1

Electromagnetic Theory

The success of electromagnetic analysis during the past century would not have been possible except for the existence of an accurate and complete theory. This chapter summarizes a number of concepts from electromagnetic field theory used in numerical formulations for scattering problems. Differential and integral equations that provide the basis for many of the computational techniques are introduced. In addition, expressions are developed for calculating the scattering cross section of a target. The presentation is intended as a review of these concepts rather than an introduction, and the reader is encouraged to study references [1–11] for an in-depth discussion of this material.

1.1 MAXWELL'S EQUATIONS

Consider a source-free region of space containing an inhomogeneity characterized by relative permittivity ε_r and permeability μ_r, both of which may be a function of position (Figure 1.1). If this region is illuminated by an electromagnetic field having time dependence $e^{j\omega t}$, the fields in the vicinity of the inhomogeneity must satisfy Maxwell's equations

$$\nabla \times \bar{E} = -j\omega\mu_0\mu_r\bar{H} \tag{1.1}$$

$$\nabla \times \bar{H} = j\omega\varepsilon_0\varepsilon_r\bar{E} \tag{1.2}$$

$$\nabla \cdot (\varepsilon_0\varepsilon_r\bar{E}) = 0 \tag{1.3}$$

$$\nabla \cdot (\mu_0\mu_r\bar{H}) = 0 \tag{1.4}$$

where $\bar{E}$ and $\bar{H}$ are the electric and magnetic fields, respectively. (More precisely, $\bar{E}$ and $\bar{H}$ are complex-valued phasors representing the vector amplitude and phase angle of the time-harmonic fields.) We have specialized these equations to a medium that is linear and isotropic.

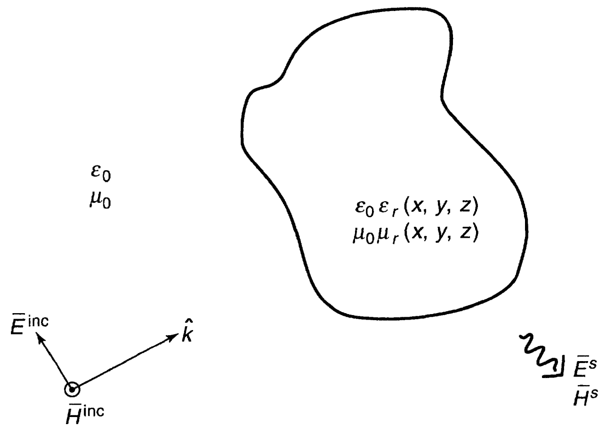

Figure 1.1 An inhomogeneity illuminated by an incident electromagnetic field.

The study of electromagnetics involves the application of Equations (1.1)–(1.4) to a specific geometry and their subsequent solution to determine the fields present within the inhomogeneity, the fields scattered in some direction by the presence of the inhomogeneity, or some similar observable quantity. The focus of this text will be the development of techniques for the numerical solution of Equations (1.1)–(1.4) or their equivalent.

Regions containing penetrable dielectric or magnetic material may be bounded by material having a very high conductivity, which is often approximated by infinite conductivity and termed a perfect electric conductor. Although ε_r and μ_r may in general be complex valued to represent conducting material, in the limit of infinite conductivity we denote the surface of a perfect electric conductor as a boundary of the problem domain. On such a boundary, the electric and magnetic field vectors satisfy the conditions

$$\hat{n} \times \bar{E} = 0 \tag{1.5}$$

$$\hat{n} \times \bar{H} = \bar{J}_s \tag{1.6}$$

$$\hat{n} \cdot \bar{E} = \frac{\rho_s}{\varepsilon_0 \varepsilon_r} \tag{1.7}$$

$$\hat{n} \cdot \bar{H} = 0 \tag{1.8}$$

where $\hat{n}$ is the normal vector to the surface that points into the problem domain (Figure 1.2), $\bar{J}_s$ is the surface current density, and ρ_s is the surface charge density.

Along an interface between two homogeneous regions specified by relative permittivity ε_r and permeability μ_r, appropriate continuity conditions involving the electric and magnetic fields are

$$\hat{n} \times (\bar{E}_1 - \bar{E}_2) = 0 \tag{1.9}$$

$$\hat{n} \times (\bar{H}_1 - \bar{H}_2) = 0 \tag{1.10}$$

$$\hat{n} \cdot (\varepsilon_{r1} \bar{E}_1 - \varepsilon_{r2} \bar{E}_2) = 0 \tag{1.11}$$

$$\hat{n} \cdot (\mu_{r1} \bar{H}_1 - \mu_{r2} \bar{H}_2) = 0 \tag{1.12}$$

where $\hat{n}$ is normal to the interface. Although the tangential components of the $\bar{E}$ and $\bar{H}$ fields are continuous, the normal components exhibit a jump discontinuity at material interfaces.

By combining Equations (1.1) and (1.2) and eliminating one of the fields, we obtain the "curl–curl" form of the vector Helmholtz equations

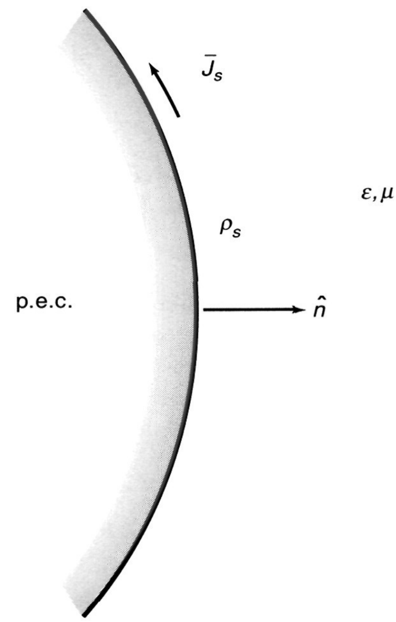

Figure 1.2 Electric current and charge density at the surface of a perfect conductor.

$$\nabla \times \left(\frac{1}{\mu_r} \nabla \times \bar{E} \right) - k^2 \varepsilon_r \bar{E} = 0 \tag{1.13}$$

$$\nabla \times \left(\frac{1}{\varepsilon_r} \nabla \times \bar{H} \right) - k^2 \mu_r \bar{H} = 0 \tag{1.14}$$

where $k^2 = \omega^2 \mu_0 \varepsilon_0$. The parameter k is known as the wavenumber of the medium (in this case, free space). We will consider several forms of these equations for application to two- and three-dimensional scattering problems.

Two-dimensional problems are those with invariance in the third dimension, such as an infinite cylindrical structure illuminated by a field that does not vary along the axis of the cylinder. Throughout this text, the term "cylinder" will be reserved for a structure whose geometry (ε_r, μ_r, and any conducting boundaries) does not vary with translation along z. If the cylinder axis lies along the $\hat{z}$ axis in a Cartesian coordinate system, it is usually convenient to separate the fields into transverse magnetic (TM) and transverse electric (TE) parts with respect to the variable z [2]. The $\hat{z}$-component of the magnetic field is absent in the TM case, while the $\hat{z}$-component of the electric field is absent in the TE case. Under the assumption of z-invariance, the $\hat{z}$-component of Equation (1.13) can be extracted and written in the form of a scalar Helmholtz equation

$$\nabla \cdot \left(\frac{1}{\mu_r} \nabla E_z \right) + k^2 \varepsilon_r E_z = 0 \tag{1.15}$$

In a two-dimensional problem, the TM part of the field can be found from the solution of Equation (1.15). Similarly, the TE part of the field can be found from

$$\nabla \cdot \left(\frac{1}{\varepsilon_r} \nabla H_z \right) + k^2 \mu_r H_z = 0 \tag{1.16}$$

Since the materials and the fields do not vary with z, all derivatives with respect to z vanish in the two-dimensional equations. After separate treatment, the TM and TE solutions can be superimposed to complete the analysis.

1.2 VOLUMETRIC EQUIVALENCE PRINCIPLE FOR PENETRABLE SCATTERERS [2, 7]

As alternatives to the differential equations presented above, integral equations are often chosen as the starting point for electromagnetic scattering analysis. To simplify the formulation of integral equations, it is convenient to convert the original scattering problem into an equivalent problem for which a formal solution may be directly written. One way of accomplishing this is to replace the inhomogeneous dielectric and magnetic material present in the problem by equivalent induced polarization currents and charges. Equations (1.1)–(1.4) can be rewritten to produce

$$\nabla \times \bar{E} = -j\omega\mu_0\bar{H} - \bar{K} \tag{1.17}$$

$$\nabla \times \bar{H} = j\omega\varepsilon_0\bar{E} + \bar{J} \tag{1.18}$$

$$\nabla \cdot (\varepsilon_0\bar{E}) = \rho_e \tag{1.19}$$

$$\nabla \cdot (\mu_0\bar{H}) = \rho_m \tag{1.20}$$

where

$$\bar{K} = j\omega\mu_0(\mu_r - 1)\bar{H} \tag{1.21}$$

$$\bar{J} = j\omega\varepsilon_0(\varepsilon_r - 1)\bar{E} \tag{1.22}$$

$$\rho_e = \varepsilon_0\varepsilon_r\bar{E} \cdot \nabla\left(\frac{1}{\varepsilon_r}\right) \tag{1.23}$$

$$\rho_m = \mu_0\mu_r\bar{H} \cdot \nabla\left(\frac{1}{\mu_r}\right) \tag{1.24}$$

Equations (1.17)–(1.20) describe the same fields as Equations (1.1)–(1.4) but appear to involve a homogeneous space characterized by permittivity ε_0 and permeability μ_0 instead of the original heterogeneous environment. The source terms compensate for the apparent difference between the two sets of equations, and we can think of these sources as replacing the dielectric or magnetic material explicit in (1.1)–(1.4). Since the two sets of equations are equivalent, we refer to the procedure of replacing the dielectric or magnetic material by induced sources as a *volumetric equivalence principle.* This type of equivalence principle will be used in the formulation of volume integral equations.

The sources of (1.21)–(1.24) radiate in free space. The task of finding electromagnetic fields in free space is much more straightforward than the original burden of solving Equations (1.1)–(1.4) directly in the inhomogeneous environment, and a solution will be presented in general form below (Section 1.4). However, at this point in the development the equivalent sources are unknowns to be determined, so the problem has not been solved by their introduction.

The expressions in Equations (1.23) and (1.24) require additional explanation. The quantities being differentiated, ε_r and μ_r, will be discontinuous at medium interfaces (including the scatterer surface). Therefore the derivatives in (1.23) and (1.24) must be interpreted in the context of generalized functions, that is, Dirac delta functions and their derivatives. Since ε_r and μ_r are constant throughout homogeneous regions of the original scatterer, Equations (1.23) and (1.24) show that there is no induced charge density in those regions. If ε_r and μ_r vary in a continuous manner and are differentiable in the classical sense, (1.23) and (1.24) produce the correct induced volume charge density. Furthermore, at an interface such as the scatterer surface, the Dirac delta function arising from the derivative signifies that there is actually a surface charge density rather than a volume charge density at that location. If (1.23) or (1.24) results in an induced surface charge density, the normal-field discontinuity produced in free space at that location can be shown to be identical to the discontinuity produced in the original scatterer by the material interface (Prob. P1.6). Therefore, when interpreted as generalized functions, the expressions in (1.23) and (1.24) conveniently account for all the possibilities. For additional information on generalized functions, the reader is referred to Chapter 1 of [12]. Throughout this text, the concepts of generalized functions are freely used wherever necessary. Problem P1.7 provides additional practice at manipulating delta functions and their derivatives.

Although we expressed $\bar{J}$ in terms of $\bar{E}$ and $\bar{K}$ in terms of $\bar{H}$ in Equations (1.21) and (1.22), we will have occasion to use the alternate forms

$$\bar{J} = \frac{\varepsilon_r - 1}{\varepsilon_r} \nabla \times \bar{H} \tag{1.25}$$

and

$$\bar{K} = -\frac{\mu_r - 1}{\mu_r} \nabla \times \bar{E} \tag{1.26}$$

These follow directly from Equations (1.21), (1.22), (1.1), and (1.2) and must also be interpreted as generalized functions.

1.3 GENERAL DESCRIPTION OF A SCATTERING PROBLEM [6, 7]

We are now in a position to describe one way of posing an electromagnetic scattering problem. Suppose the scatterer of Figure 1.1 is illuminated by a field produced by a primary source located somewhere outside the scatterer. We have shown that the inhomogeneous material can be replaced by equivalent induced sources radiating in free space. Consider splitting the fields into two parts, one associated with the primary source and another associated with the equivalent induced sources. The fields produced by the primary source in the absence of the scatterer will be denoted the *incident fields* $\bar{E}^{\text{inc}}$ and $\bar{H}^{\text{inc}}$. The secondary induced sources, which also radiate in free space, produce the *scattered fields* $\bar{E}^s$ and $\bar{H}^s$. The superposition of the incident and scattered fields yield the original fields in the presence of the scatterer. In other words, we can write

$$\bar{E} = \bar{E}^{\text{inc}} + \bar{E}^s \tag{1.27}$$

$$\bar{H} = \bar{H}^{\text{inc}} + \bar{H}^s \tag{1.28}$$

where the incident fields in the immediate vicinity of the scatterer (away from the primary source) satisfy the vector Helmholtz equations

$$\nabla^2 \bar{E}^{\text{inc}} + k^2 \bar{E}^{\text{inc}} = 0 \tag{1.29}$$

$$\nabla^2 \bar{H}^{\text{inc}} + k^2 \bar{H}^{\text{inc}} = 0 \tag{1.30}$$

and the scattered fields are solutions to the equations

$$\nabla^2 \bar{E}^s + k^2 \bar{E}^s = j\omega\mu_0 \bar{J} - \frac{\nabla\nabla \cdot \bar{J}}{j\omega\varepsilon_0} + \nabla \times \bar{K} \tag{1.31}$$

$$\nabla^2 \bar{H}^s + k^2 \bar{H}^s = -\nabla \times \bar{J} + j\omega\varepsilon_0 \bar{K} - \frac{\nabla\nabla \cdot \bar{K}}{j\omega\mu_0} \tag{1.32}$$

where $\bar{J}$ and $\bar{K}$ denote the equivalent sources defined in (1.21) and (1.22). Note that these sources are a function of the total fields $\bar{E}$ and $\bar{H}$. [As an alternate proof that the fields can be decomposed in this manner, combine Maxwell's equations for the incident field and primary source with Maxwell's equations for the scattered field and induced sources to obtain (1.1)–(1.4).]

