CHAPTER I

PRELIMINARIES

1.1. OBJECTIVE

In the pedagogy of the theory of the electromagnetic field it is customary to pay considerable attention to the plane wave. This is due primarily to the relative simplicity of plane wave solutions of Maxwell's equations; their use enables some of the important elementary physical and engineering characteristics of the electromagnetic field to be elucidated without appeal to other than quite straightforward mathematics.

On the other hand, in somewhat more advanced work, such as diffraction theory, the tradition has mainly been, in the spirit of Huyghens and Fresnel, to think of the field as generated by a distribution of localized sources. Also, of course, the standard retarded potential formulation involving volume integrals over the actual current distribution in effect simply treats each volume element as a dipole.

There is, however, the possibility of continuing to benefit from the simplicity of plane wave solutions by retaining them as the bricks from which to construct whatever more elaborate type of solution arises. This idea has a long history. Its wide exploitation, though, is comparatively recent, as is also the explicit recognition of its close association with the technique of Fourier analysis.

The object of this short book is to explain how general electromagnetic fields can be represented by the superposition of plane waves travelling in divers directions, and to illustrate the way in which this *plane wave spectrum* representation can be put to good use in attacking various characteristic problems belonging to the classical theories of radiation, diffraction and propagation.

It need hardly be said that in a book of this size the problems are not treated exhaustively. To have included alternative theoretical methods, or details of the physical background, or details of the analytical, numerical or physical nature of the solutions, would have tended to swamp the avowed didactic content.

It must also be conceded that various topics that could legitimately be embraced by the title of the book are omitted altogether. Most conspicuous by their absence are problems in which the plane wave spectrum of the field is essentially discrete. Such fields arise typically in cavities and waveguides, and these topics are so fully covered in other books that their inclusion seemed superfluous. Also omitted are problems involving fields in some sense "random" in space or time; their treatment would require the introduction of statistical concepts, which themselves are quite unconnected with the main stream of the mathematical development here presented.

On the positive side of the balance sheet the book offers a largely unified theory of a range of problems, solutions to all of which are obtained in forms at least patently capable of yielding numerical results by straightforward means. The reader is assumed to be competent at integration in the complex plane, but otherwise the discussion is virtually self-contained; the burden of the analysis is carried by the exponential function, and the sprinkling of Bessel functions does not signify the need for any great familiarity with their properties. In this way the aim is to furnish the student of electromagnetic theory with a useful technical tool and a comparatively compact account of some interesting aspects of his discipline.

1.2. MAXWELL'S EQUATIONS

The electromagnetic fields are for the most part assumed to be time-harmonic. The complex representations of the field vectors, with the time factor $\exp(i\omega t)$ understood, are used in the standard way. They satisfy the Maxwell equations

$$\operatorname{curl} \mathbf{E} = -i\omega\mathbf{B}, \tag{1.1}$$

$$\operatorname{curl} \mathbf{H} = i\omega\mathbf{D} + \mathbf{J}. \tag{1.2}$$

In (1.2), $\mathbf{J}$ is the volume current density, and it is associated with the volume charge density ϱ through the charge conservation relation

$$\operatorname{div} \mathbf{J} + i\omega\varrho = 0. \tag{1.3}$$

The divergence of (1.1) gives

$$\operatorname{div} \mathbf{B} = 0; \tag{1.4}$$

and the divergence of (1.2), together with (1.3), gives

$$\operatorname{div} \mathbf{D} = \varrho. \tag{1.5}$$

Any media involved are treated macroscopically, being described by linear constitutive relations between the field vectors, which then denote the "average" fields that would be recorded by conventional laboratory measurements.

