# 0

# Number Systems and Counting

The focus of this book is the branch of mathematical logic known as Boolean algebra or Boolean logic. Although we will spend considerable time exploring Boolean algebra in its various forms, first we should spend time thinking about numbers and the ways in which we represent them by using symbols.

## 0.1 NUMBERS: SOME BACKGROUND

A **numeral** is a single symbol that represents a quantity or number. A **number system** is a way of assigning combinations of numerals to different quantities. The act of assigning combinations of numerals to different quantities through the use of a number system is called **counting.** Human beings happen to have 10 fingers, five on each hand, which has resulted in the use of the **base 10** or **decimal** number system by most cultures. In fact, the word **digit** comes from the Latin *digitus,* meaning finger.

The decimal system has 10 numerals: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. We can express any number we want by using different combinations of these 10 numerals within the base 10 number system. Some cultures, notably the Mayas, Celts, and Aztecs, developed base 20 number systems probably from originally using both fingers and toes to count. The Sumerians and the Babylonians, for reasons lost in the mists of time, developed a base 60 number system to which we still owe our practice of dividing the hour into 60 minutes, the minute into 60 seconds, and the circle into 360°.

The most direct ancestor of our particular number system, with its rather abstract but indispensable notion of zero, and its use of the position of a given digit to determine the quantity represented by that digit, was developed in India more than a thousand years ago. During the great flourishing of Islam, it slowly spread throughout the Arab world before making its way to Europe. The great popularizer of the Hindu-Arabic number system in Europe was Leonardo of Pisa, a well-traveled mathematician whose 1202 publication *Liber Abaci* brought Europe out of the mathematical dark age of the Roman numeral.

## 0.2 THE DECIMAL SYSTEM: A CLOSER LOOK

If your niece shows you a jar full of pebbles that she gathered at the beach and tells you that she counted them and that there are exactly 608 pebbles in the jar, what exactly is the relationship between the sequence of numerals "608" and the number of pebbles in the jar? How do we make this connection?

The number 608 has three digit positions or numeral slots: the 100s position (which contains a six), the 10s position (which contains a zero), and the ones position (which contains an eight). Every base 10 number consists of numerals in such slots or digit positions. The total value represented by a given numeral in a number is determined by two things: the value of the numeral itself and the value assigned to the position that the numeral occupies in the number. To get the total value carried in each position, we multiply the numeral in the position by the value of the position itself. To get the value of the entire number, we add up the total values for all the positions. Thus, for the number 608,

$$(6 \times 100) + (0 \times 10) + (8 \times 1) = 600 + 0 + 8 = 608.$$

Table 0.1 shows the relationship between each digit position and the value we assign it. Each position represents exactly 10 times more than the position to its right and exactly one-tenth the value of the position to its left. Another way of saying this is that the positions represent successive powers of 10. If we count the positions from right to left beginning with zero (not one), each position carries a value equal to 10 raised to the power of the position number. Stated more mathematically, each position carries the value $10^x$ where $x$ is the number of the

**TABLE 0.1**

| 6 | 0 | 8 |
|---|---|---|
| 100s<br>$10^2$<br>Position 2 | 10s<br>$10^1$<br>Position 1 | 1s<br>$10^0$<br>Position 0 |

position. For example, position 2 is the $10^2$ or 100s position. Position 17 would be the $10^{17}$ or 100,000,000,000,000,000's position.

In any number the digit in the leftmost position is called the **most significant digit** because it carries the most value in the number. The rightmost digit is called the **least significant digit** because it carries the least value in the number.

Look at our example number 608 again. What about the 0 in the 10s position? It is tempting to say that there are no 10s in 608 but that is not really true; there are 60 tens (plus one-eighth of a 10, if you want to get picky). What the 0 in the 10s position really indicates is that there are no leftover 10s that cannot be lumped together into a hundred or a thousand, etc.

Likewise, the 6 in the 100s position means that there are six 100s left over that cannot be lumped together into a thousand (or some even higher power of 10). If the number were 908, there would be nine 100s that could not be lumped together into a thousand. However, if we were to add another hundred to 908, we would have 10 hundreds, which would be an even thousand with no 100s left over, or 1008.

Because each digit position only holds the leftover numbers that could not be lumped together into a higher power of 10, each digit only needs to hold at most nine. Nine is as full as a digit can get before it ''flips over'' into the next digit position.

The numeral 0 is somewhat different from the rest of the base 10 numerals, 1 through 9. It does not actually represent any value. It represents nothing, which is different than saying that it does not represent anything. It is sometimes called a place holder because it occupies space in a number without adding any value to it. Yet without the lowly 0, how could the 1 in 1,000,000 occupy the millions position?

Another property of 0 is that it invisibly occupies all digit positions to the left of the most significant digit of a number; we just do not write them all out for convenience's sake. Thus, 4 is the same as 0000000004. All those 0s mean that there are no 10s, no 100s, no 1000s, etc. When we do choose to write the meaningless 0s in the front of a number, they are sometimes called **padding zeros.** Sometimes padding zeros are necessary when filling out forms (e.g., month of birth: 03).

## 0.3 OTHER BASES

What if people had only seven fingers? There would only be seven digits: 0, 1, 2, 3, 4, 5, and 6 but no 7, 8, or 9. We would count in base 7, in which a digit ''flips over'' into the next higher position when it reaches six not nine. If you had a car with a base 7 odometer, as you drove it would climb mile after mile toward 666666, at which point it would flip back to 000000. Counting in base 7 works as shown in Table 0.2.

