Chapter 5

Minding Your Ps and Qs: Functions

In This Chapter

- Understanding math symbols
- Synthesizing compound functions
- Reverting inverse functions
- Comparing domain and range
- Analyzing functions with intervals
- Recognizing graphs of functions
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Functions are relationships between two sets of numbers. Functions can describe the relationship between the x- and y-axes on a graph, but you can also work on them in algebraic formats. If the relationship between two variables, x and y, is such that there is exactly one value of y for each value of x, you can say that y is a function of x.

Additionally, the SAT II Math test may throw in some cryptic-looking symbols to see whether you're awake. Those kinds of functions are just meant to intimidate you, like many of the other questions on the SAT test. They can't scare you, though, if you know the tricks of the trade.

A *function* is simply a rule that turns each member of one set into a member of another set. Usually, you're dealing with numbers, and a function turns one number into another number. The number you want to find the function of is called the *independent variable.* The independent variable is also called the *input,* because you put it into the function to see what happens to it when it comes out the other side of the function.

The resulting *output* for the function is called the *dependent variable.* It's called the dependent variable because, as you may have guessed, the output of the function *depends* upon what the input is as well as what the function is.

The set of all possible values of the independent variable is called the *domain.* The corresponding set of all possible values of the dependent variable is called the *range.*

One thing to keep in mind about functions is that for each value of the input (independent variable), there is one and only one output (dependent variable). When two quantities or numbers are related to one another like this, the *dependent* variable is said to be a *function* of the *independent* variable.

Symbolism: Understanding SAT II Math Symbols

Functions can be displayed in any number of ways, especially on the SAT II Math test.

Sitting in: Symbols as functions

Some functions use any number of different types of symbols to show the relationship between two sets of numbers. Some examples of these kinds of symbols are #, $, &, @, and other even stranger ones that may look more confusing than the Greek alphabet. Don't fret when you see these unusual symbols. The SAT test-makers are throwing these extra red herrings in to throw you off balance. One such expression may appear on the exam like this:

$H(x) = 3x^2 + 5$. What is $H(8)$?

Again, the strange-looking symbol in the initial expression is nothing more than a different way of saying that the function of x is to take x squared, multiply the result by 3 and then add 5. In calculating that function exercise with, say, the number 8, you simply substitute the x with 8, like so.

$$\begin{aligned} H(8) &= 3(8)^2 + 5 \\ &= 3(64) + 5 \\ &= 162 + 5 \\ &= 167 \end{aligned}$$

See, that wasn't so scary after all! Now you try one.

Find &(3) if &(x) = $2x - 1$.

Once more, all you're doing is finding the value of the function of 3 by substituting 3 for x in the equation that gives you the functional relationship. Here, the functional relationship is to take the old number, x (or 3 where x is given a value), multiply it by 2, and then subtract 1. When you input 3 into the function in place of x, your work will look something like this.

$$\begin{aligned} \&(x) &= 2x - 1 \\ \&(3) &= 2(3) - 1 \\ \&(3) &= 2(3) - 1 \\ \&(3) &= 6 - 1 = 5 \end{aligned}$$

Keeping the status quo: "Normal" functions

More commonly, you'll see functions displayed with more conventional characters, such as the letters *f, F, g, G,* and ϕ (another Greek letter, phi). Any type of letter or symbol can represent a function.

So, for example, $f(x)$ is used to indicate the function of x, and is simply stated as "*f* of *x*."

WARNING!

Just a word of caution on this convention, the expression $f(x)$ is not the same as $f \times x$. It's merely standard notation for a function.

TIP

For any function *f*, $f(x)$ indicates the value of *f* at *x;* that is, the number that *f* is equal to when *x* is equal to a given number. This means simply that if you have a value of *x* as your *input* into the function, the *f* of *x* is the value of *f* (or the *y*-value on the vertical *y*-axis) as the *output* of the function. All the possible values for *x* are known as the *domain.* All the possible values of $f(x)$ are called the *range.* Here are a few examples of various functions.

$$f(x) = 2x^2 + 17$$
$$g(c) = (4c - 1)^3 - 3c^2 + 17c - 10$$
$$f(t) = \tfrac{1}{4}(t - 7)^2$$
$$g(q) = (3q - 5) \div (q + 1)$$

While all these equations may appear to be very, very different in looks, you treat them all pretty much the same. Just do the math! You get a value for the *old* number; that is, the value for the variable in the parenthesis to the left-hand side of each of the preceding equations. Then, you simply find the *new* number, or function, by plugging the old number into the equation that's given and see what the operation kicks out.

So, for the first equation, you are looking for the f of (x), and you are told that the value of $x = 12$. Therefore, the value of f(the function of x) is:

$$f(x) = 2x^2 + 17$$
$$f(12) = 2(12)^2 + 17$$
$$f(12) = 288 + 17 = 303.$$

When you do the math on this function, you'll find that the x-value is 12, and the y-value — the $f(x)$ — is 303. If you were to graph this on a coordinate plane (see Chapter 7), the ordered pair would be (12, 303). It's that simple.

Occasionally, you may be asked to find a *piecewise function.* A piecewise function is one that is broken into pieces, or different segments. It may also be called a *split function.* A piecewise function is not much different from a "normal" function. You determine the function depending on how you define the possible values of the domain. The function in each "piece" is continuous. Here are some examples of piecewise functions.

$$f(x) = \begin{cases} 2x - 1, x \leq 1 \\ x + 7, x > 1 \end{cases}$$

$$y(x) = \begin{cases} x^2, x \leq -2 \\ -x, x > -2 \end{cases}$$

Notice that in each of these piecewise functions, the function is split into pieces. The value of the y or f(the range) is determined by the value of x (the domain) just as with normal functions, but the y or f value of a piecewise function follows a different pattern depending on which of the two rules x falls under.

So, in the first piecewise function just given, if $x = 0$, then the first rule goes into play, and $f(0) = 2(0) - 1$, or simply $f = -1$. If $x = 2$, the second rule governs, and $f(2) = 2 + 7$, or, more simply, $f = 9$.

You can also manipulate the value of y in the second piecewise function equation just given in much the same way. If $x = -2$, then $y(x) = (-2)^2$, or simply $y = 4$. On the other hand, if $x = -1$, the second rules goes into play and $y(x) = 1$.

There's also another kind of strange-looking "normal" function you need to recognize. You should be able to recognize a *paired value* when it's used in a function. Suppose you come across something like the following:

$$t(m, n) = 4m^2 - 3n$$

This looks only slightly stranger than a normal function, so you should have slightly more than no trouble figuring it out, right? In the case of a paired function, you may be asked to find the value of the function of (5, 6), or find $t(5, 6)$. Just work this like you would a normal function. Plug the two paired values right into the spots where they belong in the equation of your function and you won't go wrong.

$$\begin{aligned} t(m, n) &= 4m^2 - 3n \\ t(5, 6) &= 4(5)^2 - 3(6) \\ &= 4(25) - 18 \\ &= 100 - 18 \\ &= 82 \end{aligned}$$

Many questions regarding functions may ask you to just plug in numbers to get the answer, but sometimes you'll be asked to plug in variables as well. Consider the following algebraic functions that will take an extra step or two to solve.

If $f(x) = (x - 4)^2$, find the value of $f(4x - 4)$.

(A) $16x^2 - 16$

(B) $16x^2 + 16$

(C) $16x^2 - 32x + 64$

(D) $16x^2 - 64x + 64$

(E) $16x^2 - 64x - 64$

This is really pretty simple as long as you don't try to do this in your head. Just use a systematic and logical approach, plug in the variables and numbers where they belong, and you're there! The first thing you need to do with this exercise is substitute $4x - 4$ in place of your x in the initial definition of the function. Thus, $f(x)$ becomes $f(4x - 4)$. Then you get the following:

$$\begin{aligned} f(x) &= (x - 4)^2 \\ f(4x - 4) &= (4x - 4 - 4)^2 \\ f(4x - 4) &= (4x - 8)^2 \\ f(4x - 4) &= (4x - 8)\,(4x - 8) \\ f(4x - 4) &= 16x^2 - 32x - 32x + 64 \\ f(4x - 4) &= 16x^2 - 64x + 64 \end{aligned}$$

Your correct answer is D. If you chose A, all you did was square the two terms in the function, and you didn't multiply it out correctly. The answer B is similar, but the operation is just reversed. C is close, but you need to get the correct coefficient in the second term by adding the $32x$ twice to get the $64x$. Finally, E is wrong, because it doesn't switch the sign on the last term as must be done when multiplying two negatives together.

Here is one way to figure this one out if you're in a hurry and you don't want to mess around with all those x's and stuff. Just substitute a number into your equation for x and see what happens if you solve for the equation. You could pick an easy number like 2 and see what happens when you plug it in as the x-value in your $f(4x - 4)$. Thus, if $f(x) = (x - 4)^2$, find the value of $f(4x - 4)$. Plug the 2 in the second f expression.

$$\begin{aligned} &f(4 \times 2 - 4) \\ &f(8 - 4) = f(4) \end{aligned}$$

Now, you should plug your new number (4) into the original function.

$$f(4) = (4 - 4)^2$$
$$f(4) = (0)^2 = 0$$

Your original number 2 has become 0 if it's plugged into the original function. How do you know this is correct? You don't until you see whether this works by plugging that same number 2 into the answer choices to see which one comes up with 36. The only equation where that works, amazingly enough, is choice D. Try it.

$$16x^2 - 64x + 64$$
$$16(2)^2 - 64(2) + 64$$
$$16(4) - 128 + 64$$
$$64 - 128 + 64 = 0$$

That's probably not as much fun when your final answer is 0, is it? Here are some more function questions to see whether you're still having fun. But don't spend too much time on these on the SAT II test because they're not really that difficult. Work on them now mainly to see whether you can do what the SAT II folks want you to do. They mainly want you to plug the numbers in correctly. The answers should flow from that simple operation pretty easily.