If the fields can be regarded as generated by a current in what is otherwise a vacuum, then

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}, \tag{1.6}$$

where ε_0 and μ_0 are the vacuum permittivity and permeability. In this case eqns. (1.1) and (1.2) read

$$\operatorname{curl} \mathbf{E} = -i\omega\mu_0 \mathbf{H}, \tag{1.7}$$

$$\operatorname{curl} \mathbf{H} = i\omega\varepsilon_0 \mathbf{E} + \mathbf{J}. \tag{1.8}$$

At points where there is no current density

$$\operatorname{curl} \mathbf{E} = -i\omega\mu_0 \mathbf{H}, \tag{1.9}$$

$$\operatorname{curl} \mathbf{H} = i\omega\varepsilon_0 \mathbf{E}, \tag{1.10}$$

which imply

$$\operatorname{div} \mathbf{H} = \operatorname{div} \mathbf{E} = 0. \tag{1.11}$$

By eliminating one of **E**, **H** from (1.9), (1.10), and using (1.11), it follows that each cartesian component of **E** and **H** satisfies the time-harmonic, homogeneous wave equation

$$\nabla^2 \varphi + k_0^2 \varphi = 0, \tag{1.12}$$

where

$$k_0^2 = \omega^2 \varepsilon_0 \mu_0. \tag{1.13}$$

It is sometimes convenient to appeal to the converse of this last statement, namely that any divergence free vector each of whose cartesian components satisfy (1.12) can legitimately be identified with either **E** or **H** to specify a vacuum electromagnetic field. It is also worth noting that from any solution of (1.9) and (1.10) another can be deduced at once by the transformation

$$\mathbf{E} \to \mathbf{H}, \quad \mathbf{H} \to -\mathbf{E}, \quad \varepsilon_0 \leftrightarrow \mu_0. \tag{1.14}$$

Difficulties associated with the vector character of eqns. (1.7) and (1.8) are significantly eased in the idealized case in which the field is two-dimensional, being independent of one cartesian coordinate, z say. For the equations then separate into two independent groups, namely

$$\frac{\partial E_z}{\partial y} = -i\omega\mu_0 H_x, \quad \frac{\partial E_z}{\partial x} = i\omega\mu_0 H_y,$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega\varepsilon_0 E_z + J_z, \tag{1.15}$$

and

$$\frac{\partial H_z}{\partial y} = i\omega\varepsilon_0 E_x + J_x, \quad -\frac{\partial H_z}{\partial x} = i\omega\varepsilon_0 E_y + J_y,$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu_0 H_z; \tag{1.16}$$

and any two-dimensional field can therefore be regarded as the superposition of an *E-polarized* field, in which E_z, H_x, H_y, and J_z are the only non-zero field components, and an *H-polarized* field in which H_z, E_x, E_y, J_x and J_y are the only non-zero field components. The identification of E_z with any solution of the two-dimensional, time-harmonic, homogeneous wave equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + k_0^2\varphi = 0, \tag{1.17}$$

completely specifies an E-polarized field in a current free region; the other non-zero field components, H_x and H_y, follow at once from a knowledge of E_z through the first two equations of (1.15). The identification of H_z with any solution of (1.17) likewise specifies an H-polarized field in a current free region.

When isotropic media are considered it is assumed for the sake of simplicity, what is commonly the case in practice, that the permeability differs negligibly from the vacuum permeability μ_0. The constitutive relations are thus taken to be

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu_0\mathbf{H}, \quad \mathbf{J} = \sigma\mathbf{E}, \tag{1.18}$$

where ε and σ are the permittivity and conductivity respectively, and $\mathbf{J}$ in (1.18) of course signifies the conduction current. The substitution of (1.18) into (1.1) and (1.2) gives, at points where there is no impressed current source,

$$\operatorname{curl}\mathbf{E} = -i\omega\mu_0\mathbf{H}, \tag{1.19}$$
$$\operatorname{curl}\mathbf{H} = i\omega(\varepsilon - i\sigma/\omega)\,\mathbf{E}; \tag{1.20}$$

so that the use of the complex representation has the advantage that the effect of conductivity can be readily allowed for by working in terms of the single parameter

$$\varepsilon - i\sigma/\omega, \tag{1.21}$$

which is sometimes called the complex permittivity. The appearance of ω in (1.21) indicates explicitly what may often be the major dependence of the complex permittivity on frequency, but it must not be forgotten that ε and σ are themselves certainly frequency dependent, albeit possibly in effect constant over an appreciable range of frequencies.

