1.1 SET DEFINITIONS

The concept of sets play an important role in probability. We will define a set in the following paragraph.

**Definition of Set**

A set is a collection of objects called elements. The elements of a set can also be sets. Sets are usually represented by uppercase letters $A$, and elements are usually represented by lowercase letters $a$. Thus

$$A = \{a_1, a_2, \ldots, a_n\} \quad (1.1.1)$$

will mean that the set $A$ contains the elements $a_1, a_2, \ldots, a_n$. Conversely, we can write that $a_k$ is an element of $A$ as

$$a_k \in A \quad (1.1.2)$$

and $a_k$ is not an element of $A$ as

$$a_k \notin A \quad (1.1.3)$$

A finite set contains a finite number of elements, for example, $S = \{2, 4, 6\}$. Infinite sets will have either countably infinite elements such as $A = \{x : x$ is all positive integers$\}$ or uncountably infinite elements such as $B = \{x : x$ is real number $\leq 20\}$.

**Example 1.1.1** The set $A$ of all positive integers less than 7 is written as

$$A = \{x : x \text{ is a positive integer } < 7\} : \text{finite set.}$$

**Example 1.1.2** The set $N$ of all positive integers is written as

$$N = \{x : x \text{ is all positive integers}\} : \text{countably infinite set.}$$

**Example 1.1.3** The set $R$ of all real numbers is written as

$$R = \{x : x \text{ is real}\} : \text{uncountably infinite set.}$$
Example 1.1.4 The set $R^2$ of real numbers $x, y$ is written as

$$R^2 = \{(x, y) : x \text{ is real, } y \text{ is real}\}$$

Example 1.1.5 The set $C$ of all real numbers $x, y$ such that $x + y \leq 10$ is written as

$$C = \{(x, y) : x + y \leq 10\}: \text{ uncountably infinite set.}$$

**Venn Diagram**

Sets can be represented graphically by means of a Venn diagram. In this case we assume tacitly that $S$ is a universal set under consideration. In Example 1.1.5, the universal set $S = \{x : x \text{ is all positive integers}\}$. We shall represent the set $A$ in Example 1.1.1 by means of a Venn diagram of Fig. 1.1.1.

**Empty Set**

An empty set contains no element. It plays an important role in set theory and is denoted by $\emptyset$. The set $A = \{0\}$ is not an empty set since it contains the element 0.

**Cardinality**

The number of elements in the set $A$ is called the cardinality of set $A$, and is denoted by $|A|$. If it is an infinite set, then the cardinality is $\infty$.

Example 1.1.6 The cardinality of the set $A = \{2, 4, 6\}$ is 3, or $|A| = 3$. The cardinality of set $R = \{x : x \text{ is real}\}$ is $\infty$.

Example 1.1.7 The cardinality of the set $A = \{x : x \text{ is positive integer} < 7\}$ is $|A| = 6$.

Example 1.1.8 The cardinality of the set $B = \{x : x \text{ is a real number} < 10\}$ is infinity since there are infinite real numbers $< 10$.

**Subset**

A set $B$ is a subset of $A$ if every element in $B$ is an element of $A$ and is written as $B \subseteq A$. $B$ is a proper subset of $A$ if every element of $A$ is not in $B$ and is written as $B \subset A$.

![Venn Diagram](FIGURE_1.1.1)
Equality of Sets

Two sets $A$ and $B$ are equal if $B \subseteq A$ and $A \subseteq B$, that is, if every element of $A$ is contained in $B$ and every element of $B$ is contained in $A$. In other words, sets $A$ and $B$ contain exactly the same elements. Note that this is different from having the same cardinality, that is, containing the same number of elements.

Example 1.1.9 The set $B = \{1,3,5\}$ is a proper subset of $A = \{1,2,3,4,5,6\}$, whereas the set $C = \{x: x$ is a positive even integer $\leq 6\}$ and the set $D = \{2,4,6\}$ are the same since they contain the same elements. The cardinalities of $B$, $C$, and $D$ are 3 and $C = D$.

We shall now represent the sets $A$ and $B$ and the sets $C$ and $D$ in Example 1.1.9 by means of the Venn diagram of Fig. 1.1.2 on a suitably defined universal set $S$.

Power Set

The power set of any set $A$ is the set of all possible subsets of $A$ and is denoted by $PS(A)$. Every power set of any set $A$ must contain the set $A$ itself and the empty set $\emptyset$. If $n$ is the cardinality of the set $A$, then the cardinality of the power set $|PS(A)| = 2^n$.

Example 1.1.10 If the set $A = \{1,2,3\}$ then $PS(A) = \{\emptyset, (1,2,3), (1,2), (2,3), (3,1), (1), (2), (3)\}$. The cardinality $|PS(A)| = 8 = 2^3$.