Given $h(r) = \begin{cases} 4|r| \text{ if } r \geq 2 \\ -|r| \text{ if } r < 2 \end{cases}$, evaluate $h(-r)$ if $r = -7$.

(A) –28

(B) –14

(C) –7

(D) 7

(E) 28

Don't make the mistake of getting goofed up by the negative signs here. If $r = -7$, then $h(-r)$ is the same as saying $h(7)$.

Because 7 is greater than 2, look to the first rule of the function $h(r)$. You want to find the solution to $h(r) = 4 \times$ the absolute value of 7, or simply 4×7. Your answer for this one, then, is E. You can see that if you miss the sign here and how it translates into the function $h(r)$, you will get a negative of the correct answer, choice A, or any of the others if you follow the incorrect rule.

If $g(x) = x + 1$ and $h(x) = 2x - 1$, what is the absolute value of the difference between $g(2)$ and $h(2)$?

(A) –2

(B) –1

(C) 0

(D) 1

(E) 2

You can find the answer using simple algebra. Simply plug in the number 2 into each function to get the result of each operation. When you plug 2 into the first function, $g(x)$, you get $2 + 1 = 3$. When you plug it into the second function, $h(x)$, you get $2(2) - 1$, or 3. The difference is, quite simply, 0.

You know you can eliminate choices A and B right off the bat, because the question asks what the *absolute* value of the difference is. Absolute values are always positive. Your choices are narrowed already. Your answer choice should be 0 if you correctly subtracted 3 from 3. That was almost too easy, but you still get a gold star. If you chose D, you somehow subtracted 1 from 2, or 2 from 3. Go back and rework your math. If you ended up with E, you may have added the terms in your $h(x)$ function instead of subtracting them. Just be careful with your math.

Adding to the Fun: Compound Functions

Imagine what would happen if you could put double or even triple the "fun" into functions. What would happen if you tried to find out the function of a function? Well, that's exactly what is meant when you calculate a *compound function.* The way it works is to find the first function, and then plug that value into the second function. Take a look at these two functions.

$$f(x) = 3x + 1 \qquad g(x) = x - 4$$

Now, suppose you want to find $g(f(x))$. This function reads "find g of f of x." The way you solve this is to start with the first function you can work with. With typical algebra, you try to find the answer to f of x first, because the f of x is buried most deeply inside the parentheses. After you find the answer to (the output of) the f function, you need to plug it into the function g of x.

Take an example of a number that you may plug in. Say that $x = 2$. Now, you want to find $g(f(2))$. Broken down in steps, you first find $f(2)$:

$$f(x) = 3x + 1$$

$$f(2) = 3(2) + 1$$

$$f(2) = 6 + 1 = 7$$

You now know that $f(x) = 7$. Your next task is to find $g(f(x))$, so you are basically looking for $g(7)$. You can see that the number 7 is now the x that is your input in the g function. Plug 7 in and see what you get:

$$g(x) = x - 4$$

$$g(7) = 7 - 4$$

$$g(7) = 3$$

Easy as pie! Just remember to do the inside part of the compound function first, and then work your way to the outside. It's just a matter of finding these functions in the right order.

The preceding operation in finding $g(f(x))$ is not the same as finding $f(g(x))$. While you want to use the same two functions just given, you need to do them in the reverse order. Again, suppose that $x = 2$, and you need to find $f(g(x))$. Try it out.

$$f(g(x)) = ?$$

First, go to the inside of the parentheses and find out the g of x.

If $x = 2$, then

$$g(2) = 2 - 4 = -2$$

Now, you have the first part done, and see how easy that was! You're on a roll, so go for the second half. Because you've discovered that $g(x)$ is –2, you want to plug –2 as the *new* input or the new x into the second function $f(x)$.

$$f(x) = 3x + 1$$

$$f(-2) = 3(-2) + 1$$

$$f(-2) = (-6) + 1 = -5$$

So, your final answer is $f(g(x)) = -5$. This is quite different from $g(f(x)) = 3$. We can't stress enough how important it is to keep the two functions straight when you're doing these calculations for compound functions. Try one or two yourself.

If $t(x) = 3x - 12$ and $r(x) = (\frac{1}{3})x + 4$, what is $t(r(x))$ where $x = -1$?

(A) –15

(B) –2

(C) –1

(D) 2

(E) $3\frac{2}{3}$

This problem requires you to do the function formula twice. First, you go to the inside parentheses. You need to figure out what the answer to $r(x)$ is and, from there, it goes pretty smoothly with a couple of quick algebraic equations.

$$r(x) = (\tfrac{1}{3})x + 4$$

$$r(-1) = (\tfrac{1}{3})(-1) + 4$$

$$r(-1) = -\tfrac{1}{3} + 4 = 3\tfrac{2}{3}$$

You now have half the problem solved. To find out $t(r(x))$, simply plug the number $3\frac{2}{3}$ as a new x-value into the $t(x)$ function.

$$t(x) = 3x - 12$$

$$t(3\tfrac{2}{3}) = 3(3\tfrac{2}{3}) - 12$$

$$t(3\tfrac{2}{3}) = 11 - 12 = -1$$

Your correct answer is –1, so you should have chosen C. If you chose A, you found only $t(x)$, and not $t(r(x))$. If you chose B or D, you got the inverse of an incorrect calculation. If you chose E, you merely found $r(x)$, and not $t(r(x))$.

In this last problem, you end up with the same number you started with. If you had to find the function $(r(t(x))$, you may be surprised to find out that you would come out with, of all things, –1. This tells you that the two functions, $t(x)$ and $r(x)$ are *inverse* functions, and the subject of the next section.

Doing the Opposite: Inverse Functions

An *inverse function* is a function that does just the opposite of an ordinary function. There are a couple of different ways you can show an inverse function.

- If the function f is the inverse of the function g, and if $y = g(x)$, then $x = f(y)$. In other words, if you put x into the original function and you get y as an outcome, you can put y into the inverse function and get x as your outcome.
- Another way of showing inverse functions is to use the following notation. The inverse of the function $g(x)$ is the same as $g^{-1}(x)$. Take a look at how this works in practice.

$$g(x) = 4x - 3 \qquad g^{-1}(x) = (x + 3) \div 4$$

You can see what's happening here. When you take an inverse of a function, you do just the opposite of what you did to get the ordinary function. Try to prove it.

Suppose you are given the preceding function, $g(x) = 4x - 3$ and that $x = 10$. To solve, you use the rule for the function as follows.

$$g(x) = 4x - 3$$
$$g(10) = 4(10) - 3$$
$$g(10) = 40 - 3 = 37$$

Now take the output of the original function (37) and plug that number into the inverse of the original function. Thus, you have $g^{-1}(x) = (x + 3) \div 4$, where $x = 37$. Here's what you get.

$$g^{-1}(x) = (x + 3) \div 4$$
$$g^{-1}(37) = (37 + 3) \div 4$$
$$g^{-1}(x) = (40) \div 4 = 10$$

Get it? You're right back where you began before you started doing any of the functions. Simple as all that!

TIP

A function is no more than a rule, and an inverse function is no more than an undoing of that rule by going backward. Remember to undo everything by reversing the operations in exactly the opposite order. In the preceding example, the original function first multiplies by 4, and then subtracts the number 3. When doing the inverse, you need to first *add* the number 3, and *then* divide by the original coefficient 4.

WARNING!

Compare the standard notation for a function as $f(x)$ and the inverse function as $f^{-1}(x)$. Don't be fooled into thinking that the inverse function $f^{-1}(x) = 1/f(x)$. While the latter may be the same as the inverse of the *number* that you get when you find the original function of x, it's *not* the inverse function of $f(x)$. This is a common mistake; make sure you don't fall victim to this trap.

Doing inverse functions can seem complicated, but it isn't as long as you remember that you just take the opposite of the original operations, and you take them in reverse order from the original function. Here's another.

If $f(x) = (x + 17)/5$, then find $f^{-1}(x)$.

The initial function or rule requires you to add 17 and then divide by 5. To get the inverse of the original rule, you must now first multiply by 5, and then subtract 17. So, if you had $f(x) = (x + 17) \div 5$, and $x = 3$, you want to find $f^{-1}(x)$. Here's how.

$$f(x) = (x + 17) \div 5$$
$$f(3) = (3 + 17) \div 5$$
$$f(3) = (20) \div 5 = 4$$

Now find $f^{-1}(x)$, where $x = 4$. Just do the opposite of the original operations in the reverse order. First, multiply by 5, then subtract 17:

$$f^{-1}(x) = 5(x) - 17$$
$$f^{1}(4) = 5(4) - 17$$
$$f^{-1}(x) = 20 - 17 = 3$$

And that's how you get back to square 1!

These two exercises show you exactly what happens when you take the compound of two inverse functions and put them together. Here is how the combination of two inverse functions, $f(x)$ and $g(x)$, may appear as a compound function.

$$f(g(x)) = x$$

Another way to say the same thing is

$$f(f^{-1}(x)) = x$$

You can see that by placing two inverse functions together as a compound function, your result is that the two functions will *undo* each other, and you end up with the number you started with!

If $f(x) = (2x - 3) \div 2$ and $f(g(x)) = x$, what is $g(x)$?

