuum assumption, that is, on a scale sufficiently small to avoid macroscopic variations but sufficiently larger than the local discontinuities, this mass is simply proportional to the volume considered. We can thus define a locally constant quantity, namely, the *density*, as the ratio of this mass to the corresponding volume, and we will denote it as ρ. Note that the different elements of matter can obviously be in motion; in that case the preceding definition must be seen as describing the instantaneous, spatial distribution of the material mass.

A related quantity, which will play an important role when studying materials such as suspensions, is the *volume fraction* (ϕ) of a species of elements (bubbles, particles, etc.) in the remaining material. This is defined as the ratio of the volume occupied by this species to the corresponding volume of material. In some cases variables related to the mass fraction are used instead. However, in general, the volume fraction has a more straightforward rheological meaning than does mass concentration, since it provides a direct indication of the fraction of volume of material that can flow around the species in question, which is at the origin of the apparent viscosity of suspensions (see Chapter 2). However, it is worth noting that since gravity effects often play a major role in the flow properties of granular materials, mass concentration can also be a relevant parameter in view of rheological considerations for such materials. In particular, slight changes in density may induce significant changes in their rheological behavior; dry granular systems appear to suddenly liquefy at a critical density value. This effect has some similarities with the structure breakdown of pasty materials, where this is basically the local configuration of elements and not the density, which changes; since these materials are generally composed of various elements immersed in a liquid, significant variations of density rarely occur because this would imply some variations of the liquid density, which are generally negligible. Macroscopic density variations may nevertheless occur in such materials as a result of some collective migration of elements through the liquid (phase separation; see Section 3.6). Such an effect can be accounted for in the mechanical description with respect to its implications on the local rheological behavior of the material. In the following continuum mechanics treatment phase separation phenomena are assumed negligible, and if some of them occur, they will be considered as perturbating effects (see Section 3.6).

Temperature and Local Displacement

Now we can consider the possible displacements of the different elements of a material. When directly observing the small elements in the material, we can observe various motions in different directions resulting from the various interactions on different scales between the elements, from thermal agitation of the liquid or gas molecules to the motion fluctuations of the colloidal particles or grains; similar to the density, on a sufficiently small scale, the velocity fluctuates

in space and time. Once again, if we increase the spatiotemporal scales of observation (i.e., the dimensions of the elementary volume and the time over which one measures the quantity), the average motion in this volume should in principle reach a nonfluctuating value, varying only slowly in space and time because of thermomechanical constraints. The displacement of an elementary volume on this scale will be considered as the *local displacement* at a given point in the continuous material. The more local and rapid fluctuations of velocity within the volume of observation reflect the *temperature* (T) of the sample, but a more precise definition of this notion is beyond the scope of this book. In addition, some fluctuations significant on any scale of observation intermediate between that of the elementary parts and the macroscopic scale may occur under some conditions as a result of inertia effects; this is termed *turbulence* (see Section 1.2.4).

Deformation and Velocity

Here, we consider the deformation in time of a material. This deformation is the result of the relative displacements of the different elementary volumes of the material. These motions induce deformations and relative displacements of the elementary volumes that contain various such elements. In order to quantify the flow characteristics of the material, it is necessary to give a mathematical description of these motions in both space and time. Here we will follow the evolutions in time of the distribution of elementary volumes in space; this is the *material configuration*. The mathematical approach adopted is that of Coleman et al. [1].

Let us first consider the displacement of a given elementary volume of material situated in $\mathbf{x}$ at time t. We will designate $\boldsymbol{\xi}$ as the position of this elementary part at time $\vartheta < t$. At each time t, we then can define a function χ_t that gives the position $\boldsymbol{\xi}$ at ϑ of the elementary part, which is in the position $\mathbf{x}$ at t (see Figure 1.4):

$$\boldsymbol{\xi} = \chi_t(\mathbf{x}, \vartheta) \tag{1.1}$$

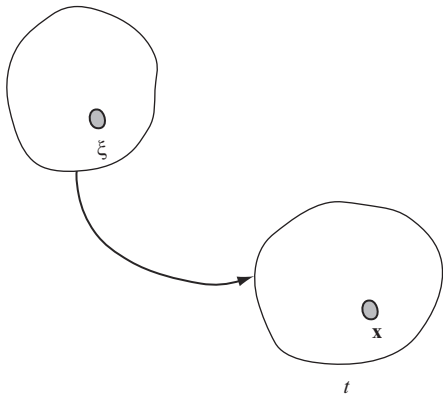

Figure 1.4 Configuration at time ϑ as a function of the configuration at the actual time t.

Now we can quantify the changes in the relative positions of the elementary parts of the material. A global description of the changes in their spatial configurations in time would be difficult. It is more appropriate and straightforward to describe these relative motions on a local scale, only around the elementary part considered, through the *relative configuration gradient*, defined as

$$\mathbf{F}_t(\mathbf{x}, \vartheta) = \nabla \chi_t(\mathbf{x}, \vartheta) \tag{1.2}$$

in which the gradient is taken relative to the actual configuration $\mathbf{x}$. It is usual to drop $\mathbf{x}$ and write $\mathbf{F}_t(\mathbf{x}, \vartheta)$ as $\mathbf{F}_t(\vartheta)$. This tensor, for example, makes it possible to determine the relative configuration change, since the time ϑ, of a small material volume represented by the vector $d\mathbf{x}$ around $\mathbf{x}$ at the time t: it is equal to $\mathbf{F}_t(\mathbf{x}, \vartheta)\, d\mathbf{x}$. Note that obviously if $\vartheta = t$, or if the material volumes do not move relative to each other around $\mathbf{x}$ between ϑ and t (i.e., $\mathbf{x} = \boldsymbol{\xi}$), we have $\mathbf{x} = \chi_t(\mathbf{x}, \vartheta)$, so that $\mathbf{F}_\vartheta(\vartheta) = \mathbf{I}$.

The instantaneous *velocity* $\mathbf{v}$ of the elementary volume situated in $\mathbf{x}$ at t can also be defined from the function χ_t:

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial}{\partial \vartheta}(\chi_t(\mathbf{x}, \vartheta))_{\vartheta = t} \tag{1.3}$$

It is worth noting that the function $\boldsymbol{\xi}(\vartheta) = \chi_t(\mathbf{x}, \vartheta)$ can be determined from knowledge of the velocity field $v(\mathbf{x}, \vartheta)$ at any time $\vartheta < t$. Indeed, by definition we have $\boldsymbol{\xi} = \chi_\vartheta(\boldsymbol{\xi}, \vartheta)$ for each ϑ, so that writing equation (1.3) for $\mathbf{x} = \boldsymbol{\xi}$ and $t = \vartheta$ gives the differential equation

$$\dot{\boldsymbol{\xi}}(\vartheta) = \mathbf{v}(\boldsymbol{\xi}(\vartheta), \vartheta) \tag{1.4}$$

from which $\boldsymbol{\xi}(\vartheta)$ is the solution with the initial condition $\boldsymbol{\xi}(t) = \mathbf{x}$.

The instantaneous *acceleration* vector $\mathbf{a}$ of this elementary volume is obtained in a similar way:

$$\mathbf{a}(\mathbf{x}, t) = \frac{\partial^2}{\partial \vartheta^2}(\chi_t(\mathbf{x}, \vartheta))_{\vartheta = t} \tag{1.5}$$

Using (1.3) and (1.4) in (1.5), we find the expression for the acceleration as a function of velocity:

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial \vartheta} + \nabla \mathbf{v} \cdot \mathbf{v} \tag{1.6}$$

It is worth noting that these definitions for the velocity and the acceleration correspond to the so-called Eulerian description, in which we follow at each time the physical properties of elementary fluid volumes as a function of their different spatial positions.

We can also define the *velocity gradient tensor* as

$$\mathbf{L}(t) = \nabla \mathbf{v}(\mathbf{x}, t) \tag{1.7}$$

which expresses the spatial variations of local velocity. Note that, as expected, since we described spatial and temporal variations of the local positions, in both cases, this velocity gradient tensor is simply equal to the time derivative of the configuration gradient history at the current instant:

$$\dot{\mathbf{F}}_{\vartheta<t}(t) = \frac{\partial}{\partial\vartheta}\mathbf{F}_t(\vartheta)_{\vartheta=t} = \nabla\left(\frac{d}{d\tau}\chi_t(\mathbf{x},\tau)_{\tau=t}\right) = \mathbf{L} \tag{1.8}$$

Thus $\mathbf{L}$ expresses the rate of configuration change of the material around $\mathbf{x}$ at t. It may be expressed as the sum of an antisymmetric tensor ($\mathbf{\Omega} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)$), which expresses the rotations of the configuration, and a symmetric tensor:

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^T) \tag{1.9}$$

This is the *strain rate tensor*, which expresses solely the deformation of the fluid around the point under consideration.

Forces

Obviously a material is not left to evolve by itself; there are various external and internal actions that condition its transformations. Each of these actions is a *force*, which is loosely defined as an action that would tend to increase the velocity of an elementary part of material if it were acting solely on that part. Different types of forces are exerted on the elements. In general, the external, long-range actions due to gravity or electromagnetic effects do not significantly vary from one position to another close position in the material, so the resulting forces over close, similar elements are similar. It follows that the resulting forces over two neighboring elementary volumes containing a similar set of various elements are similar. The total force resulting from such actions is simply proportional to the small volume of material (dV) under consideration, which we can express as

$$\mathbf{b}\,dV \tag{1.10}$$

where $\mathbf{b}$ is the force density. Note that in the presence of gravity only, the force density is $\rho\mathbf{g}$.