In a source-free homogeneous medium, Equations (1.29) and (1.30) can be obtained from (1.13) and (1.14) using the vector Laplacian

$$\nabla^2 \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla \times \nabla \times \bar{E} \tag{1.33}$$

and Maxwell's divergence equations. The derivation of Equations (1.31) and (1.32), which includes sources, is slightly more complicated and will be left as an exercise (Prob. P1.8).

Although the incident field may be arbitrary and may in fact be produced by sources immediately adjacent to the scatterer or within the scatterer, our primary interest is the case of an excitation produced by some source in the far zone. Often, we will consider the incident field to be a uniform plane wave.

Radiation conditions ensure that the fields satisfying Equations (1.31) and (1.32) propagate away from the scatterer. In a three-dimensional problem, where r is the conventional spherical coordinate variable, radiation conditions have the form

$$\lim_{r\to\infty} \hat{r} \times \nabla \times \bar{E}^s = jk\bar{E}^s \tag{1.34}$$

$$\lim_{r\to\infty} \hat{r} \times \nabla \times \bar{H}^s = jk\bar{H}^s \tag{1.35}$$

In the two-dimensional case, these simplify to a form of the Sommerfeld radiation conditions

$$\lim_{\rho\to\infty} \frac{\partial E_z^s}{\partial \rho} = -jkE_z^s \tag{1.36}$$

$$\lim_{\rho\to\infty} \frac{\partial H_z^s}{\partial \rho} = -jkH_z^s \tag{1.37}$$

for the TM and TE polarizations, respectively, where ρ is the radial variable in cylindrical coordinates.

1.4. SOURCE–FIELD RELATIONSHIPS IN HOMOGENEOUS SPACE [1–7]

There are a number of ways to approach the solution of the Helmholtz equations (1.31) and (1.32) in homogeneous, infinite space. The classical approach is to express the fields in terms of the magnetic vector potential $\bar{A}$ and the electric vector potential $\bar{F}$, according to

$$\bar{E}^s = \frac{\nabla\nabla \cdot \bar{A} + k^2 \bar{A}}{j\omega\varepsilon_0} - \nabla \times \bar{F} \tag{1.38}$$

$$\bar{H}^s = \nabla \times \bar{A} + \frac{\nabla\nabla \cdot \bar{F} + k^2 \bar{F}}{j\omega\mu_0} \tag{1.39}$$

By substitution into Maxwell's equations, it is easily demonstrated that the vector potentials satisfy

$$\nabla^2 \bar{A} + k^2 \bar{A} = -\bar{J} \tag{1.40}$$

$$\nabla^2 \bar{F} + k^2 \bar{F} = -\bar{K} \tag{1.41}$$

A solution to these equations satisfying the radiation condition can be concisely written in the form

$$\bar{A} = \bar{J} * G \tag{1.42}$$

$$\bar{F} = \bar{K} * G \tag{1.43}$$

where the scalar function G is the well-known three-dimensional Green's function

$$G = \frac{e^{-jk|\bar{r}|}}{4\pi|\bar{r}|} \tag{1.44}$$

and the asterisk ($*$) denotes three-dimensional convolution, that is,

$$\bar{A}(\bar{r}) = \iiint \bar{J}(\bar{r}') \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} d\bar{r}' \tag{1.45}$$

The convolutional property of the solution is useful in a variety of ways and will be exploited in some of the numerical formulations to be presented in subsequent chapters.

In a two-dimensional problem, the integration over the third dimension only involves the Green's function and can be performed analytically. For generality, we first present the result [13, 3.876]

$$\int_{z=-\infty}^{\infty} \frac{e^{-jk\sqrt{\rho^2+z^2}}}{4\pi\sqrt{\rho^2+z^2}} e^{-j\gamma z}\, dz = \begin{cases} \frac{1}{4j} H_0^{(2)}\left(\rho\sqrt{k^2-\gamma^2}\right) & k^2 > \gamma^2 \\ \frac{1}{2\pi} K_0\left(\rho\sqrt{\gamma^2-k^2}\right) & \gamma^2 > k^2 \end{cases} \tag{1.46}$$

which may be useful if the geometry is z invariant but the excitation is not. In Equation (1.46), H_0 and K_0 are the zero-order Hankel and modified Bessel functions of the second kind, respectively. In the limiting case, as γ vanishes, we obtain the two-dimensional Green's function

$$G = \frac{1}{4j} H_0^{(2)}(k|\bar{\rho}|) \tag{1.47}$$

For two-dimensional problems, Equation (1.47) may be used within Equations (1.42) and (1.43) as two-dimensional convolutions.

To summarize, the above procedure requires that $\bar{A}$ and $\bar{F}$ be constructed by an integration of $\bar{J}$ and $\bar{K}$ according to (1.42) and (1.43). The electric and magnetic fields can then be produced by Equations (1.38) and (1.39), which involve differentiations. Unfortunately, the *integration-followed-by-differentiation* procedure dictated by these equations is not well suited for numerical implementation. The typical integrals arising from source–field relations can seldom be evaluated in closed form but usually must be evaluated at individual observation points by numerical quadrature algorithms. Both the accuracy and the efficiency of the computation will suffer if it is necessary to implement a subsequent derivative of the integral using finite-difference operations. In three dimensions, a finite-difference implementation of the second-order vector derivatives in (1.38) and (1.39) will require every vector component of each integral to be evaluated at a minimum of seven points around the desired location.

On the other hand, the free-space Green's function is easy to differentiate analytically. For observation points outside the source region, derivatives can be brought inside the integrals and carried out analytically, changing the procedure to one of *differentiation followed by integration*. The modified procedure eliminates the error introduced by the finite-difference operations and reduces the number of quadrature evaluations to one per integral. Because of the singularity of the Green's function in the region containing the sources, however, a direct interchange of integration and differentiation is not possible without violating Leibnitz's rule. Thus, unless an integral can be evaluated in closed form in the source region, an alternate approach may be necessary.

As an alternative to the pure vector potential source–field relationship, a mixed-potential formalism can be developed by seeking a solution of the form

$$\bar{E}^s = -j\omega\mu_0\bar{A} - \nabla\Phi_e - \nabla \times \bar{F} \tag{1.48}$$

$$\bar{H}^s = \nabla \times \bar{A} - j\omega\varepsilon_0\bar{F} - \nabla\Phi_m \tag{1.49}$$

where Φ_e and Φ_m are scalar potential functions. By carrying out a solution procedure similar to that employed above (Prob. P1.11), $\bar{A}$ and $\bar{F}$ can be shown to be the identical convolution expressions appearing in Equations (1.42) and (1.43). The scalar potentials are given by

$$\Phi_e = \frac{\rho_e}{\varepsilon_0} * G \tag{1.50}$$

$$\Phi_m = \frac{\rho_m}{\mu_0} * G \tag{1.51}$$

where the asterisk again denotes multidimensional convolution. Therefore, this particular choice of scalar and vector potentials results in a complete decoupling of the contribution to the field from the electric current density, magnetic current density, electric charge density, and magnetic charge density.

Once the scalar and vector potentials are determined by integration over the given sources, Equations (1.48) and (1.49) require only a single differentiation to obtain the electromagnetic fields. Because of the lower order derivative, the mixed-potential source–field representation is often used within numerical formulations in preference to Equations (1.38) and (1.39). For direct field calculations in the source region, this procedure is still an integration-followed-by-differentiation approach with the disadvantages noted above.

A third form of the source–field relationship can be developed using an analogy between Equations (1.40) and (1.41) and their general solutions (1.42) and (1.43) and the vector Helmholtz equations appearing in (1.31) and (1.32). Formally, we can write the solutions to Equations (1.31) and (1.32) directly as

$$\bar{E}^s = \left(-j\omega\mu_0\bar{J} + \frac{\nabla\nabla\cdot\bar{J}}{j\omega\varepsilon_0} - \nabla \times \bar{K}\right) * G \tag{1.52}$$

and

$$\bar{H}^s = \left(\nabla \times \bar{J} - j\omega\varepsilon_0\bar{K} + \frac{\nabla\nabla\cdot\bar{K}}{j\omega\mu_0}\right) * G \tag{1.53}$$

without the need of intermediate potential functions [14]. These equations can also be obtained from Equations (1.38) and (1.39) by employing the property that differentiation operators commute with the convolution operation. In contrast to Equations (1.38) and (1.39), which require an integration followed by a differentiation, Equations (1.52) and (1.53) require a differentiation followed by an integration. As noted above, it is often easier to differentiate a given expression in closed form than it is to obtain a closed-form expression for the relatively complex convolution integrals of (1.42) or (1.43). Thus, Equations (1.52) and (1.53) will often permit the closed-form evaluation of the first step of the process. As discussed in Section 1.2, these derivatives must be interpreted as generalized functions. Formal rules for manipulating Dirac delta functions and their derivatives must be followed to correctly carry out their evaluation.

It should be noted that (1.52) and (1.53) are not the equations that would result from a simple interchange of integration and differentiation in Equations (1.38) and (1.39). In fact, carrying out such an interchange leads to a fourth type of source–field relationship, obtained in the form of a convolution between the sources $\bar{J}$ and $\bar{K}$ and the so-called dyadic Green's functions [2, 4, 9]. The expressions can be written

$$\bar{E}^s = \bar{J} * \bar{\bar{G}}_{ej} + \bar{K} * \bar{\bar{G}}_{ek} \tag{1.54}$$

$$\bar{H}^s = \bar{J} * \bar{\bar{G}}_{mj} + \bar{K} * \bar{\bar{G}}_{mk} \tag{1.55}$$

where $\bar{\bar{G}}_{ej}$ and $\bar{\bar{G}}_{ek}$ are the dyadic Green's functions for the electric field, symbolically denoted

$$\bar{\bar{G}}_{ej} = \frac{1}{j\omega\varepsilon_0}(\nabla\nabla + k^2\bar{\bar{I}})G \tag{1.56}$$

$$\bar{\bar{G}}_{ek} = -\nabla \times (\bar{\bar{I}}G) \tag{1.57}$$

and $\bar{\bar{G}}_{mj}$ and $\bar{\bar{G}}_{mk}$ are the dyadic Green's functions for the magnetic field,

$$\bar{\bar{G}}_{mj} = \nabla \times (\bar{\bar{I}}G) \tag{1.58}$$

$$\bar{\bar{G}}_{mk} = \frac{1}{j\omega\mu_0}(\nabla\nabla + k^2\bar{\bar{I}})G \tag{1.59}$$

As a consequence of the singularity in the Green's function, Leibnitz's rule is violated by this interchange of integration and differentiation when the source and observation regions overlap. In this situation, formal classical integration is not sufficient to evaluate the integrals required in (1.54) and (1.55). The evaluation of these integrals using "regularization" procedures is possible [15–17] but is beyond the scope of this text.

Although we will not have occasion to use the dyadic representation, the first three source–field relationships will be used throughout the text when formulating numerical schemes for solving electromagnetic scattering problems. Occasionally, it will be possible to find closed-form expressions for the vector potential functions, which readily permit their differentiation according to Equations (1.38) and (1.39). Often, the mixed-potential representation of Equations (1.48) and (1.49) will be preferred because of the lower order derivative appearing in front of the scalar potential terms. The mixed-potential representation is also preferred if the charge densities are defined separately from the current densities, as occurs when (1.23) and (1.24) are used to define the charge densities within a volumetric formulation. Equations (1.52) and (1.53) require no differentiation of the integral, which may make them preferable for situations where the integration over the sources must be performed numerically. Of course, if applied correctly, all of the above source–field relationships are equivalent and produce identical results. As will be demonstrated in chapters to follow, however, one approach is usually easier to implement than the others within a specific numerical treatment.

1.5 DUALITY RELATIONSHIPS [2]

Because the equations describing the electric and magnetic fields and the associated sources exhibit almost perfect symmetry, any relationship between the fields and sources can be used to directly arrive at a dual relationship describing the complementary fields and sources. The idea of duality is a useful aid to generating new formulas or just remembering some of the existing ones. Table 1.1 summarizes the duality relationships.

TABLE 1.1 Principle of Duality. The equations describing electromagnetic fields remain valid if all the quantities in the left column are replaced by those in the right.

$\bar{E}$	$\bar{H}$
$\bar{H}$	$-\bar{E}$
$\bar{J}$	$\bar{K}$
$\bar{K}$	$-\bar{J}$
ρ_e	ρ_m
ρ_m	$-\rho_e$
ε	μ
μ	ε
$\bar{A}$	$\bar{F}$
$\bar{F}$	$-\bar{A}$

1.6 SURFACE EQUIVALENCE PRINCIPLE [2, 7, 8]

Equations (1.17)–(1.24) demonstrate that mathematical volume sources can be used to replace dielectric and magnetic materials. Equivalent sources distributed on a surface are of similar utility. To illustrate the surface equivalence principle, consider the hypothetical situation posed in Figure 1.3. Figure 1.3 shows two regions of space separated by a mathematical surface S. Region 1 is homogeneous with ε_1 and μ_1, whereas region 2 contains inhomogeneities that may include perfectly conducting materials. A source $(\bar{J}_2, \bar{K}_2)$ located in region 2 and radiating in the presence of the inhomogeneities produces fields $\bar{E}_2$ and $\bar{H}_2$ throughout region 1. We also postulate a second source $(\bar{J}_1, \bar{K}_1)$ located in region 1 but radiating fields $\bar{E}_1$ and $\bar{H}_1$ in a homogeneous space having constitutive parameters ε_1 and μ_1. The fields of both sources satisfy the radiation condition on the boundary at infinity (S_∞).