(A) $2(2x - 3)$

(B) $2(4x - 6)$

(C) $2x + \frac{3}{2}$

(D) $(2x + 3) \div 2$

(E) $4x + 6$

Because you know that the compound function $f(g(x)) = x$, you should realize that $f(x)$ and $g(x)$ are inverse functions. From there, the answer is quite simple. The *rule* for $f(x)$ is to first multiply by 2 and subtract 3, and then divide your answer by 2. To get the inverse of that function — $g(f(x))$ — you need to first multiply by 2 and add 3, and then divide your answer by 2. This gives you an answer choice of D. Answers A, B, and E don't have any division in them, so they cannot be an inverse of an original function that involves multiplication. Choice C is close, but results when x is multiplied by 2, instead of adding 3 and *then* dividing by 2; the product is added to the *quotient* of 3 divided by two. It looks similar on its face, but the two sets of operations yield very different results.

If $g(x) = 2x^2 - 7x - 8$, find $g^{-1}(7)$.

(A) –2

(B) 2

(C) 4

(D) 5

(E) 7

Because the function $g^{-1}(7)$ is the inverse of $g(x)$, you should know that when you calculate the function $g(x)$, your answer is 7.

The best way to find the correct answer is by plugging your various options from the list of answers into $g(x)$, and then seeing what answer is produced. When you come up with the one that results in 7, you've hit the jackpot. It may well take you a couple of tries with the answer choices, but ultimately, you'll come up with 5 as the number to plug into the initial function that gives you 7. Therefore, your correct answer is D. The other answers just won't work.

Setting Limits: Domain and Range

Sometimes, the SAT II Math test may ask you questions that test your knowledge of how to determine what the extent of the domain or the range is for certain functions. These questions are not difficult, but you need to be aware of some basic rules of algebra in order to determine the proper limits of the domain and range.

Domain

The *domain* of a function is the set of all numbers that could possibly be an input (or *argument*) of a function. You normally think of the domain of a function including all real numbers, unless it is limited in some artificial way. The only limitation on whether a number is a real number is if it follows the rules of real numbers.

There are only a few limitations on what a real number cannot be.

- A real number can't be a fraction where the denominator is 0. Otherwise the number is undefined.
- A real number can't be an even-numbered root of a negative number. Otherwise, as you know, it's an imaginary number. Any number when squared or else taken to some other even power cannot result in a negative number.

What does this all mean? Simply that the domain of real numbers cannot contain a fraction where the denominator is 0 or a square or other even-numbered root of a negative number.

How does this work with functions? Suppose you have the following function.

$$f(x) = \frac{x+4}{x-2}$$

Normally, the domain of x in a function can contain an unlimited number of values. In the preceding example, though, there is a fraction in the function having the variable x in the denominator. Because your denominator can't add up to 0, the denominator of $x - 2$ cannot be equal to 0. This means that x cannot be equal to 2. In terms of functions, the domain of $f(x)$ is $\{x \neq 2\}$.

Look at another.

$$f(x) = \frac{x-3}{5x+1}$$

You have pretty much the same kind of situation. Because the value of the denominator cannot be 0, the value of $5x + 1$ cannot equal 0. This means that $5x$ cannot be equal to -1, and therefore, x cannot equal $-\frac{1}{5}$. In terms of functions, you would say that the domain of $f(x)$ is $\{x \neq -\frac{1}{5}\}$.

Take a look at functions involving square roots.

$$g(n) = 3\sqrt[4]{n+2}$$

In the preceding function, you have an even-numbered radical sign with the variable *x* under it. You know that the root of an even-numbered radical — in this case, the 4th root — cannot be a negative number. Otherwise, you won't have a real number as your final answer. That said, the number under the radical sign cannot be less than 0. So *n* must be greater than or equal to –2. The result is the domain of the function *g*(*n*) is $\{n \geq -2\}$.

One more example of how the domain is limited by what's under the radical sign

$$g(y) = 7\sqrt{-2y}$$

What's that, you say? There's *already* a negative number under the radical sign. Well, yes there *seems* to be, but take heart. In order to have a proper domain, you need to make sure the *product* of –2 and *y* is not negative. If *y* is a positive number, the number under the radical would be negative, and you don't want that. Because two negative numbers multiplied together become positive, the *y* in this case needs to be negative to make the domain viable. The resulting domain of the function *g*(*y*) is $\{y \leq 0\}$.

Sometimes, your function will be a bit more complicated, and you just need to do another step or two to figure what values to eliminate from the range. Consider the following function.

$$f(x) = \frac{3}{x^2 - 4x - 21}$$

In the preceding function, you need to make sure that nothing in the fraction causes the denominator to equal 0.

TIP

This problem is not as easy to figure out as the first two functions just given. The best way to figure out the range in this kind of function is to use good old factoring. This tells you how to find out what values of *x* can make each factor in the denominator be 0. And those are the terms you want to avoid in the domain.

$$f(x) = \frac{3}{x^2 - 4x - 21}$$

$$f(x) = \frac{3}{(x+3)(x-7)}$$

You can easily see that the two values of *x* that give you a 0 in the denominator are –3 and 7. If any one of the factors in the denominator is 0, then the entire denominator is 0. This means that the domain of *f*(*x*) is $\{x \neq -3, 7\}$.

There's another example where a polynomial in the function makes it a bit tougher to spot what's wrong with the potential domain. When you have a polynomial under a radical sign, be prepared. The following function is one such example.

$$t(a) = \sqrt{a^2 + 5a - 50}$$

You need to find out what values of *a* under the radical sign turn the entire value into a negative number. Remember, a negative number can't be a square root and still be a real number. So factoring saves the day.

$$t(a) = \sqrt{a^2 + 5a - 50}$$

$$t(a) = (a+10)(a-5)$$

That wasn't too bad, was it? You have to compare the variable a with two numbers, –10 and 5.

This process isn't quite the same as finding out how to make the factors become 0 as you did in the previous two functions involving fractions. It may be tempting to say that the range does not include the numbers –10 and 5. Nothing could be more fatal for the SAT II math test, though.

Keep in mind that when you multiply the two factors together, you cannot come up with a product that is a negative number. That means that the factors *must* either be both positive or both negative. The only way to make that happen is to make sure that a is either less than or equal to –10 or greater than or equal to 5. Thus, the domain of $t(a)$ is $\{a \leq -10\}$ or $\{a \geq 5\}$.

What is the domain of $f(x) = \sqrt[3]{-2x^2 + 5}$?

(A) $x \geq 0$

(B) $x \leq 0$

(C) All real numbers

(D) $-1.71 < x < 1.71$

(E) There is no solution for the domain of $f(x)$

Because your function gives you an odd-numbered root, you can have either a positive or a negative value under the radical sign. Thus, the laws of math do not limit you except that you need a real number to plug in as your domain.

If you decided on C, all real numbers, you would be in good company with those who got this answer correct. If you chose A or B, you are artificially limiting the domain of the function as it may be if you were looking for the square root instead of the cube root. If you answered D, you are getting hung up on finding the cube root of 5 and artificially limiting the domain. Finally, if you chose E, you simply gave up too soon and were looking for an easy way out.

Determine the domain of the function $f(x) = \dfrac{4}{x^2 - x - 2}$.

(A) $\{x \neq -1, 2\}$

(B) $\{x \neq 1, -2\}$

(C) $\{x = -1, 2\}$

(D) $\{x = -4, 2\}$

(E) $\{x \neq -4, 2\}$

This is simple algebra. You know the denominator cannot equal 0, so solve for x in that trinomial. You do that by factoring:

$$(x^2 - x - 2)$$

$$(x + 1)(x - 2)$$

$$x = -1, 2$$

Simple enough, right? If you went on to choose C as your answer, your factoring would have been absolutely right, but your answer would be 100% wrong. C gives you only the factors in the polynomial expression in the denominator. You're not finished.

–1 and 2 are the values of x that make the denominator equal to 0 and, therefore, they cannot be values in the domain. Thus, your correct answer is A. If you chose B, you had the factors switched around with the incorrect sign in front of them. If you chose D, you found the correct factors of the denominator, and then divided the numerator by each fraction. Again, not correct. Finally, if you chose E, you did a hybrid; you divided the numerator by the solution to the factors of the denominator. You also said your domain was not equal to either of those numbers, but this again misses the boat that the solutions to the factors themselves are the numbers you need to exclude from the domain.

Range

The *range* of a function is the set of all numbers that could possibly be an output of a function. In other words, if you think of the domain as the set of all possible independent variables to be put into a function, the range is the set of all possible dependent variables that can come out of any particular function.

TIP

Just as the domain of a function is limited by certain laws of mathematics, so, too, is the range. The rules of math that limit the range are few.

- An absolute value of a real number cannot be a negative number.
- An even exponent or power cannot produce a negative number.

How does that work in real life? Check out some situations where these rules come into play. Look at the following functions.

$$g(x) = x^2 \qquad g(x) = |x|$$

The one thing these two functions have in common is that the range of both of them is limited to the laws of mathematics. Each of these functions can only give you an output that is a positive number. Therefore, in each case, the domain of the function of g is greater than or equal to 0.

The notation format for this expression can be any of the following.

- The range of $\{g(x) \geq 0\}$
- The range of $g(x)$ is the set $\{g: g \geq 0\}$
- The range of $g(x)$ is $\{y: y \geq 0\}$

Suppose the SAT folks decide to throw a 180° curveball at you and do something like this.

$$g(x) = -x^2 \qquad g(x) = -|x|$$

Does this kind of negativity make you go berserk and want to storm out of the test center? We hope not!

TIP

All you need to realize is that the negative sign simply flips around the potential values of the range in the opposite direction. Thus, the new range of these functions is g is less than or equal to 0. This can be expressed as follows.

