chapter 1

General Considerations

The analysis of power systems usually implies the computation of network voltages and currents under a given set of conditions. In many cases the computation is organized to give a particular kind of information for a special purpose. For example, it may be desirable to determine the current flowing through the ground in a particular situation to facilitate the setting of a ground relay. Figure 1.1 presents an organization for power system computation. The class of problems to the left in the figure are called "steady state" because they are to be solved by algebraic equations. This is not to imply that the system is static or unchanging at the moment in time for which a solution is sought. On the contrary, the system may be undergoing very rapid changes, for example, in a faulted condition. The point is that algebraic equations are much easier to solve than differential equations. We therefore have learned to make good use of the steady state solutions in system planning and for determining system protection. This is like taking a set of photographs of the system behavior under certain specified conditions. From these photographs we can design system additions and protective schemes and can learn a great deal about the system strengths and weaknesses.

The dynamic problems shown to the right in Figure 1.1 are what we usually call "stability problems" in power system work. Here we solve a set of differential equations to determine the behavior of voltages, currents, and other variables as functions of time.

Both the steady state and dynamic problems are usually of large dimension for power system work. Networks of a few hundred nodes are quite common and many machines may be represented in dynamic problems. Thus in both kinds of problems we must be oriented toward computer solutions of some kind.

The emphasis in this text is on the faulted network, both balanced and unbalanced. As will be shown later, the techniques used for unbalanced faulted networks will work equally well for unbalanced normal networks. In most cases we will consider the system to be linear and will use algebraic equations exclusively to take "photographs" of voltage-current relationships under certain given conditions. There is much we can do with these photographs, and these ideas will be explored in some depth.

Unbalanced system operation is one we would usually prefer to ignore. Often when dealing with normal or near-normal operation, this is precisely what we do—ignore it. That is, we assume balanced operation, solve the network on a per phase basis, and extrapolate to obtain information for the remaining two

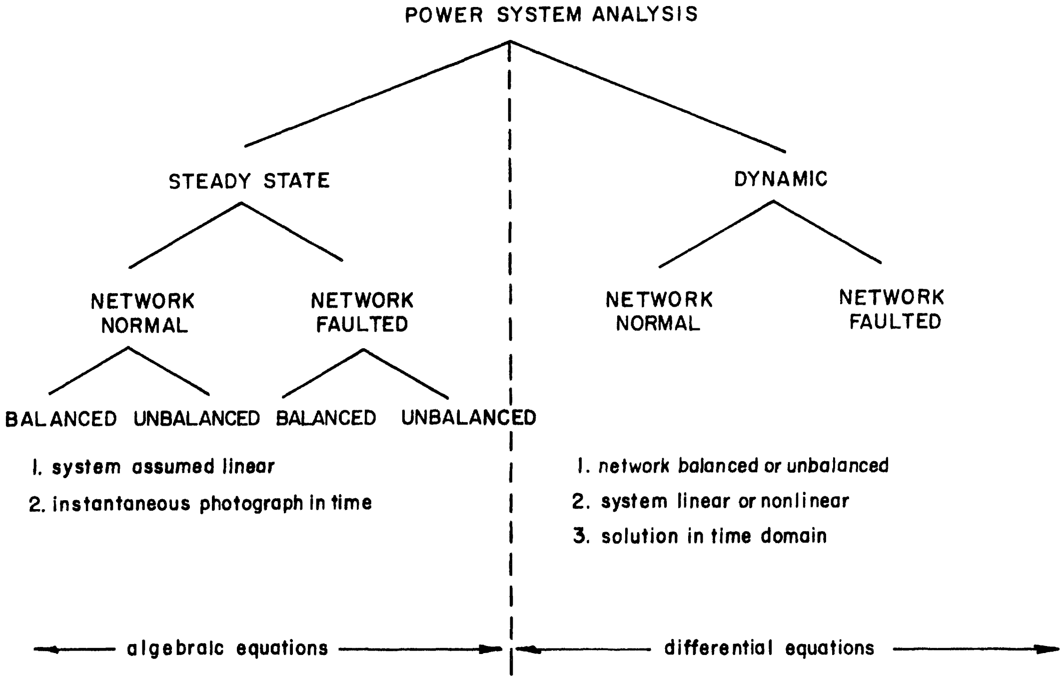

Fig. 1.1. An organization of power system analysis problems.

phases.[1] This results in a great saving in time and effort and usually gives results of reasonable accuracy if the system is nearly normal, i.e., nearly balanced. When the system is obviously unbalanced, other methods must be used. The method generally favored is that of "symmetrical components" as proposed by Fortescue in 1918 [1]. This method permits us to extend the per phase analysis to systems with unbalanced loads or with unbalanced termination of some kind, such as a short circuit or fault. The method is of limited value when the system itself is unbalanced. Our analysis will make extensive use of symmetrical components and will do so with the aid of matrix notation whenever possible. In doing so, we will assume a balanced system ($Z_a = Z_b = Z_c$) with unbalanced loading unless special note is made of a system or network unbalance. The only reason for doing this is to extend the simplicity of the per phase representation to unbalanced system terminations. It may be possible in the future by the use of a large computer to completely represent all three phases of a system and to handle unbalances in either the system or the load in a more direct manner.