In parallel there exist short-range forces between the elements, due to intermolecular or interparticle (electrochemical or frictional) forces. Existing theoretical expressions of these forces [2] generally concern simple situations (spherical or platelet particles, uniform ionic strength, etc.) but give an idea of typical variations of the force intensity with distance. These forces decrease rapidly with distance; thus, as a first approximation, their effects can be considered as restricted to actions between neighboring elements. The net force exerted between the elements from one side of a surface and those on the other side is as a first approximation simply equal to the sum of the combined forces between neighboring elements on both sides of this surface. It follows that the net force between two small ensembles of elementary volumes of material through some surface

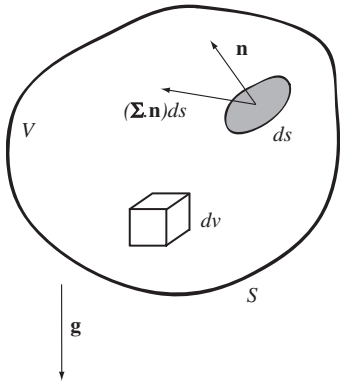

Figure 1.5 Volume and surface elements in a material.

is proportional to the extent of this surface (ds). It is then possible to show (see, e.g., Ref. 3) that, in the absence of internal torque acting on the elements, this force may be expressed as $(\boldsymbol{\Sigma} \cdot \mathbf{n})\, ds$, where $\boldsymbol{\Sigma}$ is the stress tensor, which is symmetric ($\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^T$), and $\mathbf{n}$ is the normal vector to the elementary surface under consideration. The stress vector $\mathbf{t}$ on this surface element (Figure 1.5) is thus defined as follows:

$$\mathbf{t} = \boldsymbol{\Sigma} \cdot \mathbf{n} \tag{1.11}$$

1.2.4 Conservation Laws

The three variables (density, velocity, temperature) provide a spatiotemporal description of the distribution of the elements of material in statistical terms (due to the averaging over the elementary volumes). The stress tensor and the density of external forces provide information concerning the different actions exerted on these elements. The objective of thermodynamics is to predict the evolution of the abovementioned variables in time and space under the action of these forces. In this frame we can set up three principles of conservation of physical quantities: mass, momentum, and energy. Here, only the basic formulas useful for the developments in this book are directly presented; the reader being referred to the text by Bird et al. [4] for a much more complete approach.

Mass Conservation

The *mass conservation* principle expresses the fact that in the absence of any source of matter, the amount of matter remains constant during any displacement, possibly with some density change. More precisely, let us consider a fixed volume V with an external surface S; a net flux of matter through this volume during a short time dt must be balanced by a variation in the local density within the volume V, given by

$$\frac{d}{dt}\left(\int_V \rho\, dv\right) = -\int_S (\rho \mathbf{v} \cdot \mathbf{n})\, ds = -\int_V \nabla \cdot \rho \mathbf{v}\, dv \tag{1.12}$$

This is the integral form of the mass conservation principle. The validity of this equation in each elementary volume of the material implies that we have the following formula locally:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \tag{1.13}$$

This is the local form of the mass conservation. For a gas, density variations under flow may be wide, but for a liquid, a solid, or a solid–liquid mixture, these variations are often negligible, so equation (1.13) or (1.12) generally reduces to

$$\nabla \cdot \mathbf{v} = 0 \quad \text{or, equivalently,} \quad \int_S (\mathbf{v} \cdot \mathbf{n})\, ds = 0 \tag{1.14}$$

Momentum Equation

The *momentum equation* expresses the fact that the fundamental principle of dynamics acts on each material element, that is, that the acceleration of a material element is proportional to the sum of forces acting on it. Let us apply this principle to a volume of material V of external surface S. The sum of forces exerted on its different components is equal to the sum of external forces acting on all the elements composing it and the sum of all surface forces acting on all the elementary surfaces in this volume. For each elementary surface surrounded by material (i.e., not situated along the boundary), the sum of surface forces acting from one side to the other (and reciprocally) cancels, so that only the surface forces acting along the external surface of the volume remain. Finally the momentum equation in integral form is expressed as follows:

$$\int_V \rho \mathbf{a}(\mathbf{x}, t)\, dv = \int_V \left(\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \nabla \mathbf{v} \cdot \mathbf{v} \right) dv = \int_V \rho \mathbf{b}\, dv + \int_S (\boldsymbol{\Sigma} \cdot \mathbf{n})\, ds \tag{1.15}$$

The validity of this equation for each material volume leads to the local expression of the momentum equation:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \nabla \mathbf{v} \cdot \mathbf{v} = \rho \mathbf{b} + \operatorname{div} \boldsymbol{\Sigma} \tag{1.16}$$

Note that in the absence of inertia effects (see Section 3.9) the left-hand side of (1.15) or (1.16) can be neglected.

In rheometry the material is often submitted to a torque and its flow characteristics are symmetric by rotation around an axis (see Section 1.4). Here, in the absence of inertia effects, and if the volume force can be taken to be homogeneous ($\mathbf{b}$ = constant), equation (1.16) may be rewritten

$$\int_S \mathbf{r} \times (\boldsymbol{\Sigma} \cdot \mathbf{n})\, ds = 0 \tag{1.17}$$

which provides a simplified form of the kinetic momentum theorem.

Energy Conservation

There exist some solicitations, such as heat fluxes or chemical reactions, which generally influence the material temperature. These processes are taken into account in the first principle of thermodynamics, which expresses the balance between the time variation of the total energy (e) of the system, which includes all types of element motion (fluctuations and average displacements), and the heat and work imparted to the material by external actions:

$$\frac{de}{dt} = \mathrm{Tr}(\boldsymbol{\Sigma} \cdot \mathbf{D}) + r - \nabla \cdot \mathbf{q} \tag{1.18}$$

in which r is the volume density of internal heat (as a result, e.g., of chemical reactions) and $\mathbf{q}$ is the heat density flux. The first term of the right-hand side [$\mathrm{Tr}(\boldsymbol{\Sigma} \cdot \mathbf{D})$] results solely from relative motions between material elements so that it corresponds to the energy stored or dissipated by mechanical actions per unit time within the material. In the absence of elastic effects, it is generally referred to as *energy dissipation* (per unit time). The corresponding total energy dissipation in a volume V per unit time gives

$$P = \int_V \mathrm{Tr}(\boldsymbol{\Sigma} \cdot \mathbf{D})\, ds \tag{1.19}$$

We should add the second law of thermodynamics to this equation. However, in rheometry, we are concerned primarily with the determination of mechanical behavior under isothermal conditions. Thermal processes may play a role as an artifact or simply because the temperature constitutes one parameter of the constitutive equation (see Section 3.8) but, in practice, thermodynamic laws are rarely used for determining the constitutive equation.

Boundary Conditions

The boundary conditions provide the values of the flow characteristics at the frontiers of the sample. They are absolutely necessary for solving a flow problem since the conservation equations retain a given form only over a volume in which the material characteristics do not change. In order to solve a given set of differential equations, it is thus necessary to know some specific values assumed by the variables at some place and time. This corresponds to the boundary conditions. In practice we need to know three independent components of $\mathbf{u}$ and $\boldsymbol{\sigma} \cdot \mathbf{n}$ along the boundary of the material (surface Σ).

Typical boundary conditions are (1) the stress tensor at the interface between a gas and a more complex fluid is known, whereas the velocity is unknown (this follows from conservation equations); for example, when gas motions are negligible, we have

$$\boldsymbol{\Sigma} \cdot \mathbf{n} = -p_0 \mathbf{n} \tag{1.20}$$

along the boundary (in which p_0 is the gas pressure); and (2) at the interface between a solid and a fluid, the stress is unknown but the velocity is continuous

($\mathbf{v}_F = \mathbf{v}_S$, in which $\mathbf{v}_S$ is given along the boundary). Note that in case (1) there may be some difference between the stress components at the interface as a result of surface tension effects (see Section 3.4). Similarly, in case (2), there may be some velocity variations at the interface if wall slip occurs (see Section 3.2).

Constitutive Equation

If we disregard possible thermodynamic changes, the flow characteristics should be determined from the set of equations including the mass conservation and the momentum equation along with boundary and initial conditions. However, although the external forces $\mathbf{b}$ generally are known, this is not the case for the stress tensor. This means that the number of unknown variables in this set of equations is equal to 1 (density) + 3 (velocity) + 6 (stress tensor) = 10, which is much larger than the number of equations (four). Knowledge of some additional relation between the stress tensor and the velocity, or more generally the history of deformation, thus appears to be necessary in order to solve the problem. This is the main objective of rheometry to determine at best this relation, in fact the *constitutive equation* of the material, and we discuss its origin and expected structure in detail in the next section.

Turbulence

We have seen above that over small scales of space or time the local velocity fluctuates because of the basically discontinuous nature of matter but, in practice, measurements are generally carried out over sufficiently long durations, so these effects are not observed. However, the conservation equations as expressed above in fact apply for the instantaneous, local values of the variables. In previous developments, it has thus been implicitly assumed possible to extend the validity of these equations to the nonfluctuating (averaged) part of the different variables. In fact only the linear terms remain unchanged after averaging, and thus the mass conservation remains valid whatever the importance of fluctuating terms. This is nevertheless not the case for the second (inertia) term on the left-hand side of the momentum equation.

Let us express a local variable in the form of the sum of its average and fluctuating values. For the local velocity this gives $\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}'$, in which $\langle \mathbf{v} \rangle$ is the time-averaged value of $\mathbf{v}$ and $\mathbf{v}'$ is its fluctuating part. By definition, we have $\langle \mathbf{v}' \rangle = 0$. In equations (1.13) and (1.15), the different terms, and possibly space and time derivatives of all the local and instantaneous variables, transform into similar terms involving the averaged values, except for the inertia term since $\langle \mathbf{v} \cdot \nabla \mathbf{v} \rangle = \langle \mathbf{v} \rangle \cdot \langle \nabla \mathbf{v} \rangle + \langle \mathbf{v}' \cdot \nabla \mathbf{v}' \rangle$. As a consequence, the local and averaged expressions for the momentum equation are similar only if the additional term $\langle \mathbf{v}' \cdot \nabla \mathbf{v}' \rangle$ is negligible. This situation corresponds to what we call *laminar flows*. On the contrary, when this term is significant, we are dealing with *turbulent flows*. Note that for turbulent flows, it is also possible to have similar local and average forms for the momentum equation by including the term $\rho \langle \mathbf{v}' \cdot \nabla \mathbf{v}' \rangle$ in the stress expression so as to obtain a turbulent stress tensor. However, in contrast to laminar flows, the relationship between this turbulent stress tensor and flow history depends on boundary and initial conditions.