- The range of $\{g(x) \leq 0\}$
- The range of $g(x)$ is the set $\{g: g \leq 0\}$
- The range of $g(x)$ is $\{y: y \leq 0\}$

That last item reads: "The range of g of x is the set of all numbers y, such that y is less than or equal to 0." You see the variable y used to substitute for the function $f(x)$ or $g(x)$, because when you graph the function, you generally have an x- and a y-value.

Now, suppose you see a function that looks a bit scarier and has several operations involved.

$$f(x) = \frac{-x^2 - 7 \cdot |-x|}{-8}$$

Just look at this logically and ask yourself what happens to the x when it gets put into the function. You look at x^2 and know it can't have a negative value, so the lowest possible value of the range of $f(x)$ at this point is 0. But the negative sign in front of x^2 means that it now can't have a positive value. Whatever positive value it had before, the negative sign took care of that and flipped it around. So at this point, the range of $f(x)$ becomes *less than* or equal to 0.

Moving right along, you can see that the absolute value of $-x$ must be positive, so you subtract 7 times what must be a positive number from your initial negative value of $-x^2$. Therefore, the numerator must be a negative number for any value of x. Now you divide the whole thing by the denominator, –8. Of course, when you divide a negative number by a negative number, you end up with a positive number.

Go ahead and plug in a couple of different values for x. Whether you choose a positive or negative value for x, your final answer comes out positive The lowest non-negative number you could possibly get is 0. This means that the lowest possible value of the range of $f(x)$ is either 0 or some positive number.

The way to express this range is

the range of $f(x)$ is $\{f(x) \geq 0\}$

You may well be asked on the SAT II Math test to pick out a possible range from a given function. Try this one on for size.

What is the range of the function $g(x) = 1 - \sqrt{x - 2}$?

(A) $g(x) \geq -2$

(B) $g(x) \leq -2$

(C) $g(x) \geq 2$

(D) $g(x) \geq -1$

(E) $g(x) \leq 1$

It's very easy to get tricked and start looking for the *domain* when you should be finding the *range.* If you chose C, you were thinking about the range of x. If you chose A or B, you were hung up on the negative sign when trying to get the number under the radical to be a non-negative number. If you chose D, you somehow got the number 2 out of the radical and subtracted 1. The correct answer is E, where $g(x) \leq 1$.

First, make sure you have figured out how to make the radical a real number. You must have at least 0 under the square root sign to have a real number, so x must be at least 2. If x is 2, the function would be 1 – 0, or simply 1. Any higher value for x results in a lower value for the output of the function. Thus, $g(x) \leq 1$.

What is the range of $f(x) = -\left|-x^3\right| - x^2$ if the domain of x is all negative numbers?

(A) $\{f(x) < 0\}$

(B) $\{f(x) > 0\}$

(C) $\{f(x) \geq 0\}$

(D) $\{f(x) \leq 0\}$

(E) $\{f(x) = \text{all real numbers}\}$

This can almost be considered a trick question. Remember that 0 is neither negative nor positive, so the *domain* of x must be less than 0. That said, it's easy to figure out the range of the function as long as all those negative signs in the expression don't terrify you. It doesn't matter how great the exponent or how many negative signs there are inside the absolute value symbol. When you see that symbol, everything inside becomes positive. The negative sign outside the absolute value turns the term after it into a negative. Looking at what follows, you know that the x^2 can only be a positive number. So the bottom line is, you are subtracting a positive number from a negative number, and your result will be something less than 0. This gives you the correct answer A.

If you chose B, you got fouled up with all the negative signs along the way. Just remember to take a deep breath when you see a pile of negative signs and be systematic about it. Start with the inside number or variable and work your way out. If you chose C as your answer, you not only got fouled up with all the signs, you ignored the fact that the range cannot equal 0 because your domain is less than 0. Remember, 0 is not a negative number. And if you chose D, you again did not exclude 0 from the domain and range. E is wrong because the limitation on your domain to negative numbers has limited your range accordingly.

If $f(x) = (-x^2 - 7) \div 10$, what is the range of f/x? (Consider using a graphing calculator for this problem.)

(A) $\{y: y = -0.7\}$

(B) $\{y: y = 0.7\}$

(C) $\{y: y \leq -0.7\}$

(D) $\{y: y \geq 0.7\}$

(E) $\{y: y \leq -0.8\}$

You know that x^2 must be a positive number because any number squared yields a positive result (or 0). The negative sign in front of x^2 turns the positive number into a negative number (or 0). After you subtract 7, your numerator is now certainly negative, no greater than –7. And when you divide the negative numerator by the positive denominator 10, you end up with a negative number, –0.7 at the most. This means that the range of the function is either less than or equal to –0.7, and your correct answer is C.

If you chose A as your answer, you may have been thinking that you were trying to solve for the greatest value of y without solving for the range. If you chose B, you thought you were solving for the lowest value of y, because you did not flip the sign correctly in the numerator. If you chose D, you got a range value, but you did not flip the sign correctly. And if you chose E, you probably did not consider 0 as a possible value of x and ended up going to the next integer, thus throwing your answer off by 1.

Functions with intervals

You may occasionally come across a question on the SAT II Math test that gives you a set of values, or an *interval,* for the variables that can be included in the domain of the function. You then have to come up with the set of values for the range of that function. The following question illustrates the type of problem you may encounter that gives the interval of the domain and asks you to find the interval of the range.

> A person can earn \$5.00 an hour doing janitorial work. What is the set of values that represents how much money that person can earn in any one day?

In this question, you can say that the amount of money that the person earns is a function of the number of hours in the day that he or she works. Thus:

$$f(x) = 5x$$

This means that x is the variable representing the hours that a person works, and for every hour (x) that the person works, he or she earns \$5.00. The fact that there are only 24 hours in a day means that the set of values in the domain is limited to the set $\{0 \le x \le 24\}$, which is an interval of x-values.

Another way of asking to find the range within a certain interval of the domain is

If $f(x) = 5x$ for $[0, 24]$, then what is the set that represents the range of $f(x)$?

The question asks you to find the different values of the range when you are given the interval of values from 0 to 24 inclusive as your domain. Finding the set that includes the values of the range is a simple matter of multiplying 5 (that is, \$5.00/hour) times the different amounts of hours in a day (the x-value) that the person could possibly work, which is the interval or artificial limitation on the domain. The answer is that the person can earn between \$0 and \$120 in a day. You get that when you plug the upper and lower limits of the hours in a day into the function that multiplies those values times 5. You express the interval for the range of this function like so:

$$\{y: 0 \le y \le 24\}$$

The preceding example is a simple illustration of how you can find and express the values of the range as limited by the interval of values for the domain. The questions on the SAT II Math test will probably not be nearly so easy, which means more work for you. It's not too bad, though, because these types of exercises build on what you already know about the limitations on domains and ranges.

If $f(x) = \dfrac{1}{x^2 - 5x - 14}$ for $[3, 10]$, then what is the range of $f(x)$?

(A) $\{y: -0.03 \le y \le 0.03\}$

(B) $\{y: -0.03 \le y \le 0.05\}$

(C) $\{y: -0.05 \le y \le 0.03\}$

(D) $\{y: -0.14 < y < 0.50\}$

(E) $\{y: -0.50 < y < 0.14\}$

This question gives you a red herring. Because you have a fraction, your natural inclination may be to solve for x by factoring $x^2 - 5x - 14$ to find out what's going to give you a 0 in the denominator. When you factor the expression in the denominator, you get $(x + 7)$ and $(x - 2)$. You may be thinking to yourself that you should not have $x = -7$ or 2, because either of those numbers would result in one of your factors being 0, and then you would have a 0 in the denominator. This strategy may be okay if you were trying to find the *domain* of x, but in this case you are looking for the *range*. In fact, the information in the problem tells you that the lower and upper bounds of the domain are 3 and 10, respectively. If you took the bait and went barking up that tree, you may have decided to eliminate $-\frac{1}{7}$ and $\frac{1}{2}$ from the range. If you had done so, you may have chosen either D or E as your answer because those two choices are variations on the theme of eliminating $-\frac{1}{7}$ and $\frac{1}{2}$ from the range.

Your best bet is to simply plug the upper and lower ends of the given range into the function.

First, plug the lowest value of the domain (3) into the function, and you get

$$f(x) = \frac{1}{x^2 - 5x - 14}$$

$$f(x) = \frac{1}{3^2 - 5(3) - 14}$$

$$f(x) = \frac{1}{9 - 15 - 14}$$

$$f(x) = -\frac{1}{20}$$

This value of the function, $-\frac{1}{20}$, can also be expressed as –0.05. Your next task is to plug in the high value of the domain of $x(10)$ and you end up with

$$f(x) = \frac{1}{x^2 - 5x - 14}$$

$$f(x) = \frac{1}{10^2 - 5(10) - 14}$$

$$f(x) = \frac{1}{100 - 50 - 14}$$

$$f(x) = \frac{1}{36}$$

You have now figured out your "high" value, which in this function turns out to be $\frac{1}{36}$, or 0.03. Not a large number, but at least it's a positive number and it's higher than the low range value you got by plugging in the number 3. Thus, the range y is greater than –0.015, and it is also less than 0.03. A very slim range, but one that is called for by the problem. The correct choice is, therefore, C. If you chose A or B, you found a variation on this theme, but the signs somehow got flipped.

Lining Up: Graphing Functions

You should be able to use algebra to recognize and determine functions along with their domains and ranges, but you should also be familiar with functions as they relate to graphs of functions. By looking at a graph of a function, you should be able to tell something about the function itself. You should also be able to look at the graph of a function and determine something about the domain or range of the same function. You may be asked to look at the graph of a function and determine whether a statement about the function is true or false.