The method of symmetrical components has led to the development of useful apparatus to measure or use symmetrical component variables. One example is the negative sequence relay used to protect generators from overheating in the event of unbalanced loading. The positive sequence segregating network is sometimes used to supply the sensing voltage to generator voltage regulators. Certain connections of instrument current and potential transformers develop zero sequence quantities that are used in protective ground relaying schemes.

Finally, a comment concerning the use of digital computers is necessary.

[1] A three-phase system is assumed here since they are by far the most common. When unspecified, a three-phase system will always be implied.

Since power systems are really very large electric networks, it is usually impossible to perform a complete solution by hand computation because of the size of the problem. Thus any solution or technique which we devise must be applicable to the digital computer. The power system engineer should be capable of bridging this gap between theory and computer utilization, and this subject will be emphasized throughout the text.

1.1 Per Unit Calculations

The calculation of system performance conveniently uses a per unit representation of voltage, current, impedance, power, reactive power, and apparent power (voltampere).[2] The numerical per unit value of any quantity is its ratio to the chosen base quantity of the same dimensions. Thus a per unit quantity is a "normalized" quantity with respect to a chosen base value.

Any quantity can be converted to a per unit quantity by dividing the numerical value by a selected base quantity of the same dimensions. In an electrical network five quantities are usually involved in the calculation of networks. These are shown in Table 1.1 together with their dimensions. Our use of the term "dimension" here is admittedly loose since I and V can be specified dimensionally in terms of force, length, time, and charge, for example. The meaning is clear, however. In calculations of steady state phenomena the time dimension is suppressed in phasor notation. Of the five remaining quantities one is dimensionless and the other four are completely described by only two dimensions. Thus an *arbitrary* choice of two quantities as base quantities will automatically fix the other two. In power system calculations the nominal voltage of lines and equipment is almost always known, so the voltage is a convenient base value to choose. A second base is usually chosen to be the apparent power (voltampere). In equipment this quantity is usually known and makes a convenient base. In a system study the voltampere base can be selected to be any convenient value such as 100 MVA, 200 MVA, etc.

Table 1.1. Electrical Quantities and Their Dimensions

Quantity	*Symbol*	*Dimension*
Current, A	I	$[I]$
Voltage, V	V	$[V]$
Voltamperes, S	$S = P + jQ$	$[VI]$
Impedance, Ω	$Z = R + jX$	$[V/I]$
Phase angle	ϕ, θ, etc.	dimensionless
Time, sec	t	$[T]$

The same voltampere base is used in all parts of the system. One base voltage is selected arbitrarily. All other base voltages must be related to the arbitrarily selected one by the turns ratios of the connecting transformers. Special treatment must be given to load tap changing transformers and to network loops wherein the net product of turns ratios around the loop is not equal to unity. This matter is treated in Chapter 7. In three-phase networks the turns ratios used to relate the several base voltages are those of Y-Y or equivalent Y-Y transformers.

[2]Portions of this section are derived from [2], prepared by several authors, particularly W. B. Boast, J. E. Lagerstrom, and J. W. Nilsson.

If we designate a base quantity by the subscript B, we have on a per phase basis

$$\text{base voltamperes} = S_B \quad \text{VA} \tag{1.1}$$

and

$$\text{base voltage} = V_B \quad \text{V} \tag{1.2}$$

Then the base current and base impedance are computed as

$$\text{Base } I = I_B = \frac{S_B}{V_B} \quad \text{A} \tag{1.3}$$

$$\text{Base } Z = Z_B = \frac{V_B}{I_B} = \frac{V_B^2}{S_B} \quad \Omega \tag{1.4}$$

Similarly, we define a base admittance as

$$\text{Base } Y = Y_B = \frac{S_B}{V_B^2} \quad \text{mho} \tag{1.5}$$

Having defined the base quantities, we can normalize any system quantity by dividing by the base quantity of the same dimension. Thus the per unit impedance Z is defined as

$$Z = \frac{Z \text{ ohm}}{Z_B} \quad \text{pu} \tag{1.6}$$

Note that the dimensions (ohm) cancel, and the result is a dimensionless quantity whose units are specified as per unit or pu. At this point one might question the advisability of using the same symbol, Z,. for both the per unit impedance and the ohmic impedance. This is no problem, however, since the problem solver always knows the system of units he is using and it is convenient to use a familiar notation for system quantities. Usually we will remind ourselves that a solution is in per unit by affixing the "unit" pu as in equation (1.6). Furthermore, where there may be a question of units, we will always identify a *system* quantity by adding the apprppriate units, such as the notation (Z ohm) of equation (1.6).