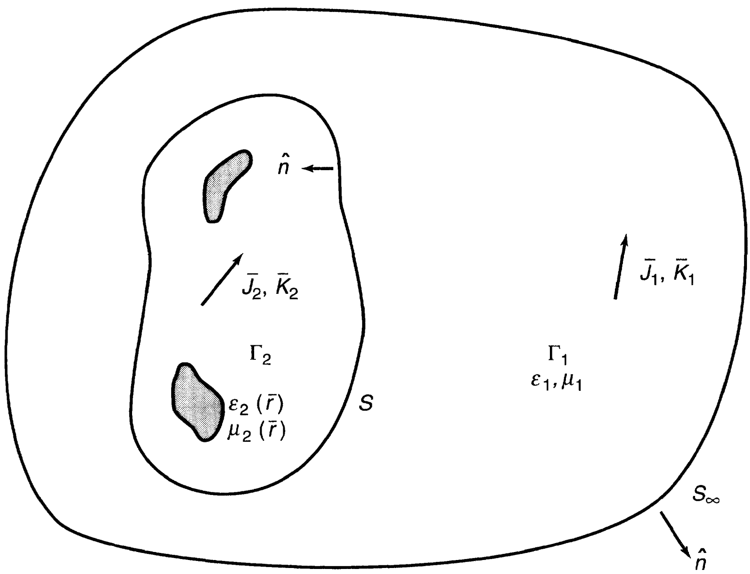

Figure 1.3 Two regions separated by a surface S.

Throughout region 1, Maxwell's curl equations can be written

$$\nabla \times \bar{E}_1 = -j\omega\mu_1\bar{H}_1 - \bar{K}_1 \tag{1.60}$$

$$\nabla \times \bar{H}_1 = j\omega\varepsilon_1\bar{E}_1 + \bar{J}_1 \tag{1.61}$$

$$\nabla \times \bar{E}_2 = -j\omega\mu_1\bar{H}_2 \tag{1.62}$$

$$\nabla \times \bar{H}_2 = j\omega\varepsilon_1\bar{E}_2 \tag{1.63}$$

Therefore, in region 1 we can construct the following equations:

$$\bar{H}_2 \cdot \nabla \times \bar{E}_1 = -j\omega\mu_1\bar{H}_2 \cdot \bar{H}_1 - \bar{H}_2 \cdot \bar{K}_1 \tag{1.64}$$

$$\bar{E}_2 \cdot \nabla \times \bar{H}_1 = j\omega\varepsilon_1\bar{E}_2 \cdot \bar{E}_1 + \bar{E}_2 \cdot \bar{J}_1 \tag{1.65}$$

$$\bar{H}_1 \cdot \nabla \times \bar{E}_2 = -j\omega\mu_1\bar{H}_1 \cdot \bar{H}_2 \tag{1.66}$$

$$\bar{E}_1 \cdot \nabla \times \bar{H}_2 = j\omega\varepsilon_1\bar{E}_1 \cdot \bar{E}_2 \tag{1.67}$$

By combining these equations, we obtain

$$\bar{H}_2 \cdot \nabla \times \bar{E}_1 - \bar{E}_1 \cdot \nabla \times \bar{H}_2 + \bar{E}_2 \cdot \nabla \times \bar{H}_1 - \bar{H}_1 \cdot \nabla \times \bar{E}_2 = \bar{E}_2 \cdot \bar{J}_1 - \bar{H}_2 \cdot \bar{K}_1 \tag{1.68}$$

which is equivalent to

$$\nabla \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) = \bar{E}_2 \cdot \bar{J}_1 - \bar{H}_2 \cdot \bar{K}_1 \tag{1.69}$$

Equation (1.69) is a form of the *Lorentz reciprocity theorem* [2]. Integrating both sides of Equation (1.69) over region 1 and applying the divergence theorem

$$\iiint_{\Gamma_1} \nabla \cdot \bar{Q}\, dv = \iint_S \bar{Q} \cdot \hat{n}\, dS + \iint_{S_\infty} \bar{Q} \cdot \hat{n}\, dS \tag{1.70}$$

where $\hat{n}$ is the normal vector on the surface pointing out of region 1, produce

$$\iint_S (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) \cdot \hat{n}\, dS = \iiint_{\Gamma_1} (\bar{E}_2 \cdot \bar{J}_1 - \bar{H}_2 \cdot \bar{K}_1)\, dv \tag{1.71}$$

(The integral over the surface at infinity vanishes as a consequence of the radiation condition.) Vector identities dictate that

$$\bar{E}_1 \times \bar{H}_2 \cdot \hat{n} = -\bar{E}_1 \cdot (\hat{n} \times \bar{H}_2) \tag{1.72}$$

and

$$\bar{E}_2 \times \bar{H}_1 \cdot \hat{n} = -\bar{H}_1 \cdot (\bar{E}_2 \times \hat{n}) \tag{1.73}$$

Therefore, Equation (1.71) can be rewritten as

$$\iint_S [\bar{E}_1 \cdot (-\hat{n} \times \bar{H}_2) - \bar{H}_1 \cdot (-\bar{E}_2 \times \hat{n})]\, dS = \iiint_{\Gamma_1} (\bar{E}_2 \cdot \bar{J}_1 - \bar{H}_2 \cdot \bar{K}_1)\, dv \tag{1.74}$$

Equation (1.74) is a generalized statement of reciprocity.

Now, let us suppose that the sources in region 1 are

$$\bar{J}_1 = \hat{u}\delta(\bar{r} - \bar{r}') \tag{1.75}$$

and

$$\bar{K}_1 = 0 \tag{1.76}$$

where $\bar{r}$ denotes the source point in region 1 and $\bar{r}'$ represents the integration variable in (1.74). For these sources, Equation (1.74) can be written as

$$\hat{u} \cdot \bar{E}_2|_{\bar{r}} = \iint_S [\bar{E}_1 \cdot (-\hat{n} \times \bar{H}_2) - \bar{H}_1 \cdot (-\bar{E}_2 \times \hat{n})]\, dS' \tag{1.77}$$

where $\bar{E}_1$ and $\bar{H}_1$ are the fields produced at location $\bar{r}'$ in an infinite homogeneous space by sources $\bar{J}_1$ and $\bar{K}_1$ located at $\bar{r}$. These fields can be expressed in terms of the first source–field relationship derived in Section 1.4, to obtain

$$\bar{E}_1(\bar{r}') = \frac{\nabla'\nabla' \cdot + k^2}{j\omega\varepsilon_1} \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \right) \tag{1.78}$$

$$\bar{H}_1(\bar{r}') = \nabla' \times \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \right) \tag{1.79}$$

where $k = \omega(\mu_1\varepsilon_1)^{1/2}$. Note that the derivatives are taken with respect to the primed coordinates. Because of the symmetry of the Green's function, however, it is easily shown that

$$\nabla'\nabla' \cdot \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \right) = \nabla\nabla \cdot \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \right) \tag{1.80}$$

and

$$\nabla' \times \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \right) = -\nabla \times \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \right) \tag{1.81}$$

Therefore, (1.77) becomes

$$\hat{u} \cdot \bar{E}_2|_{\bar{r}} = \iint_S \left[\frac{\nabla\nabla \cdot + k^2}{j\omega\varepsilon_1} \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \right) \cdot (-\hat{n} \times \bar{H}_2) + \nabla \times \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r} - \bar{r}'|} \right) \cdot (-\bar{E}_2 \times \hat{n}) \right] dS' \tag{1.82}$$

The integration is to be performed in primed coordinates over the surface S. Note that the derivatives appearing in Equation (1.82) are now taken with respect to unprimed coordinates, while $\bar{E}_2$ and $\bar{H}_2$ functions of primed variables. Therefore, the first term in (1.82) can be modified using

$$(-\hat{n} \times \bar{H}_2) \cdot \nabla\nabla \cdot \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \right)$$

$$= \sum_{i=1}^{3} \hat{x}_i \cdot (-\hat{n} \times \bar{H}_2) \frac{\partial}{\partial x_i} \frac{\partial}{\partial u} \left(\frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \right)$$

$$= \frac{\partial}{\partial u} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\hat{x}_i \cdot (-\hat{n} \times \bar{H}_2) \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \right)$$

$$= \hat{u} \cdot \nabla\nabla \cdot \left((-\hat{n} \times \bar{H}_2) \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \right) \tag{1.83}$$

where $\{x_i\}$ denote the three Cartesian variables and u is a variable defined along $\hat{u}$. Furthermore, the second term in (1.82) can be converted using vector identities and the fact that $\hat{u}$ is constant to produce

$$\nabla \times \left(\hat{u} \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \right) \cdot (-\bar{E}_2 \times \hat{n})$$

$$= \nabla \left(\frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \right) \times \hat{u} \cdot (-\bar{E}_2 \times \hat{n})$$

$$= -\hat{u} \cdot \nabla \left(\frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \right) \times (-\bar{E}_2 \times \hat{n})$$

$$= -\hat{u} \cdot \nabla \times \left((-\bar{E}_2 \times \hat{n}) \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \right) \tag{1.84}$$

After these results are substituted into (1.82), the derivatives (taken with respect to unprimed variables) can be moved outside the integrals to produce

$$\hat{u} \cdot \bar{E}_2|_{\bar{r}} = \hat{u} \cdot \frac{\nabla\nabla \cdot +k^2}{j\omega\varepsilon_1} \iint_S (-\hat{n} \times \bar{H}_2) \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \, dS'$$

$$-\hat{u} \cdot \nabla \times \iint_S (-\bar{E}_2 \times \hat{n}) \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \, dS' \tag{1.85}$$

Equation (1.85) is a statement that the field produced by $(\bar{J}_2, \bar{K}_2)$ at some location outside of region 2 can be expressed in the form of an integration over tangential fields on the surface of region 2. In fact, by comparing Equation (1.85) with Equation (1.38), it is immediately apparent that the field is equivalent to that produced by surface current densities

$$\bar{J}_s = -\hat{n} \times \bar{H}_2 \tag{1.86}$$

and

$$\bar{K}_s = -\bar{E}_2 \times \hat{n} \tag{1.87}$$

located on the surface S and radiating in a homogeneous space having constitutive parameters ε_1 and μ_1. This property is a fundamental theorem of electromagnetics generally known as *Huygens' surface equivalence principle.* [Note that the normal vector $\hat{n}$ in Equations (1.86) and (1.87) points into region 2, i.e., into the closed surface S. This is actually the opposite of our usual convention, which requires that the normal vector point out of the region containing the sources. In the following discussion, we revert back to the usual convention.]

We will now state the surface equivalence principle in a slightly different form. Consider Figure 1.4, which shows a source in region 1 radiating in the presence of inhomogeneities located in region 2. Fields produced in region 1 are denoted $\bar{E}_1$ and $\bar{H}_1$; those produced in region 2 are denoted $\bar{E}_2$ and $\bar{H}_2$. Now, consider equivalent sources $\bar{J}_s$ and $\bar{K}_s$ located on the mathematical surface S and satisfying

$$\bar{J}_s = \hat{n} \times \bar{H}_1 \tag{1.88}$$

$$\bar{K}_s = \bar{E}_1 \times \hat{n} \tag{1.89}$$

where $\hat{n}$ is the outward normal vector. According to the equivalence principle, the combination of the original source and the equivalent sources produces fields $\bar{E}_1$ and $\bar{H}_1$ in region 1 identical to that of the original problem illustrated in Figure 1.4. The fields in region 2 are not identical to those of the original problem; in fact, null fields are produced throughout region 2 by the combination of the original and equivalent sources. (This result is sometimes known as the *extinction theorem.*) The modified problem is illustrated in Figure 1.5.

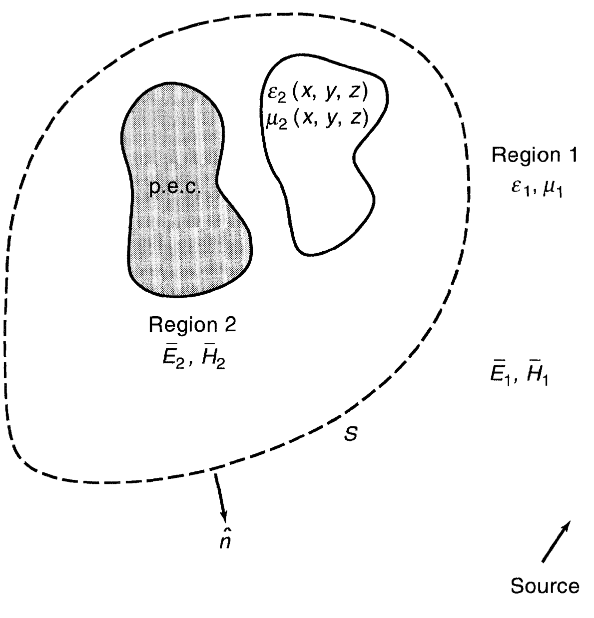

Figure 1.4 Electromagnetic source in region 1 radiating in the presence of inhomogeneities located in region 2. The surface S separates the two regions.