**TABLE 0.2**

| Base 10 | Base 7 |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 5 |
| 6 | 6 |
| 7 | 10 |
| 8 | 11 |
| 9 | 12 |
| 10 | 13 |
| 11 | 14 |
| 12 | 15 |
| 13 | 16 |
| 14 | 20 |
| 15 | 21 |
| ... | ... |
| 46 | 64 |
| 47 | 65 |
| 48 | 66 |
| 49 | 100 |
| 50 | 101 |
| 51 | 102 |
| 52 | 103 |

The digit positions in base 7 do not represent the same values that they do in base 10. In base 10, each digit position represents a value 10 times as great as that represented in the position to its right. In base 7 each digit position represents a value seven times as great as that represented by the position to its right and one-seventh as great as that of the position to its left. That is, digit positions represent successively greater powers of seven. There is a ones position at the far right, followed by a sevens position, followed by a 49s ($7^2$) position, followed by a 343s ($7^3$) position, and so forth. The values assigned to each of the first five base 7 digit positions are shown in Table 0.3.

**TABLE 0.3**

| Position 4 | Position 3 | Position 2 | Position 1 | Position 0 |
|---|---|---|---|---|
| $7^4$ | $7^3$ | $7^2$ | $7^1$ | $7^0$ |
| 2401 | 343 | 49 | 7 | 1 |

## EXERCISE 0.1

A. Write out (in base 10) the values assigned to the first five digit positions of the following bases, beginning with the least significant digit position (zero) and working toward the most significant digit position (four):

1. 9
2. 5
3. 4
4. 3
5. 2

## 0.4 CONVERTING FROM BASE 7 TO BASE 10

If we have a number in base 7 e.g., 4625, how do we know how big it is? Because we are not used to working in the base 7 number system, for it to mean anything to us we must be able to convert it to base 10.

To convert a number from base 7 to base 10, we multiply the numeral in each digit position by that digit position's value then add the resulting products together. For example, to find the base 10 representation of the base 7 number 4625, we note that there is a four in the 343s position, a six in the 49s position, a two in the sevens position, and a five in the ones position, as shown in Table 0.4.

Based on this observation, we calculate:

$$(4 \times 7^3) + (6 \times 7^2) + (2 \times 7^1) + (5 \times 7^0) = 4 \times 343 + 6 \times 49 + (2 \times 7) + (5 \times 1) = 1372 + 294 + 14 + 5 = 1685.$$

To avoid confusion in places where confusion is likely, we will put a subscript on numbers indicating their base. Thus, $4625_7 = 1685_{10}$. Naturally, the subscripted number itself, indicating the base, is in base 10.

Consider for a moment the quantity $25_{10}$. It represents the same quantity

**TABLE 0.4**

| 4 | 6 | 2 | 5 |
|---|---|---|---|
| $7^3$ | $7^2$ | $7^1$ | $7^0$ |
| 343 | 49 | 7 | 1 |

that $34_7$ does. To illustrate this, look at Figures 0.1 and 0.2. Each contains the same number of chili peppers, but in each group the chili peppers are arranged differently to reflect the different ways of looking at the same quantity depending on the number system being used.

Figure 0.1 represents 25 chili peppers grouped in a way that makes sense in base 10: two groups of 10 chili peppers each and one group of five chili peppers left over. Figure 0.2 shows the same number of chili peppers but grouped in a way that reflects a base 7 method of counting them: three groups of seven chili peppers each and one group of four chili peppers left over.