TIP

When you graph a function $f(x)$ on the coordinate plane, you'll notice that the x-value of the function (the input or the domain of the function) goes along the x- or horizontal axis. And you chart the $f(x)$ value of the function along the y- or vertical axis. Anytime you see a coordinate pair that represents a function, for example (x, y), the x-value is the domain, or input, of the function, and the y-value is the output, or range, of the function.

The vertical line test

Remember that a function is a distinct relationship between the x (input) value and the y or $f(x)$ (output) value of the function. For every x-value, there is a distinct y-value that will be different from the y-output value of any other x-value. The vertical line test is one way to look at a graph and tell whether it is a graph of a function. The *vertical line test* says that no vertical line intersects the graph of a function at more than one point.

For example, the graphs in Figure 5-1 show a line that passes the vertical line test and is, therefore, a function.

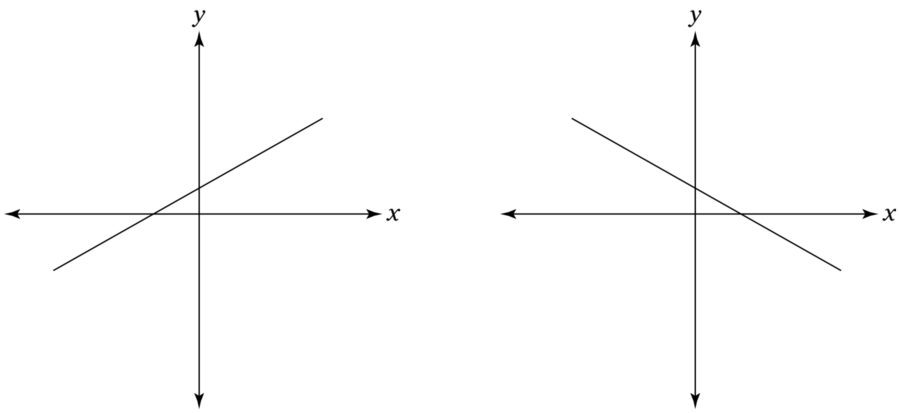

Figure 5-1: Vertical line test #1.

The two lines in Figure 5-1 go on in both directions to infinity. For each graph, you could draw a vertical line and the line would only intersect the line in the graph at one point. Thus, the lines in these graphs are functions because they pass the vertical line test. For every *x*-value along the line in each of these graphs, there is a separate and distinct *y*-value that corresponds to it. You know that they are both functions.

TIP

You probably already knew that most lines are functions when you think about the equation of a line, $y = mx + b$, but now you can see it for yourself graphically.

The difference between the two lines in Figure 5-2 is obvious. One line is horizontal, while the other is vertical. The graph of the horizontal line is $y = 2$, and the graph of the vertical line is $x = 2$. While these are both technically lines, only one of them is a function. Can you guess which one? Remember the vertical line test. The second graph in Figure 5-2 with the vertical line fails that test (miserably!). There are bazillions of *y*-values along the line having only one *x*-value. The vertical line in Figure 5-2 is not a function.

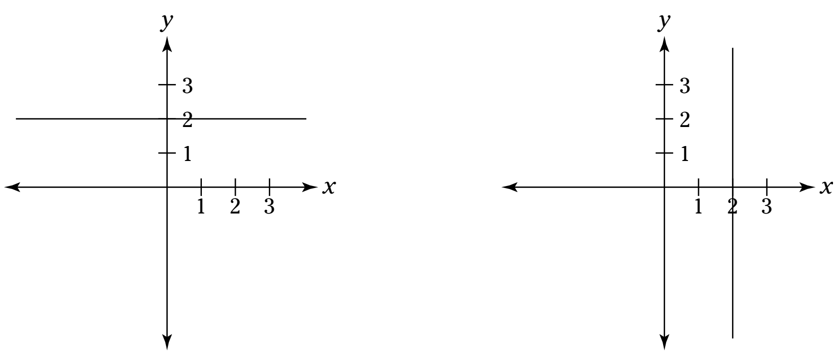

Figure 5-2: Vertical line test #2.

Take a look at the graphs in Figure 5-3 and decide which sets of points are functions.

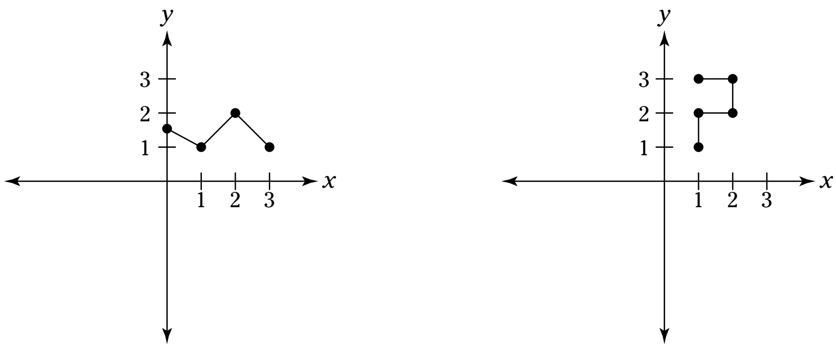

Figure 5-3: Vertical line test #3.

As you can see, the first graph in Figure 5-3 shows a function, because each *x*-value has a separate and distinct *y*-value. However, the second graph shows several points that have more than one *y*-value for a discrete *x*-value. Thus, the second graph in Figure 5-3 fails the vertical line test and is therefore not a function.

Look at the curves in Figure 5-4. Which one passes the vertical line test?

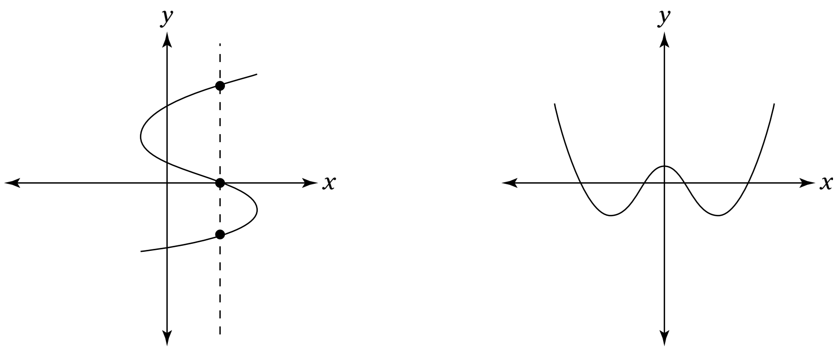

Figure 5-4: Vertical line test #4.

The first graph in Figure 5-4 obviously does not pass the vertical line test. It shows where a vertical line crosses the curve in three separate places. The second graph in Figure 5-4, on the other hand, shows a curve wherein, while the *y*-value may be repeated in places, there is still only one distinct *y*-value for every *x*-value on the graph. Thus, the second graph is a function, and it certainly passes the vertical line test.

Take a gander at a couple more graphs of curves in Figure 5-5.

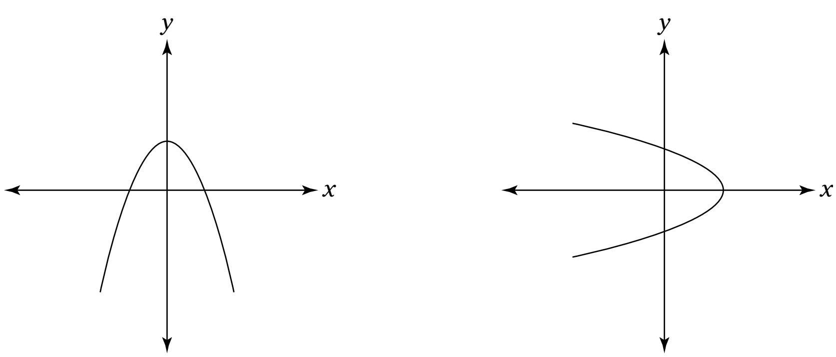

Figure 5-5: Vertical line test #5.

The curves in Figure 5-5 look like parabolas. The first curve opens downward, so it goes on to infinity downward and outward. For every *x*-value on that curve, there is a separate and distinct *y*-value, although there is an upper limit to the *y*-values at the vertex of the parabola. Thus, it passes the vertical line test and is, therefore, a function. The second curve is almost like the first one, except that it opens sideways. This means that a vertical line can cross the path of this curve in an infinite number of places. Therefore, this curve is not a function.

The first picture in Figure 5-6 is a function. It has one distinct *y*-value for each *x*-value, although the link between each piece is broken. This is a graph of a piecewise function. The value for the dot that is filled in is a point on the graph, while the value for the point with the hollow dot represents a point that is excluded from the function. The second graph in Figure 5-6 is not a function. Although nearly every point that represents a value for *x* has only one corresponding *y*-value, there is one place where the *x*-value has two corresponding *y*-values. This means the second curve in Figure 5-6 does not pass the vertical line test.

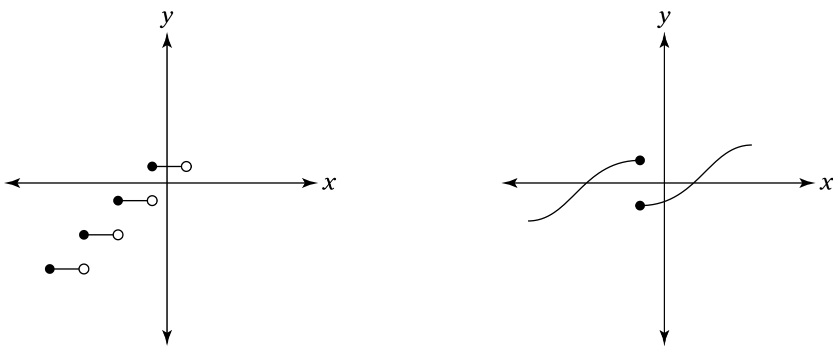

Figure 5-6: Vertical line test #6.