Since the fields throughout region 2 of the modified problem vanish, any inhomogeneities present in region 2 may be replaced at will without affecting the fields in region 1. Figure 1.6 shows one possibility, that of removing all the inhomogeneities from region 2 and

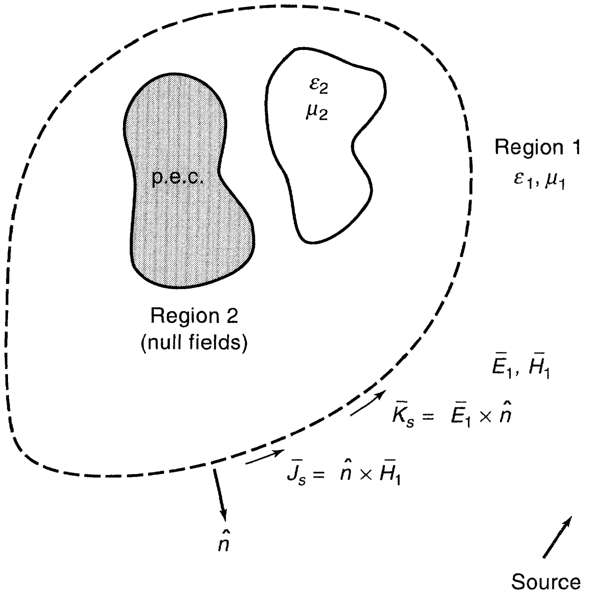

Figure 1.5 Intermediate step in the construction of the equivalent exterior problem associated with Figure 1.4. Sources $\bar{J}_s$ and $\bar{K}_s$ are introduced on the surface S and replicate the original fields in region 1. Null fields are produced throughout region 2.

leaving a homogeneous medium with the same constitutive parameters as region 1. This is often the approach followed in practice, since it effectively replaces the original problem involving complicated inhomogeneous media with a problem involving only sources radiating in homogeneous space. The effect is that the original scattering geometry has been replaced by an *equivalent exterior problem.* As was the case in the volume equivalence principle described previously, we have not solved the electromagnetic field problem by the introduction of equivalent sources. In fact, $\bar{J}_s$ and $\bar{K}_s$ are unknowns that remain to be determined. However, we have converted the problem from one requiring the solution of

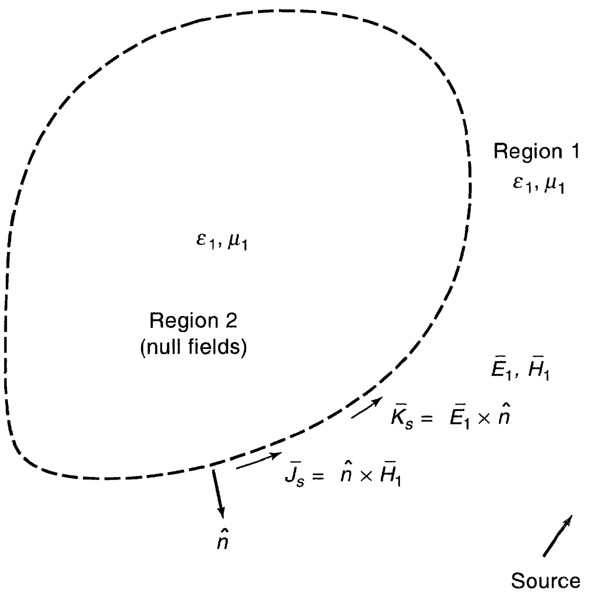

Figure 1.6 Equivalent exterior problem constructed from Figure 1.4. The inhomogeneities within region 2 have been replaced by a homogeneous medium identical to that of region 1. The combination of the original source and the equivalent sources $\bar{J}_s$ and $\bar{K}_s$ produce the original fields throughout region 1 and null fields throughout region 2.

Equations (1.1)–(1.4) to one requiring the solution of Equations (1.17)–(1.20), with equivalent surface currents and charges defined respectively by Equations (1.88) and (1.89) and the continuity equations

$$\rho_e = \frac{-1}{j\omega}\nabla_s \cdot \bar{J}_s \tag{1.90}$$

$$\rho_m = \frac{-1}{j\omega}\nabla_s \cdot \bar{K}_s \tag{1.91}$$

where ∇_s is the surface divergence operator.

1.7. SURFACE INTEGRAL EQUATIONS FOR PERFECTLY CONDUCTING SCATTERERS

Figure 1.7 shows a scatterer of perfect electric conducting (p.e.c.) material illuminated by a source. Consider a mathematical surface enclosing the scatterer, over which equivalent sources $\bar{J}_s$ and $\bar{K}_s$ are defined according to Equations (1.88) and (1.89). If the mathematical surface is permitted to shrink until it coincides with the surface of the perfect conductor, Equation (1.5) dictates that the tangential electric field must vanish on the surface. It follows that equivalent sources

$$\bar{J}_s = \hat{n} \times \bar{H} \tag{1.92}$$

$$\bar{K}_s = 0 \tag{1.93}$$

located on the surface of the p.e.c. scatterer will produce the correct scattered fields in the exterior region. The equivalent problem is depicted in Figure 1.8.

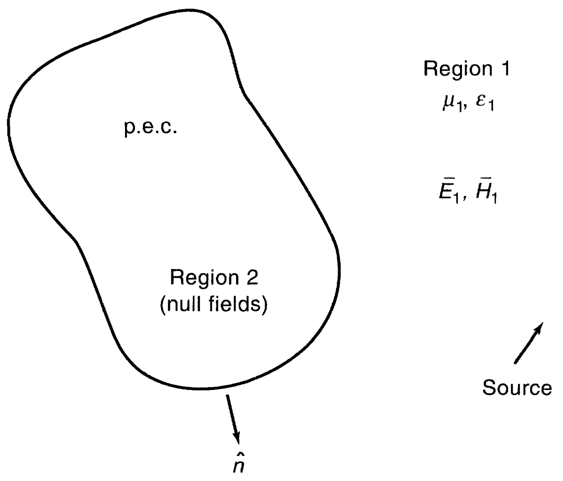

Figure 1.7 Original problem involving a p.e.c. scatterer.

We now combine the surface equivalence principle, the source–field relationships, and the boundary conditions discussed previously in order to formulate integral equations for the unknown equivalent sources. Assuming that the incident fields $\bar{E}^{\text{inc}}$ and $\bar{H}^{\text{inc}}$ are specified, the source–field relationships from Equations (1.38) and (1.39) may be combined

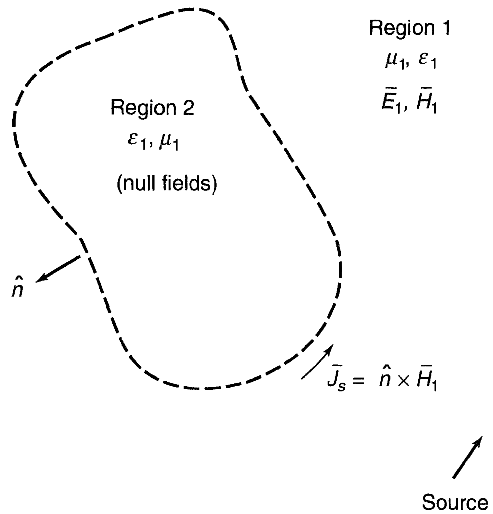

Figure 1.8 Equivalent exterior problem associated with Figure 1.7. An equivalent source $\bar{J}_s$ is introduced along the location of the conducting surface, and the conductor is replaced by a homogeneous medium with the same constitutive parameters as the exterior region.

with the equivalent sources from (1.92) and (1.93) to produce

$$\bar{E}^{\mathrm{inc}}(\bar{r}) = \bar{E}(\bar{r}) - \frac{\nabla\nabla \cdot \bar{A} + k^2 \bar{A}}{j\omega\varepsilon_0} \tag{1.94}$$

$$\bar{H}^{\mathrm{inc}}(\bar{r}) = \bar{H}(\bar{r}) - \nabla \times \bar{A} \tag{1.95}$$

At present, these relationships simply define the fields $\bar{E}$ and $\bar{H}$ produced by the excitation in the presence of the p.e.c. scatterer and hold throughout the exterior region (in this case they hold throughout the interior region also, since the fields within a p.e.c. vanish). However, if the boundary condition of Equation (1.5) is imposed on the surface of the scatterer, Equation (1.94) becomes

$$\hat{n} \times \bar{E}^{\mathrm{inc}} = -\hat{n} \times \left\{ \frac{\nabla\nabla \cdot \bar{A} + k^2 \bar{A}}{j\omega\varepsilon_0} \right\}_S \tag{1.96}$$

which is an integro-differential equation for the unknown equivalent surface current density $\bar{J}_s$. Equation (1.96) holds only for points on the surface S of the scatterer and is one form of the *electric field integral equation* (EFIE). If Equation (1.6) is combined with Equation (1.95), we obtain the *magnetic field integral equation* (MFIE)

$$\hat{n} \times \bar{H}^{\mathrm{inc}} = \bar{J}_s - \{\hat{n} \times \nabla \times \bar{A}\}_{S^+} \tag{1.97}$$

Equation (1.97) is also an integro-differential equation for the unknown surface current $\bar{J}_s$ and is enforced an infinitesimal distance outside the scatterer surface (S^+). It is common practice to refer to these equations as integral equations rather than integro-differential equations. Note that any of the source–field relationships presented in Section 1.4 could be employed as alternatives, producing equivalent equations.

In principle, either of Equations (1.96) or (1.97) can be solved to produce the unknown equivalent source $\bar{J}_s$. Once $\bar{J}_s$ is determined, the electric and magnetic fields everywhere in space may be found from the source–field relationships presented previously, superimposing the incident field with the scattered fields produced by $\bar{J}_s$. In deriving the EFIE and MFIE, we imposed only one of the conditions (1.5) and (1.6). Because of this, there are scatterers for which the solution of these equations is not unique. The uniqueness issue will be discussed in Chapter 6.

Suppose that, instead of a closed body, we wish to treat scattering from an infinitesimally thin open p.e.c. shell, strip, or plate (Figure 1.9). The surface equivalence principle can be applied in the same fashion as in the case of a solid scatterer. However, if the surface S collapses to the scatterer surface, the equivalent current densities on either side of the scatterer become superimposed at the location of the thin shell. The equations are unable to distinguish between the two equivalent sources, and we are forced to work with a single equivalent source that represents the sum of the sources on either side. Since the boundary condition of (1.5) remains valid for infinitesimally thin p.e.c. structures, however, an EFIE of the form of Equation (1.96) can be used to treat this type of scattering problem.

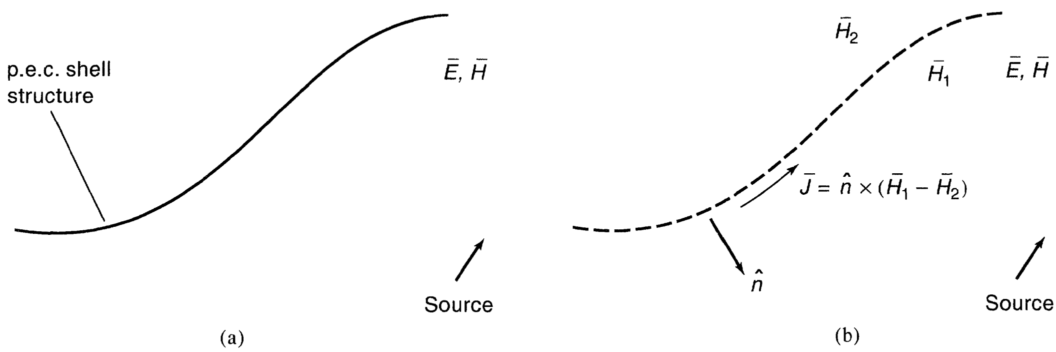

Figure 1.9 (a) Original problem involving an infinitesimally thin p.e.c. scatterer; (b) Equivalent problem. The p.e.c. material is replaced by a single source $\bar{J}_s$ that represents the superposition of the electric currents on both surfaces of the thin scatterer.

Although the EFIE can be employed to model a thin shell, the MFIE of Equation (1.97) is based on a boundary condition that is not valid for extremely thin geometries. Equation (1.6) is actually a special case of the general boundary condition

$$\hat{n} \times (\bar{H}_1 - \bar{H}_2) = \bar{J}_s \tag{1.98}$$

For closed bodies or other situations where the magnetic field vanishes on one side of the surface, Equation (1.98) reduces to (1.6). For infinitesimally thin structures with nonzero fields on both sides of the surface, however, Equation (1.6) is not equivalent to (1.98) and does not actually constitute a boundary condition. Consequently, the MFIE of (1.97) is restricted to closed bodies and cannot be used to describe scattering from an infinitesimally thin p.e.c. structure.

1.8. VOLUME INTEGRAL EQUATIONS FOR PENETRABLE SCATTERERS

The volumetric equivalence principle from Section 1.2 can be used to construct integro-differential equations describing the interaction of electromagnetic fields with penetrable scatterers. Combining the sources from Equations (1.21)–(1.24) with the source–field relationships from Equations (1.48) and (1.49), we obtain

$$\bar{E}^{\text{inc}}(\bar{r}) = \bar{E}(\bar{r}) + jk\eta\bar{A} + \nabla\Phi_e + \nabla\times\bar{F} \tag{1.99}$$

$$\bar{H}^{\text{inc}}(\bar{r}) = \bar{H}(\bar{r}) - \nabla\times\bar{A} + j\frac{k}{\eta}\bar{F} + \nabla\Phi_m \tag{1.100}$$

where $\eta = (\mu_0/\varepsilon_0)^{1/2}$. (Throughout the text, we will frequently interchange $\omega\mu_0 = k\eta$ and $\omega\varepsilon_0 = k/\eta$.) Equations (1.99) and (1.100) can be thought of as an EFIE and MFIE, respectively, although it is noteworthy that there are some differences between these equations and those of the previous section describing the p.e.c. scatterer. Specifically, these are volume equations that hold everywhere throughout the penetrable scatterer rather than just on the scatterer surface. Instead of designating the equivalent sources as the primary unknowns to be determined, it is usually more convenient to pose the problem directly in terms of the internal $\bar{E}$ or $\bar{H}$ fields. By using Equations (1.22), (1.23), and (1.26) with the EFIE, all equivalent sources can be defined as functions of $\bar{E}$. Similarly, with the MFIE it is convenient to employ Equations (1.21), (1.24), and (1.25) to define all quantities in terms of $\bar{H}$.