Since we may write $Z = R + jX$ in ohms, we may divide both sides of this equation by Z_B with the result

$$Z = R + jX \quad \text{pu} \tag{1.7}$$

where

$$R = \frac{R \text{ ohm}}{Z_B} \quad \text{pu} \tag{1.8}$$

and

$$X = \frac{X \text{ ohm}}{Z_B} \quad \text{pu} \tag{1.9}$$

In a similar manner we write $S = P + jQ$ voltampere, and dividing through by S_B, we have

$$S = P + jQ \quad \text{pu} \tag{1.10}$$

where

$$P = \frac{P \text{ watt}}{S_B} \quad \text{pu} \tag{1.11}$$

and

$$Q = \frac{Q \text{ var}}{S_B} \quad \text{pu} \tag{1.12}$$

1.2 Change of Base

The question sometimes arises that given a pu impedance referred to a given base, what would be its pu value if referred to a new base? To answer this question, we first substitute equation (1.4) into (1.6) with the result

$$Z = (Z \text{ ohm}) \frac{S_B}{V_B^2} \quad \text{pu} \tag{1.13}$$

Two such pu impedances, referred to their respective base quantities, are now written, using the subscript o for old and n for new.

$$Z_o = (Z \text{ ohm}) \frac{S_{Bo}}{V_{Bo}^2}$$

$$Z_n = (Z \text{ ohm}) \frac{S_{Bn}}{V_{Bn}^2} \tag{1.14}$$

But the system or ohmic value must be the same no matter what the base. Equating the (Z ohm) quantities in (1.14), we have

$$Z_n = Z_o \left(\frac{V_{Bo}}{V_{Bn}}\right)^2 \left(\frac{S_{Bn}}{S_{Bo}}\right) \quad \text{pu} \tag{1.15}$$

Equation (1.15) is very important since it permits us to change base without knowledge of the ohmic value (Z ohm). Note that the pu impedance varies directly as the chosen (new) voltampere base and inversely with the square of the voltage base.

1.3 Base Value Charts

In most power system problems the nominal transmission line voltages are known. If these nominal voltages are chosen as the base voltages, an arbitrary choice for the S_B will fix the base current, base impedance, and base admittance. These values are tabulated in Table 1.2 for several values of system voltage and for several convenient MVA levels. A more extensive list of base values is given in Appendix B.

1.4 Three-Phase Systems

The equation derived for pu impedance (1.13), or its reciprocal for pu admittance, is correct only for a single-phase system. In three-phase systems, however, we often prefer to work with three-phase voltamperes and line-to-line voltages. We investigate this problem by rewriting (1.13) using the subscript LN for "line-to-neutral" and 1ϕ for "per phase," with the result

Table 1.2. Base Current, Base Impedance, and Base Admittance for Common Transmission Voltage Levels and for Selected MVA Levels

	Base Kilovolts	Base Megavolt-Amperes							
		5.0	*10.0*	*20.0*	*25.0*	*50.0*	*100.0*	*200.0*	*250.0*
Base current in amperes	34.5	83.67	167.35	334.70	418.37	836.74	1673.48	3346.96	4183.70
	69.0	41.84	83.67	167.35	209.19	418.37	836.74	1673.48	2091.85
	115.0	25.10	50.20	100.41	125.51	251.02	502.04	1004.09	1255.11
	138.0	20.92	41.84	83.67	104.59	209.18	418.37	836.74	1045.92
	161.0	17.93	35.86	71.72	89.65	179.30	358.60	717.21	896.51
	230.0	12.55	25.10	50.20	62.76	125.51	251.02	502.04	627.55
	345.0	8.37	16.74	33.47	41.84	83.67	167.35	334.70	418.37
	500.0	5.77	11.55	23.09	28.87	57.74	115.47	230.94	288.68
Base impedance in ohms	34.5	238.05	119.03	59.51	47.61	23.81	11.90	5.95	4.76
	69.0	952.20	476.10	238.05	190.44	95.22	47.61	23.81	19.04
	115.0	2645.00	1322.50	661.25	529.00	264.50	132.25	66.13	52.90
	138.0	3808.80	1904.40	952.20	761.76	380.88	190.44	95.22	76.18
	161.0	5184.20	2592.10	1296.05	1036.84	518.42	259.21	129.61	103.68
	230.0	10580.00	5290.00	2645.00	2116.00	1058.00	529.00	264.50	211.60
	345.0	23805.00	11902.50	5951.25	4761.00	2380.50	1190.25	595.13	476.10
	500.0	50000.00	25000.00	12500.00	10000.00	5000.00	2500.00	1250.00	1000.00
Base admittance in micromhos	34.5	4200.80	8401.60	16803.19	21003.99	42007.98	84015.96	168031.93	210039.91
	69.0	1050.20	2100.40	4200.80	5251.00	10502.00	21003.99	42007.98	52509.98
	115.0	378.07	756.14	1512.29	1890.36	3780.72	7561.44	15122.87	18903.59
	138.0	262.55	525.10	1050.20	1312.75	2625.50	5251.00	10502.00	13127.49
	161.0	192.89	385.79	771.58	964.47	1928.94	3857.88	7715.75	9644.69
	230.0	94.52	189.04	378.07	472.59	945.18	1890.36	3780.72	4725.90
	345.0	42.01	84.02	168.03	210.04	420.08	840.16	1680.32	2100.40
	500.0	20.00	40.00	80.00	100.00	200.00	400.00	800.00	1000.00

$$Z = \frac{S_{\text{B-1}\phi}}{V^2_{\text{B-LN}}} (Z \text{ ohm}) \text{ pu} \tag{1.16}$$

and

$$Y = \frac{V^2_{\text{B-LN}}}{S_{\text{B-1}\phi}} (Y \text{ mho}) \text{ pu} \tag{1.17}$$