Rheometry is relevant only in the case of laminar flows because, despite some local fluctuations, this is solely the average resistance to displacement of the different elements relative to each other, which gives rise to stresses. Thus, if we measure this resistance under some specific flow conditions but for all types and rates of relative displacements, we can use the corresponding data for predicting flow characteristics under any other conditions in the laminar regime. In the turbulent regime, a significant fraction of stresses also results from the transport of momentum due to motion fluctuations, but in contrast with laminar flows, the local expressions of these "turbulent" stress terms depend on the boundary and initial conditions of the flow in a complex manner. As a consequence, a simple rheometrical study under specific conditions cannot provide sufficient information for predicting turbulent flows under other conditions. The prediction of turbulent flow characteristics requires sophisticated modeling developments [5].

1.3 CONSTITUTIVE EQUATION

Here, for the sake of simplicity, we will consider mainly the constitutive equation in mechanical terms, leaving aside the effects of temperature changes on flow characteristics. This in particular means that the effect, in the constitutive equation, of the variations of fluctuating motions of elementary volumes during flow can be neglected.

1.3.1 Physical Origin

As soon as the elements in a material are submitted to a stress and/or displace relative to each other as a result of a macroscopic force or deformation applied to it, the local forces acting between the material elements can vary in time and space. Then the distribution of forces within the material evolves as a function of the configuration and state of these elements (e.g., the deformations of polymer molecules or bubbles). Since the actual configuration depends on the complete history of configuration and state changes, the actual force distribution is a function of both flow history and intrinsic properties of the material. Considering the problem on the local scale and accounting for each elementary force and its time evolution as a result of configuration changes would lead to an extremely complex set of equations, and for most materials such an approach is not realistic. This, added to the already mentioned difficulty in precisely expressing the local surface forces in real materials composed of various particle dimensions and types, led us to assume that a *constitutive equation* (i.e., a function relating the local stress tensor over an elementary volume to some average of the history of motions experienced by the material elements in this volume) applies for that material. This is in fact another, more physical, way to reach our conclusion in Section 1.2 regarding the need for a specific relation between stress and velocity fields, which in that case followed from a direct, continuous description of the mechanical properties of the material.

1.3.2 General Characteristics

Although the constitutive equation depends on the specific physical properties of each material, some general, basic trends of the constitutive equations have been derived from a *rational* approach of mechanics [6]. First it can reasonably be assumed that the actual stress tensor ($\boldsymbol{\Sigma}(\mathbf{x}, t)$) is given in a unique way as a function of the previous motions of material elements in a given sample; this is the *determinism principle*. We thus expect a relationship of the type $\boldsymbol{\Sigma}(\mathbf{x}, t) = \mathsf{H}_{0<\vartheta<t;\mathbf{y}\in\Omega}(\chi_t(\mathbf{y}, \vartheta))$ in which Ω is the sample volume and H is a function depending on material properties. In this expression we assume that the observation of the system begins at the instant $\vartheta = 0$ for which its configuration is $\chi_t(\mathbf{y}, 0)$.

Since the stress tensor directly derives from the interactions of neighboring elements in contact (see Section 1.2), the configuration history has an impact on the local stress tensor only through the configuration changes that occur near the small material volume under consideration. This is the *principle of local action*. A possible way to express this in practice consists in considering that the stress tensor $\boldsymbol{\Sigma}$ depends only on the history of the local, relative configuration gradient, $\boldsymbol{\Sigma}(\mathbf{x}, t) = \mathsf{H}_{0<\vartheta<t}(\chi_t(\mathbf{x}, 0), \mathbf{F}_t(\vartheta))$, in which H is another function depending on material properties. Note that as soon as we consider solely configuration changes, we have to keep some information concerning the initial state; this is the origin of the presence of $\chi_t(\mathbf{x}, 0)$ in the function. Using the stress tensor at the initial time $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}(\mathbf{x}, 0) = \mathsf{H}_{\vartheta=0}(\chi_0(\mathbf{x}, 0))$ and after inversion of this relation, we can rewrite the preceding expression in the general form:

$$\boldsymbol{\Sigma}(\mathbf{x}, t) = \mathsf{H}_{0<\vartheta<t}(\boldsymbol{\Sigma}_0, \mathbf{F}_t(\vartheta)) \tag{1.21}$$

These behavior characteristics are those of a *simple material*. A final rational, assumption is that the material behavior does not vary with a change in frame of observation; this is the principle of *material objectivity* or *frame indifference*. The implications of the principle of material objectivity on the characteristics of the constitutive equation will be considered later, after we have examined the effects on stresses and deformations of a change of frame of observation.

1.3.3 Effect of Change in Frame of Observation

If we view the system in another frame of observation (while keeping the same referential), from the current position we obtain

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{q}) \tag{1.22}$$

where $\mathbf{q}$ is a constant vector. The first term of the right-hand side, the vector $\mathbf{c}(t)$, expresses the displacement of the origin of the frame of observation while the second term expresses the frame rotation. It is necessary that $\mathbf{Q}$ be an orthogonal tensor, such that for any t, we obtain

$$\mathbf{Q}(t)\mathbf{Q}(t)^T = 1 \tag{1.23}$$

Obviously we also have $\boldsymbol{\xi}^* = \mathbf{c}(\vartheta) + \mathbf{Q}(\vartheta)(\boldsymbol{\xi} - \mathbf{q})$ since (1.22) is valid at any time and any point within the material. The laws of composition of derivatives reveal that the relative configuration gradient (1.2) in the new frame of observation can be expressed as

$$\mathbf{F}_t^*(\vartheta) = \nabla \chi_t^*(\mathbf{x}^*, \vartheta) = \nabla_{(\mathbf{x}^*)} \boldsymbol{\xi}^* = [\nabla_{(\boldsymbol{\xi})} \boldsymbol{\xi}^*][\nabla_{(\mathbf{x})} \boldsymbol{\xi}(\mathbf{x})] \lfloor \nabla_{(\mathbf{x}^*)} \mathbf{x} \rfloor \tag{1.24}$$

in which $\nabla_{(\mathbf{x})} \boldsymbol{\xi}$ is the gradient of $\boldsymbol{\xi}$ relative to $\mathbf{x}$. From (1.2), (1.22), and (1.24) it follows that

$$\mathbf{F}_t^*(\vartheta) = \mathbf{Q}(\vartheta) \mathbf{F}_t(\vartheta) \mathbf{Q}(t)^T \tag{1.25}$$

Let us now consider the stress vector $\mathbf{t}^*$ along a material surface element of normal vector $\mathbf{n}^*$. In the new frame this is expressed as $\mathbf{t}^* = \boldsymbol{\Sigma}^* \cdot \mathbf{n}^*$. From (1.22) we also have $\mathbf{t}^* = \mathbf{Q}(t)\mathbf{t}$ and $\mathbf{n}^* = \mathbf{Q}(t)\mathbf{n}$. From these relations along with (1.11), we find the relation between the stress tensor expressions in the two frames:

$$\boldsymbol{\Sigma}^* = \mathbf{Q}(t) \boldsymbol{\Sigma} \mathbf{Q}(t)^T \tag{1.26}$$

It follows from (1.26) that $\boldsymbol{\Sigma} \cdot \mathbf{u} = \lambda \mathbf{u} \Rightarrow \boldsymbol{\Sigma}^* \cdot (\mathbf{Q} \cdot \mathbf{u}) = \lambda (\mathbf{Q} \cdot \mathbf{u})$, which means that $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^*$ have the same eigenvalues and the same diagonal tensorial expressions in the corresponding bases of eigenvectors [respectively $(\mathbf{u}_{i=1,2,3})$ and $(\mathbf{Q} \cdot \mathbf{u}_{i=1,2,3})$]. Since they are simple functions of eigenvalues $\mathrm{Tr}(\boldsymbol{\Sigma})$, $\mathrm{Tr}(\boldsymbol{\Sigma}^2)$ and $\det(\boldsymbol{\Sigma})$ are thus independent of the frame of observation and we can define three *invariants* of the stress tensor $\boldsymbol{\Sigma}$ as follows:

$$\Sigma_{\mathrm{I}} = \mathrm{Tr}\boldsymbol{\Sigma}; \Sigma_{\mathrm{II}} = \tfrac{1}{2}[(\mathrm{Tr}\boldsymbol{\Sigma}) - \mathrm{Tr}(\boldsymbol{\Sigma}^2)]; \Sigma_{\mathrm{III}} = \det(\boldsymbol{\Sigma}) \tag{1.27}$$

From (1.25) we also have

$$\mathbf{L}^* = \frac{\partial}{\partial \vartheta} \left(\mathbf{F}_t^*(\vartheta) \right)_{\vartheta = t} = \mathbf{Q}'(t) \mathbf{Q}(t)^T + \mathbf{Q}(t) \mathbf{L} \mathbf{Q}(t)^T \tag{1.28}$$

and since from (1.23), $\mathbf{Q}'(t)\mathbf{Q}(t)^T + \mathbf{Q}(t)\mathbf{Q}'(t)^T = 0$, we deduce the following from (1.9):

$$\mathbf{D}^* = \mathbf{Q}(t) \mathbf{D} \mathbf{Q}(t)^T \tag{1.29}$$

As a consequence, we may use definitions similar to (1.27) for three invariants (D_{I}; D_{II}; D_{III}) of the strain rate tensor. Note that from the mass conservation (1.14), the first invariant ($D_{\mathrm{I}} = \nabla \cdot \mathbf{u}$) is equal to zero when density variations are negligible. These invariants of the stress and strain rate tensor express global amounts of deformation or stress, and it is practical to use them as parameters of constitutive equations since they provide expressions directly following the frame indifference principle. However, this is relevant only for isotropic materials since the different directions play a similar role in these invariants.