In the special case in which the body in question is composed entirely of dielectric material, terms involving equivalent magnetic currents and charges drop out, leaving

$$\bar{E}^{\text{inc}}(\bar{r}) = \bar{E}(\bar{r}) + jk\eta\bar{A} + \nabla\Phi_e \tag{1.101}$$

$$\bar{H}^{\text{inc}}(\bar{r}) = \bar{H}(\bar{r}) - \nabla\times\bar{A} \tag{1.102}$$

If the scatterer is composed entirely of magnetic material, terms involving electric current and charge density vanish, producing

$$\bar{E}^{\text{inc}}(\bar{r}) = \bar{E}(\bar{r}) + \nabla\times\bar{F} \tag{1.103}$$

$$\bar{H}^{\text{inc}}(\bar{r}) = \bar{H}(\bar{r}) + j\frac{k}{\eta}\bar{F} + \nabla\Phi_m \tag{1.104}$$

When simultaneously treating dielectric and magnetic materials with (1.99) and (1.100), fewer unknowns are required if the internal $\bar{E}$ or $\bar{H}$ field instead of the equivalent sources is designated the primary unknown.

These volume integral equations are suitable for the analysis of inhomogeneous material. If the penetrable scatterer under consideration is homogeneous with constant μ_r and ε_r, the problem can be formulated with either volume integral equations or surface integral equations. Surface integral equations are usually more manageable for numerical solution since the unknowns to be determined are confined to the scatterer surface rather than distributed throughout the scatterer volume.

1.9. SURFACE INTEGRAL EQUATIONS FOR HOMOGENEOUS SCATTERERS

Figure 1.10 depicts a homogeneous, penetrable body illuminated by an incident electromagnetic field. Region 1 is free space and region 2 is characterized by a constant μ_r and ε_r. The terms $\bar{E}_1$ and $\bar{H}_1$ denote the fields in region 1, and $\bar{E}_2$ and $\bar{H}_2$ denote the fields throughout region 2. Using the surface equivalence principle, we wish to define equivalent sources on the scatterer surface that replicate the original fields in both regions.

The equivalent exterior problem, as shown in Figure 1.11, is constructed in a manner identical to the general situation presented in Section 1.6. Equivalent sources $\bar{J}_1$ and $\bar{K}_1$ have been placed on a surface coinciding with the original scatterer. These sources have been defined so that

$$\bar{J}_1 = \hat{n} \times \bar{H}_1 \tag{1.105}$$

$$\bar{K}_1 = \bar{E}_1 \times \hat{n} \tag{1.106}$$

where $\hat{n}$ is the outward normal vector at points on the surface. These sources, radiating in conjunction with the original source, replicate the original fields $\bar{E}_1$ and $\bar{H}_1$ throughout region 1. Null fields are produced in region 2, allowing us to replace medium 2 with free space without changing the fields in region 1. Thus, the exterior part of the original problem of Figure 1.10 is equivalent to the problem of Figure 1.11, which only involves sources radiating in free space.

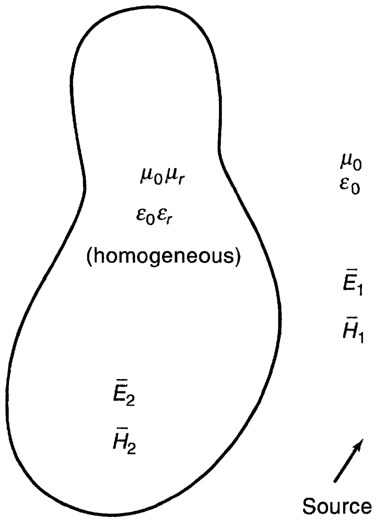

Figure 1.10 Original problem, involving a homogeneous body.

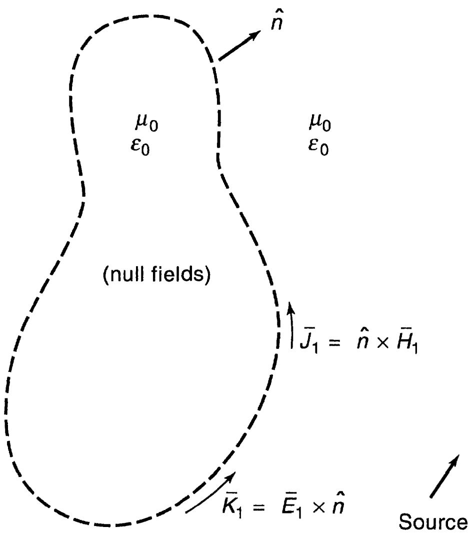

Figure 1.11 Equivalent exterior problem associated with Figure 1.10.

To describe the interior problem, a second equivalence relationship must be constructed. The equivalent interior problem is depicted in Figure 1.12. Sources $\bar{J}_2$ and $\bar{K}_2$ are defined according to

$$\bar{J}_2 = (-\hat{n}) \times \bar{H}_2 \tag{1.107}$$

$$\bar{K}_2 = \bar{E}_2 \times (-\hat{n}) \tag{1.108}$$

where $\hat{n}$ is still the normal vector pointing into region 1 (out of the scatterer). Radiating *in the absence* of the original source, these equivalent sources replicate the original fields throughout region 2 and produce null fields throughout the entirety of region 1. Since the region 1 fields vanish, we are free to insert material having $\mu = \mu_2$ and $\varepsilon = \varepsilon_2$ throughout region 1 in order to convert the problem to one involving infinite, homogeneous space. Thus, the equivalent interior problem involves sources $\bar{J}_2$ and $\bar{K}_2$ radiating in homogeneous space characterized by permittivity ε_2 and permeability μ_2.

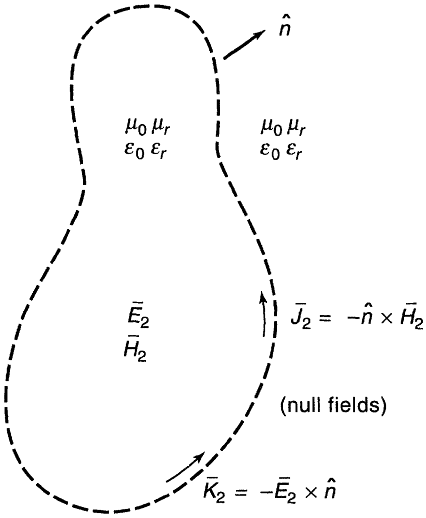

Figure 1.12 Equivalent interior problem associated with Figure 1.10.

The continuity of the tangential $\bar{E}$ and $\bar{H}$ fields at the dielectric interface dictates that

$$\bar{J}_1 = -\bar{J}_2 \tag{1.109}$$

and

$$\bar{K}_1 = -\bar{K}_2 \tag{1.110}$$

Therefore, it suffices to work with $\bar{J}_1$ and $\bar{K}_1$ as the primary unknowns to be determined. Since we have two equivalent problems and two unknown sources, we must employ a system of two coupled equations. The source–field relationships from Equations (1.38) and (1.39) can be combined with conditions (1.105)–(1.108) to produce the coupled EFIEs

$$\hat{n} \times \bar{E}^{\text{inc}} = -\bar{K}_1 - \hat{n} \times \left\{ \frac{\eta_1}{jk_1}(\nabla\nabla \cdot \bar{A}_1 + k_1^2 \bar{A}_1) - \nabla \times \bar{F}_1 \right\}_{S^+} \tag{1.111}$$

$$0 = \bar{K}_1 - \hat{n} \times \left\{ \frac{\eta_2}{jk_2}(\nabla\nabla \cdot \bar{A}_2 + k_2^2 \bar{A}_2) - \nabla \times \bar{F}_2 \right\}_{S^-} \tag{1.112}$$

where

$$\bar{A}_1 = \bar{J}_1 * \frac{e^{-jk_1 r}}{4\pi r} \tag{1.113}$$

$$\bar{F}_1 = \bar{K}_1 * \frac{e^{-jk_1 r}}{4\pi r} \tag{1.114}$$

$$\bar{A}_2 = \bar{J}_1 * \frac{e^{-jk_2 r}}{4\pi r} \tag{1.115}$$

$$\bar{F}_2 = \bar{K}_1 * \frac{e^{-jk_2 r}}{4\pi r} \tag{1.116}$$

$k_1 = \omega(\mu_1 \varepsilon_1)^{1/2}$, $k_2 = \omega(\mu_2 \varepsilon_2)^{1/2}$, $\eta_1 = (\mu_1/\varepsilon_1)^{1/2}$, and $\eta_2 = (\mu_2/\varepsilon_2)^{1/2}$ (the subscript on $\bar{A}$ and $\bar{F}$ indicates the medium into which the sources $\bar{J}_1$ and $\bar{K}_1$ radiate). Equation (1.111) is evaluated an infinitesimal distance *outside* the scatterer surface (S^+), while Equation

(1.112) is evaluated an infinitesimal distance *inside* the surface (S^-). As an alternative, coupled MFIEs are obtained as

$$\hat{n} \times \bar{H}^{\text{inc}} = \bar{J}_1 - \hat{n} \times \left\{ \nabla \times \bar{A}_1 + \frac{\nabla\nabla \cdot \bar{F}_1 + k_1^2 \bar{F}_1}{jk_1\eta_1} \right\}_{S^+} \tag{1.117}$$

$$0 = -\bar{J}_1 - \hat{n} \times \left\{ \nabla \times \bar{A}_2 + \frac{\nabla\nabla \cdot \bar{F}_2 + k_2^2 \bar{F}_2}{jk_2\eta_2} \right\}_{S^-} \tag{1.118}$$

Either of these formulations could be used to represent homogeneous scatterers. The extension of these equations to treat layered homogeneous regions will be left as an exercise (Prob. P1.20). In common with the EFIE and MFIE for p.e.c. scatterers, the coupled surface integral equations do not always guarantee a unique solution (see Chapter 6).

1.10. SURFACE INTEGRAL EQUATION FOR AN APERTURE IN A CONDUCTING PLANE

Consider the problem of scattering from an aperture in an infinite p.e.c. plane. Figures 1.13 and 1.14 show the geometry. The source is located in region 1. We will develop an integral equation formulation based on two equivalent problems, representing regions 1 and 2, respectively.

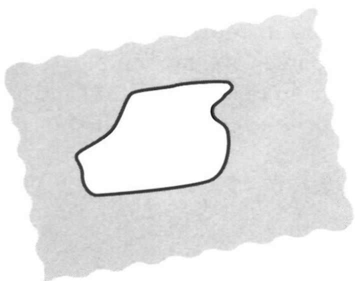

Figure 1.13 Aperture in an infinite p.e.c. plane.

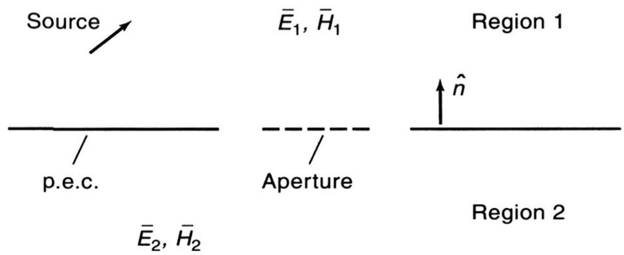

Figure 1.14 Side view of an aperture in an infinite p.e.c. plane.

An equivalent representation for region 1 can be constructed by placing a mathematical surface S on the region 1 side of the conducting plane and introducing equivalent sources

$$\bar{J}_1 = \hat{n} \times \bar{H}_1 \tag{1.119}$$

$$\bar{K}_1 = \bar{E}_1 \times \hat{n} \tag{1.120}$$

on S. These sources, radiating together with the original source, will replicate the original fields in region 1 and create null fields throughout region 2. Although $\bar{J}_1$ is nonzero over the entire surface, the tangential electric field vanishes on the p.e.c. part of the plane, and therefore the magnetic source $\bar{K}_1$ is nonzero only over the aperture. The fact that $\bar{K}_1$ is confined to the aperture motivates its use as the primary unknown within an integral equation formulation.

Since the fields in region 2 of the equivalent problem vanish, we are free to modify the material present without changing the fields in region 1. Previous examples have employed this property in order to remove p.e.c. material. In this situation, suppose instead that we introduce additional p.e.c. material to completely close the aperture and create a uniform p.e.c. plane. The equivalent problem now involves the original source and the sources $\bar{J}_1$ and $\bar{K}_1$ radiating in front of the infinite conducting plane. As a second step, the method of images [2, 3, 7] can be employed to remove the p.e.c. plane. The image of a magnetic current over a perfect electric conductor is the mirror image; the image of an electric source is the negative mirror image. For tangential sources immediately adjacent to the plane, the image of the electric source cancels the original, while that of the magnetic source adds to the original. Thus, application of image theory eliminates the p.e.c. plane and the electric source $\bar{J}_1$, leaving only an equivalent magnetic source $2\bar{K}_1$ located in the original aperture. Consequently, the superposition of the original source, the image of the original source, and an equivalent source $2\bar{K}_1$ radiating in free space replicate the original fields in region 1 (Figure 1.15). These sources produce nonzero fields in region 2 as well, but these fields differ from the original fields in region 2 and the equivalence only holds for region 1.