But using LL to indicate "line-to-line" and 3ϕ for "three-phase," we write for a balanced system

$$V_{\text{B-LN}} = \frac{V_{\text{B-LL}}}{\sqrt{3}} \text{ V} \tag{1.18}$$

and

$$S_{\text{B-1}\phi} = \frac{S_{\text{B-3}\phi}}{3} \text{ VA} \tag{1.19}$$

Making the appropriate substitutions, we compute

$$Z = \frac{S_{\text{B-3}\phi}}{V^2_{\text{B-LL}}} (Z \text{ ohm}) \text{ pu} \tag{1.20}$$

and

$$Y = \frac{V^2_{\text{B-LL}}}{S_{\text{B-3}\phi}} (Y \text{ mho}) \text{ pu} \tag{1.21}$$

These equations are seen to be identical with the corresponding equations derived from per phase voltampere and line-to-neutral voltages. This is fortunate since it makes the formulas easy to recall from memory.

A still more convenient form for (1.20) and (1.21) would be to write voltages in kV and voltamperes in MVA.

Thus we compute

$$Z = \frac{\text{Base MVA}_{3\phi}}{(\text{Base kV}_{\text{LL}})^2} (Z \text{ ohm}) \text{ pu} \tag{1.22}$$

The admittance formula may be expressed two ways, the choice depending upon whether admittances are given in micromhos or as reciprocal admittances in megohms. From (1.21) we compute

$$Y = \frac{(\text{Base kV}_{\text{LL}})^2 \ (Y \,\mu\text{mho})}{(\text{Base MVA}_{3\phi})\ (10^6)} \text{ pu} \tag{1.23}$$

or

$$Y = \frac{(\text{Base kV}_{\text{LL}})^2 \ (10^{-6})}{(\text{Base MVA}_{3\phi})\ (Z \text{ M}\Omega)} \text{ pu} \tag{1.24}$$

Equations (1.23) and (1.24) are both useful in transmission line calculations where the shunt susceptance is sometimes given in micromhos per mile and sometimes in megohm-miles.

Usually the subscripts LL and 3ϕ can be omitted without ambiguity, but it is wise to use caution in these simple calculations. Many a system study has been made useless by a simple error in preparing data, and the cost of such errors can easily run into thousands of dollars.

For transmission lines it is possible to further simplify equations (1.22) to (1.24). In this case the quantities usually known from a knowledge of wire size and spacing are

1. the resistance R in ohm/mile at a given temperature
2. the inductive reactance X_L in ohm/mile at 60 Hz
3. the shunt capacitive reactance X_C in megohm-miles at 60 Hz

We make the following arbitrary assumptions:

$$\text{Base MVA}_{3\phi} = 100 \text{ MVA}$$
$$\text{line length} = 1.0 \text{ mi} \tag{1.25}$$

Values thus computed are on a per mile basis but can easily be multiplied by the line length. Likewise, for any base MVA other than 100 the change of base formula (1.15) may be used to correct the value computed by the method given here. Thus for one mile of line

$$Z = \frac{(Z\ \Omega/\text{mi})\ (\text{Base MVA}_{3\phi})}{(\text{Base kV}_{\text{LL}})^2} = (Z\ \Omega/\text{mi})K_Z \quad \text{pu} \tag{1.26}$$

$$K_Z = \frac{\text{Base MVA}_{3\phi}}{(\text{Base kV}_{\text{LL}})^2} = \frac{100}{(\text{Base kV}_{\text{LL}})^2} \tag{1.27}$$

Similarly, we compute

$$B = \frac{(\text{Base kV}_{\text{LL}})^2\ (10^{-6})}{(\text{Base MVA}_{3\phi})\ (X_C\ \text{M}\Omega\text{-mi})} = \frac{K_B}{X_C} \quad \text{pu} \tag{1.28}$$

where

$$K_B = \frac{(\text{Base kV}_{\text{LL}})^2\ (10^{-6})}{100} = 10^{-8}\,(\text{Base kV}_{\text{LL}})^2 \tag{1.29}$$

The constants K_Z and K_B may now be tabulated for commonly used voltages. Several values are given in Table 1.3.

1.5 Converting from Per Unit Values to System Values

Once a particular computation in pu is completed, it is often desirable to convert some quantities back to system values. This conversion is quite simple and is performed as follows:

$$(\text{pu } I)\ (\text{Base I}) = I \quad \text{A}$$
$$(\text{pu } V)\ (\text{Base V}) = V \quad \text{V}$$
$$(\text{pu } P)\ (\text{Base VA}) = P \quad \text{W}$$
$$(\text{pu } Q)\ (\text{Base VA}) = Q \quad \text{var} \tag{1.30}$$

Usually there is no need to convert an impedance back to ohms, but the procedure is exactly the same.