### 1.2 SET OPERATIONS

**Union**

Let $A$ and $B$ be sets belonging to the universal set $S$. The union of sets $A$ and $B$ is another set $C$ whose elements are those that are in either $A$ or $B$, and is denoted by $A \cup B$. Where there is no confusion, it will also be represented as $A + B$:

$$A \cup B = A + B = \{x: x \in A \text{ or } x \in B\} \quad (1.2.1)$$
Example 1.2.1 The union of sets $A = \{1,2,3\}$ and $B = \{2,3,4,5\}$ is the set $C = A \cup B = \{1,2,3,4,5\}$.

Intersection

The intersection of the sets $A$ and $B$ is another set $C$ whose elements are the same as those in both $A$ and $B$ and is denoted by $A \cap B$. Where there is no confusion, it will also be represented by $AB$.

$$A \cap B = AB = \{x : x \in A \text{ and } x \in B\} \quad (1.2.2)$$

Example 1.2.2 The intersection of the sets $A$ and $B$ in Example 1.2.1 is the set $C = \{2,3\}$ Examples 1.2.1 and 1.2.2 are shown in the Venn diagram of Fig. 1.2.1.

Mutually Exclusive Sets

Two sets $A$ and $B$ are called mutually exclusive if their intersection is empty. Mutually exclusive sets are also called disjoint.

$$A \cap B = \emptyset \quad (1.2.3)$$

One way to determine whether two sets $A$ and $B$ are mutually exclusive is to check whether set $B$ can occur when set $A$ has already occurred and vice versa. If it cannot, then $A$ and $B$ are mutually exclusive. For example, if a single coin is tossed, the two sets, \{heads\} and \{tails\}, are mutually exclusive since \{tails\} cannot occur when \{heads\} has already occurred and vice versa.

Independence

We will consider two types of independence. The first is known as functional independence [42]. Two sets $A$ and $B$ can be called functionally independent if the occurrence of $B$ does not in any way influence the occurrence of $A$ and vice versa. The second one is statistical independence, which is a different concept that will be defined later. As an example, the tossing of a coin is functionally independent of the tossing of a die because they do not depend on each other. However, the tossing of a coin and a die are not mutually exclusive since any one can be tossed irrespective of the other. By the same token, pressure and temperature are not functionally independent because the physics of the problem, namely, Boyle’s law, connects these quantities. They are certainly not mutually exclusive.

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*Numbers in brackets refer to bibliographic entries in the References section at the end of the book.
Cardinality of Unions and Intersections

We can now ascertain the cardinality of the union of two sets $A$ and $B$ that are not mutually exclusive. The cardinality of the union $C = A \cup B$ can be determined as follows. If we add the cardinality $|A|$ to the cardinality $|B|$, we have added the cardinality of the intersection $|A \cap B|$ twice. Hence we have to subtract once the cardinality $|A \cap B|$ as shown in Fig. 1.2.2. Or, in other words

$$|A \cup B| = |A| + |B| - |A \cap B|$$  \hspace{1cm} (1.2.4a)

In Fig. 1.2.2 the cardinality $|A| = 9$ and the cardinality $|B| = 11$ and the cardinality $|A \cup B|$ is $11 + 9 - 4 = 16$.

As a corollary, if sets $A$ and $B$ are mutually exclusive, then the cardinality of the union is the sum of the cardinalities; or

$$|A \cup B| = |A| + |B|$$  \hspace{1cm} (1.2.4b)

The generalization of this result to an arbitrary union of $n$ sets is called the *inclusion–exclusion* principle, given by

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots$$

$$- (\pm 1)^n \sum_{i \neq j \neq k \ldots \neq n} |A_i \cap A_j \cap A_k \cdots \cap A_n|$$  \hspace{1cm} (1.2.5a)

If the sets $\{A_i\}$ are mutually exclusive, that is, $A_i \cap A_j = \emptyset$ for $i \neq j$, then we have

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|$$  \hspace{1cm} (1.2.5b)

This equation is illustrated in the Venn diagram for $n = 3$ in Fig. 1.2.3, where if $|A \cup B \cup C|$ equals, $|A| + |B| + |C|$ then we have added twice the cardinalities of $|A \cap B|$, $|B \cap C|$ and $|C \cap A|$. However, if we subtract once $|A \cap B|$, $|B \cap C|$, and $|C \cap A|$ and write

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A|$$

then we have subtracted $|A \cap B \cap C|$ thrice instead of twice. Hence, adding $|A \cap B \cap C|$ we get the final result:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| - |A \cap B \cap C|$$  \hspace{1cm} (1.2.6)

**FIGURE 1.2.2**
Example 1.2.3 In a junior class, the number of students in electrical engineering (EE) is 100, in math (MA) 50, and in computer science (CS) 150. Among these 20 are taking both EE and MA, 25 are taking EE and CS, and 10 are taking MA and CS. Five of them are taking EE, CS, and MA. Find the total number of students in the junior class.