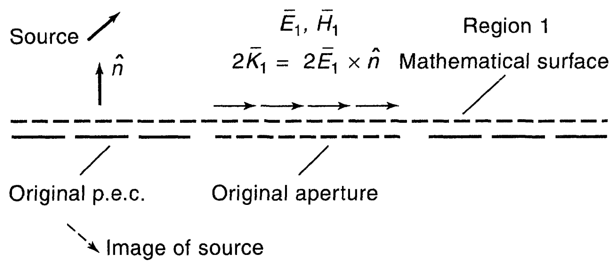

Figure 1.15 Equivalent problem for region 1 associated with Figure 1.14.

An equivalent problem for region 2 can be constructed in a similar manner and is depicted in Figure 1.16. A mathematical surface is introduced on the region 2 side of the plane, containing equivalent sources

$$\bar{J}_2 = (-\hat{n}) \times \bar{H}_2 \qquad (1.121)$$

$$\bar{K}_2 = \bar{E}_2 \times (-\hat{n}) \qquad (1.122)$$

where $\hat{n}$ is still the normal vector pointing into region 1. These sources replicate the original fields in region 2 and produce null fields throughout region 1. Again, the aperture can be closed with p.e.c. material, and image theory can be employed to reduce the problem to one involving just an equivalent magnetic source $2\bar{K}_2$ radiating in free space. This source is confined to the original aperture and, radiating in the absence of any other source, replicates the original fields throughout region 2. (The source also produces nonzero fields throughout region 1 that differ from the original fields in region 1.)

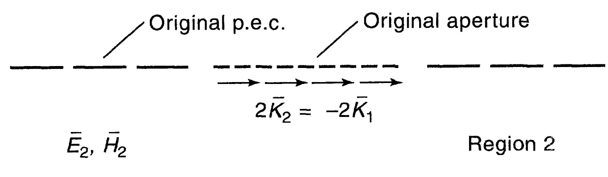

Figure 1.16 Equivalent problem for region 2 associated with Figure 1.14.

Because of the continuity of the original tangential electric field through the aperture, $\bar{K}_1 = -\bar{K}_2$, and it suffices to work with $\bar{K}_1$ as the primary unknown. Relationships similar to Equations (1.117) and (1.118) can be written involving the tangential magnetic field on either side of the aperture. Combining these two expressions to eliminate the aperture $\bar{H}$-field yields the integral equation

$$\hat{n} \times \bar{H}^{\text{inc}} = -4\hat{n} \times \left\{ \frac{\nabla\nabla \cdot \bar{F} + k^2 \bar{F}}{j\omega\mu_0} \right\}_S \tag{1.123}$$

where $\bar{H}^{\text{inc}}$ is the field produced by the original source *and its image* and $\bar{F}$ is the electric vector potential produced by $\bar{K}_1$ radiating in free space. The term $\bar{H}^{\text{inc}}$ can also be thought of as the field produced by the original source *radiating in the presence of the infinite p.e.c. plane* (aperture closed). Equation (1.123) holds in the original aperture and can be solved in principle to find $\bar{K}_1$.

1.11. SCATTERING CROSS SECTION CALCULATION FOR TWO-DIMENSIONAL PROBLEMS

A useful characterization of the scattering properties of an electromagnetic target is given by the *bistatic scattering cross section.* This quantity is an equivalent area proportional to the apparent size of the target in a particular direction (with the apparent size determined by the amount of power scattered in that direction in response to an excitation that may be incident from some other direction). More precisely, it is the area that, if multiplied by the power flux density of the incident field, would yield sufficient power to produce by isotropic radiation the same intensity in a given direction as that actually produced by the scatterer. In a two-dimensional problem, the scattering cross section (sometimes known as the "echo width") can be defined in a similar fashion as an equivalent width proportional to the apparent size of the scatterer in a particular direction.

Consider the two-dimensional situation involving a TM plane wave of the form

$$E_z^{\text{inc}}(x, y) = e^{-jk(x\cos\phi^{\text{inc}} + y\sin\phi^{\text{inc}})} \tag{1.124}$$

impinging upon an infinite, cylindrical geometry. The only electric field component present in the problem is E_z. The two-dimensional bistatic scattering cross section can be expressed as

$$\sigma_{\text{TM}}(\phi, \phi^{\text{inc}}) = \lim_{\rho\to\infty} 2\pi\rho \frac{|E_z^s(\rho, \phi)|^2}{|E_z^{\text{inc}}(0, 0)|^2} \tag{1.125}$$

where (ρ, ϕ) are ordinary polar coordinates. The scattered electric field can be found from Equation (1.38), which simplifies for the TM polarization to

$$E_z^s(x, y) = -jk\eta A_z - \frac{\partial F_y}{\partial x} + \frac{\partial F_x}{\partial y} \tag{1.126}$$

where

$$A_z(x, y) = \iint J_z(x', y') \frac{1}{4j} H_0^{(2)}(kR)\, dx'\, dy' \tag{1.127}$$

$$\bar{F}(x, y) = \iint \bar{K}(x', y') \frac{1}{4j} H_0^{(2)}(kR)\, dx'\, dy' \tag{1.128}$$

and

$$R = \sqrt{(x - x')^2 + (y - y')^2} \tag{1.129}$$

Since the observation point (x, y) is in the far field, it is convenient to work in cylindrical coordinates, that is,

$$R = \sqrt{(\rho \cos\phi - x')^2 + (\rho \sin\phi - y')^2} \tag{1.130}$$

which can be rearranged and written as

$$R = \rho\sqrt{1 - \frac{2}{\rho}(x' \cos\phi + y' \sin\phi) + \frac{(x')^2 + (y')^2}{\rho^2}} \tag{1.131}$$

As $\rho \to \infty$, the third expression under the radical is negligible compared to the others and may be omitted. The approximation

$$\sqrt{1+\alpha} \cong 1 + \frac{1}{2}\alpha \tag{1.132}$$

can be used for small α to simplify Equation (1.131) to the "far-field" form

$$R \cong \rho - x' \cos\phi - y' \sin\phi \quad \text{as } \rho \to \infty \tag{1.133}$$

This result can be obtained from a purely geometrical argument, as illustrated in Figure 1.17.

To further simplify the calculation, the large-argument asymptotic form of the Hankel function

$$H_0^{(2)}(\alpha) \approx \sqrt{\frac{2j}{\pi\alpha}} e^{-j\alpha} \quad \text{as } \alpha \to \infty \tag{1.134}$$

may be employed. Substituting (1.133) and (1.134) into the previous expressions, we obtain

$$A_z(\rho, \phi) = \frac{1}{4j}\sqrt{\frac{2j}{\pi k\rho}} e^{-jk\rho} \iint J_z(x', y') e^{jk(x' \cos\phi + y' \sin\phi)}\, dx'\, dy' \tag{1.135}$$

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{-k}{4}\sqrt{\frac{2j}{\pi k\rho}} e^{-jk\rho} \iint (K_y \cos\phi - K_x \sin\phi) e^{jk(x' \cos\phi + y' \sin\phi)}\, dx'\, dy' \tag{1.136}$$

which may be combined with Equation (1.126) to produce

$$\sigma_{\text{TM}}(\phi, \phi^{\text{inc}}) = \frac{k}{4}\left| \iint (\eta J_z + K_x \sin\phi - K_y \cos\phi) e^{jk(x' \cos\phi + y' \sin\phi)}\, dx'\, dy' \right|^2 \tag{1.137}$$

As shown for emphasis, the scattering cross section is a function of the direction of the incident field and the far-zone observation angle. Although Equation (1.137) contains a double integral over volume sources defined throughout the scatterer, the integral collapses in an obvious way to a surface integral if the equivalent sources are confined to surfaces. For instance, in the special case of a p.e.c. cylinder represented by electric sources, the scattering cross section is given by

$$\sigma_{\text{TM}}(\phi, \phi^{\text{inc}}) = \frac{k\eta^2}{4}\left| \int J_z(t') e^{jk(x(t') \cos\phi + y(t') \sin\phi)}\, dt' \right|^2 \tag{1.138}$$

where t is a parametric variable defined along the contour of the cylinder surface.

In a two-dimensional TE problem, the magnetic field has only a $\hat{z}$-component. For a plane-wave excitation of the form

$$H_z^{\text{inc}}(x, y) = e^{-jk(x \cos\phi^{\text{inc}} + y \sin\phi^{\text{inc}})} \tag{1.139}$$

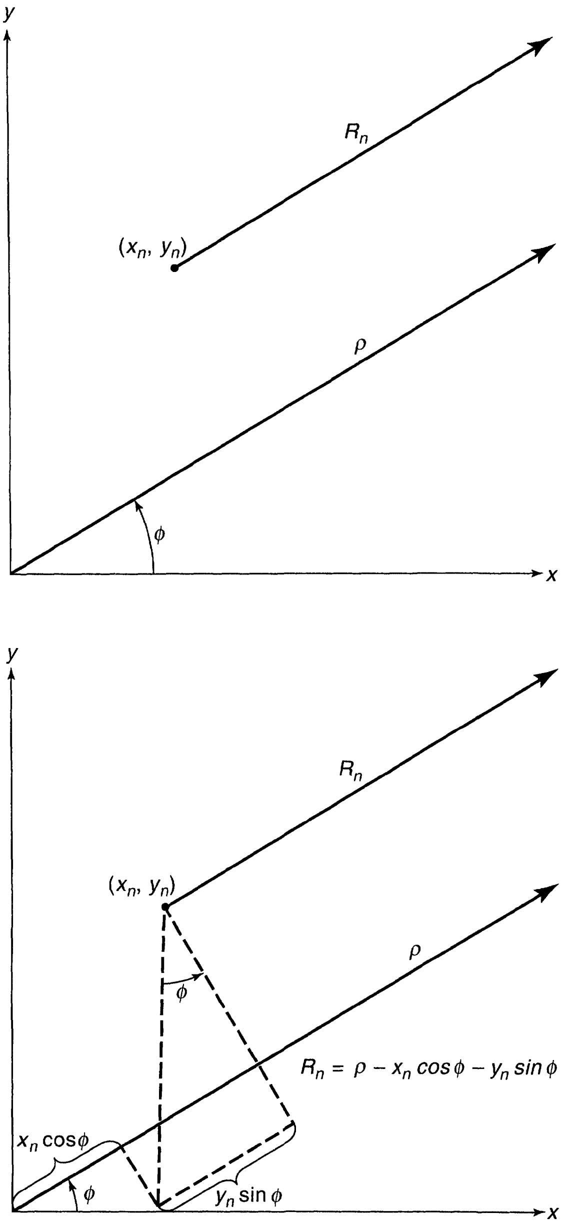

Figure 1.17 Relative path lengths for the far-field approximation.

the bistatic scattering cross section may be determined from

$$\sigma_{\mathrm{TE}}(\phi, \phi^{\mathrm{inc}}) = \lim_{\rho\to\infty} 2\pi\rho \frac{|H_z^s(\rho, \phi)|^2}{|H_z^{\mathrm{inc}}(0,0)|^2} \tag{1.140}$$

The scattered magnetic field can be found from Equation (1.39), which simplifies in the TE case to

$$H_z^s(x, y) = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} - j\frac{k}{\eta}F_z \tag{1.141}$$

The far-zone approximations from Equations (1.133) and (1.134) may be used to obtain

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{-k}{4}\sqrt{\frac{2j}{\pi k \rho}}e^{-jk\rho} \iint (J_y \cos\phi - J_x \sin\phi) e^{jk(x'\cos\phi + y'\sin\phi)}\, dx'\, dy' \tag{1.142}$$

$$F_z(\rho, \phi) = \frac{1}{4j}\sqrt{\frac{2j}{\pi k \rho}}e^{-jk\rho} \iint K_z(x', y') e^{jk(x'\cos\phi + y'\sin\phi)}\, dx'\, dy' \tag{1.143}$$

Therefore, the scattering cross section is given by

$$\sigma_{\mathrm{TE}}(\phi, \phi^{\mathrm{inc}}) = \frac{k}{4}\left|\iint \left(J_x \sin\phi - J_y \cos\phi - \frac{K_z}{\eta}\right) e^{jk(x'\cos\phi + y'\sin\phi)}\, dx'\, dy'\right|^2 \tag{1.144}$$

The general expressions for σ_{TM} and σ_{TE} will be specialized to a variety of specific examples in the chapters to follow.