1.6 Examples of Per Unit Calculations

To clarify the foregoing procedures some simple examples are presented.

Table 1.3. Values of K_Z and K_B for Selected Voltages

Base kV	K_Z	K_B
2.30	18.903592	0.0529×10^{-6}
2.40	17.361111	0.0576
4.00	6.250000	0.1600
4.16	5.778476	0.1731
4.40	5.165289	0.1936
4.80	4.340278	0.2304
6.60	2.295684	0.4356
6.90	2.100399	0.4761
7.20	1.929012	0.5184
11.00	0.826446	1.2100
11.45	0.762762	1.3110
12.00	0.694444	1.4400
12.47	0.643083	1.5550
13.20	0.573921	1.7424
13.80	0.525100	1.9044
14.40	0.482253	2.0736
22.00	0.206612	4.8400
24.94	0.160771	6.2200
33.00	0.091827	10.8900
34.50	0.084016	11.9025
44.00	0.051653	19.3600
55.00	0.033058	30.2500
60.00	0.027778	36.0000
66.00	0.022957	43.5600
69.00	0.021004	47.6100
88.00	0.012913	77.4400
100.00	0.010000	100.0000
110.00	0.008264	121.0000
115.00	0.007561	132.2500
132.00	0.005739	174.2400
138.00	0.005251	190.4400
154.00	0.004217	237.1600
161.00	0.003858	259.2100
220.00	0.002066	484.0000
230.00	0.001890	529.0000
275.00	0.001322	756.2500
330.00	0.000918	1089.0000
345.00	0.000840	1190.2500
360.00	0.000772	1296.0000
362.00	0.000763	1310.4400
420.00	0.000567	1764.0000
500.00	0.000400	2500.0000
525.00	0.000363	2756.2500
550.00	0.000331	3025.0000
700.00	0.000204	4900.0000
735.00	0.000185	5402.2500
750.00	0.000178	5625.0000
765.00	0.000171	5852.2500
1000.00	0.000100	10000.0000
1100.00	0.000083	12100.0000
1200.00	0.000069	14400.0000
1300.00	0.000059	16900.0000
1400.00	0.000051	19600.0000
1500.00	0.000044	$22500.0000 \times 10^{-6}$

Example 1.1

Power system loads are usually specified in terms of the absorbed power and reactive power. In circuit analysis it is sometimes convenient to represent such a load as a constant impedance. Two such representations, parallel and series, are possible as shown in Figure 1.2. Determine the per unit R and X values for both the parallel and series connections.

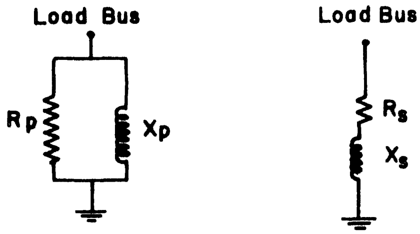

Fig. 1.2. Constant impedance load representation: left, parallel representation; right, series representation.

Solution

Let

$$P = \text{load power in W}$$
$$Q = \text{load reactive power in var}$$
$$R_p \text{ or } R_s = \text{load resistance in } \Omega$$
$$X_p \text{ or } X_s = \text{load reactance in } \Omega$$
$$V = \text{load voltage in V}$$

Parallel Connection. From the parallel connection we observe that the power absorbed depends only upon the applied voltage, i.e.,

$$P = V^2/R_p \tag{1.31}$$

From equation (1.13) we have

$$R_u = \frac{R_p\ (S_B)}{(V_B)^2} \quad \text{pu} \tag{1.32}$$

where the value subscripted u is a pu value. Substituting R_p from (1.31), we compute

$$R_u = (V/V_B)^2\ (S_B/P) = V_u^2/P_u \quad \text{pu} \tag{1.33}$$

and we note that (1.33) is the same as (1.31) except that all values are pu. Similarly, we find the expression for pu X to be

$$X_u = (V/V_B)^2\ (S_B/Q) = V_u^2/Q_u \quad \text{pu} \tag{1.34}$$

Series Connection. If R and X are connected in series as in Figure 1.2 b, the problem is more difficult since the current in X now affects the absorbed power P. In terms of system quantities, $I = V/(R_s + jX_s)$. Thus

$$P + jQ = VI* = \frac{VV*}{R_s - jX_s} = \frac{|V|^2}{R_s - jX_s} \tag{1.35}$$

Multiplying (1.35) by its conjugate, we have

$$P^2 + Q^2 = \frac{|V|^4}{R_s^2 + X_s^2} \tag{1.36}$$

Also, from (1.35)

$$P + \mathrm{j}Q = \frac{|V|^2 (R_s + \mathrm{j}X_s)}{R_s^2 + X_s^2} \tag{1.37}$$