1.3.4 Solids and Fluids

Generalities

From a physical perspective, three types of materials are generally considered [7]:

- *Solids*, in which the molecules, forming a long-range ordered structure, are close to each other and embedded in deep potential wells from which they can barely escape
- *Liquids*, in which the molecules are close to each other but form a disordered structure and can relatively easily escape from their potential wells
- *Gas*, in which the molecules are far from each other and move almost freely relative to each other

Since a gas or a liquid usually can deform at will and retain its physical properties whereas a solid, or more precisely its ordered structure, tends to be irreversibly modified or breaks beyond a critical deformation, it is natural to first separate these materials, from a mechanical perspective, into two corresponding classes, say, "fluids" and "solids." This description applies only for materials with a simple structure (i.e., composed of one or few molecule types), and cannot a priori encompass the wide variety of industrial or natural materials containing more complex element networks.

In mechanics the concept of a simple material as defined above seems sufficiently general to encompass most industrial and natural materials, which might initially be considered either as solids, fluids, or, perhaps, hybrid materials. Then, applying the preceding classification derived from physical arguments, as well as a natural (inherent) classification based on their mathematical properties [6], rational mechanics has suggested the separation of simple materials into two broad classes: *solids* and *(simple) fluids*. According to these definitions, *solids* are materials that have a preferential, natural configuration, on the basis of which their behavior must be defined. More precisely, at any given time the current stress tensor can be determined from knowledge of the current deformation relative to the preferential configuration. *Fluids* are materials that do not have a preferential configuration; they may be deformed at will but then always relax. This means, for example, that if after any type of flow the material remains at rest for a sufficient time, it eventually completely "forgets" its previous deformations, and thus its final configuration does not depend on the previous flow history.

Although these definitions make it possible to propose a straightforward (rheological) analysis of some particular flows (simple shear; see discussion below), they do not encompass the various behavior types of real materials. This is precisely the case for pasty or granular materials, which are, for example, capable of behaving partly as solids, because under some flow conditions they have a preferential configuration, and partly as fluids, because they are capable of definitively forgetting this configuration after a particular deformation history or may recover this initial configuration after a period of rest. It thus appears necessary to distinguish a third class of materials, "jammed systems" (see below), which borrow their behavior partly from fluids and partly from solids.

Fluids

Because of its potentially interesting simplifications of the treatment of constitutive equations, theoretical rheometry (or viscometry) [1] has been dealt with so far within the frame of fluids, which have a vanishing memory. These materials are such that at a sufficient time (lapse) after some temporal point of reference they will have completely forgotten their history prior to this reference time. This may be expressed as follows. For any flow history and two different states of reference ($\boldsymbol{\Sigma}_i$ and $\boldsymbol{\Sigma}_i^*$), there exists a time ϑ_c (possibly negative) in the past such that the actual state of the material is independent of the state (with respect to the two above mentioned states) at this time:

$$\mathsf{H}_{\vartheta_c<\vartheta<t}(\boldsymbol{\Sigma}_\mathrm{i}, \mathbf{F}_\mathrm{t}(\vartheta)) = \mathsf{H}_{\vartheta_c<\vartheta<t}(\boldsymbol{\Sigma}_\mathrm{i}^*, \mathbf{F}_\mathrm{t}(\vartheta))$$

As a consequence, if there is no previous time limit, the constitutive equation is independent of the initial state, which is expressed formally as

$$\boldsymbol{\Sigma}(\mathbf{x}, t) = \mathsf{H}_{\vartheta<t}(\mathbf{F}_\mathrm{t}(\vartheta)) \tag{1.30}$$

which no longer contains any reference to a specific state at a particular time.

Let us examine some implications of such a constitutive equation. From the principle of objectivity we have $\boldsymbol{\Sigma}^*(\mathbf{x}, t) = \mathsf{H}_{\vartheta<t}(\mathbf{F}_\mathrm{t}^*(\vartheta))$ in any other frame of observation. Using this relation and substituting (1.25) in (1.26), we deduce a relation that must be respected by the constitutive equation of a simple fluid for any history of $\mathbf{F}_t(\vartheta)$ and any tensor $\mathbf{Q}(\vartheta)$:

$$\mathbf{Q}(t)\lfloor\mathsf{H}_{\vartheta<t}\mathbf{F}_t(\vartheta)\rfloor\mathbf{Q}(t)^T = \mathsf{H}_{\vartheta<t}\left(\mathbf{Q}(\vartheta)\mathbf{F}_t(\vartheta)\mathbf{Q}(t)^T\right) \tag{1.31}$$

Let us assume, for example, that the material undergoes the *rest history* defined by $\mathbf{F}_t(\vartheta) = \mathbf{I}(\vartheta) = \mathbf{I}$. With a constant rotation of the frame, defined by the fixed orthogonal tensor $\mathbf{Q}$, equation (1.31) becomes

$$\mathbf{Q}\lfloor\mathsf{H}_{\vartheta<t}\mathbf{I}(\vartheta)\rfloor\mathbf{Q}^T = \mathsf{H}_{\vartheta<t}\left(\mathbf{I}(\vartheta)\right) \tag{1.32}$$

Since (1.32) is valid for any orthogonal tensor $\mathbf{Q}$, it may be demonstrated mathematically that the function $\mathsf{H}_{\vartheta<t}(\mathbf{I}(\vartheta))$ is necessarily proportional to $\mathbf{I}$. This means that the stress tensor in a simple fluid that has experienced nothing but rigid rotations is isotropic: $\boldsymbol{\Sigma} = -p\mathbf{I}$. This important result follows from application of the principle of material objectivity in the specific case of materials with vanishing memory. A corollary is that such materials cannot support indefinitely a nonisotropic stress tensor "without eventually giving way to it" [1]. Indeed, if the material never flowed, we would have $\mathbf{F}_t(\vartheta) = \mathbf{I}$, and the stress tensor would necessarily be isotropic.

Solids

According to the definition above, solids appear to have a constitutive equation of the general form (1.21), in which the dependence of the function H on the state

at the initial time plays a critical role. In that case the principle of objectivity leads to

$$\mathbf{Q}(t)\lfloor \mathsf{H}_{0<\vartheta<t}(\Sigma_0, \mathbf{F}_t(\vartheta))\rfloor \mathbf{Q}(t)^T = \mathsf{H}_{0<\vartheta<t}(\mathbf{Q}(0)\Sigma_0\mathbf{Q}(0)^T; \mathbf{Q}(\vartheta)\mathbf{F}(\vartheta)\mathbf{Q}(t)^T) \tag{1.33}$$

which does not provide further information on the general characteristics of the constitutive equation.

Jammed Systems

The inability to support a nonisotropic stress tensor indefinitely without flowing appears to be a property in disagreement with common observations of pasty or granular materials, which, when submitted to gravity only can maintain indefinitely on our scale of observations a shape significantly different from that expected from a simple hydrostatic pressure for a simple liquid, specifically, a horizontal free surface. This is, for example, the case for various concentrated suspensions, emulsions, or foams such as paints, inks, cement pastes, fresh concrete, muds, purees, toothpaste, and mayonnaise. When these products are for example, stored in closed containers, they can remain undeformed over durations longer that our typical, maximum time of observation (years). All such materials are composed of numerous elements (particles, droplets, bubbles, etc.) interacting by means of different types of surface forces. Since Brownian effects are not sufficient, some elements can remain jammed between other elements as long as the force applied to them remains insufficient. We will refer to them as "jammed materials," a term originating in the physics literature [8]. The corresponding term in rheology is *yield stress fluids*, but this term assumes a specific behavior type, namely, the simple yielding behavior, which does not appear to be sufficiently general, particularly with respect to time-dependent behavior and solid–fluid transition. From a physical perspective these materials are intermediate between liquids and solids since their elements are in disarray (e.g., as molecules in a liquid), but, in a pattern similar to that found in a solid they are embedded in potential wells, from which they can rarely escape in the absence of a sufficient force.

A solid is characterized by the fact that if its actual configuration deviates significantly from the configuration of reference, the solid can never return to this initial reference point under the usual mechanical constraints, and thus will never recover its initial behavior. On the contrary, jammed systems can undergo any type of deformation but can recover their initial configurations (on average) under some specific constraints, for example, after some degree of flow and a sufficient time at rest. Solids are thus usually studied under small deformations, even if some theories do account for their irreversible changes of behavior for large deformations. According to these definitions, jammed systems can appear as "solid" under some limited range of conditions; under small deformations they may behave as solids, that is, their rheological behavior will be similar to that of solid, and when they stray too far from their corresponding preferential configuration, they break and become "fluid," but they can recover their initial

configurations and rheological behavior, for example, after a sufficient period of rest or after a similar preparation. An example illustrating this concept is modeling paste. When lying on a table, the paste is slightly elastically deformed like a solid; if squeezed between two plates, it may flow like a liquid. Then, if it is stretched, it may separate into two parts like a fractured solid; however, it will recover its initial "mechanical appearance" if the two parts are mixed back together under sufficient force (are modeled).

Let us now express the general characteristics of jammed systems from a mechanical point of view. Depending on flow history we expect their behavior to be that of either a solid or a simple fluid type. For such materials the constitutive equation should include a "solid regime" described by an equation such as (1.21) for which the state of reference plays a critical role, and a "liquid regime" described by a relation such as (1.30). In this context the set of constitutive relations for such materials must also include an additional relation describing the conditions for the transition from the solid to the liquid regimes as a function of flow characteristics. The general formulation of such constitutive equations appears to be extremely difficult.

