CHAPTER 1

STATIC FIELDS AND SOURCES

This chapter is a brief review of the basic laws governing static electric and magnetic fields. The defining relations between static fields and their sources are examined. Definitions and units of measurements for electric and magnetic field strength, flux, flux density, and force are summarized. For a more comprehensive study of static electric and magnetic fields, refer to any elementary text on electromagnetics such as Ramo, Whinnery, and Van Duzer [1]; Kraus and Carver [2]; Plonsey and Collin [3]; or Paul and Nasar [4].

1.1 POINT CHARGE—COULOMB'S LAW

> Charles Augustin de Coulomb (1736–1806) was a French physicist, inventor and army engineer. He made many fundamental contributions in the fields of friction, electricity and magnetism, including the formulation of *Coulomb's law*. The unit for electric charge was named in his honor.[1]

The source of the static electric field is stationary charge. The simplest source is a point charge Q as shown in Fig. 1.1. If a unit positive test charge q is placed in the vicinity of Q, a force $\mathbf{F}$ is exerted on the test charge which is given by Coulomb's law as

$$\mathbf{F} = \frac{Qq}{4\pi\varepsilon R^2}\mathbf{a}_R \tag{1.1}$$

[1] Much of the biographical data in Chapters 1 and 2 is from the *World Book Encyclopedia*, Free Enterprises Educational Corp., Chicago, 1973.

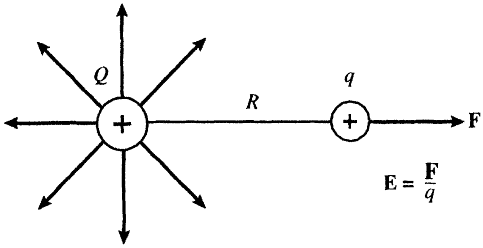


Figure 1.1 Radial electric field from a point charge.

where **F** force in newtons, N
Q charge in coulombs, C
q unit positive test charge, one coulomb
$\varepsilon = \varepsilon_r \varepsilon_o$ permittivity of the medium, farads/meter (F/m)
ε_r relative permittivity (or dielectric constant) of the medium
ε_o permittivity of free space, 8.854×10^{-12} farads/meter
R distance between charges in meters (m)
$\mathbf{a}_R$ unit vector in the radial direction.

By definition, the electric field strength **E** is

$$\mathbf{E} \equiv \frac{\mathbf{F}}{q} \qquad \text{newtons per coulomb (N/C).} \tag{1.2}$$

For a point charge then,

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon R^2}\mathbf{a}_R \qquad \text{volts per meter (V/m).} \tag{1.3}$$

In the International System of Units (SI), the units of volts per meter and newtons per coulomb are equivalent.

Note that the radial electric field from a point charge falls off as $1/R^2$.

1.2 ELECTRIC FLUX DENSITY AND GAUSS'S LAW

Karl Friedrich Gauss (1777–1855) was a German mathematician. Often referred to as the Prince of Mathematics, he is considered one of the greatest mathematicians of all time, ranked with Archimedes and Newton. A child prodigy, he became famous for his work in number theory, geometry, astronomy, and for important contributions to the mathematical theory of electromagnetism. His inventions include the bifilar magnetometer and the electric telegraph.

Electric field strength **E** has dimensions of volts per meter and is a measure of the *intensity* of the field.

Electric flux density **D** is defined as

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{1.4}$$

and has dimensions of coulombs/m^2, or charge per unit area. Thus the term flux *density*. **D** is sometimes called the electric displacement vector.

While **E** is dependent on the permittivity of the medium, **D** is independent of the medium (assuming the medium is isotropic) and depends only on the sources of charge.

For the point charge in Fig. 1.1, we have from (1.3) and (1.4)

$$\mathbf{D} = \frac{Q}{4\pi R^2}\mathbf{a}_R \qquad \text{coulombs per square meter (C/m}^2\text{)}.$$

Now consider the more general case of a distribution of charges.

This could be a distribution of point charges q_i, a line charge distribution with density ρ_l, a surface charge distribution with density ρ_s, a volume charge distribution with density ρ_v, or a combination of these. If these sources are contained within an arbitrary closed surface S as indicated in Fig. 1.2, Gauss's law states that

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enclosed}}. \tag{1.5}$$

That is, the integral of the normal component of the electric flux density over any closed surface is equal to the net charge enclosed.

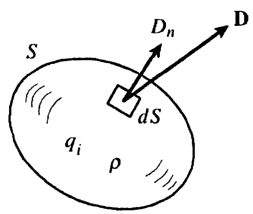


Figure 1.2 Illustration of Gauss's law.

The charge enclosed is given by, in general,

$$Q_{\text{enclosed}} = \sum q_i + \int_l \rho_l dl + \int_s \rho_s\, ds + \int_v \rho_v\, dv.$$

1.3 ELECTRIC FLUX

The electric flux ψ passing through a surface **S** is defined as the product of the normal flux density D_n and the surface area S, assuming that **D** is uniform over S. More generally, when $\mathbf{D}$ is not uniform over the surface (Fig. 1.3), ψ is the surface integral of the scalar product of $\mathbf{D}$ and $d\mathbf{S}$, or

$$\psi = \int_S \mathbf{D} \cdot d\mathbf{S} \quad \text{coulombs (C).}$$

An interesting aside—The surface exists in space and contains no charges. If ψ is a time-varying function, the quantity $d\psi/dt$ has dimensions of coulombs per second, or amperes. That is, a time rate of change of electric flux is associated with a current. This is called displacement current, a concept introduced by Maxwell.