Substituting (1.36) into (1.37), we compute

$$P + \mathrm{j}Q = \frac{(R_s + \mathrm{j}X_s)(P^2 + Q^2)}{|V|^2}$$

Rearranging,

$$R_s + \mathrm{j}X_s = \frac{|V|^2}{P^2 + Q^2}(P + \mathrm{j}Q)\ \Omega \tag{1.38}$$

Equation (1.38) is the desired result, but it is not in pu. Substituting into (1.13), we have

$$R_u + \mathrm{j}X_u = \frac{(R_s + \mathrm{j}X_s) S_{\mathrm{B}}}{V_{\mathrm{B}}^2}$$

Then we compute from (1.38)

$$R_u = \frac{V_u^2\ S_{\mathrm{B}}\ (P \text{ watt})}{P^2 + Q^2}\ \text{pu} \tag{1.39}$$

$$X_u = \frac{V_u^2\ S_{\mathrm{B}}\ (Q \text{ var})}{P^2 + Q^2}\ \text{pu} \tag{1.40}$$

Example 1.2

Given the two-machine system of Figure 1.3, we select, quite arbitrarily, a base voltage of 161 kV for the transmission line and a base voltampere of

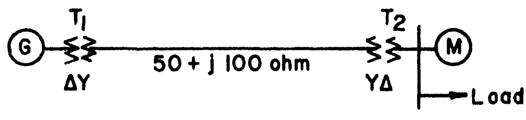

Fig. 1.3. A two-machine system.

20 MVA. Find the pu impedances of all components referred to these bases. The apparatus has ratings as follows:

Generator: 15 MVA, 13.8 kV, $x = 0.15$ pu
Motor: 10 MVA, 13.2 kV, $x = 0.15$ pu
T1: 25 MVA, 13.2-161 kV, $x = 0.10$ pu
T2: 15 MVA, 13.8-161 kV, $x = 0.10$ pu
Load: 4 MVA at 0.8 *pf lag*

Solution

Using equation (1.15), we proceed directly with a change in base for the apparatus.

$$\text{Generator: } x = (0.15)\left(\frac{20}{15}\right)\left(\frac{13.8}{13.2}\right)^2 = 0.2185 \text{ pu}$$

$$\text{Motor: } x = (0.15)\left(\frac{20}{10}\right)\left(\frac{13.2}{13.8}\right)^2 = 0.2745 \text{ pu}$$

$$\text{T1: } x = (0.10)\left(\frac{20}{25}\right)\left(\frac{161}{161}\right)^2 = 0.08 \text{ pu}$$

$$\text{T2: } x = (0.10)\left(\frac{20}{15}\right)\left(\frac{161}{161}\right)^2 = 0.1333 \text{ pu} \qquad (1.41)$$

For the transmission line we must convert from ohmic values to pu values. We do this either by dividing by the base impedance or by application of equation (1.22). Using the latter method,

$$Z = \frac{(50 + \text{j}100 \text{ ohm})(20)}{(161)^2} = 0.0386 + \text{j}0.0771 \text{ pu} \qquad (1.42)$$

For the load a parallel R-X representation may be computed from equations (1.32) and (1.34)

$$\begin{aligned} S = P + \text{j}Q &= |S|(\cos\theta + \text{j}\sin\theta) \\ &= 4(0.8 + \text{j}0.6) \\ &= 3.2 + \text{j}2.4 \text{ MVA} \end{aligned}$$

Then

$$R_u = \frac{V_u^2 S_B}{P} = \frac{V_u^2(20)}{3.2} = 6.25\, V_u^2 \quad \text{pu} \qquad (1.43)$$

Similarly, $X_u = 8.33\, V_u^2$ pu.

Example 1.3

Suppose in Example 1.2 that the motor is a synchronous machine drawing 10 MVA at 0.9 *pf lead* and the terminal voltage is 1.1 pu. What is the voltage at the generator terminals?

Solution

First we compute the total load current. For the motor, with its voltage taken as the reference, i.e., $V = 1.1 + \text{j}0$, we have

$$I_M = \frac{P - \text{j}Q}{V^*} = \frac{9 - \text{j}(-10\sin 25.9°)}{20(1.1)} = 0.409 + \text{j}0.1985 \text{ pu}$$

For the static load

$$I_L = \frac{3.2 - \text{j}2.4}{20(1.1)} = 0.1455 - \text{j}0.109 \text{ pu}$$

Then the total current is $I_M + I_L$ or

$$I = 0.5545 + \text{j}0.0895 \text{ pu} \qquad (1.44)$$

From Example 1.2 we easily find the total pu impedance between the buses to be the total of T1, T2, and Z (line); $Z = 0 + j0.213$ pu. Note that the transmission line impedance is negligible because the base is small and the line voltage high for the small power in this problem. Thus the generator bus voltage is

$$\begin{aligned} V_g &= 1.1 + j0 + (0 + j0.213)(0.5545 + j0.0895) \\ &= 1.1 - 0.0191 + j0.118 = 1.08 + j0.118 \text{ pu} \\ &= 1.087 \underline{/6.24^\circ} \text{ pu on 13.2 kV base} \\ &= 14.32 \text{ kV} \end{aligned}$$

1.7 Phasor Notation

In this book we will deal with voltages and currents which are rms *phasor* quantities. This implies that all signals are pure sine waves of voltage or current but with the time variable suppressed. Thus only the magnitude (amplitude) and relative phase angle of the sine waves are preserved. To do this we use a special definition.