Pressure

Now we can comment that for most fluids, in particular those consisting of particles or elements dispersed in a liquid or a gas, the influence of an isotropic stress applied onto the material, namely, a *pressure*, on the flow characteristics is negligible. As a consequence, the stress tensor may be expressed to within an indeterminate isotropic tensor, which can generally be determined from the momentum equation and boundary conditions. By convention we express this in the form

$$\boldsymbol{\Sigma} = -p\mathbf{I} + \mathbf{T} \tag{1.34}$$

where $p = -\mathrm{Tr}\boldsymbol{\Sigma}/3$ is the pressure and $\mathbf{T}$ the deviatoric part of $\boldsymbol{\Sigma}$, which verifies $\mathrm{Tr}\ \mathbf{T} = 0$. In that case this is the deviating part $\mathbf{T}$, which reflects the constitutive equation of the material and can be expressed in the form of equation (1.21) or (1.30).

This approach is useful when the role of the pressure in the second term of the right-hand side of equation (1.34) is effectively negligible. However, this hypothesis appears to fail for granular materials, whose behavior may, under certain circumstances, result from that of a continuous network of grains in contact. In that case, the shear stress in simple shear is approximately proportional to some normal component of the stress tensor (see Section 3.1) which is not independent of the pressure.

1.3.5 Simple Shear and Viscometric Flows

Simple Shear

The constitutive equation in its general form given by equation (1.21) cannot easily be determined experimentally. Indeed, the stress tensor depends on an

unknown function of a great number of variables, including all the values of the components of the relative configuration gradient at different times. Determining this relation would require measurement of the values of the deformation and stress components at any given time and position in the material, which, in practice is unrealistic. The solution found in rheometry consists in studying certain aspects of the constitutive equation via specific flows for which the configuration gradient history has a simple form.

In this context, the basic flow is the *simple shear*, which is such that in some Cartesian coordinate system in which the position $\mathbf{x}$ is defined as a function of its components (x, y, z), the components of the velocity have the form

$$v_x = \dot{\gamma} y, v_y = 0, v_z = 0 \tag{1.35}$$

in which $\dot{\gamma}$, called the *shear rate*, is a constant. In that case the deformation undergone by the fluid after the time Δt is expressed as $\gamma = \dot{\gamma}\Delta t$. The motion history is given by the solution of the differential equation (1.4) with the initial condition $\boldsymbol{\xi}(t) = \mathbf{x}$, which yields

$$\xi = x + \dot{\gamma} y(\vartheta - t); \eta = y, \zeta = z \tag{1.36}$$

where (ξ, η, ζ) are the coordinates of $\boldsymbol{\xi}$. Then the relative configuration gradient tensor is:

$$\mathbf{F}_t(\vartheta) = \begin{bmatrix} 1 & \dot{\gamma}(\vartheta - t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or equivalently

$$\mathbf{F}_t(\vartheta) = \mathbf{I} + (\vartheta - t)\mathbf{M} \tag{1.37}$$

where $\mathbf{M}$ is the tensor, defined as

$$M = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.38}$$

It follows from (1.37) that $\dot{\gamma}$ is the only parameter appearing in the right-hand side of (1.21), which means that the components of $\boldsymbol{\Sigma}$ depend only on $\dot{\gamma}$ and a state of reference. The shear rate may also depend on time (*generalized simple shear*), in which case a similar result is found—the components of $\boldsymbol{\Sigma}$ are functions of the history of $\dot{\gamma}$ and the state of reference. This is the fundamental result for simple shear of any type of material (solids, fluids, jammed systems).

Simple Shear with a Fluid

Let us now assume that the material is a simple fluid. Here, in simple shear, from equations (1.30) and (1.37), the stress tensor is

$$\boldsymbol{\Sigma} = \mathsf{H}_{\vartheta < t}(\mathbf{I} + (\vartheta - t)\mathbf{M}) \tag{1.39}$$

which means that the components of $\mathbf{\Sigma}$ depend only on $\dot{\gamma}$. We can apply the identity (1.31) in that case using

$$\mathbf{Q}(\vartheta) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and find

$$\begin{aligned} \mathbf{Q}\lfloor \mathsf{H}_{\vartheta<t}(\mathbf{I} + (\vartheta - t)\mathbf{M})\rfloor \mathbf{Q}^T &= \mathsf{H}_{\vartheta<t}\left[\mathbf{Q}(\mathbf{I} + (\vartheta - t)\mathbf{M})\mathbf{Q}^T\right] \\ &= \mathsf{H}_{\vartheta<t}(\mathbf{I} + (\vartheta - t)\mathbf{M}) \end{aligned} \tag{1.40}$$

which means that

$$\mathbf{\Sigma} = \mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T \tag{1.41}$$

Expressing the components of the stress tensor as

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

we deduce from (1.41) that

$$\sigma_{xz} = 0 = \sigma_{yz} \tag{1.42}$$

Moreover, since the components of $\mathbf{\Sigma}$ only depend on $\dot{\gamma}$, the four following *material functions* completely define the stress tensor in simple shear:

- *The shear stress*:

$$\tau(\dot{\gamma}) = \sigma_{xy} \tag{1.43}$$

- *The first normal stress difference*:

$$N_1(\dot{\gamma}) = \sigma_{xx} - \sigma_{yy} \tag{1.44}$$

- *The second normal stress difference*:

$$N_2(\dot{\gamma}) = \sigma_{yy} - \sigma_{zz} \tag{1.45}$$

- *The pressure*:

$$p = \tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \tag{1.46}$$

The *apparent viscosity* is then defined as follows:

$$\eta(\dot{\gamma}) = \frac{\tau}{\dot{\gamma}} \tag{1.47}$$

In this book we will, however, use the notation μ for fluids assumed to be Newtonian, that is, the viscosity of which is constant.

Viscometric Flows

Now we are interested in *viscometric flows*, motions that at any time are a simple shear in a particular frame that may vary in time. This means that at each time the relation (1.37) is valid with both a frame of observation and a value $\dot{\gamma}$ possibly depending on time: $\mathbf{F}_t{}^*(\vartheta) = \mathbf{I} + (\vartheta - t)\mathbf{M}(\dot{\gamma}(t))$. Let us consider the corresponding frame at time ϑ seen from the actual frame at time t; it may be deduced from the actual frame by the orthogonal tensor $\mathbf{Q}(\vartheta)$, which is such that $\mathbf{Q}(t) = \mathbf{I}$. Under these conditions the relative configuration gradient at any time in the actual frame is deduced from (1.25):

$$\mathbf{F}_t(\vartheta) = \mathbf{Q}(\vartheta)(\mathbf{I} + (\vartheta - t)\mathbf{M}(\dot{\gamma}(t))) \tag{1.48}$$

Let us now apply the principle of objectivity (1.31) for the flow history $\mathbf{I} + (\vartheta - t)\mathbf{M}$ and a change of frame defined by $\mathbf{Q}$:

$$\begin{aligned} \mathsf{H}_{0<\vartheta<\mathrm{t}}(\boldsymbol{\Sigma}_0, \mathbf{I} + (\vartheta - t)\mathbf{M}) &= \mathsf{H}_{0<\vartheta<t}(\mathbf{Q}(0)\boldsymbol{\Sigma}_0\mathbf{Q}(0)^T, \mathbf{Q}(\vartheta)(\mathbf{I} + (\vartheta - t)\mathbf{M})) \\ &= \mathsf{H}_{0<\vartheta<\mathrm{t}}(\mathbf{Q}(0)\boldsymbol{\Sigma}_0\mathbf{Q}(0)^T, \mathbf{F}_t{}^*(\vartheta)) \end{aligned} \tag{1.49}$$

For a fluid, or for a jammed material in its liquid regime, we get

$$\mathsf{H}_{\vartheta<\mathrm{t}}(\mathbf{I} + (\vartheta - t)\mathbf{M}) = \mathsf{H}_{\vartheta<\mathrm{t}}(\mathbf{F}_t{}^*(\vartheta)) = \boldsymbol{\Sigma}(\mathbf{x}, \mathrm{t}) \tag{1.50}$$

which means that the actual stress tensor is the same as the stress tensor associated with the simpler history $\mathbf{I} + (\vartheta - t)\mathbf{M}$, that is, without factoring in the time changes of the frame in which we have instantaneous simple shear. On the left-hand side (LHS) of (1.50) the function depends only on the different values of $\dot{\gamma}$ in time, which implies that *the stress tensor is simply a function of the history of* $\dot{\gamma}$. This result is in particular valid for a simple shear in a fixed frame but with time variations of $\dot{\gamma}$. On the contrary, for a solid or a jammed material in its solid regime, equation (1.49) does not provide a straightforward result concerning viscometric flows except if $\mathbf{Q} = \mathbf{I}$, which is merely a simple shear with a time-dependent shear rate.

Practical Approach for Simple Shear of Jammed Systems

Because it is difficult to express the constitutive equation of jammed systems in the different possible regimes in a general way, it seems preferable in practice to maintain the dependence on the initial state of the material, even under conditions for which the material widely deforms, like a fluid. This makes it possible to account for the possible solid–liquid transition and the possible nonvanishing memory of pastes or granular materials under some conditions. Under these conditions, in generalized simple shear, the stress tensor of a jammed system may be expressed as

$$\boldsymbol{\Sigma}(t) = \mathsf{H}_{0<\vartheta<t}(\boldsymbol{\Sigma}_0, \dot{\gamma}(\vartheta)) \tag{1.51}$$

In practice it often appears useful to express the actual shear rate as a function of the stress history. By inverting (1.51) we can deduce

$$\dot{\gamma}(t) = \mathsf{H}_{0<\vartheta<t}(\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}(\vartheta)) \tag{1.52}$$

Note that a specificity of jammed systems is that in steady state, when $\dot{\gamma}$ and $\boldsymbol{\Sigma}$ are constant, the relationship between these two variables may still depend on the initial state of the material $\boldsymbol{\Sigma}_0$. In particular, for the shear stress, we deduce from (1.51) a steady-state dependence of the type

$$\tau = \varsigma_{\Sigma_0}(\dot{\gamma}) \tag{1.53}$$

in which ς is a function depending on $\boldsymbol{\Sigma}_0$.