1.12. SCATTERING CROSS SECTION CALCULATION FOR THREE-DIMENSIONAL PROBLEMS

For a three-dimensional geometry where all components of the electric and magnetic field are present, the bistatic scattering cross section can be expressed for plane-wave incidence as

$$\sigma(\theta, \phi, \theta^{\mathrm{inc}}, \phi^{\mathrm{inc}}) = \lim_{r\to\infty} 4\pi r^2 \frac{\left|\bar{E}^s(\theta, \phi)\right|^2}{\left|\bar{E}^{\mathrm{inc}}(0, 0)\right|^2} \tag{1.145}$$

In the far zone, the scattered electric field has the form

$$\bar{E}^s \cong \hat{\theta} E^s_\theta + \hat{\phi} E^s_\phi \tag{1.146}$$

Since Equation (1.145) involves the expression

$$|\bar{E}^s|^2 \cong |E^s_\theta|^2 + \left|E^s_\phi\right|^2 \tag{1.147}$$

it is sufficient to compute the θ- and ϕ-components separately. In the far zone, these can be obtained from

$$E^s_\theta \cong -jk\eta A_\theta + \frac{\partial F_\phi}{\partial r} \tag{1.148}$$

$$E^s_\phi \cong -jk\eta A_\phi - \frac{\partial F_\theta}{\partial r} \tag{1.149}$$

where the potential functions are defined in Equations (1.42) and (1.43). Because the sources of the scattered field are often described in Cartesian coordinates, it may be necessary to transform to the spherical system using

$$A_\theta = \cos\theta\cos\phi A_x + \cos\theta\sin\phi A_y - \sin\theta A_z \tag{1.150}$$

$$A_\phi = -\sin\phi A_x + \cos\phi A_y \tag{1.151}$$

For three-dimensional analysis, the argument of the Green's function within the vector potentials is given by

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \tag{1.152}$$

After converting the observation point (x, y, z) to spherical coordinates and grouping terms according to powers of r, we obtain

$$R = r\sqrt{1 - \frac{2}{r}(x' \sin\theta \cos\phi + y' \sin\theta \sin\phi + z' \cos\theta) + \frac{(x')^2 + (y')^2 + (z')^2}{r^2}} \quad (1.153)$$

Approximations similar to those employed to derive Equation (1.133) produce the far-field expression

$$R \cong r - x' \sin\theta \cos\phi - y' \sin\theta \sin\phi - z' \cos\theta \quad (1.154)$$

It follows that far-field forms of the vector potential functions are

$$\bar{A}(r, \theta, \phi) = \frac{e^{-jkr}}{4\pi r} \iiint \bar{J}(x', y', z') e^{jk(x' \sin\theta \cos\phi + y' \sin\theta \sin\phi + z' \cos\theta)}\, dx'\, dy'\, dz' \quad (1.155)$$

and

$$\bar{F}(r, \theta, \phi) = \frac{e^{-jkr}}{4\pi r} \iiint \bar{K}(x', y', z') e^{jk(x' \sin\theta \cos\phi + y' \sin\theta \sin\phi + z' \cos\theta)}\, dx'\, dy'\, dz' \quad (1.156)$$

Consequently, the scattering cross section can be written as

$$\sigma(\theta, \phi, \theta^{\text{inc}}, \phi^{\text{inc}}) = \sigma_\theta(\theta, \phi) + \sigma_\phi(\theta, \phi) \quad (1.157)$$

where

$$\sigma_\theta(\theta, \phi) = \frac{k^2}{4\pi} \left| \iiint (\eta J_x \cos\theta \cos\phi + \eta J_y \cos\theta \sin\phi - \eta J_z \sin\theta - K_x \sin\phi + K_y \cos\phi) e^{jk(x' \sin\theta \cos\phi + y' \sin\theta \sin\phi + z' \cos\theta)}\, dx'\, dy'\, dz' \right|^2 \quad (1.158)$$

$$\sigma_\phi(\theta, \phi) = \frac{k^2}{4\pi} \left| \iiint (-\eta J_x \sin\phi + \eta J_y \cos\phi + K_x \cos\theta \cos\phi + K_y \cos\theta \sin\phi - K_z \sin\theta) e^{jk(x' \sin\theta \cos\phi + y' \sin\theta \sin\phi + z' \cos\theta)}\, dx'\, dy'\, dz' \right|^2 \quad (1.159)$$

and where we have assumed that the magnitude of the incident electric field is unity. Equations (1.158) and (1.159) are written in terms of triple integrals over volume sources; as in the two-dimensional case the integrals collapse to surface integrals in an obvious way if the equivalent sources are confined to surfaces.

As shown for emphasis in Equation (1.157), σ is a function of the direction of the incident field and the far-zone observation angle. The bistatic scattering cross section is also a function of the polarization of the incident wave. To explicitly characterize the scatterer as a function of polarization, the scattering cross section data can be obtained for two orthogonal polarizations and arranged in the form of a scattering matrix such as

$$\Sigma = \begin{bmatrix} \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix} \quad (1.160)$$

The entries of this 2×2 matrix remain a function of the direction of the incident field and the far-zone observation angle.

1.13 APPLICATION TO ANTENNA ANALYSIS

Although preceding sections of this chapter have dealt exclusively with scattering problems, it is important to note that most antenna radiation problems can be analyzed using identical techniques. To illustrate the connection between scattering formulations and antenna analysis in detail, consider a monopole antenna of radius a and height $L/2$ radiating over a p.e.c. ground plane (Figure 1.18). The monopole is coincident with the z-axis of a cylindrical coordinate system and is fed by a coaxial transmission line with outer radius b. The ground plane is located at $z = 0$, and for simplicity we assume that the aperture of the transmission line contains only a transverse electromagnetic mode with electric field distribution

$$\bar{E}(\rho, \phi) = \hat{\rho}\frac{E_0}{\rho \ln(b/a)} \tag{1.161}$$

Following the procedure discussed in Section 1.10, we introduce an equivalent magnetic current density defined at the transmission line aperture. The method of images is used to remove the ground plane, leaving a dipole antenna of length L (Figure 1.19) illuminated by a "magnetic frill" source that can be expressed by the volume current density

$$\bar{K}(\rho, \phi, z) = -\hat{\phi}\frac{2E_0}{\rho \ln(b/a)} p(\rho; a, b)\delta(z) \tag{1.162}$$

where the pulse function

$$p(\rho; a, b) = \begin{cases} 1 & a < \rho < b \\ 0 & \text{otherwise} \end{cases} \tag{1.163}$$

serves as a window to identify the original aperture location. The equivalent dipole produces the same fields in the upper half space as the original monopole.

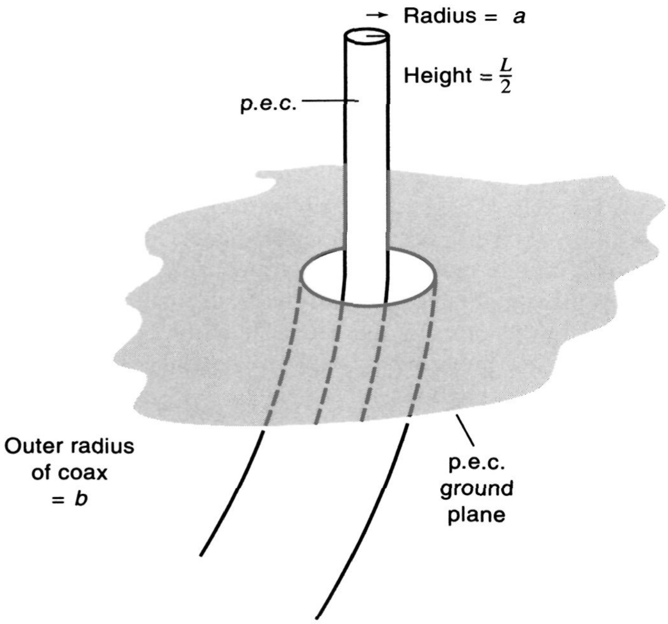

Figure 1.18 Monopole antenna radiating over a p.e.c. ground plane.

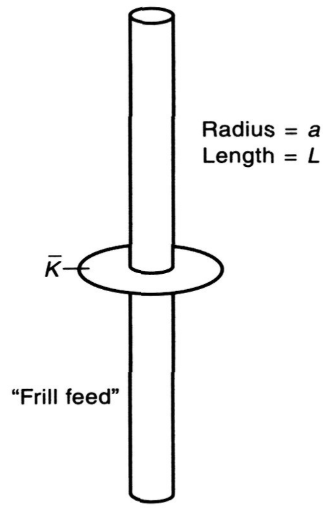

Figure 1.19 Equivalent problem of a dipole fed by a magnetic frill source.

At this point in the development, the problem is that of a p.e.c. scatterer (the dipole) illuminated by an incident field (produced by the magnetic frill source). The electric field produced by a frill source is the topic of Prob. P1.17, and the z-component of the incident field produced on the surface of the monopole is

$$E_z^{\text{inc}}(a, \phi, z) = \frac{2E_0}{\ln(b/a)} \int_{\phi'=0}^{2\pi} \left(\frac{e^{-jkR_1}}{4\pi R_1} - \frac{e^{-jkR_2}}{4\pi R_2} \right) d\phi' \tag{1.164}$$

where R_1 and R_2 are defined in Prob. P1.17. Thus, the antenna currents can be determined from a surface integral equation, as discussed in Section 1.7. (A specific EFIE for the dipole antenna will be presented in Chapter 8.)

In general, the primary difference between antenna analysis and the scattering formulations considered earlier is that the primary source in an antenna geometry is located immediately adjacent to the scatterer, rather than an infinite distance away. In fact, antennas actually function by scattering the energy emitted by the primary feed. For instance, the arms of the dipole in Figure 1.19 scatter (or focus) the energy radiated by the magnetic frill source in order to produce the characteristic dipole radiation pattern. Yagi antennas contain parasitic elements that clearly act as scatterers to enhance the radiation pattern. Reflector antennas focus the energy from a primary feed horn to achieve a narrow radiation beam. Almost all types of antennas can be thought of as scatterers and posed in terms of differential equation or integral equation formulations.

1.14 SUMMARY

Chapter 1 has reviewed concepts from electromagnetic theory that play an important role in the numerical procedures of interest. Of particular importance are the source–field relations summarized in Section 1.4 and the equivalence principles presented in Sections 1.2 and 1.6. These ideas are central to the formulation of integral equations for scatterers or antennas.

Integral equations for conducting bodies, penetrable bodies, and aperture problems have been developed in Sections 1.7–1.10. These equations will be specialized to a variety of situations in the chapters to follow. For instance, Chapter 2 considers integral equation formulations for two-dimensional (infinite cylinder) geometries. Differential equations have also been presented and will provide the foundation for alternate numerical solution methods. Chapter 3 presents several ways of using the scalar Helmholtz equation to treat two-dimensional open-region geometries. Subsequent chapters extend these procedures to three dimensions and to a variety of other situations.

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PROBLEMS

P1.1 (a) Derive Equations (1.13) and (1.14).

(b) In a homogeneous region with $\varepsilon_r = 1$ and $\mu_r = 1$, containing electric and magnetic sources $\bar{J}$ and $\bar{K}$, respectively, Equations (1.1) and (1.2) become

$$\nabla \times \bar{E} = -j\omega\mu_0\bar{H} - \bar{K} \qquad \nabla \times \bar{H} = j\omega\varepsilon_0\bar{E} + \bar{J}$$

Under these conditions, show that the equivalent curl–curl equation for $\bar{E}$ can be obtained as

$$\nabla \times \nabla \times \bar{E} - k^2\bar{E} = -j\omega\mu_0\bar{J} - \nabla \times \bar{K}$$

Find the corresponding equation for $\bar{H}$.

P1.2 Repeat Prob. P1.1(a) for the case where ε_r and μ_r are replaced by tensors to represent anisotropic material.

P1.3 Under the assumption that the fields in some region do not depend on z, show (using Maxwell's equations) that the field components E_z, H_x, and H_y (the TM part) are completely independent from H_z, E_x, and E_y (the TE part).

P1.4 (a) Under the assumption that the z-dependence of the electric field is $e^{-j\gamma z}$, show that the z-component of Equation (1.13) can be expressed as

$$\nabla_t \cdot \left(\frac{1}{\mu_r}\nabla_t E_z\right) + k^2\varepsilon_r E_z = \frac{1}{\mu_r}\gamma^2 E_z - j\gamma\nabla_t\left(\frac{1}{\mu_r\varepsilon_r}\right)\cdot\varepsilon_r\bar{E}_t$$

where ∇_t is the transverse part of the operator, for instance,

$$\nabla_t E_z = \hat{x}\frac{\partial E_z}{\partial x} + \hat{y}\frac{\partial E_z}{\partial y}$$

(b) Using duality (Table 1.1), write down the analogous equation involving the magnetic field $\bar{H}$.

(c) In certain cases, the z-component of the vector Helmholtz equation can be used instead of the complete vector Helmholtz equation to produce a solution. In general, this occurs whenever the z-components of the field decouple from the transverse components, so that the equations in (a) and (b) coincide with the scalar Helmholtz equations in (1.15) and (1.16). Identify two situations where the equations obtained in (a) and (b) constitute a sufficient description of the fields E_z and H_z. (*Hint:* Obviously, the case $\gamma = 0$ is one answer. What is the other situation?)

(d) What is the physical interpretation of zero γ? Of nonzero γ?

P1.5 (TM–TE Decomposition) In a homogeneous region with fields having z-dependence $e^{-j\gamma z}$, show that the transverse-to-z field components can be expressed as a function of the z-components according to

$$\bar{E}_t = \frac{1}{k^2 - \gamma^2}\left(-j\gamma \nabla_t E_z + j\omega\mu_0 \hat{z} \times \nabla_t H_z\right)$$

$$\bar{H}_t = \frac{1}{k^2 - \gamma^2}\left(-j\omega\varepsilon_0 \hat{z} \times \nabla_t E_z - j\gamma \nabla_t H_z\right)$$

P1.6 (a) Assuming that ε_r is sufficiently differentiable, derive Equation (1.23) using the identity

$$\nabla\left(\frac{1}{\varepsilon_r}\right) = -\left(\frac{1}{\varepsilon_r}\right)^2 \nabla\varepsilon_r$$

(b) At a jump discontinuity in the permittivity, the normal component of $\bar{E}$ must behave according to Equation (1.11), that is,

$$(1) \qquad 0 = \varepsilon_0\varepsilon_1 E_1^{\text{nor}} - \varepsilon_0\varepsilon_2 E_2^{\text{nor}}$$

In the equivalent problem constructed by replacing the dielectric material with induced sources $\bar{J}$ and ρ_e, the proper behavior at the location of the original dielectric interface is

$$(2) \qquad \rho_{\text{es}} = \varepsilon_0 E_1^{\text{nor}} - \varepsilon_0 E_2^{\text{nor}}$$

where ρ_{es} represents a surface charge density, and the normal direction points from region 2 into region 1. Demonstrate that Equation (1.23) is consistent with (1) and (2) and therefore produces the proper surface charge density at an interface when the permittivity has a jump discontinuity. Because of the generalized function interpretation of (1.23), the volume charge density in that equation will appear as a Dirac delta function in the three-dimensional space in order to represent a surface charge density.