What's different about the two graphic curves in Figure 5-7? They both may represent some humongous math equations, but the first curve in Figure 5-7 is *not* a function. Why? The very first part of the first curve shows some area that is virtually straight up and down. Hence, it doesn't pass the straight-line test. That's what keeps it from being a function. The curve on the right, in fact, *is* a function.

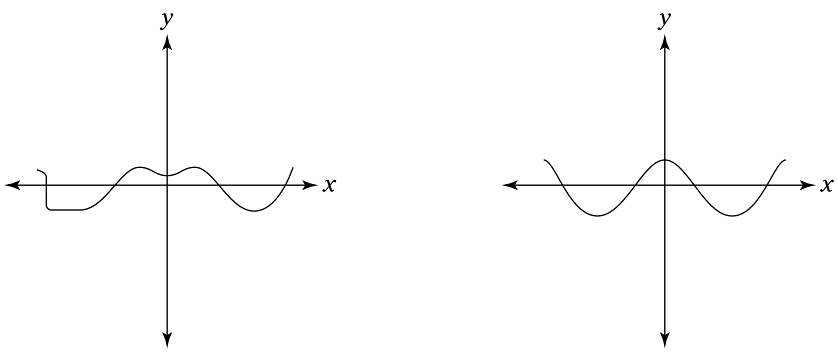

Figure 5-7: Vertical line test #7.

Which of the following graphs in Figure 5-8 is not a graph of a function?

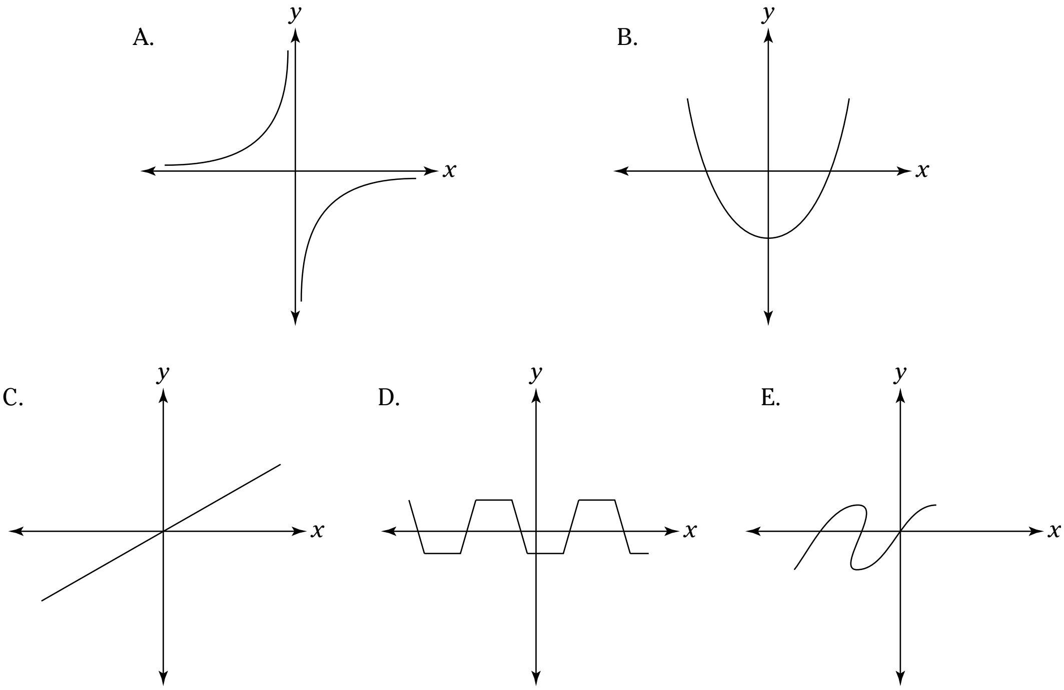

Figure 5-8: Vertical line test #8.

This kind of question is relatively easy, and you just have to use the vertical line test to see which graph fills the bill. Choice E is the correct answer, because it is a curve that sort of doubles back from right to left. It's possible for a vertical line to intersect that curve at more than one point. The other graphs in this question show curves, lines, or some other function where a vertical line never touches the figure at more than one point. Thus, they all pass the test and, of course, are functions.

Which of the following graphs in Figure 5-9 is not a graph of a function?

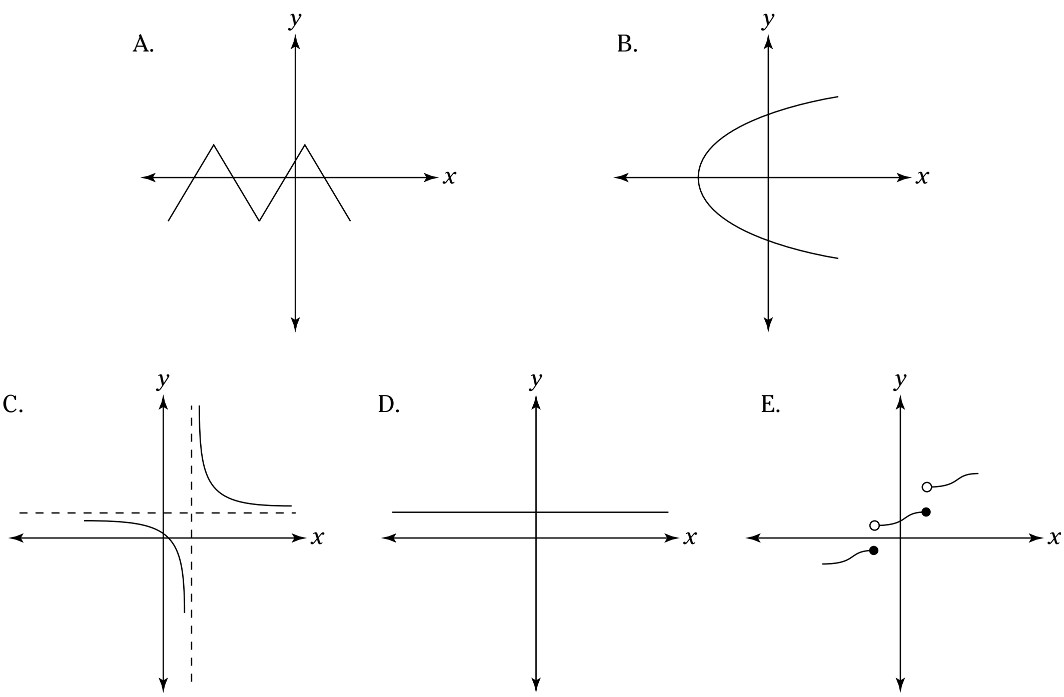

Figure 5-9: Vertical line test #9.

TIP

You need to dust off the old eyeglasses and see which one of these graphs does not have a vertical line cutting through more than one point on the curve or line. While some of these look pretty close, there is only one graph where a vertical line can cut through it in more than one place, and that's choice B. While a parabola that opens upward or downward can be a function, if you have one that opens to the left or right, it can never be a function. There are an infinite number of points that the vertical line can cut through the graph. So it stands out from the other graphs in this exercise as not being a function.

Finding domain and range from a graph of a function

You should be able to look at a graph of a function and have a pretty good idea of what the domain and the range of that particular function are. The SAT II Math test may occasionally show you a graph of a function, and you're expected to determine the effective domain and range. Take a look at some of these graphs.

In Figure 5-10, you can see that the function is a parabola, and its vertex is the coordinate point (0, 2). The graph extends outward from side to side so this function contains all possible values of x and, therefore, its domain is "all real numbers." The graph also extends downward to infinity, but because the y-value in this function is limited on the upward side and does not extend above the point (0, 2), its range is $\{y: y \leq 2\}$

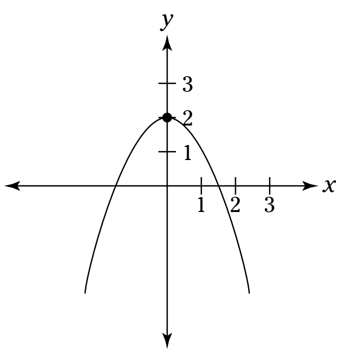

Figure 5-10: Finding domain and range #1.

Take a look at another graph and see whether you can find the possible domain and range. You can see from Figure 5-11 that the straight line goes on forever from left to right. This line also extends upward on the left side to infinity and downward to infinity on the right-hand side. Thus, the domain and range of this linear function is "all real numbers." There is no artificial limit to the x- and y-values in this graph.

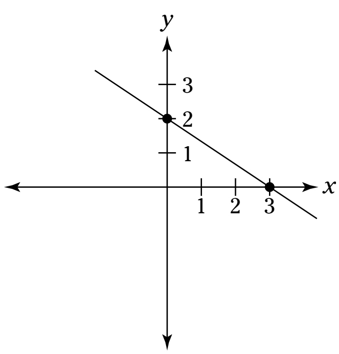

Figure 5-11: Finding domain and range #2.

In Figure 5-12, the horizontal line goes off into infinity from right to left, but it has only one value on the vertical or *y*-axis. Its *y*-value is limited to –3. Of course the equation for this line is $y = -3$. Because the line goes on forever from left to right, it includes every *x*-value there can possibly be. Thus, the domain of this linear function is "all real numbers." The range is limited to simply $\{y: y = -3\}$.