Note that in some cases it may be assumed that all stress components except the tangential one in the direction of shear play a negligible role in expression (1.51) which, for a constant shear stress, leads to

$$\dot{\gamma}(t) = \chi_{\Sigma_0}(\tau, t) \tag{1.54}$$

Finally, we note that in the rest of the book, when dealing with flow problems, we will use the generic terms “material” or “fluid” to describe a jammed system under flow.

1.4 VISCOMETRIC FLOWS

The practical interest in the viscometry of flows is that this method in principle makes it possible to determine the resistance to flow of a fluid from simple macroscopic measurements. Indeed, if we have been capable of devising flows with appropriate properties of symmetry such that the fluid layers glide over each other parallel to some boundary, we can expect that a measure of the relative velocity between two solid boundaries in contact with the fluid, and the total force exerted on one of these boundaries, will provide the values of the local material functions (including its apparent viscosity). For simple fluids it has been shown that in that case, which corresponds to that of equation (1.30), the four material functions completely define the stress tensor. In fact, even in that case, there is a direct correspondence between the macroscopic measurements and the local material functions of the fluid only when the shear rate is homogeneous within the fluid volume. However, this situation is expected only for very small gaps [4], but large gaps are often required with pasty or granular materials for the continuum assumption to be valid, at least in terms of the density. Moreover, even for small gaps, the theory of viscometric flows as established by Coleman et al. [1] is valid only for steady-state flows of simple fluids, capable of completely forgetting their initial state. As we have seen above, within the frame of jammed systems (pastes or granular materials), this assumption may not be valid.

In fact, the problem of describing the specific flow characteristics of a material is rather complex because, even for viscometric flows, a sufficient knowledge of the constitutive equation of the material is required. Moreover, it would be necessary to determine the exact boundary conditions, such as possible edge effects and wall slip. Here we will separate the problems and simply assume that the flow a priori possesses the basic, qualitative, ideal characteristics generally assumed for viscometric flows (for simple fluids) and that jamming properties (leading to yielding and thixotropy) or perturbing effects act to modify, to a certain extent, the quantitative characteristics of this ideal flow without changing its qualitative aspects. The possible consequences of these additional effects will be examined in detail in Chapter 3.

To summarize, our approach consists in first assuming some ideal form of the velocity field and deducing the corresponding stress field and then considering the perturbing effects or jamming properties that play a role within this simple frame. Such a procedure obviously constitutes a nonconventional, approximate, mechanical approach. However, with regard to the difficulty of determining a priori the strain–stress field with no knowledge of the material behavior, this appears to be a good method for dealing with viscometric flows of pastes and granular materials in a sufficiently simple way and at the same time taking into account the specificities of these flows.

In the sections that follow we will make the following, ideal assumptions: (1) the material remains homogeneous and constant (there is no irreversible chemical transformation); (2) there are no edge effects; (3) the ambient fluid induces only a uniform pressure p_0 in the fluid located along the interface—that is, the effects of surface tension effects and of the flow of the ambient fluid are neglected; (4) inertia effects are negligible; (5) there are no thermal effects; (6) the flow remains laminar; (7) there is no relative velocity between the fluid and the solid along the rigid boundaries (no wall slip); and (8) there are no flow instabilities, that is, the flow field retains the simplest form, verifying the conservation equations and the boundary conditions. We start by considering the simplest viscometric flow that may be obtained in practice, namely, free surface flow over an inclined plane (Section 1.4.1), then we turn to the viscometric flows that can be obtained from usual laboratory rheometers: flow between parallel disks (Section 1.4.2), between a cone and a plate (Section 1.4.3), between two coaxial cylinders (Section 1.4.4), and through a straight cylindrical conduit (Section 1.4.5).

1.4.1 Free Surface Flow over a Plane

We consider the flow of a material over a solid inclined plane of a slope i with respect to the horizontal and describe it in terms of (x, y, z) coordinates as represented in Figure 1.6. Neglecting edge effects is equivalent to the plane surface and the free surface being infinite. This implies that all positions within each plane parallel to them are equivalent so that the velocity and the stress tensor at each point in the material depends only on their distance y from the plate.

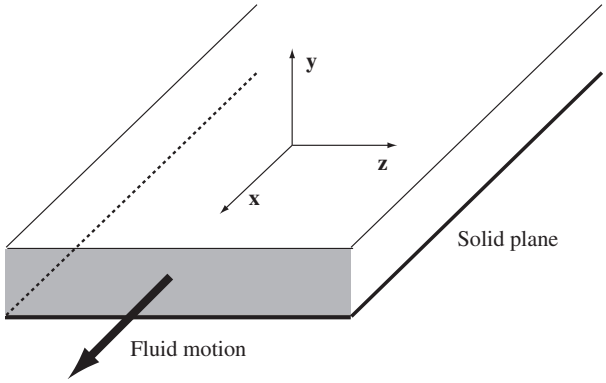

Figure 1.6 Geometric characteristics of inclined plane flow.

Under such conditions the mass conservation (1.14) gives $\partial v_y/\partial y = 0$, which means that the velocity component perpendicular to the planes is uniform, and in particular is equal to the normal velocity along the solid plane, specifically, zero. In addition, for reasons of symmetry, the velocity component in the direction z is zero everywhere; a fluid element at any point in each plane (x, y) "sees" a similar situation on each side of this plane, so that it has no particular reason for moving in one direction rather than in the other, and we have $v_z = 0$. Note that these results concerning the velocity components in the y and z directions are valid only in the absence of mechanical or thermal flow instabilities possibly leading to the formation of vortices with a particular fluid length scale. Finally, for a stable flow the only nonzero component of the velocity is v_x, which depends only on y. This means that the fluid motion takes the form of planar layers sliding over each other in the direction of steepest slope (x).

Let us now apply the momentum equation in integral form (1.15) to a fluid portion limited by the free surface, the planar surface situated at a height y and two sections perpendicular to the solid plane at a distance L (see Figure 1.7). Since the flow is uniform, there are no flux variations between these two sections and the normal forces acting over them balance. The momentum equation finally reduces to the balance of the gravity force acting on the volume and the viscous force resulting from the contact between the two local, adjacent, fluid layers at the height y. In projection along the direction x the momentum equation provides the expression for the shear stress

$$\sigma_{xy} = \tau(y) = \rho g(h - y) \sin i \tag{1.55}$$

in which h is the thickness of the fluid layer. This equation shows that the shear stress varies linearly in the fluid layer and assumes its maximum value

$$\tau_p = \sigma_{xy}(0) = \rho g h \sin i \tag{1.56}$$

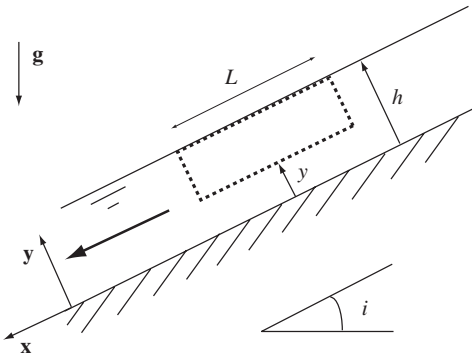

Figure 1.7 Fluid portion in a free surface flow over an inclined plane.

along the interface with the plate. The momentum equation applied to this fluid portion in projection along the y direction yields

$$\sigma_{yy} = p_0 + \rho g(h - y)\cos i \tag{1.57}$$

Since the flow consists in the relative motion of parallel planar layers of a material along the x direction, we have a local simple shear of intensity possibly varying in time. More precisely, the history of the relative configuration gradient at a given point is a function of the gradient of velocity at a fixed distance from the axis, namely, the shear rate:

$$\dot{\gamma} = \frac{\partial v_x}{\partial y} \tag{1.58}$$

which is a priori a function of the height, time, and initial state, but for the sake of simplicity we will simply write it $\dot{\gamma}(y, t)$. The stress components of the fluid at a level y above the plane are functions of the history of $\dot{\gamma}(y, t)$ and of the state of the material at the initial time. For this flow the stress distribution, and in particular the shear stress at the wall, is known as soon as the fluid depth has been measured. On the contrary, the corresponding shear rate, which depends entirely on the constitutive equation, varies with the position in the fluid and cannot be determined easily from macroscopic measurements. In practice, two quantities can be measured readily: the flow rate (by unit surface) (q) through a cross section and the velocity of the fluid at the free surface ($\mathbf{U} = U\mathbf{x}$), which is also the maximum velocity of the fluid. These quantities can be related to the velocity distribution in the fluid, and thus to the distribution of shear rate, through simple integrations:

$$q(t) = \int_0^h v_x(y, t)\, dy = \int_0^h (h - y)\dot{\gamma}(y, t)\, dy \tag{1.59}$$

$$U(t) = v_x(h,t) = \int_0^h \dot{\gamma}(y,t)\,dy \tag{1.60}$$

These relations are valid at any time in the flow for any fluid type as long as our initial assumptions are valid.

Here we consider the specific case of steady-state flows. From equation (1.55) the shear stress at any height above the plane is fixed. If the other stress components play a negligible role in (1.51), the shear rate and shear stress are linked by a relation of the type

$$\dot{\gamma} = \chi_{\Sigma_0}(\tau) \tag{1.61}$$

simply derived from equation (1.54). We can then use the shear stress, $\tau = \tau_p(h-y)/h$, instead of the depth in the integrals above, so as to obtain (dropping the index Σ_0 for the sake of clarity)

$$q = \frac{h^2}{\tau_p^2}\int_0^{\tau_p} \tau\,\chi(\tau)\,d\tau \tag{1.62}$$

$$U = \frac{h}{\tau_p}\int_0^{\tau_p} \chi(\tau)\,d\tau \tag{1.63}$$

Thus we find that the flow rate and the fluid velocity at the free surface are simple functions of the fluid depth, the plane slope, and an integral that by derivation may yield the shear rate along the plane. These equations may be differentiated relative to the fluid depth or the plane slope such that,

$$\left(\frac{dq}{dh}\right)_i = h\chi(\tau_p) \tag{1.64}$$

$$\left(\frac{dU}{dh}\right)_i = \chi(\tau_p) \tag{1.65}$$

With the help of these equations the constitutive equation of the material in the form of shear rates associated with different shear stresses can be deduced from an ensemble of measurements of the flow rate or the surface velocity as a function of the flow depth. In practice, this generally requires variation of the flow rate starting from the material prepared under fixed conditions and, at each level, waiting for the steady state before measuring the corresponding fluid depth.