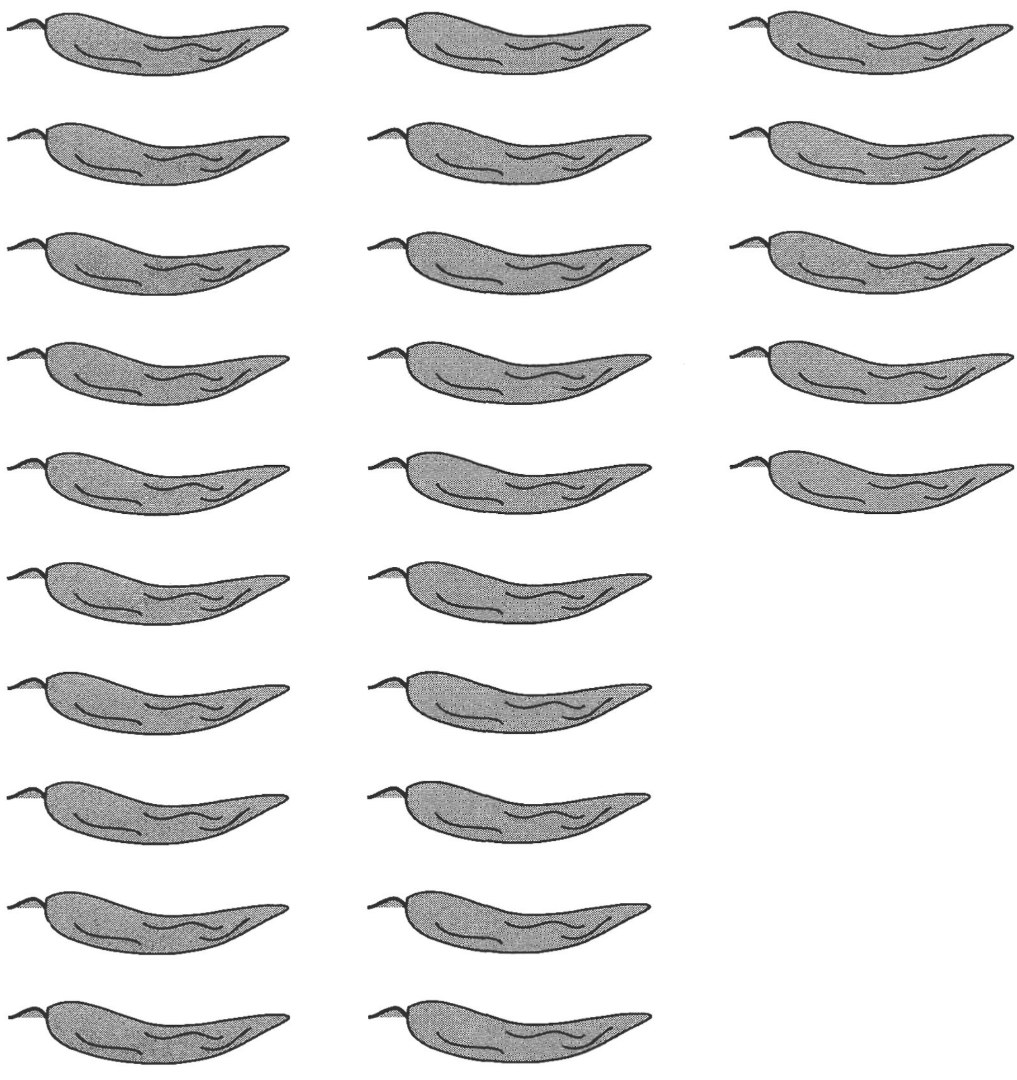

**Figure 0.1** Base 10 style grouping of 25 chili peppers.

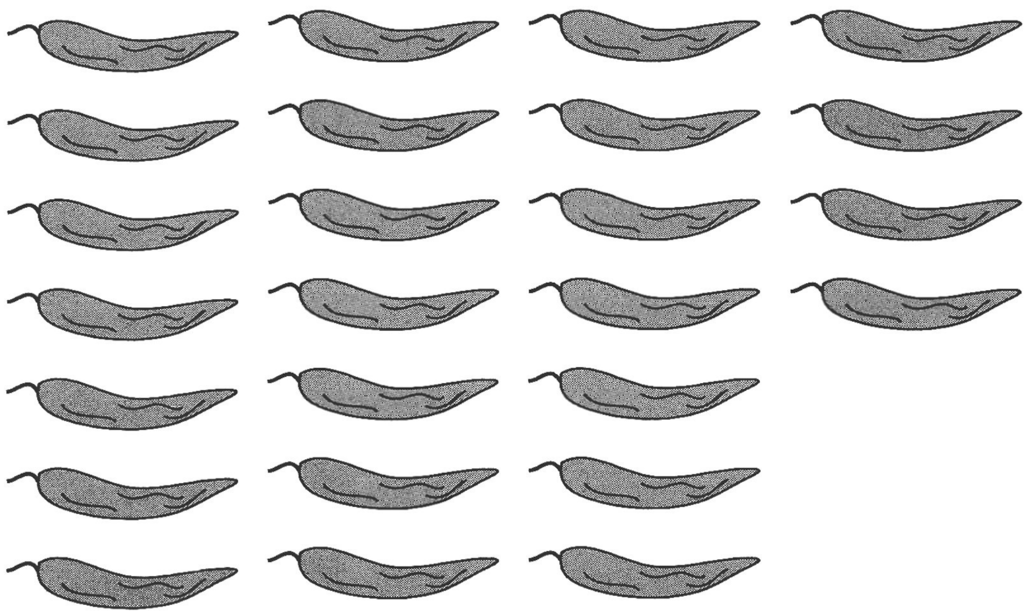

**Figure 0.2** Base 7 style grouping of 25 chili peppers.

## 0.5 CONVERTING FROM BASE 10 TO BASE 7

To convert a number from base 10, which we are used to, to another base (e.g., base 7), we need to rearrange the number from groupings of powers of 10 to powers of 7. We do not draw out chili peppers and move them around as we did in the last example. We regroup the number mathematically using division.

Recall that each digit in a base 7 number can be thought of as ''leftover,'' quantities that were left over after forming groups of greater powers of seven. This concept of ''leftovers'' is the same as the concept of **remainders** in long division. To find the base 7 representation of a number whose base 10 representation is known to us, we keep dividing the number by 7 according to the normal rules of long division. The remainder that results from each division gives us one more digit in our base 7 number, working right to left, from the least significant digit to the most significant digit.

For example, to find out what $967_{10}$ is in base 7, first we divide 967 by seven: $967 \div 7 = 138$, with a remainder of one. (If we had 967 ball bearings and we tried to form them on the ground in neat groups of seven, there would be one left over.) Thus, there is a one in the ones column of our base 7 number. Next, we take the quotient we got before, 138, and divide it by 7. This is equivalent to grouping the groups of seven ball bearings we had before into groups of seven, making groups of 49 ball bearings each: $138 \div 7 = 19$, with a remainder of five. Thus, there are five groups of seven ball bearings that do not fit evenly into any of the groups of 49 ball bearings. Therefore, the digit in the sevens position in the base 7 number is five. To find the digit in the 49s position, we again divide our quotient by 7: $19 \div 7 = 2$, with a remainder of five, therefore a five occupies the 49s position in our base 7 number. Finally, $2 \div 7 = 0$ with a remainder of two, so a two occupies the 343s position in the base 7 number. Now we are done because our number has melted away to zero, and we have nothing more to divide by seven: $967_{10} = 2551_7$. We can check this result by reconverting $2551_7$ into base 10:

$$(2 \times 7^3) + (5 \times 7^2) + (5 \times 7^1) + (1 \times 7^0) =$$

$$(2 \times 343) + (5 \times 49) + (5 \times 7) + (1 \times 1) =$$

$$686 + 245 + 35 + 1 =$$

$$967.$$

Now let us consider base 5, in which the only digits are 0, 1, 2, 3, and 4 and the digit positions have values of successive powers of five, as shown in Table 0.5.