P1.7 (a) Consider the subsectional "triangle" function

$$t(x) = \begin{cases} \dfrac{x+\Delta}{\Delta} & -\Delta < x < 0 \\ \dfrac{\Delta - x}{\Delta} & 0 < x < \Delta \\ 0 & \text{otherwise} \end{cases}$$

Show that the second derivative with respect to x is

$$t''(x) = \frac{d^2 t}{dx^2} = \frac{1}{\Delta}\delta(x+\Delta) - \frac{2}{\Delta}\delta(x) + \frac{1}{\Delta}\delta(x-\Delta)$$

Sketch t, t', and t''.

(b) A "sinusoidal triangle" function can be defined

$$s(x) = \begin{cases} \dfrac{\sin(kx + k\Delta)}{\sin(k\Delta)} & -\Delta < x < 0 \\ \dfrac{\sin(k\Delta - kx)}{\sin(k\Delta)} & 0 < x < \Delta \\ 0 & \text{otherwise} \end{cases}$$

Show that

$$\begin{aligned} &\frac{d^2 s}{dx^2} + k^2 s(x) \\ &= \frac{k}{\sin(k\Delta)}\delta(x + \Delta) - \frac{2k\cos(k\Delta)}{\sin(k\Delta)}\delta(x) + \frac{k}{\sin(k\Delta)}\delta(x - \Delta) \end{aligned}$$

P1.8 (a) In a homogeneous medium, use Maxwell's equations

$$\nabla \times \bar{E} = -j\omega\mu_0\bar{H} - \bar{K} \qquad \nabla \times \bar{H} = j\omega\varepsilon_0\bar{E} + \bar{J}$$
$$\nabla \cdot (\varepsilon_0\bar{E}) = \rho_e \qquad \nabla \cdot (\mu_0\bar{H}) = \rho_m$$

to derive the equations of continuity

$$\nabla \cdot \bar{J} = -j\omega\rho_e \qquad \nabla \cdot \bar{K} = -j\omega\rho_m$$

(b) Using the preceding results and the vector Laplacian

$$\nabla^2\bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla \times \nabla \times \bar{A}$$

derive Equations (1.31) and (1.32).

P1.9 Show that the three-dimensional Green's function

$$G = \frac{e^{-jkr}}{4\pi r}$$

satisfies the scalar Helmholtz equation

$$\nabla^2 G + k^2 G = -\delta(r)$$

(*Hint:* The calculation for $r \neq 0$ is straightforward. In the vicinity of $r = 0$, integrate the equation throughout a sphere of radius r, and use the divergence theorem to obtain

$$\iiint_V \nabla \cdot \nabla G \, dv = \iint_S \nabla G \cdot \hat{r} \, ds = -1$$

as $r \to 0$.)

P1.10 A subsectional "pulse" function can be defined as

$$p\left(x; -\frac{\Delta}{2}, \frac{\Delta}{2}\right) = \begin{cases} 1 & -\dfrac{\Delta}{2} < x < \dfrac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$

Show that the convolution

$$\begin{aligned} &p\left(x; -\frac{\Delta}{2}, \frac{\Delta}{2}\right) * p\left(x; -\frac{\Delta}{2}, \frac{\Delta}{2}\right) \\ &= \int_{x'=-\Delta/2}^{\Delta/2} p\left(x - x'; -\frac{\Delta}{2}, \frac{\Delta}{2}\right) dx' \\ &= \Delta t(x) \end{aligned}$$

where $t(x)$ is the subsectional triangle function defined in Prob. P1.7.

P1.11 The potential functions in Equations (1.48) and (1.49) satisfy the Lorentz gauge conditions

$$\nabla \cdot \bar{A} = -j\omega\varepsilon_0\Phi_e \qquad \nabla \cdot \bar{F} = -j\omega\mu_0\Phi_m$$

Using these conditions, substitute (1.48) and (1.49) into Maxwell's equations to show that $\bar{A}$, $\bar{F}$, Φ_e, and Φ_m can be decoupled from one another to produce

$$\nabla^2\bar{A} + k^2\bar{A} = -\bar{J} \qquad \nabla^2\bar{F} + k^2\bar{F} = -\bar{K}$$

$$\nabla^2\Phi_e + k^2\Phi_e = -\frac{\rho_e}{\varepsilon_0} \qquad \nabla^2\Phi_m + k^2\Phi_m = -\frac{\rho_m}{\mu_0}$$

P1.12 Using properties of the Fourier transform integral (see, e.g., Chapter 7), demonstrate that differentiation and convolution operations commute. Specifically, for two differentiable functions $a(x)$ and $b(x)$, show that

$$\frac{d}{dx}[a(x) * b(x)] = \frac{da}{dx} * b(x) = a(x) * \frac{db}{dx}$$

This concept provides an alternative way of deriving Equations (1.52) and (1.53).

P1.13 A rectangular waveguide of dimension $a \times b$ radiates through an aperture in an infinite p.e.c. ground plane into the half-space $z > 0$. Assume that the only fields present in the aperture are those associated with the TE_{10} mode.

(a) Identify equivalent surface currents located at $z = 0^+$ that, radiating in the presence of an infinite ground plane (no aperture) at $z = 0$, reproduce the fields in the region $z > 0$. (Give an explicit expression for $\bar{J}_s$ and $\bar{K}_s$ in the location of the original aperture, and comment on their values away from the aperture.)

(b) Use the method of images to remove the p.e.c. ground plane and provide expressions for the equivalent surface currents that, radiating in free space, reproduce the $z > 0$ fields. (*Hint:* The image of a magnetic current over a p.e.c. is the mirror image, while the image of an electric source is the negative mirror image.)

P1.14 Repeat P1.13 for the fields of the TE_{11} mode in the aperture of a circular waveguide having radius a, radiating in the presence of an infinite p.e.c. ground plane.

P1.15 Section 1.4 presents several alternative expressions for the fields produced by sources. In particular, given an equivalent source density $\bar{J}$ radiating in free space, the following are just three of the possible formulas:

$$(a) \qquad \bar{E}^s = \frac{\nabla\nabla \cdot +k^2}{j\omega\varepsilon}\bar{A}$$

$$(b) \qquad \bar{E}^s = -j\omega\mu\bar{A} - \nabla\Phi_e$$

$$(c) \qquad \bar{E}^s = \left(\frac{\nabla\nabla \cdot +k^2}{j\omega\varepsilon}\bar{J}\right) * G$$

where $\bar{A}$, Φ_e, and G are defined in Section 1.4. For the case in which $\bar{J}$ is a sphere of uniform current density, that is,

$$\bar{J}(r, \theta, \phi) = \hat{x}\,p(r; 0, a) = \hat{x}\begin{cases} 1 & r < a \\ 0 & \text{otherwise} \end{cases}$$

the electric field at the center of the sphere has been obtained using dyadic Green's functions (D. E. Livesay and K. M. Chen, *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, Dec. 1974) in the form

$$\bar{E}^s(0, 0, 0) = \hat{x}\frac{-1}{j\omega\varepsilon}[1 - \tfrac{2}{3}e^{-jka}(1 + jka)]$$

(a) Derive this result using Equation (c) above. [*Hint:* You should obtain

$$\nabla \cdot \bar{J} = -J_r(\theta, \phi)\delta(r-a) = -\sin\theta\cos\phi\delta(r-a)$$
$$\hat{x} \cdot (\nabla\nabla \cdot \bar{J}) = -\delta'(r-a)\sin^2\theta\cos^2\phi - \tfrac{\delta(r-a)}{r}(\cos^2\theta\cos^2\phi + \sin^2\phi)$$

as intermediate results.]

(b) Why is it difficult to derive the result using Equations (a) or (b)? Explain.

P1.16 The current density along a thin linear center-fed dipole antenna of length 2Δ is often approximated by a sinusoidal function with support confined to the dipole axis. If written in terms of a volume current, this function has the form

$$\bar{J}(\rho, z) = \hat{z} I_0 s(z)\delta(\rho)$$

where s is the sinusoidal triangle function defined in Prob. P1.7. Following a procedure similar to that employed in Prob. P1.15, find the z-component of the electric field at a general location (ρ, ϕ, z) produced by this current density in free space.

P1.17 The magnetic current density

$$\bar{K}(\rho, z) = \hat{\phi}\frac{-1}{\rho\ln(b/a)}p(\rho; a, b)\delta(z)$$

obtained from the aperture transverse electromagnetic field of an open-ended coaxial cable is often used as a "magnetic frill" feed model for a dipole antenna. Using Equation (1.52), show that the z-component of the electric field produced by this magnetic current density radiating in free space can be expressed as

$$E_z(\rho, \phi, z) = \frac{1}{\ln(b/a)}\int_{\phi'=0}^{2\pi}\left(\frac{e^{-jkR_1}}{4\pi R_1} - \frac{e^{-jkR_2}}{4\pi R_2}\right)d\phi'$$

where

$$R_1 = \sqrt{z^2 + \rho^2 + a^2 - 2\rho a\cos\phi'}$$

and

$$R_2 = \sqrt{z^2 + \rho^2 + b^2 - 2\rho b\cos\phi'}$$

P1.18 The surface integral equations for p.e.c. scatterers, (1.96) and (1.97), are readily specialized to the two-dimensional case using

$$\bar{A}(t) = \int_S \bar{J}(t')\frac{1}{4j}H_0^{(2)}\left(k\sqrt{[x(t) - x(t')]^2 + [y(t) - y(t')]^2}\right)dt'$$

(a) For the TM polarization, identify the components of $\bar{E}$, $\bar{H}$, and $\bar{J}$ present and write down a scalar form of the EFIE.

(b) Repeat part (a) for the TE case, producing a scalar form of the MFIE.

P1.19 Equations (1.96) and (1.97) are obtained using the tangential-field boundary conditions $\hat{n} \times \bar{E} = 0$ and $\hat{n} \times \bar{H} = \bar{J}_s$ on the scatterer surface. Similar relationships obtained from the normal-field boundary conditions $\hat{n} \cdot \bar{E} = \rho_s/\varepsilon_0$ and $\hat{n} \cdot \bar{H} = 0$ can be expressed as

$$\hat{n} \cdot \bar{E}^{\text{inc}} = \frac{\rho_s}{\varepsilon_0} - \hat{n} \cdot \left\{\frac{\nabla\nabla \cdot + k^2}{j\omega\varepsilon_0}\bar{A}\right\}_{S^+}$$
$$\hat{n} \cdot \bar{H}^{\text{inc}} = -\hat{n} \cdot \{\nabla \times \bar{A}\}_{S^+}$$

Are these valid equations? Can they be used instead of (1.96) and/or (1.97)? Discuss their possible utility for both two-dimensional and three-dimensional problems.

P1.20 Figure 1.20 depicts a layered dielectric scatterer illuminated by an incident wave. We wish to extend the procedure of Section 1.9 in order to produce a system of coupled surface integral equations describing this problem.

(a) By introducing equivalent sources ($\bar{J}_1$, $\bar{K}_1$) on S_1 and ($\bar{J}_2$, $\bar{K}_2$) on S_2, identify three equivalent problems involving sources radiating in homogeneous space that reproduce the original fields in region A, region B, and the exterior of the scatterer.

(b) For each equivalent problem in (a), construct a surface EFIE. Develop a notation that clearly indicates the medium employed within each equation, the surface over which each integral is evaluated, and the surface on which each equation is enforced. (*Hint:* Use four surfaces: S_1^+, S_1^-, S_2^+, and S_2^-.)

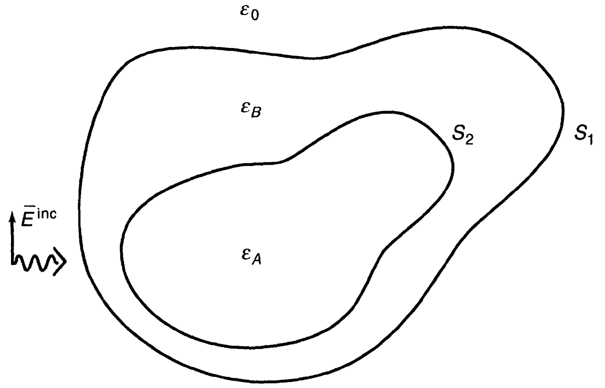

Figure 1.20 Layered dielectric scatterer (for Prob. P1.20).

P1.21 (a) Specialize the two-dimensional form of Equations (1.38) and (1.39) in order to obtain formulas valid in the far field of the sources $\bar{J}$ and $\bar{K}$. Retain only the dominant-order terms as $\rho \to \infty$.

(b) Repeat part (a) for the three-dimensional situation.

P1.22 A wave having the form of Equation (1.124) produces scattered fields $E_z^s(\phi)$ and $H_\phi^s(\phi)$ on a circular boundary of radius a enclosing a two-dimensional obstacle. Express the two-dimensional scattering cross section $\sigma_{TM}(\phi)$ in terms of an integral over E_z^s and H_ϕ^s on the circular contour.

P1.23 Repeat Prob. P1.22 for the three-dimensional case by obtaining an expression for the scattering cross section in the form of an integral over tangential electric fields on the surface of a sphere of radius a.

P1.24 Exterior to the circular contour defined in Prob. P1.22, E_z^s can be written as a Fourier series of the form

$$E_z^s(\rho, \phi) = \sum_{n=-\infty}^{\infty} j^{-n} A_n H_n^{(2)}(k\rho) e^{jn\phi}$$

where

$$A_n = \frac{1}{2\pi j^{-n} H_n^{(2)}(ka)} \int_{\phi'=0}^{2\pi} E_z^s(a, \phi') e^{-jn\phi'} \, d\phi'$$

and where H_n denotes the nth-order Hankel function. Using the asymptotic approximation

$$H_n^{(2)}(k\rho) \approx \sqrt{\frac{2j}{\pi k \rho}} j^n e^{-jk\rho}$$

to simplify your result, find an expression for the two-dimensional scattering cross section σ_{TM} as a function of the coefficients $\{A_n\}$.