1.4.2 Flow between Parallel Disks

We consider a material contained and sheared between two parallel, circular plates in relative rotation at a velocity Ω around their common axis, and separated by

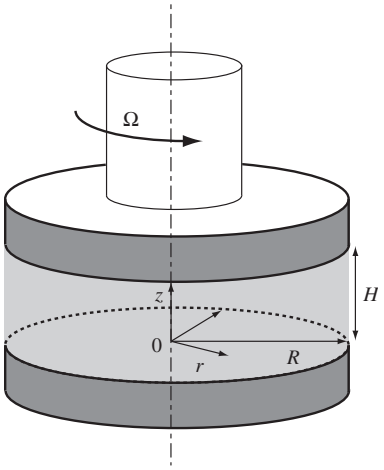

Figure 1.8 Geometric characteristics of parallel-plate flow.

a distance H. We describe the flow in the cylindrical coordinates attached to the lower plate (O, r, θ, z) (see Figure 1.8). We assume that the peripheral interface of the material with the ambient fluid situated at a distance R from the axis is a constant, straight cylinder centered around the disk axis.

Because of the symmetry of the problem by rotation around the central axis, the components of the velocity and the stress tensor a priori do not depend on θ. Moreover, the radial and axial components of the velocity would tend to push the fluid out of its initial volume. If the velocity had a nonzero radial or axial component somewhere, there would be some vortices within the material in order to respect the mass conservation and the constant boundaries. This situation, typical of an instability, will not be considered here. Under these conditions the only nonzero component of velocity is the tangential one, $v_\theta = v_\theta(r, z)$. Now we can state that the boundary conditions are $v_\theta(r, 0) = 0$ (fixed lower plate) and $v_\theta(r, z_0) = \Omega r$. The simplest velocity field satisfying these boundary conditions corresponds to a linear variation of the velocity between *0* and H at any distance r: $v_\theta(r, z) = \Omega r z/H$. This means that the fluid layers parallel to the disk slide over each other by rotating around the central axis. The local shear rate simply corresponds to the spatial rate of variation of the tangential velocity:

$$\dot{\gamma} = \frac{\Omega r}{H} \tag{1.66}$$

It follows that the flow consists in the relative motion of disks of material around their common axis and the history of the relative configuration gradient at a given point is described entirely by the history of the local shear rate.

Now, we can formulate the conservation of torque (see Section 1.2.3) over a fluid portion limited by the upper disk surface and a parallel surface within the fluid (Figure 1.8). We find that the torque applied to the upper disk may be

expressed as follows:

$$M = \int_0^R 2\pi r^2 \tau(r)\, dr \tag{1.67}$$

The specificity of this flow is that the shear stress is well known as a function of the rotation velocity but the resulting shear rate is strongly heterogeneous, varying from zero at the axis to the maximum value $\dot{\gamma}_R = \Omega R/H$ at the periphery. It follows that the torque and the rotation velocity provide only global information concerning heterogeneous radial distributions of the shear stress and the shear rate. Apparently a specific technique for interpretation of a series of different torque–rotation velocity values is needed.

Let us consider a transient flow under a given rotation velocity Ω (leading to a constant shear rate distribution) from a given initial state. We can transform (1.67) by a change of variables using (1.53) and (1.66) (again dropping Σ_0):

$$M(t) = \frac{2\pi}{(\Omega/H)^3} \int_0^{\dot{\gamma}_R} \dot{\gamma}^2 \varsigma(\dot{\gamma}, t)\, d\dot{\gamma} \tag{1.68}$$

This equation may be differentiated relative to $\dot{\gamma}_R$:

$$\tau_R = \varsigma(\dot{\gamma}_R, t) = \frac{M(t)}{2\pi R^2} \left[3 + \frac{\dot{\gamma}_R}{M} \frac{\partial M(t)}{\partial \dot{\gamma}_R} \right] \tag{1.69}$$

Thus the relation between the shear stress and the shear rate at the periphery at each time may be obtained through equation (1.69) from a sufficient set of experimental values of the pair $(M(t), \Omega)$. The second term of the sum in brackets in (1.69) is a priori not negligible, since for a Newtonian fluid it is equal to 1 and is equal to n for a power-law fluid, but for yield stress fluids it becomes very small when the maximum stress (at the periphery) is not too far from the yield stress.

A further analysis of the flow equations [1] gives a similar equation for the first normal stress difference:

$$(2\sigma_1 - \sigma_2)\pi R^2 = 2N + \dot{\gamma}_R \frac{dN}{d\dot{\gamma}_R} \tag{1.70}$$

Here, N is the total normal force (including the atmospheric pressure) exerted onto the upper plate and $\sigma_1 = \sigma_{zz} - \sigma_{rr}$ and $\sigma_2 = \sigma_{\theta\theta} - \sigma_{rr}$. Thus the analysis of a series of pairs $(N, \dot{\gamma}_R)$ makes it possible to derive the value of the function $2\sigma_1 - \sigma_2$.

1.4.3 Flow between a Cone and a Plate

We consider the flow between a cone and a plate such that the cone axis is perpendicular to the plate and the cone vertex falls exactly onto the plate surface. We describe the flow in the spherical coordinates attached to the plate (r, θ, φ)

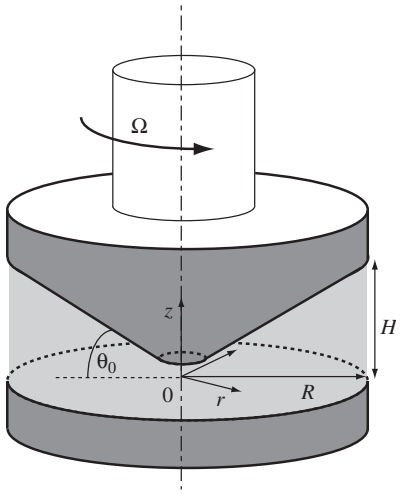

Figure 1.9 Geometric characteristics of cone and plate flow.

(see Figure 1.9). Here the cone is rotating around its axis at a rotation velocity Ω. The angle θ_0 between the cone and the plate is assumed to be small so that the ratio of the fluid thickness to the distance from the axis remains much smaller than 1. The material remains between the cone and the plate and its peripheral free surface is assumed to be constant. Under such conditions, because of the symmetry of the problem, the velocity and the stress tensor do not depend on φ. A simple velocity field with only a lateral component (v_φ) is the simplest situation that may be expected. As for the parallel disks, some radial or tangential motion somewhere within the fluid would indicate the occurrence of some flow instability. Moreover, the flows in all such fluid portions are similar since edge effects are negligible and the boundary conditions are similar; the geometry is similar (with constant cone angle) and both the material thickness and the relative velocity of the solid surfaces are proportional to the distance. It results that the material velocity is simply proportional to the distance r from the cone submit. Finally the only nonzero component of the velocity is v_φ, which may be expressed as: $v_\varphi = r\omega(\theta)$, where ω is an unknown function. The flow associated with such a velocity field consists in the relative motion of conical material layers rotating around the central axis and sliding one over each other. At a given point the history of the relative configuration gradient is thus given entirely by the history of the relative velocity of the local conical layers or equivalently the gradient of the tangential velocity at a fixed distance from the submit, that is, the shear rate:

$$\dot{\gamma} = \frac{1}{r}\frac{dv_\varphi}{d\theta} = \omega'(\theta) \tag{1.71}$$

If the flow is homogeneous, we obtain the relation $\dot{\gamma} = \Omega/\theta_0$ from (1.71).

Let us now apply the torque conservation in a conical portion of fluid with one interface falling along the cone and the other inside the fluid. It follows that if the peripheral free surface is situated at a distance R, the total torque applied by the cone onto the material is expressed as follows:

$$M = \int_0^R 2\pi r^2 (\cos\theta)^2 \tau \, dr = \frac{2}{3}\pi R^3 (\cos\theta)^2 \tau(\dot{\gamma}) \tag{1.72}$$

Thus for a given torque value the shear stress varies in material thickness, and the maximum relative difference is equal to

$$\frac{\Delta\tau}{\tau} = \frac{\tau(\theta_0) - \tau(0)}{\tau(0)} = (\tan\theta_0)^2$$

For a cone angle smaller than 4°, this relative difference is slightly smaller than 0.5%, so that the impact of these variations on the determination of the constitutive equation will often be negligible as compared to that of perturbing effects or time changes of material behavior.

In addition, a detailed description of flow characteristics leads [1] to the following equation for the normal stress difference:

$$N = K + \frac{\pi R^2 \sin\theta_0}{2}[\sigma_1 - \sigma_2] \tag{1.73}$$

in which $\sigma_1 = \sigma_{\theta\theta} - \sigma_{rr}$, $\sigma_2 = \sigma_{\varphi\varphi} - \sigma_{rr}$ and $K = \pi R^2 \sin\theta_0 \sigma_{\varphi\varphi}(R) + \int_0^R 2\pi r \sin\theta_0 p(r)\, dr$. Note that K may be considered as constant for a series of tests within the same sample so that the value of the function $\sigma_1 - \sigma_2$ can be directly found from equation (1.73) by observing the evolution of N with the rotation velocity.

Note that in practice the cone should preferably be truncated in order to avoid any contact between the cone and the plate leading to an additional and probably uncontrolled frictional force. This truncation implies that the flow is not as assumed in the preceding theory at the approach of the central axis. The resulting effect is even smaller as the truncation diameter is small. Nevertheless, in order to avoid particle jamming with suspensions, this truncation must be sufficiently large for the central gap to be several times larger than the diameter of the largest particles.