Definition: A phasor A is a complex number which is related to the time domain sinusoidal quantity by the expression[3]

$$a(t) = \mathcal{R}e(\sqrt{2}A\,e^{j\omega t}) \tag{1.45}$$

If we express A in terms of its magnitude $|A|$ and angle α, we have $A = |A|e^{j\alpha}$ and

$$a(t) = \mathcal{R}e(\sqrt{2}\,|A|e^{j(\omega t+\alpha)}) = \sqrt{2}\,|A|\cos(\omega t + \alpha) \tag{1.46}$$

Thus equations (1.45) or (1.46) convert the rms phasor (complex) quantity to the actual time domain variable. We seldom have occasion to do this since it is generally preferable to work exclusively with complex rms phasor quantities.

Note the simplicity of the definition. The phasor is *not* a "rotating vector." It is simply a complex number having the same dimensions (ampere, volt) as the time domain quantity. Note that a single constant frequency (constant ω) is implied by the definition. Since we are dealing with steady state solutions of networks, ω is constant and impedance is a constant ratio of V/I and is itself a phasor (complex) quantity. There is, however, no occasion to convert impedance to the time domain since this is meaningless.

From our phasor definition we may write

$$\begin{aligned} \sqrt{2}\,|A|e^{j(\omega t+\alpha)} &= \sqrt{2}\,|A|\cos(\omega t + \alpha) + j\sqrt{2}\,|A|\sin(\omega t + \alpha) \\ &= a(t) + jb(t) \end{aligned} \tag{1.47}$$

If we differentiate (1.46) with respect to time, we have

$$\dot{a}(t) = -\omega\sqrt{2}\,|A|\sin(\omega t + \alpha)$$

or from (1.47),

$$\dot{a}(t) = -\omega b(t) \tag{1.48}$$

Substituting $b(t)$ from (1.48) into (1.47), we have the differential equation

$$\dot{a}(t) + j\omega a(t) = j\omega\sqrt{2}A e^{j\omega t}$$

[3]The definition could just as well use the imaginary part of $(\sqrt{2}Ae^{j\omega t})$, but either definition should be used consistently.

which has the solution

$$a(t) = \left[a(0) - \frac{A}{\sqrt{2}}\right] e^{-j\omega t} + \frac{A}{\sqrt{2}} e^{j\omega t}$$

This can be rearranged to the form

$$A = \frac{a(t) - a(0)\, e^{-j\omega t}}{j\sqrt{2}\sin \omega t} \tag{1.49}$$

which is a formula for computing a phasor A, based on definition (1.45) and given the time domain sinusoid $a(t)$.

It is convenient to think of (1.49) as a "phasor transformation" which we designate by the script letter $\mathcal{P}$ according to the relation[4]

$$\mathcal{P}[a(t)] = \mathcal{P}[\sqrt{2}\,|A|\cos(\omega t + \alpha)] = |A| e^{j\alpha} = A \tag{1.50}$$

Then the inverse transformation is

$$\mathcal{P}^{-1}[A] = \mathcal{R}e(\sqrt{2} A\, e^{j\omega t}) = a(t) \tag{1.51}$$

Either the symbolic notation of (1.50) or the exact formula (1.49) may be used. Note that the expression (1.50) is a linear transformation [4].

1.8 The Phasor a or a-Operator

In working with three-phase quantities, it is convenient to have a simple phasor operator which will add 120° to the phase angle of a phasor and leave its magnitude unchanged. A common notation for this is to define

$$a = 1\underline{/120^\circ} = e^{j2\pi/3} \tag{1.52}$$

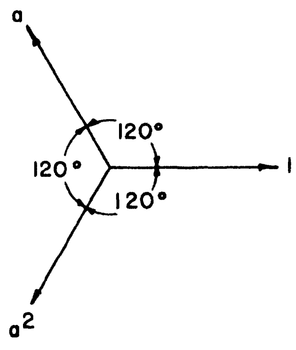

Fig. 1.4. Phasor diagram of the a-operator.

It is convenient to think of the a-operator as one which rotates a phasor by +120° and the a^2-operator as one which rotates a phasor by −120°. It is also clear that (see Fig. 1.4)

$$\begin{aligned} &a^2 = 1\underline{/-120^\circ}, \quad &&a^3 = 1\underline{/0^\circ}, \quad &&a^4 = a \\ &a^5 = a^2, &&a^6 = 1, &&\text{etc.} \end{aligned} \tag{1.53}$$

At times it is convenient to know various linear combinations of a-operators. Some familiar combinations are given in Table 1.4.

[4]This notation is due to Nilsson [3].