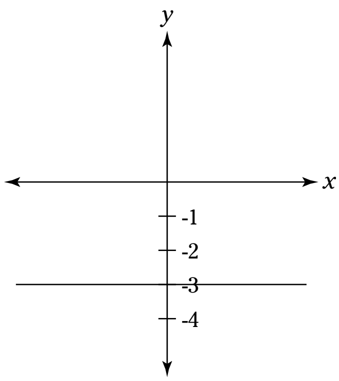

Figure 5-12: Finding domain and range #3.

Evaluating asymptote lines

The equation for the function in Figure 5-13 is $f(x) = \frac{1}{x-1}$. As you can see, this hyperbola approaches two asymptote lines. An *asymptote* is a straight line that the function comes very close to but never quite touches. One asymptote line in this function is the *x*-axis, while the other asymptote is $x = 1$. As a consequence, both the domain and range in this function are limited. The *x* in the function (the domain) never equals 1, and the *f*(*x*) — the *y*-value or the range — never equals 0. Thus, the domain is $\{x: x \neq 1\}$. The range is $\{y: y \neq 0\}$.

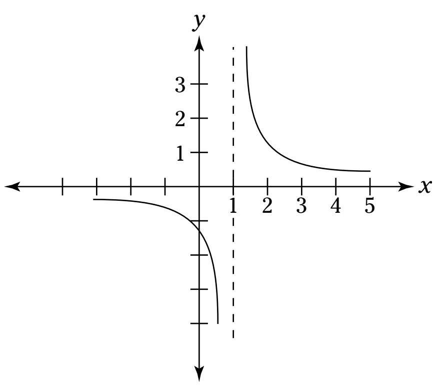

Figure 5-13: Asymptote line.

TIP

Here are some tips for working with asymptotes:

- If you have trouble figuring out an asymptote, try using your graphing calculator. After all, if you're allowed to bring them in for the test, put 'em to use.
- You can also plug in the numbers from your answer choices to see which ones fill the bill.
- If there is a vertical asymptote, the *x*-value is limited, and your domain is likewise limited. You have to exclude the *x*-value of the asymptote from your domain.

- If there is a horizontal asymptote, the *y*-value is limited, and that means your range is likewise limited. You must exclude that *y*-value from your range.
- If a function can go "to infinity and beyond" in any direction, its domain and range consist of "all real numbers."
- If a function cannot go on forever in any direction, its domain or range is limited.

EXAMPLE

Which of the following could be the range of the function of the graph shown in Figure 5-14?

(A) $\{y: y \leq 0\}$

(B) $\{y: y \geq 0\}$

(C) $\{y: y = 0\}$

(D) $\{y: y \neq 0\}$

(E) $\{y: y < 0\}$

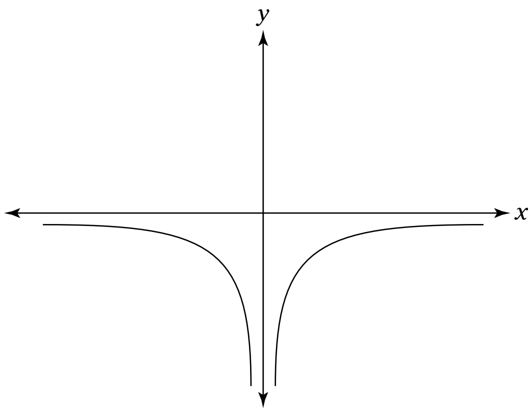

Figure 5-14: Determine the range of the function for this graph.

TIP

Just by looking at the graph of this function you know that the range (the *y*-value) is going to end up being less than 0.

That knowledge easily eliminates answers B and C. (C is the value of the horizontal asymptote line in this function.) If D were true, the graph would extend both above and below, but not touch, the *x*-axis. So that answer is out of the question as well. That leaves either A or E. Because the *x*-axis is an asymptote line for this function, the range does not include the *y*-value of 0. Thus, you've eliminated A from your choices. You are left with E, and, of course, the range in this function is less than 0.

EXAMPLE

Which of the following could be the domain of the function of the graph shown in Figure 5-15?

(A) $\{x: x \neq 0\}$

(B) $\{x: x \neq 3\}$

(C) $\{x: x = 0\}$

(D) $\{x: x \leq 3\}$

(E) $\{x: x < 0 > x\}$

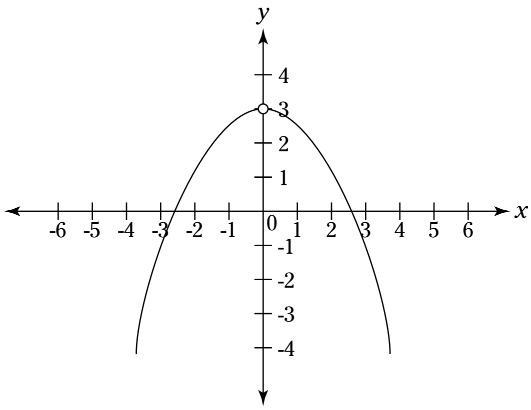

Figure 5-15: Determine the domain of the function for this graph.

Remember that this question is asking for the *domain* and not the *range,* so the fact that the upper limit of the *y*-value is just shy of 3 should not distract you from looking for all the possible *x*-values that make up the domain. You should drop any answer choice that references the value 3 like a hot potato. So get rid of answers B and D right away. They are red herrings. The circle around the point located at (3, 0) on the graph means that you don't count that point in your answer. This means that choice C is exactly the opposite of what you are looking for. C limits your domain to only one value, and that is 0. In this function, the value of 0 is, in fact, *excluded* from the function, so throw out C. Choice E is nonsensical from just about any standpoint. Set your sights on A as the answer of the hour here. The domain, or *x*-value, is not equal to 0.

Which of the following lines is an asymptote of the graph of function $y = 1 \div (x - 2)$?

I. $x = 2$

II. $y = 0$

III. $y = 1$

(A) I only

(B) II only

(C) III only

(D) I and II only

(E) I and III only

Your graphing calculator comes in very handy here, or you can draw a graph of the function if you have time. This function looks something like Figure 5-16.

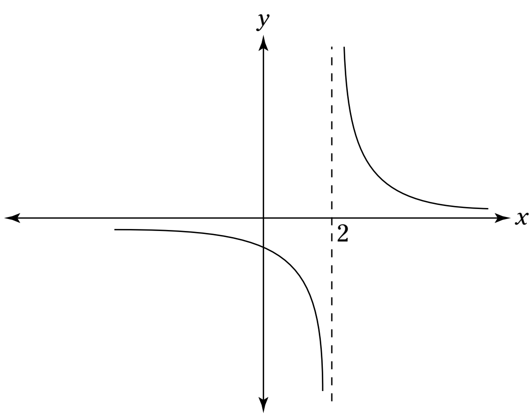

Figure 5-16: Graph of the function.

This kind of function, which is basically a quotient involving polynomials in the fraction is called a *rational function.* Seeing this function in graphic form, it's very easy to see that the your asymptote lines are $x = 2$ and $y = 0$. The short answer to the problem is D. To solve the problem algebraically, just plug in some of the answers and see what works. If you plug in 2 for x, you'll find that the denominator of the function is equal to 0, which is undefined. That means you have your asymptote line that limits your domain. The correct answer choice must have I as one of the solutions. Exclude choices B and C.

TIP

Now try giving x a very large value, such as 100, and then plugging it into the function. You can see that y begins to approach 0, so you can eliminate any answer that does not have II as a solution, and out go choices A, C, and E. The only one left for you to pick is D. You're on your way to the next one. Aren't these getting terribly easy?

Finding the roots of a function

Every once in awhile, you may be asked to find the roots of a function in a graph. The *root* of a function is the solution to the function that makes the equation equal to 0. Roots are also called *zeroes.* A zero or root of a function is a point where the graph of the equation or function intersects with the x-axis. At any such point, the y-value is 0, and that's why they're called zeroes.

When you see a function written out in algebraic form, you can "solve for x," and find out where the graph of the line or curve crosses the x-axis. Thus, you can match up a graph with a particular function on the SAT II Math test.

Suppose you get hit with the following function and need to recognize the graph of such function.

$$f(x) = x^3 - 3x^2 - x + 3$$

The first thing you should do is factor the function and see what you get.

$$f(x) = x^3 - 3x^2 - x + 3$$
$$f(x) = x^2(x - 3) - 1(x - 3)$$
$$f(x) = (x^2 - 1)\ (x - 3)$$
$$f(x) = (x - 1)\ (x + 1)\ (x - 3)$$

Now that your initial equation is factored into something more workable, you can more easily associate this function with the following graph.

See how the graph in Figure 5-17 of the preceding function intersects with the x-axis at three distinct points? Those points of intersection are the roots or zeroes of the function, that are, coincidentally, the solution you get when you "solve for x" in the equation. This means that the points of intersection with the x-axis give you the solution $y = 0$. As you can see, the curve crosses the x-axis at the points –1, 1, and 3, which are the *roots* of the function.

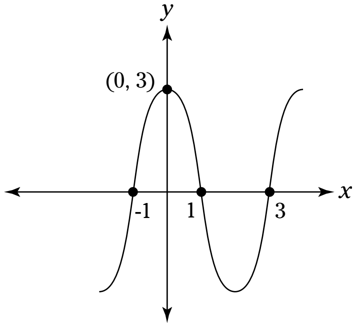

Figure 5-17: Looking for distinct roots.

How many distinct roots does the function $f(x) = x^3 - 6x^2 + 32$ have?

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Your first inclination may be to just say there are three distinct roots because this function is a polynomial of the third degree. Thus, you may choose D as your answer. Big mistake!