To convert a base 5 number to base 10, we follow the same method we used for base 7: multiply each numeral by the value its digit position represents. For example, to convert $413_5$ to base 10, we calculate as follows:

$$(4 \times 5^2) + (1 \times 5^1) + (3 \times 5^0) =$$

$$(4 \times 25) + (1 \times 5) + (3 \times 1) =$$

$$100 + 5 + 3 = 108_{10}.$$

To convert back to base 5, we use the same sequence of steps for converting base 10 to base 7:

$108_{10} \div 5 = 21$, remainder 3 (3 in 1's position)

$21_{10} \div 5 = 4$, remainder 1 (1 in 5's position)

$4 \div 5 = 0$, remainder 4 (4 in 25s position)

The two methods for converting from base 10 to another base and back

**TABLE 0.5**

| Position 5 | Position 4 | Position 3 | Position 2 | Position 1 | Position 0 |
|---|---|---|---|---|---|
| $5^5$ | $5^4$ | $5^3$ | $5^2$ | $5^1$ | $5^0$ |
| 3125 | 625 | 125 | 25 | 5 | 1 |

again are general and can be applied to any base. Such a distinct sequence of steps is called an **algorithm.**

What is the smallest base we could have? Because any number system has digits 0 through one less than the base of the number system itself, we could not really have a base 1 because it would only have one digit, 0. With only one symbol, we could not use our method of using the digit position to impart value to a numeral because we would have no place holder. However, we can have a base 2 with digits 0 and 1. The base 2 or **binary** number system would be extremely unwieldy to use when balancing your checkbook, for example, but it has some interesting properties that make it of crucial concern for the rest of this book.

On the other hand, could we have a base greater than 10? We could, but only if we invent new digits to represent single-digit numbers greater than nine. By convention, people who need to use such number systems (and there are more of them than you might think) usually use letters to represent these digits. Accordingly, counting in base 16 works as shown in Table 0.6.

The base 16 or **hexadecimal** number system has $16_{10}$ digits: 0 through F ($F_{16} = 15_{10}$).

**TABLE 0.6**

| Base 10 | Base 16 |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 5 |
| 6 | 6 |
| 7 | 7 |
| 8 | 8 |
| 9 | 9 |
| 10 | A |
| 11 | B |
| 12 | C |
| 13 | D |
| 14 | E |
| 15 | F |
| 16 | 10 |
| 17 | 11 |
| 18 | 12 |
| 19 | 13 |

## EXERCISE 0.2

**A.** Count from one to $20_{10}$ in the following bases:

1. 8
2. 6
3. 5
4. 3
5. 2

**B.** Convert the following numbers to their base 10 equivalents:

6. $3124_7$
7. $12332_4$
8. $15366_9$
9. $4000_5$
10. $7122_8$
11. $77_8$
12. $333_4$
13. $111111_2$

**C.** Convert the following base 10 numbers to the indicated base:

14. $413_{10}$ to base 5
15. $128_{10}$ to base 8
16. $3000_{10}$ to base 6
17. $963_{10}$ to base 3
18. $67_{10}$ to base 2

**D.** Perform the indicated base conversions:

19. $54_8$ to base 5
20. $312_4$ to base 7
21. $520_6$ to base 7
22. $12212_3$ to base 9
23. $100_8$ to base 2
24. What generalizations can you draw about converting a number from one base to a power of that base, e.g., from base 3 to base 9 ($3^2$) or from base 2 to base 4 ($2^2$) or base 8 ($2^3$)?

## 0.6 ADDITION IN OTHER BASES

Adding numbers in other bases is much like adding numbers in base 10. Care must be taken, however, because it is easy to forget that you are working in another base. For instance, in base 6, $4 + 3$ does not equal 7, as it does in base 10 because there is no numeral 7 in the base 6 number system. In base 6, $4 + 3 = 11$; a one in the sixes position and a one in the ones position, $11_6 = 7_{10}$.

What about adding multidigit numbers? The familiar rules of carrying apply. To perform the following addition in base 4:

$$\begin{array}{r} 32113 \\ \underline{+11323} \end{array}$$

We begin with the first column and add the numbers in it together: $3_4 + 3_4 = 12_4$. We would then drop the 2 into the first column of the sum and carry the 1:

$$\begin{array}{r} 1\phantom{3} \\ 32113 \\ \underline{+11323} \\ 2 \end{array}$$

Next, we add the numbers in the second column, including the carry: $1_4 + 1_4 + 2_4 = 10_4$. We drop the 0 and carry the 1. We continue in this vein until we have

$$\begin{array}{r} 1111\phantom{3} \\ 32113 \\ \underline{+11323} \\ 110102 \end{array}$$

We can check that $32113_4 + 11323_4 = 110102_4$ by converting all three numbers to base 10 performing the addition on the first two numbers in base 10 and making sure that the sum we get is equal to the third number.