1.4.4 Flow between Two Coaxial Cylinders

Let us consider the flow of a material contained and sheared between two concentric cylinders. Such a flow is often called a "Couette flow." We describe the fluid motion in the cylindrical coordinates (O, r, θ, z) attached to the outer cylinder (see Figure 1.10). Here the inner cylinder rotates around its axis at a velocity Ω. Assuming negligible edge effects is equivalent to assuming that the length of the cylinders is infinite, and in that case the variables of the flow do not depend

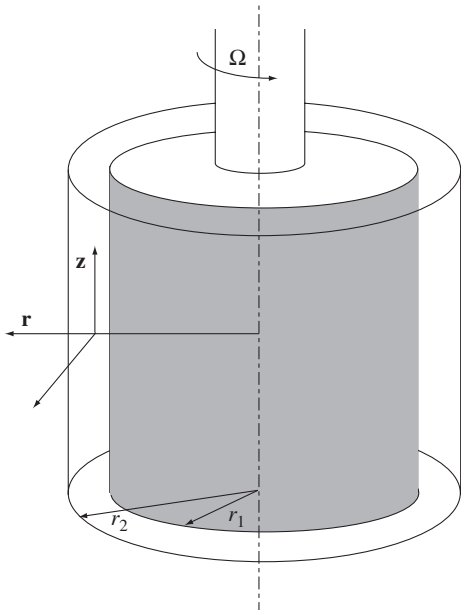

Figure 1.10 Geometric characteristics of concentric coaxial cylinders.

on z. Moreover, the flow is symmetric by rotation around the axis; that is, the variables do not depend on θ. Under these conditions the simplest velocity field corresponds to radial and axial velocity components equal to zero. If the radial or axial component of the velocity deviated from zero somewhere, there would be a net outward, radial or vertical, flow, associated with some flow instability. For a stable flow it follows that only the tangential velocity component is different from zero, and it depends only on r. We will express this in the following form

$$v_\theta = r\omega(r) \tag{1.74}$$

in which ω is the local rotation velocity of the fluid layer situated at a distance r from the central axis. A velocity field as given by equation (1.74) shows that the flow consists in the relative motion of concentric cylindrical layers of material rotating around the central axis. At a given point the history of the relative configuration gradient is given entirely by the history of the relative velocity of the local concentric layers or equivalently the gradient of the relative velocity of these fictive cylinders: $\omega'(r)$. Here, the shear rate corresponds to the local velocity induced by this gradient:

$$\dot{\gamma} = r\omega'(r) \tag{1.75}$$

In the absence of inertia effects, the torque applied by the inner cylinder onto the material may be found by integration of (1.17) over any cylindrical surface of radius r:

$$M = 2\pi h r^2 \tau(r) \tag{1.76}$$

This means that the stress in the gap ($\delta = r_2 - r_1$) separating the cylinders is not uniform; it depends on the distance from the cylinder axis. The amplitude of this heterogeneity depends on the ratio of the gap to one of the cylinder radii since the ratio of the maximum to the minimum stresses is $(r_2/r_1)^2 = (1 + \delta/r_1)^2$.

We can compute the expression for the rotation velocity as a function of the material motion along a radius:

$$\Omega = \int_{r_i}^{r_e} \frac{\dot{\gamma}}{r}\, dr \tag{1.77}$$

This means that there is no straightforward relation between the mean shear rate and the rotation velocity. In the limit of a small gap to radius ratio, the shear stress is almost constant throughout the gap, and we can expect the shear rate to be approximately constant so that we have

$$\Omega \approx \dot{\gamma} \frac{r_2 - r_1}{r_1} \tag{1.78}$$

which provides a simple relation between the rotation velocity and the shear rate in the fluid. In that case the shear stress–shear rate data are directly found from torque–rotation velocity data with the help of equations (1.76) and (1.78). In other cases one needs to interpret a set of torque–rotation velocity data in terms of a relation between the shear stress and the shear rate.

Assume that we applied a constant torque M to a material for which the actual shear rate depends only on the tangential stress component [equation (1.54)]. In the case of Couette flow this is in fact a very reasonable hypothesis with regard to boundary conditions (see Section 4.3.1). Using this relation and (1.76), we can transform (1.77) into

$$\Omega = \int_{\tau_1}^{\tau_2} \frac{\chi_{\Sigma_0}(\tau, t)}{\tau}\, d\tau \tag{1.79}$$

in which $\tau_1 = M/2\pi h r_1^2$ and $\tau_2 = M/2\pi h r_2^2$. Dropping Σ_0 for the sake of clarity and differentiating (1.79) relative to M, we get

$$2M \frac{\partial \Omega(t)}{\partial M} = \chi(\tau_1, t) - \chi(\tau_2, t) \tag{1.80}$$

As a consequence, the shear rate at the wall may be obtained by summing a series of such relations with successive decreasing torques in a ratio $\beta = (r_1/r_2)^2$. Since

a zero shear rate is necessarily associated with a zero torque we eventually obtain

$$\dot{\gamma}(\tau_1, t) = \sum_{p=0}^{\infty} \left(2M \frac{\partial \Omega}{\partial M} \right)_{\beta^p \tau_1, t} \tag{1.81}$$

The shear rate can thus be deduced by determining the different terms of this series [the right-hand side (RHS) term of (1.81)], associated with the different values of $\beta^p \tau_i$ by varying either (1) the radii of the cylinders and computing the derivative for the same torque value or (2) the torque and keeping the same geometry. In any case it is necessary to determine the function $\Omega(M, t)$ in a generally wide range in order to accurately determine the slope of this function for specific values of M. Note that for an infinitely wide outer cylinder ($r_2 \to \infty$) we have $\dot{\gamma}(\tau_2 = 0) = 0$, so from equation (1.80) we can derive the simpler shear rate expression:

$$\dot{\gamma}(\tau_1, t) = 2M \left(\frac{\partial \Omega}{\partial M} \right)_t \tag{1.82}$$

1.4.5 Flow in a Cylindrical Conduit (Poiseuille Flow)

We consider the uniform flow of a material in cylindrical coordinates (r, θ, z) such that $\mathbf{z}$ is in the direction of the conduit axis (see Figure 1.11). We assume that the flow is uniform, which means that the flow characteristics are identical in each cross section, that is, the velocity field is independent of z. It is also

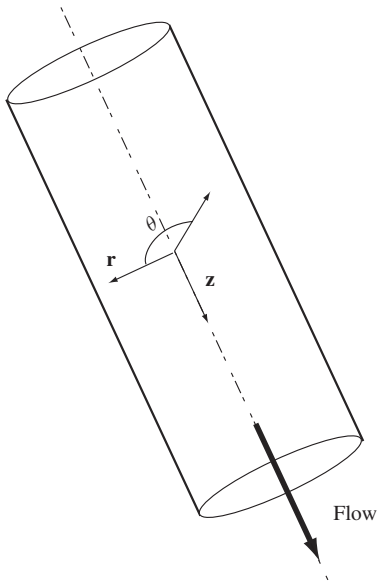

Figure 1.11 Geometric characteristics of Poiseuille flow.

symmetric around the conduit axis, so that all variables are independent of θ. However, some stress components may depend on z because the driving force is the pressure difference between the extremities of the conduit. In the absence of any flow instability only the longitudinal component of the velocity deviates from zero and depends only on r:

$$v_z = f(r) \tag{1.83}$$

A velocity field of this type means that the flow consists in the relative motion of concentric, cylindrical, material layers parallel to the central axis. At a given point the history of the relative configuration gradient is given entirely by the history of the relative velocity of the local cylindrical layers, namely, the shear rate:

$$\dot{\gamma} = f'(r) \tag{1.84}$$

Let us consider a fluid portion limited by two sections perpendicular to the cylinder axis and separated by a distance L, and a cylinder of radius r within the fluid. Along the air–fluid interfaces at the boundaries ($z = 0$ and $z = L$) the stress tensor reduces to an isotropic pressure independent of r. It is thus natural to assume that within the fluid the normal stress σ_{zz} does not depend on r but only on the distance from the boundaries (z). Under these conditions, in the absence of inertia, the momentum equation (1.15) applied to such a fluid portion and in projection along the axial direction yields

$$(\sigma_{zz}(L) - \sigma_{zz}(0))\pi r^2 + 2\pi rL\sigma_{rz}(r) = 0 \tag{1.85}$$

from which we deduce

$$\tau = \sigma_{rz} = \frac{A}{2}r \tag{1.86}$$

with $A = (\sigma_{zz}(0) - \sigma_{zz}(L))/L$. Since τ depends only on r, we can conclude from equation (1.83) that A, which could depend only on z, is in fact constant.

This shear rate distribution is far from being homogeneous in a conduit flow, as the fluid is submitted to a stress history that significantly varies from the wall to the conduit axis. As a consequence, there is no simple relation between the applied stress, via the pressure drop by unit length A, and some specific or mean shear rate in the material. We can nevertheless compute the flow rate (total fluid discharge through a cross section):

$$Q = 2\pi \int_0^R rv_z\, dr = -2\pi \int_0^R \frac{r^2}{2}\frac{dv_z}{dr}dr \tag{1.87}$$

Assuming again a behavior of the type (1.54), we can further analyze the flow under constant pressure drop A. Equation (1.87) becomes

$$Q(t) = \frac{8\pi}{A^3}\int_0^{RA/2} \chi_{\Sigma_0}(\tau, t)\tau^2\, d\tau \tag{1.88}$$

Differentiating (1.88) relative to A, we get the expression for the shear rate at the wall as a function of the variation of the flow rate with the pressure drop by unit length:

$$\dot{\gamma}_R = \chi_{\Sigma_0}\left(\frac{-RA}{2}, t\right) = \frac{1}{\pi R^3 A^2}\left(\frac{d(QA^3)}{dA}\right)_t \tag{1.89}$$

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