Table 1.4. a-Operator Functions

Function	*Polar*	*Rectangular*
a	$1\angle 120^\circ$	$-0.5 + j0.866$
a^2	$1\angle 240^\circ$	$-0.5 - j0.866$
a^3	$1\angle 0^\circ$	$1.0 + j0$
a^4	$1\angle 120^\circ$	$-0.5 + j0.866$
$1 + a = -a^2$	$1\angle 60^\circ$	$0.5 + j0.866$
$1 + a^2 = -a$	$1\angle -60^\circ$	$0.5 - j0.866$
$1 - a$	$\sqrt{3}\angle -30^\circ$	$1.5 - j0.866$
$1 - a^2$	$\sqrt{3}\angle 30^\circ$	$1.5 + j0.866$
$a - 1$	$\sqrt{3}\angle 150^\circ$	$-1.5 + j0.866$
$a^2 - 1$	$\sqrt{3}\angle -150^\circ$	$-1.5 - j0.866$
$a - a^2$	$\sqrt{3}\angle 90^\circ$	$0.0 + j1.732$
$a^2 - a$	$\sqrt{3}\angle -90^\circ$	$0.0 - j1.732$
$a + a^2$	$1\angle 180^\circ$	$-1.0 + j0$
$1 + a + a^2$	0	0

Problems

1.1. Convert all values to pu on a 10 MVA base with 100 kV base voltage on the line.

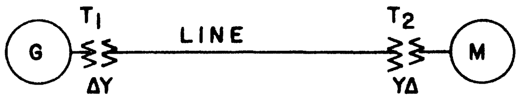

Fig. P1.1.

Generator: 15 MVA, 13.8 kV, $X = 0.15$ pu
Motor: 10 MVA, 12 kV, $X = 0.07$ pu
T1: 20 MVA, 14-132 kV, $X = 0.10$ pu
T2: 15 MVA, 13-115 kV, $X = 0.10$ pu
Line: $200 + j500\ \Omega$

1.2. Prepare a per phase schematic of the system shown and give all impedances in pu on a 100 MVA, 154 kV transmission base.

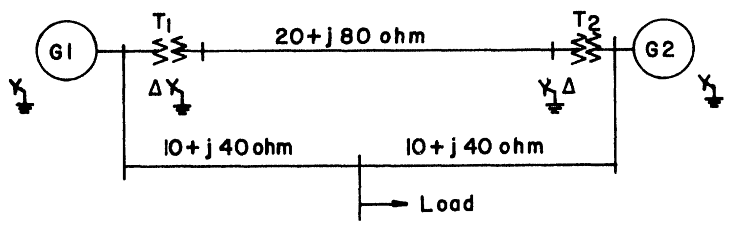

Fig. P1.2.

G1: 50 MVA, 13.8 kV, $X = 15\%$
G2: 20 MVA, 14.4 kV, $X = 15\%$
T1: 60 MVA, 13.2-161 kV, $X = 10\%$
T2: 25 MVA, 13.2-161 kV, $X = 10\%$
Load: 15 MVA, 80% *pf lag*

1.3. Draw a per phase impedance diagram for the system shown. Assume that the load impedance is entirely reactive and equal to j1.0 pu. Find the Thevenin equivalent, looking

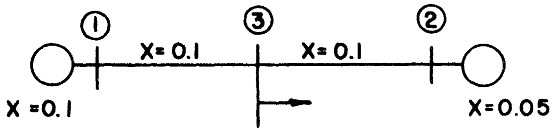

Fig. P1.3.

into this system from an external connection at bus 3 if

(a) Generated voltages V_1 and V_2 are equal.
(b) Generated voltages V_1 and V_2 are not equal.

1.4. The following table of values has been prepared for the various line sections in a small electric system. Find the total pu impedance and shunt susceptance of each line on a 10 MVA base, using the line nominal voltage as a voltage base.

Nominal Voltage (kV)	*Line Length (mi)*	*Wire Size*	*R* (Ω/*mi*)	*X* (Ω/*mi*)	X_C (MΩ-*mi*)
13.8	5.0	4/0 cu	0.278	0.690	0.160
13.8	2.0	4 cu	1.374	0.816	0.193
13.8	3.9	4/0 A	0.445	0.711	0.157
13.8	6.2	336.4 A	0.278	0.730	0.172
13.8	7.3	556.5 A	0.088	0.330	0.142
69.0	10.0	4/0 A	0.445	0.711	0.157
69.0	25.0	336.4 A	0.278	0.730	0.172

1.5. Verify equation (1.49) and compute A for the following time domain sinusoids:

(a) $A_m \cos \omega t$
(b) $A_m \cos(\omega t + \alpha)$
(c) $B_m \sin \omega t$
(d) $B_m \sin(\omega t + \beta)$

1.6. Prove that the transformation $\mathcal{P}\,[a(t)]$ is a linear transformation.

1.7. Given that $a(t)$ is a sinusoid given by equation (1.46), find the phasor transform of $b(t)$ where $b(t)$ is the derivative of $a(t)$ with respect to time.

1.8. Compare the phasor transform $\mathcal{P}$ to the Laplace transform $\mathcal{L}$.