From the problem we can identify the sets $A = \text{EE students}$, $B = \text{MA students}$, and $C = \text{CS students}$, and the corresponding intersections are $A \cap B$, $B \cap C$, $C \cap A$, and $A \cap B \cap C$. Here, $|A| = 100$, $|B| = 50$, and $|C| = 150$. We are also given $|A \cap B| = 20$, $|B \cap C| = 10$, and $|C \cap A| = 25$ and finally $|A \cap B \cap C| = 5$. Using Eq. (1.2.6) the total number of students are $100 + 50 + 150 - 20 - 10 - 25 = 250$.

Complement

If $A$ is a subset of the universal set $S$, then the complement of $A$ denoted by $\overline{A}$ is the elements in $S$ not contained in $A$; or

$$\overline{A} = S - A = \{ x : x \notin A \subset S \text{ and } x \in S \} \quad (1.2.7)$$

Example 1.2.4 If the universal set $S = \{1,2,3,4,5,6\}$ and $A = \{2,4,5\}$, then the complement of $A$ is given by $\overline{A} = \{1,3,6\}$.

Difference

The complement of a subset $A \subset S$ as given by Eq. (1.2.7) is the difference between the universal set $S$ and the subset $A$. The difference between any two sets $A$ and $B$ denoted by $A - B$ is the set $C$ containing those elements that are in $A$ but not in $B$ and, $B - A$ is the set $D$ containing those elements in $B$ but not in $A$:

$$C = A - B = \{ x : x \in A \text{ and } x \notin B \} \quad (1.2.8)$$

$$D = B - A = \{ x : x \in B \text{ and } x \notin A \}$$

$A - B$ is also called the relative complement of $B$ with respect to $A$ and $B - A$ is called the relative complement of $A$ with respect to $B$. It can be verified that $A - B \neq B - A$.

Example 1.2.5 If the set $A$ is given as $A = \{2,4,5,6\}$ and $B$ is given as $B = \{5,6,7,8\}$, then the difference $A - B = \{2,4\}$ and the difference $B - A = \{7,8\}$. Clearly, $A - B \neq B - A$. 

![Figure 1.2.3](image-url)
Symmetric Difference

The symmetric difference between sets \( A \) and \( B \) is written as \( A \triangle B \) and defined by

\[
A \triangle B = (A \setminus B) \cup (B \setminus A) = \{ x : x \in A \text{ and } x \notin B \} \cup \{ x : x \in B \text{ and } x \notin A \}
\] (1.2.9)

**Example 1.2.6** The symmetric difference between the set \( A \) given by \( A = \{2, 4, 5, 6\} \) and \( B \) given by \( B = \{5, 6, 7, 8\} \) in Example 1.2.5 is \( A \triangle B = \{2, 4\} \cup \{7, 8\} = \{2, 4, 7, 8\} \).

The difference and symmetric difference for Examples 1.2.5 and 1.2.6 are shown in Fig. 1.2.4.

Cartesian Product

This is a useful concept in set theory. If \( A \) and \( B \) are two sets, the Cartesian product \( A \times B \) is the set of ordered pairs \((x, y)\) such that \( x \in A \) and \( y \in B \) and is defined by

\[
A \times B = \{(x, y) : x \in A, y \in B\}
\] (1.2.10)

When ordering is considered the cartesian product \( A \times B \neq B \times A \). The cardinality of a Cartesian product is the product of the individual cardinalities, or \( |A \times B| = |A| \cdot |B| \).

**Example 1.2.7** If \( A = \{2, 3, 4\} \) and \( B = \{5, 6\} \), then \( C = A \times B = \{(2, 5), (2, 6), (3, 5), (3, 6), (4, 5), (4, 6)\} \) with cardinality \( 3 \times 2 = 6 \). Similarly, \( D = B \times A = \{(5, 2), (5, 3), (5, 4), (6, 2), (6, 3), (6, 4)\} \) with the same cardinality of 6. However, \( C \) and \( D \) do not contain the same ordered pairs.