First number:

$32113_4 = (3 \times 4^4) + (2 \times 4^3) + (1 \times 4^2) + (1 \times 4^1) + (3 \times 4^0) =$

$(3 \times 256) + (2 \times 64) + (1 \times 16) + (1 \times 4) + (3 \times 1) =$

$768 + 128 + 16 + 4 + 3 =$

$919_{10}$.

Second number:

$11323_4 =$

$(1 \times 4^4) + (1 \times 4^3) = (3 \times 4^2) + (2 \times 4^1) + (3 \times 4^0) =$

$(1 \times 256) + (1 \times 64) + (3 \times 16) + (2 \times 4) + (3 \times 1) =$

$256 + 64 + 48 + 8 + 3 =$

$379_{10}$.

Sum:

$110102_4 =$

$(1 \times 4^5) + (1 \times 4^4) + (0 \times 4^3)$

$+ (1 \times 4^2) + (0 \times 4^1) + (2 \times 4^0) =$

$(1 \times 1024) + (1 \times 256) + (0 \times 64) + (1 \times 16) + (0 \times 4)$

$+ (2 \times 1) = 1024 + 256 + 0 + 16 + 0 + 2 =$

$1298_{10}$.

$919_{10} + 379_{10} = 1298_{10}$.

### EXERCISE 0.3

**A.** Perform the following additions in the indicated bases. Check your work by converting each number added to base 10, performing the addition, then converting the resulting base 10 sum back into the indicated base.

1. $16_8 + 5_8$
2. $24_5 + 132_5$
3. $524_6 + 312_6$
4. $713_9 + 238_9$
5. $442_6 + 115_6$
6. $265_6 + 333_6$

## 0.7 COUNTING

**Combinatorics** is the science of counting different combinations of things. We are interested in counting combinations of numerals. In this pursuit, we think of numerals as symbols only, distinguishable from one another, but not necessarily as numbers representing quantities. Given this understanding of numerals, 00 and 03 are two different two-digit numbers. How many different two-digit base 10 numbers are there? When we list them all from the smallest, 00, to the largest, 99, we see that the numbers 01 to 99 count themselves. There are 99 numbers from 01 to 99, plus one more for 00. Thus there are exactly 100, or $10^2$, two-digit base 10 numbers.

Similarly, there are a million ($10^6$) six-digit base 10 numbers, 000000 through 999999. That is, there are 999999 numbers from 000001 to 999999, plus one more for 000000. For any base, to find how many different numbers there are in that base of a certain number of digits, we raise the number of the base to the power of the number of digits. Put more mathematically, there are always $x^n$ different $n$-digit base $x$ numbers.

How many different five-digit base 4 numbers are there? Our formula says there should be $4^5$, or $1024_{10}$, or $100000_4$ of them. If we were to list them from the smallest five-digit base 4 number to the largest, the list would look like this: 00000, 00001, 00002, . . . , 33331, 33332, 33333. As before with the base 10 numbers, the numbers from 00001 to 33333 count themselves, so there are $33333_4$ of them. Adding one more for the digit combination 00000 yields $33333_4 + 1_4 = 100000_4 = 1024_{10}$ different five-digit base 4 numbers. This means, in general, that there are $1024_{10}$ ways of arranging five of any symbol that can take on four distinct values.

Imagine that you have a large bag of jelly beans. There are six different colors of jelly beans: green, red, blue, brown, tan, and orange. If, without looking, you reach into the bag and take out three jelly beans and lay them out in a row, how many different combinations of colors could you have? Since there are six different colors, we can imagine that each color represents a different base 6 numeral (0–5), and our problem becomes the same as asking how many different three-digit base 6 numbers there are. Thus there are $6^3 = 1000_6 = 216_{10}$ possible different color combinations in a row of three randomly chosen jelly beans.

As you might imagine, combinatorics is a branch of mathematics that has powerful applications. It is closely related to the mathematical field called **statistics,** which deals with probabilities of given events occurring or not occurring. Card sharks in Las Vegas and Atlantic City train themselves in the techniques of combinatoric analysis to determine the likelihood of certain combinations of cards appearing in their hands and in those of their opponents across the table.

## EXERCISE 0.4

1. How many four digit base 5 numbers are there? Give your answer in base 10 and in base 5.
2. Plot a chart with the numbers 2 to 10 on the horizontal axis ($x$), and 0 to 100 in units of 10 along the vertical axis ($y$). For each value of $x$ (2 through 10) plot a point showing how many two-digit base $x$ numbers there are. Connect the ten resulting points.
3. Why did the $x$ values in the previous exercise begin with two and not zero or one?
4. If seven people in a row on an airplane are each given the choice of coffee, tea, or milk by the flight attendant, how many different combinations of people and drinks could there be in that row of people, assuming each person has something to drink?
5. How many combinations of people and drinks could there be in the situation described in the previous exercise if we consider the possibility that any of the seven people in the row could have declined the offer of a drink?
6. How many combinations of people and drinks could there be (including the possibility that any passenger may not drink at all) on the entire plane if there are 213 passengers? (Do not write out the entire resulting number—represent it using exponents).