Anisotropic media are not treated extensively in this book, but some consideration is given to media that can be characterized, for time-harmonic fields, by the linear constitutive relations

$$\mathbf{D} = \varepsilon_0 \mathscr{K}\mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}, \tag{1.22}$$

where $\mathscr{K}$ is a tensor. The tensor form of the relation between **D** and **E** means that the two vectors are in general no longer parallel. The relation takes account of all current due to the average motion of the charged particles constituting the medium, in the same sort of way as (1.21), and $\mathscr{K}$ can be frequency dependent and complex. For a lossless medium $\mathscr{K}$ is Hermitian; that is, its i, j element κ_{ij} is identical with the complex conjugate κ_{ji}^* of the j, i element. This result follows from a statement of energy balance, consequent on Maxwell's equations, which deserves brief mention.

The interpretation as power flux density of the Poynting vector $\mathbf{E} \wedge \mathbf{H}$, where **E** and **H** momentarily stand for the actual electric and magnetic fields, is well known. For time-harmonic fields it is commonly only the time-averaged power flux density that is of interest, and this is conveniently obtained in terms of the complex representation of the field from the form $\mathrm{Re}\,\frac{1}{2}\mathbf{E} \wedge \mathbf{H}^*$. With (1.22), eqns. (1.1) and (1.2) read

$$\operatorname{curl} \mathbf{E} = -i\omega\mu_0 \mathbf{H}, \tag{1.23}$$

$$\operatorname{curl} \mathbf{H} = i\omega\varepsilon_0 \mathscr{K}\mathbf{E}. \tag{1.24}$$

Thus the mathematical identity

$$\operatorname{div}(\mathbf{E} \wedge \mathbf{H}^*) = \mathbf{H}^* . \operatorname{curl} \mathbf{E} - \mathbf{E} . \operatorname{curl} \mathbf{H}^*$$

gives

$$\operatorname{div}(\mathbf{E} \wedge \mathbf{H}^*) = -i\omega\mu_0 \mathbf{H} . \mathbf{H}^* + i\omega\varepsilon_0 (\mathscr{K}^*\mathbf{E}^*) . \mathbf{E}. \tag{1.25}$$

Now in a lossless medium the time-averaged power flux has zero divergence at any point where there is no impressed current source; that is, the real part of (1.25) is zero. The necessary and sufficient condition that this be so is evidently that $(\mathscr{K}^*\mathbf{E}^*).\mathbf{E}$ be real; or, introducing suffix notation and the summation convention, and equating the expression to its complex conjugate, that

$$\varkappa_{ij}^* E_j^* E_i = \varkappa_{ij}\, E_j E_i^*.$$

If on one side of this relation the dummy suffixes i and j are interchanged, it appears that the condition is indeed $\varkappa_{ji} = \varkappa_{ij}^*$.

1.3. FOURIER INTEGRAL ANALYSIS

There are many ways of expressing the integral representations associated with the names of Fourier and Laplace; these differ in degree of generality, in outlook, in interpretation and in notation. The purpose of this section is merely to record the particular formulation adopted in this book, introducing only those few simple examples that are required subsequently.

The basic concept is the representation of any function $f(\xi)$ of a real variable ξ in the form

$$f(\xi) = \int_{-\infty}^{\infty} F(\eta)\, e^{i\xi\eta}\, d\eta. \tag{1.26}$$

The path of integration is initially presumed to run along the real axis, although distortions permitted by the rules of contour integration may legitimately be introduced later. The spectrum function $F(\eta)$ must therefore at least be defined for effectively all real values of η, and the essence of the Fourier theorem is that for such values

$$F(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, e^{-i\eta\xi}\, d\xi. \tag{1.27}$$

The case of paramount importance in the present context is

$$F(\eta) = \frac{1}{2\pi i(\eta - \eta_0)}. \tag{1.28}$$

Suppose, first, that η_0 has a non-zero imaginary part. Then the path of integration in (1.26) can be closed by an infinite semicircle, above the real axis when $\xi > 0$ and below when $\xi < 0$, without altering the value of the integral. The behaviour as $|\eta| \to \infty$ of the exponential factor in the integrand ensures that there is no contribution from the semicircular part of the path, a standard result which it is not difficult to establish rigorously. Once the path has been closed the value of the integral can be written down from Cauchy's residue theorem; then (1.26) gives

$$f(\xi) = \begin{cases} 0 & \text{for } \xi < 0, \\ e^{i\eta_0\xi} & \text{for } \xi > 0, \end{cases} \tag{1.29}$$

when the imaginary part of η_0 is positive; and

$$f(\xi) = \begin{cases} -e^{i\eta_0\xi} & \text{for } \xi < 0, \\ 0 & \text{for } \xi > 0, \end{cases} \tag{1.30}$$

when the imaginary of η_0 is negative. It requires but a trivial direct integration to confirm that the substitution of (1.29) or (1.30) into the inverse formula (1.27) does indeed correctly recover (1.28).