Partition

A partition is a collection of disjoint sets \( \{A_i, i = 1, \ldots, n\} \) of a set \( S \) such that \( \bigcup A_i \) over all \( i \) equals \( S \) and \( A_i \cap A_j \) with \( i \neq j \) empty. Partitions play a useful role in conditional probability and Bayes’ theorem. Figure 1.2.5 shows the concept of a partition.
Example 1.2.8  If set $S = \{1,2,3,4,5,6,7,8,9,10\}$, then the collections of sets $\{A_1, A_2, A_3, A_4\}$ where $A_1 = \{1,3,5\}, A_2 = \{7,9\}, A_3 = \{2,4,6\}, A_4 = \{8,10\}$ is a partition of $S$ as shown in Fig. 1.2.6.

A tabulation of set properties is shown on Table 1.2.1. Among the identities, De Morgan’s laws are quite useful in simplifying set algebra.

### 1.3 SET ALGEBRAS, FIELDS, AND EVENTS

#### Boolean Field

We shall now consider a universal set $S$ and a collection $\mathcal{F}$ of subsets of $S$ consisting of the sets $\{A_i : i = 1,2,\ldots,n,n+1,\ldots\}$. The collection $\mathcal{F}$ is called a Boolean field if the
following two conditions are satisfied:

1. If $A_i \in \mathcal{F}$, then $\overline{A_i} \in \mathcal{F}$.
2. If $A_1 \in \mathcal{F}$ and $A_2 \in \mathcal{F}$, then $A_1 \cup A_2 \in \mathcal{F}$.
3. If $\{A_i, i = 1, \ldots, n\} \in \mathcal{F}$, then $\bigcup_{i=1}^n A_i \in \mathcal{F}$.

Let us see the consequences of these conditions being satisfied. If $A_1 \in \mathcal{F}$ and $\overline{A_1} \in \mathcal{F}$, then $A_1 \cup (\overline{A_1}) = S \in \mathcal{F}$. If $S \in \mathcal{F}$, then $S = \emptyset \in \mathcal{F}$. If $A_1 \cup A_2 \in \mathcal{F}$, then by condition 1 $\overline{A_1} \cup \overline{A_2} \in \mathcal{F}$, and by De Morgan’s law, $\overline{A_1} \cup \overline{A_2} \in \mathcal{F}$. Hence $A_1 \cap A_2 \in \mathcal{F}$. Also $A_1 - A_2 \in \mathcal{F}$ and $(A_1 - A_2) \cup (A_2 - A_1) \in \mathcal{F}$. Thus the conditions listed above are sufficient for any field $\mathcal{F}$ to be \textit{closed} under all set operations.

\textbf{Sigma Field}

The preceding definition of a field covers only finite set operations. Condition 3 for finite additivity may not hold for infinite additivity. For example, the sum of $1 + 2 + 2^2 + 2^3 + 2^4 = (2^5 - 1)/(2 - 1) = 31$, but the infinite sum $1 + 2 + 2^2 + 2^3 + 2^4 + \cdots$ diverges. Hence there is a need to extend finite additivity concept to infinite set operations. Thus, we have a $\sigma$ field if the following extra condition is imposed for infinite additivity:

4. If $A_1, A_2, \ldots A_n, \ldots \in \mathcal{F}$ then $\bigcup_{i=1}^\infty A_i \in \mathcal{F}$

Many $\sigma$ fields may contain the subsets $\{A_i\}$ of $S$, but the smallest $\sigma$ field containing the subsets $\{A_i\}$ is called the \textit{Borel $\sigma$ field}. The smallest $\sigma$ field for $S$ by itself is $\mathcal{F} = \{S, \emptyset\}$.

\textbf{Example 1.3.1} We shall assume that $S = \{1,2,3,4,5,6\}$. Then the collection of the following subsets of $S$, $\mathcal{F} = \{S, \emptyset, (1,2,3), (4,5,6)\}$ is a field since it satisfies all the set operations such as $(1,2,3) \cup (4,5,6) = S$, $(1,2,3) = (4,5,6)$. However, the collection of the following subsets of $S$, $\mathcal{F}_1 = \{S, \emptyset, (1,2,3), (4,5,6), (2)\}$ will not constitute a field because $(2) \cup (4,5,6) = (2,4,5,6)$ is not in the field. But we can adjoin the missing sets and make $\mathcal{F}_1$ into a field. This is known as \textit{completion}. In the example above, if we adjoin the collection of sets $\mathcal{F}_2 = \{(2,4,5,6), (1,3)\}$ to $\mathcal{F}_1$, then the resulting collection $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 = \{S, \emptyset, (1,2,3), (4,5,6), (2), (2,4,5,6), (1,3)\}$ is a field.

\textbf{Event}

Given the universal set $S$ and the associated $\sigma$ field $\mathcal{F}$, all subsets of $S$ belonging to the field $\mathcal{F}$ are called \textit{events}. All events are subsets of $S$ and can be assigned a probability, but not all subsets of $S$ are events unless they belong to an associated $\sigma$ field.