7. When we **shift** a number one position to the right, we move each digit in the number one place to the right, allowing the least significant digit to disappear off the end. Thus 6377 shifted one place to the right becomes 637. Similarly, when we shift a number to the left, we move each digit in the number to the left, appending zeros to the least significant digit as needed. Thus, 6377 shifted one place to the left becomes 63770. When a base 10 number is shifted to the left, what implications does that have for the actual quantity represented by the number? What happens to a base 10 number when it is shifted to the right?
8. Give the base 10 representation of $422_5$. Shift $422_5$ to the left one place and two places, and to the right one and two places. Give the base 10 representation of all four resulting numbers. Characterize the effect of shifting a base 5 number to the left or right.
9. Shift $11010010_2$ to the right, converting to base 10 after each shift, until there is nothing left. What is the relationship between the base 10 numbers in the resulting sequence?
10. In general, how does shifting a base $x$ number $y$ places to the right or left affect the quantity represented?

## 0.8 THE BINARY NUMBER SYSTEM

Of particular interest is the base 2 or binary number system. In the binary system, there are only two digits, 0 and 1. Binary digits are called **bits** (for **b**inary dig**it**). The value of each digit position in a binary number is exactly double that of the position to its right and exactly one-half that of the position to its left.

Binary numbers get big quickly as one counts. For example, $256_{10} = 2^8 = 100000000_2$. Counting in base 2 is shown in Table 0.7 (for a longer list see Appendix A).

**TABLE 0.7**

| Base 10 | Base 2 |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 10 |
| 3 | 11 |
| 4 | 100 |
| 5 | 101 |
| 6 | 110 |
| 7 | 111 |
| 8 | 1000 |
| 9 | 1001 |
| 10 | 1010 |

**TABLE 0.8**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 128 | 64 | 32 | 16 | 8 | 4 | 2 | 1 |
| $2^7$ | $2^6$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |

Listed in Table 0.8 are the first eight powers of two in base 10. For a larger list of the powers of two, see Appendix B.

In all of the other bases we have seen, a numeral in a particular digit position told us how many of that digit position's units there were that could not be lumped into a higher power of that base. The 3 in the fives ($5^1$) digit position in $42231_5$ tells us that there are three leftover fives that can not be lumped together into a $5^2$ or a higher power of five.

In the binary number system, however, there are only zeros and ones. So the question of how many of a particular power of two there are that cannot be lumped together into a higher power of two has two possible answers: zero (none) or one. Thus we can rephrase the question from: ''*How many* fours ($2^2$) are there in this number that cannot be lumped into an eight ($2^3$) or a higher power of two?'' to ''*Is there or is there not* a four in this number that can be lumped together into an eight or a higher power of two?'' If there is a one in the fours digit position (as there is in the binary number 10010100), the answer is yes, there is a four that cannot be lumped into a higher power of two. If there is a zero in that position (as in the number 1101001001) the answer is no.

This is why the binary system is of particular interest to us. It is more than a number system that tells us ''how many?'' although it works perfectly well as such. It is also a logic system that tells us ''is there?'' Bits are more than numerals. They can represent **numerical quantities,** as numerals in other bases can, but they also represent **logical qualities.** A 0 and a 1 can mean the numerals 0 and 1, or 0 can mean false and 1 can mean true. 0 can mean off and 1 can mean on. The interpretations we may assign to the symbols 0 and 1 are endless.

Using the techniques we learned previously, we can see that the largest possible eight-bit number is 11111111, the smallest is 00000000, and there are $100000000_2 = 2^8 = 256_{10}$ different eight-bit binary numbers (00000001 through 11111111 ($255_{10}$) plus one more for 00000000 makes exactly $100000000_2$ eight-bit numbers in all).

Similarly, there are $2^2$ or four different two-bit numbers. They range from 00 to 11. There are few enough of them that we can list them: 00, 01, 10, 11. These four represent every possible combination of two bits.

Given two bits, let the first bit represent the color of your shoes, brown = 0 or black = 1, and the second bit represent the pattern on your socks, plain = 0, argyle = 1. Given these choices, how many different combinations of shoes and socks could you wear? We know that the answer is four, because we have two choices of two options each, or two bits. Again, we could list the four combinations: brown shoes and plain socks (00), brown shoes and argyle socks (01), black shoes and plain socks (10), or black shoes and argyle socks (11).

Given a choice of black or brown shoes, plain or argyle socks, and a white or pink shirt, how many total combinations are there? We have added another bit (bit 1: white shirt = 0, pink shirt = 1; bit 2: brown shoes = 0, black shoes = 1; bit 3: plain socks = 0, argyles = 1). We already know that three bits can range from 000 to 111 and that there are $2^3$ or eight possible combinations: 000, 001, 010, 011, 100, 101, 110, 111. Thus, there are eight different combinations of shoes, socks, and shirts. Look at the list of combinations of three bits again. If you ignore the first bit, the last two bits seem to cycle through the same four combinations twice—once with the first bit off, then again with the first bit on. By adding the complication of the choice of shirts to the problem of shoes and socks, we doubled the number of combinations because now we consider all of the original four combinations of shoes and socks twice: once with a white shirt, and again with a pink shirt.

## 0.9 COMBINATORIC EXAMPLES

The following examples illustrate the power of the binary number system with regard to combinatoric analysis of a variety of situations.

### 0.9.1 U.S. Presidential Election Example

Given that in a presidential election, each state in United States votes Democrat or Republican, how many different ways could all 50 states vote (ignoring for the moment such complications as independent candidates or the peculiarities of the electoral college)? That is, from a Democrat shutout in which all 50 states voted Democrat, to a Republican shutout and all combinations in between of different states voting Democrat and Republican, how many different combinations of voting patterns are there for all 50 states taken together?