Your safest bet is to factor this polynomial into its component factors:

$$f(x) = x^3 - 6x^2 + 32$$
$$f(x) = (x^2 - 8x + 16)\,(x + 2)$$
$$f(x) = (x - 4)^2\,(x + 2)$$

This gives you two separate roots: –2 and 4. The correct answer is C, as there are two *distinct* roots. Choice E is impossible because a third degree function or polynomial may have a maximum of 3 roots, but it cannot have *more* than 3 roots. And if you chose A or B, you did not factor the polynomial down far enough.

Symmetry

You should also know that certain kinds of functions result in graphs that have symmetry in relation to the *x*- and *y*-axes. *Symmetry* means that the graph of a function has a mirror image of itself on either side of an axis. A parabola having its vertex on the *y*-axis is said to be symmetrical with the *y*-axis. Figure 5-18 shows these kinds of functions:

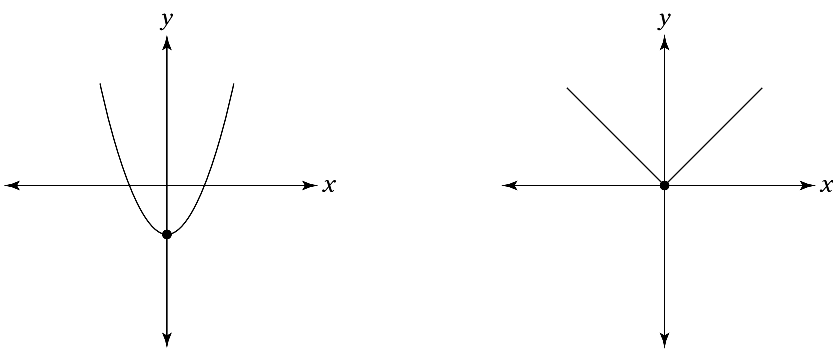

Figure 5-18: Symmetry.

A function that is symmetrical across the *y*-axis is an *even function,* because it has an even-numbered exponent. An example of an even function is $f(x) = x^2$ or $f(x) = \cos x$. An even function has the following property.

$$f(x) = f(-x)$$

If you plug any number into the *x* position of an even function, the *y*-value is going to come out the same regardless of whether you use a negative or positive number for your *x*-value.

An *odd function,* on the other hand, is not just some witty name you have for all the functions you've been studying. An odd function is one that has an odd-numbered exponent, such as $f(x) = x^3$ or $f(x) = \sin x$. An odd function has the following property:

$$f(x) = -f(x)$$

The bottom line of an odd function is that for any x-value put into the function, the result will be an opposite y-value. Odd functions have symmetry with the origin. Figure 5-19 illustrates some odd functions with origin symmetry:

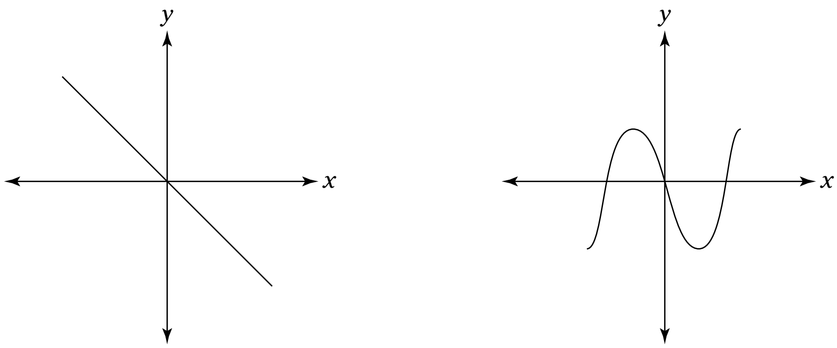

Figure 5-19: Odd functions.

As you can see the graph of these functions pictured in Figure 5-19 show a mirror image across the origin.

Occasionally, you may see a graph of an equation that appears symmetrical across the x-axis. This type of equation cannot be a function, because it would fail the vertical line test for functions. Any graph that has symmetry with the x-axis would have numerous places where a vertical line could intersect it at more than one point.

Periodic functions

LEVEL IIC
IIC

You may also possibly encounter a *periodic function* on the SAT II Math test, but only if you are taking the Level IIC test. A periodic function is simply a function that keeps repeating the same values over and over again. The measure of how often the function repeats itself is called a *period.* Figure 5-20 shows a periodic function. Coincidentally, this function also has origin symmetry:

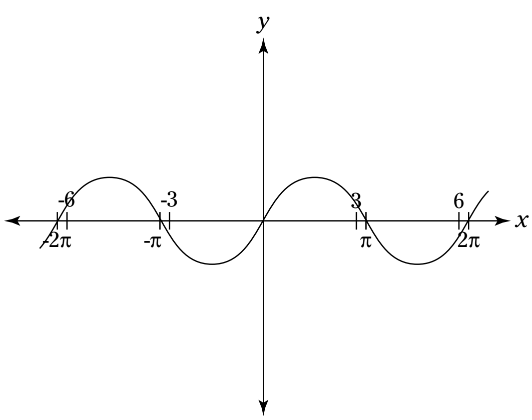

Figure 5-20: Periodic functions.

The equation of the periodic function in figure 5-20 is $y = \sin x$. It repeats itself in wave after wave, and the period of this function is 2π. On the Level IIC test, you should be able to spot the length of the period in a periodic function and address questions related to the period and its pattern.

Degrees of Functions

The *degree* of a function is the highest power (exponent) of any variable or term that occurs in the function. So for example, the function $f(x) = 2x^3 + 3x^2 + 6x - 2$ is a third-degree function. The function $g(x) = x^2 - 3x + 2$ is a polynomial function of degree 2.

The main thing you really need to know (and this is strictly for the Level IIC test), is that the *degree of the function* tells you *at most how many roots,* or zeroes, the function has. Keep in mind that the roots of the function can be distinct roots or they can be the same. To have distinct roots, the polynomial function needs to have separate and distinct solutions to the function.

The following polynomial function is a fifth-degree function having 5 roots, but it has only 3 *distinct* roots.

$$g(x) = 5x^5 - 10x^4 + 36x^3 - 54x^2 + 27x$$

That polynomial function looks a bit too large to attack all at once, so break it down into its component factors, like so:

$$g(x) = (x - 3)^3 (x - 1) x$$

In this more manageable state, you can see that there are a total of five roots to this function, but there are only three *separate and distinct* roots, while three of its roots equal 3:

$$g(x) = 0 \text{ when } x = 0, 1, 3, 3, \text{ or } 3$$

The graph of this function would look a lot different than it would if it had five distinct roots.

Another thing to be aware of is that the degree of a function tells you much about the shape of the function. A first-degree function; that is, a linear function, does not have any exponents greater than 1. Thus, a first-degree function is simply a straight line. It doesn't have a *minimum* or *maximum* value like, for example, a parabola. The graph of the first-degree function simply goes from one side of the graph to the other.

A parabola, on the other hand, is a second-degree function, based on a quadratic equation where the highest degree is the power of 2 in any one term. A parabola and other second-degree functions have a low point or high point. The *vertex* of the parabola is called the *extremum,* which is the minimum or maximum point for the function.

Figure 5-21 shows a parabola with an extremum that is a maximum value of the function. As you can also see, there is only one extremum with a parabola. This value is called the *global maximum* for the function. The graph of a second-degree function can have no more than one extremum. The graph of a third-degree extremum can have no more than two extrema. Two more extrema are often called *local extrema.* So, for example, the function in Figure 5-22 has a local minimum and a local maximum.

A *local maximum* is a point where the value of the function is higher than its immediate surrounding points. A *local minimum* is a point where the value of the function is lower than its surrounding points.

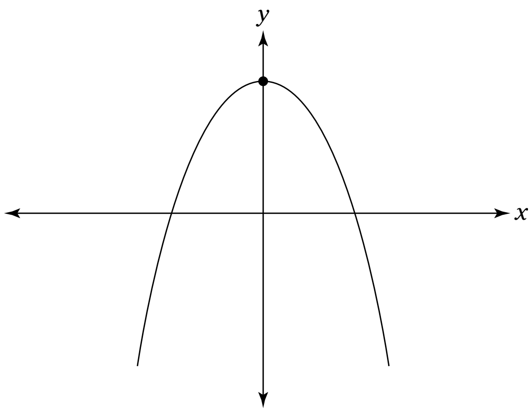

Figure 5-21: Parabola.

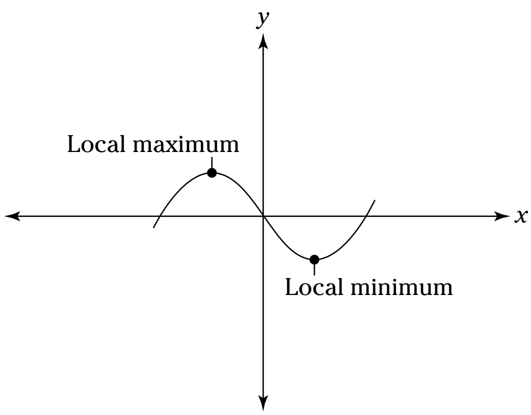

Figure 5-22: Local minimum and local maximum.

A fourth-degree function can have up to, but no more than, three extreme values.

Figure 5-23 illustrates what could be at least a fourth-degree function. You can see that it has three extreme values, with two local minimums and a local maximum. You may also notice that this function has four distinct zeroes; that is, four points where the graph of the function crosses the *x*-axis.

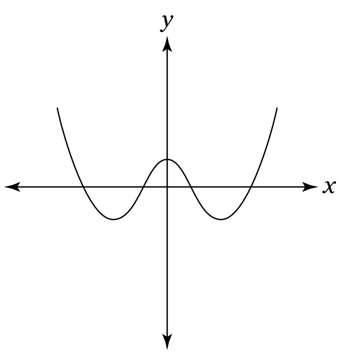

Figure 5-23: Fourth-degree function.