These results need only be expressed in a slightly different way to cater for the case when η_0 is real. It is then necessary to indent the path of integration in (1.26) so that it avoids the pole at η_0. If the path is chosen to pass above η_0, then $f(\xi)$ is given by (1.30); if below, by (1.29). Several immediate deductions from this case are now listed.

By putting $\eta_0 = 0$ it is established that the *unit step function*

$$f(\xi) = \begin{cases} 0 & \text{for } \xi < 0, \\ 1 & \text{for } \xi > 0, \end{cases} \tag{1.31}$$

has spectrum

$$F(\eta) = \frac{1}{2\pi i\eta}, \tag{1.32}$$

with the η path of integration passing below the origin.

A trivial generalization of (1.31), (1.32) is that

$$f(\xi) = \begin{cases} 0 & \text{for } \xi < \xi_0, \\ 1 & \text{for } \xi > \xi_0, \end{cases} \tag{1.33}$$

has spectrum function

$$F(\eta) = \frac{e^{-i\xi_0\eta}}{2\pi i\eta}, \tag{1.34}$$

with the η path of integration passing below the origin.

By subtracting the unit step function (1.33) for which $\xi_0 = a$ from that which for $\xi_0 = -a$ it is established that the *rectangular* *pulse*

$$f(\xi) = \begin{cases} 0 & \text{for } \xi < -a \\ 1 & \text{for } -a < \xi < a \\ 0 & \text{for } \xi > a \end{cases} \tag{1.35}$$

has spectrum function

$$F(\eta) = \frac{\sin(a\eta)}{\pi\eta}. \tag{1.36}$$

Here, of course, $F(\eta)$ has no singularity at $\eta = 0$, and the η path of integration runs undisturbed along the real axis.

Finally it is remarked that free use will be made of the concept of the delta function. This is written $\delta(\xi)$, and is as usual attributed with the formal definition

$$\delta(\xi) = 0 \quad \text{for} \quad \xi \neq 0, \qquad \int_{-\infty}^{\infty} \delta(\xi)\, d\xi = 1. \tag{1.37}$$

On replacing $f(\xi)$ in (1.27) by $\delta(\xi)$ it appears that the spectrum function of $\delta(\xi)$ is simply $1/(2\pi)$, so that it has the formal representation

$$\delta(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi\eta}\, d\eta. \tag{1.38}$$

A convenient way of thinking of the delta function in the present context is as the limit as $a \to 0$ of $1/(2a)$ times the rectangular pulse (1.35), since, loosely stated, this gives a rectangular pulse of unit area and zero width. The limit as $a \to 0$ of $1/(2a)$ times (1.36) of course recovers the spectrum $1/(2\pi)$.

The correctness of the relations (1.26) and (1.27) has been readily established for the rectangular pulse (1.35) and the associated spectrum (1.36) It is instructive to appreciate that this result can be made the basis of a heuristic demonstration of the validity of the relations for an effectively arbitrary function $f(\xi)$, in the following way. As just noted, (1.36) implies that $\delta(\xi)$ has spectrum $1/(2\pi)$. But $f(\xi)$ can be expressed as a superposition of delta functions; formally

$$f(\xi) = \int_{-\infty}^{\infty} f(\xi')\, \delta(\xi - \xi')\, d\xi'. \tag{1.39}$$

Hence the spectrum $F(\eta)$ of $f(\xi)$ is the corresponding superposition of the functions $\exp(-i\xi'\eta)/2\pi$), these being the spectra of $\delta(\xi - \xi')$; that is,

$$F(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi')\, e^{-i\xi'\eta}\, d\xi'.$$