Assign one bit to each state in order, and let each state's bit be 0 if that state votes Democrat, and 1 if it votes Republican. Then the entire country may be thought of as one big 50-bit binary number, and there are $2^{50}$ (an almost impossibly huge number) different combinations of state-level outcomes in a national election.

### 0.9.2 Pizza Example

If you were at a pizza parlor that sold one size of pizza and seven toppings were available (mushroom, sausage, pepperoni, onion, pepper, anchovies, and extra cheese), how many different kinds of pizza could you order?

Imagine one bit for each topping, seven bits in all. We can represent any particular kind of pizza by a seven-bit binary number, in which each topping's bit tells us whether that topping is present on the pizza (1 = topping is on the pizza, 0 = topping is off the pizza). Thus, all seven bits off (0000000) represents a plain cheese pizza with nothing on it, while all seven bits on (1111111) represents a pizza with the works. It should be clear that there are as many different kinds of pizza as there are combinations of seven bits, or $2^7 = 10000000_2 = 128_{10}$ (see Table 0.9).

**TABLE 0.9** This Combination of Topping Bits Represents a Pizza with Mushroom, Pepperoni, and Extra Cheese

| 1 | 0 | 1 | 0 | 0 | 0 | 1 |
|---|---|---|---|---|---|---|
| mushroom | sausage | pepperoni | onion | pepper | anchovy | extra cheese |

### 0.9.3 Hypercube Example

As a less mundane example, consider the mind-bending hypercube. A hypercube is an abstract mathematical construct which is essentially a square or cube that exists in an arbitrary number of dimensions. To begin with, look at the line segment of length 1 on the number line shown in Figure 0.3.

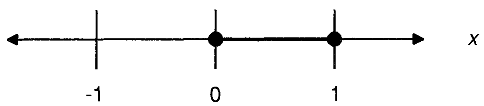


**Figure 0.3** Line segment (one-dimensional hypercube).

This line exists in only one dimension. It has length, but no width or depth. It is considered a one-dimensional hypercube. It has two end points, at 0 and 1, so to specify any endpoint on the hypercube completely, we need only specify one bit (let us call it $x$). Its end points are called $x = 0$ and $x = 1$.

Look at the square in Figure 0.4, graphed on a Cartesian grid. Each of its sides is of length 1. This is a two-dimensional hypercube. It has length and width, but no depth. There are four endpoints (only now we will call them **vertices** or **corners**). To specify a particular vertex we need two bits. The four vertices are shown on the graph and are at the points $(x, y) = (0, 0)$, $(x, y) = (0, 1)$, $(x, y) = (1, 0)$, and $(x, y) = (1, 1)$. Note that the four vertices comprise all possible combinations of two bits, just as the two endpoints on the one-dimensional hypercube ($x = 0$ and $x = 1$) comprised all possible combinations of one bit.

Now look at the hypercube in three dimensions in Figure 0.5, which we know as the cube. This figure has eight vertices, and each requires three bits to specify it completely.

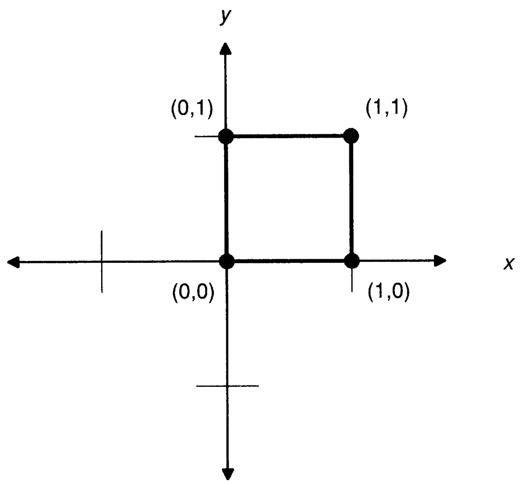


**Figure 0.4** Square (two-dimensional hypercube).

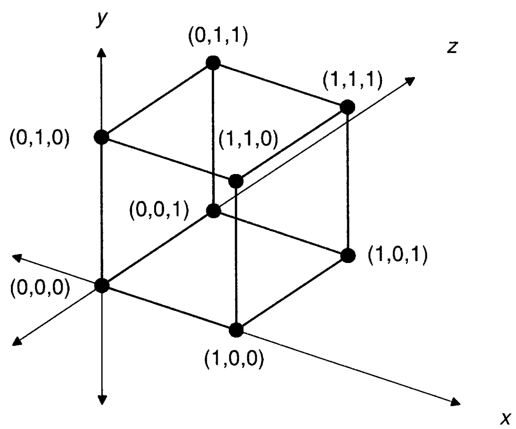


**Figure 0.5** Cube (three-dimensional hypercube).

Based on these examples, we may make some generalizations about hypercubes in any dimension. While most sane people have a hard time visualizing a four-dimensional object, it is safe to say that if there was such a thing as four-dimensional space, a hypercube in that space would have 16 vertices. Each of these vertices would require four bits to specify it completely, one bit for the coordinate along each axis on its graph. In general, for any number $n$, in $n$ dimensions a hypercube has $2^n$ vertices, each of which requires $n$ bits to specify completely. These results are summarized in Table 0.10.

**TABLE 0.10**

| Number of Dimensions | Number of Vertices | Number of Bits Needed to Specify a Vertex |
|---|---|---|
| 1 | 2 | 1 |
| 2 | 4 | 2 |
| 3 | 8 | 3 |
| 4 | 16 | 4 |
| n | $2^n$ | n |

Far from being just a brain toy for mathematics professors, the hypercube in dimensions greater than three has some useful real-world applications, even though it does not actually exist. For instance, in some supercomputers it has been found that an effective way to connect different computing components is as if they were vertices on a hypercube and the communications channels between them were the sides of the hypercube. This way, the efficiency of communication between the large number of computing components is maximized.

### 0.9.4 Binary Trees

A **binary tree** (Fig. 0.6) a diagram used in a number of disciplines. It consists of circles, dots, or boxes called **nodes** and lines called **branches** that connect the nodes. It starts at the top with a single **root node,** which is at level zero. Two branches descend from the root node, one to the left, and one to the right. Each of these branches leads to another node. Each of these two nodes is at level one. They are said to be children of the root node; the root node is their parent. Throughout the binary tree, each node is branched to by a node in the level above it (its parent) and has branches to two nodes in the level below it (its children). A binary tree could go on forever, but generally they end sometime with child

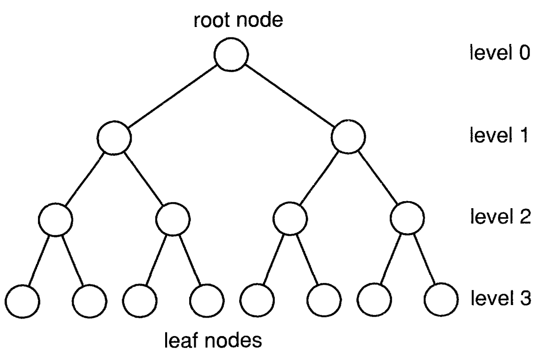


**Figure 0.6** A four-level binary tree.

nodes that have no children of their own. These nodes are called **leaves** or **leaf nodes.**

Binary trees have several interesting properties. Each level has exactly twice the number of nodes as the level above it. Beginning with the root node at level zero, level $n$ has $2^n$ nodes in it, and the entire tree above level $n$ (i.e., levels 0 through $n - 1$) has $2^n - 1$ nodes in it or one less node that level $n$ has. For example, letting $n$ in the previous sentence be 13, level 13 alone has $2^{13}$ nodes in it (8192), and there are $2^{13} - 1$ (8191) nodes in the entire tree in rows 0 through 12.

The terms ''child'' and ''parent'' suggest a genealogical table, although a binary tree only describes successive generations if each person (node) has exactly two children. However, if we think of parents in the binary tree as human children and vice versa, (conceptually turning the tree upside down) a binary tree exactly describes your relationship to your ancestors. If you are the root node, your parents are the two level-one nodes, your grandparents are the four level-two nodes, etc. How many great-great-great-great-great-great-great-great grandparents do you have? There are eight ''great''s there, so that means these are the level 10 ancestors on the binary tree. Thus are $2^{10} = 1024_{10}$ of them.

Binary trees are also sometimes called **decision trees.** Each node can be thought of as a yes/no decision. A yes (1) makes you take the branch to the right. A no (0) makes you take the branch to the left. At each level, you make a choice, starting at the root node. By the time you get to level $n$, you have made $n$ decisions, and whatever node you are on can only be reached by making the exact sequence of choices that you made. Thus, starting at the root node and making only eight yes/no decisions, you find yourself at level eight, at one node among 255 others at that level, each of which could have been reached with a different combination of eight yes/no choices.

If you wrote down a 0 or a 1 reflecting the choice you made at each of the eight levels (0–7) as you ''walked'' a binary tree in this way down to level eight, you would have an eight-bit binary number that uniquely represented the particular sequence of choices that you had made, as well as which uniquely labeled the particular level-eight node you had reached by that sequence of choices. Anyone who has played the game 20 Questions will realize that virtually anything in the world may be singled out on the basis of 20 yes/no answers. Theoretically, then, everything in the world could be given a unique 20-bit identification number, allowing for $2^{20}$ things in the world (about a million). This, however, ignores the fact that the questions asked in the game of 20 questions change on the basis of previous answers.

Tennis tournament charts are arranged in binary tree form. All players start out as leaves at the bottom of the tree and only move up to the level above if they win their match against their ''sibling'' node. Each round of matches eliminates half of the players. Finally, only the player who wins all her matches gets to the root node. Since a binary tree with eight levels of decisions in it (a nine-level tree) has 256 leaves, it takes only eight rounds of matches to whittle 256 ($2^8$) players down to the one champion.

## EXERCISE 0.5

1. How many combinations of heads and tails are possible if a coin is flipped seven times? How many combinations if it is flipped $n$ times?
2. If a given radioactive substance has a half-life of one year, then each year that passes leaves the substance half as radioactive as it had been a year before and twice as radioactive as it will be one year hence. Rounded up to the nearest year, after how many years would such a substance be one three-hundredth as radioactive as it was to begin with? How long would it take for such a substance to lose all of its radioactivity?
3. Arman and Mike review movies on television. Each of them gives each of the five movies reviewed a thumbs up or a thumbs down. On a particular show, how many different ways could the reviews turn out?
4. Describe as well as you can a zero-dimension hypercube.
5. How many combinations of options are possible on a car that has 37 optional features?