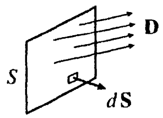


Figure 1.3 Electric flux density.

1.4 CONSERVATION OF ENERGY

If a test charge is moved around any closed path in a static electric field, no net work is done. Since the charge returns to its starting point, the forces encountered on one part of the path are exactly offset by opposite forces on the remainder of the path. The mathematical statement for conservation of energy in a static electric field is

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0. \tag{1.6}$$

This statement is not true for time varying fields, in which case Faraday's law applies.

1.5 POTENTIAL DIFFERENCE

The potential difference between two points a and b immersed in an electric field E (see Fig. 1.4) is defined as the work required to move a unit positive test charge from a to b and is given by

$$V_{ab} = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \quad \text{joules per coulomb (J/C) or volts (V).} \tag{1.7}$$

The potential difference is independent of the path taken from a to b. That is, it depends only on the endpoints. The negative sign in (1.7)

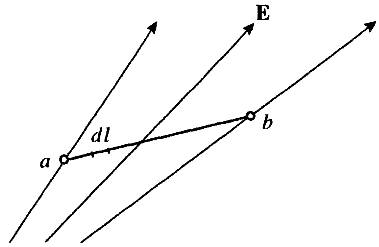


Figure 1.4 Potential difference.

indicates that the field does work on the positive charge in moving from a to b and there is a fall in potential (negative potential difference).

1.6 FIELD FROM LINE AND SURFACE CHARGES

Refer to Fig. 1.5a. The radial electric field from a uniform line charge of infinite extent is

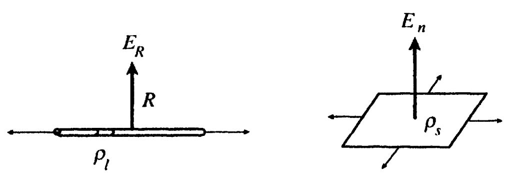
$$E_R = \frac{\rho_l}{2\pi \varepsilon_o R} \tag{1.8}$$

where E_R radial electric field strength, V/m
ρ_l line charge density, coulombs/m
R radial distance, m
ε_o permittivity of free space.

The transverse component of the electric field is zero. Note that the radial component falls off as $1/R$. This solution has applications in transmission line problems, including overhead power distribution and transmission lines.

The field normal to a surface of infinite extent having a uniform surface charge density ρ_s (Fig. 1.5b) is

$$E_n = \frac{\rho_s}{\varepsilon_o} \tag{1.9}$$



(a) Line charge (b) Surface charge

Figure 1.5 Line and surface charges.

and

$$D_n = \rho_s \tag{1.10}$$

where E_n normal electric field, V/m
D_n normal flux density, coulombs/m^2
ρ_s surface charge density, coulombs/m^2.

The field above the surface is constant because the surface is infinite in extent. Again, there is no transverse field component.

1.7 STATIC E-FIELD SUMMARY

A summary of important static electric field relations is provided in Table 1.1. The electric fields from a point charge, an infinite line charge, and an infinite surface charge are given in Table 1.2.

Table 1.1 Summary of Static Electric Field Relations

	Definition	Units
Coulomb's law	$\mathbf{F} = \dfrac{Qq}{4\pi\varepsilon R^2}\mathbf{a}_R$	newtons (N)
E Field	$\mathbf{E} \equiv \dfrac{\mathbf{F}}{q}$	V/m or N/C
Flux density	$\mathbf{D} = \varepsilon\mathbf{E}$	C/m^2
Gauss's law	$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q_{\text{enclosed}}$	C
Electric flux	$\psi = \int_S \mathbf{D}\cdot d\mathbf{S}$	C
Potential difference	$V_{ab} = -\int_a^b \mathbf{E}\cdot d\mathbf{l}$	V or J/C

Table 1.2 Fields from Various Sources

Source	*E* Field
Point charge	$\mathbf{E} = \dfrac{Q}{4\pi\varepsilon R^2}\mathbf{a}_R$
Infinite line charge	$E_R = \dfrac{\rho_l}{2\pi\varepsilon_o R}$
Infinite surface charge	$E_n = \dfrac{\rho_s}{\varepsilon_o}$

1.8 LINE CURRENT—BIOT-SAVART LAW

Biot and Savart, in 1820, established the basic experimental laws relating magnetic field strength to electric currents. They also established the law of force between two currents.

The source of static magnetic fields is charge moving at a constant velocity, namely, direct current (also referred to as steady current or stationary current). By definition, the current through any cross-sectional area is equal to the time rate at which electric charge passes through the area, or

$$I = \frac{dq}{dt} \quad \text{coulombs per second (C/sec) or amperes (A).} \tag{1.11}$$

The current gives rise to a magnetic field. For example, consider an infinitesimal current element $I\, d\mathbf{l}$ as shown in Fig. 1.6. I is the magnitude of the current element, and $d\mathbf{l}$ is a unit vector that defines the direction.

The differential magnetic field strength $d\mathbf{H}$ in vector notation is given by

$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} \quad \text{amperes per meter (A/m)} \tag{1.12}$$

and is one form of the Biot-Savart law. (This law is sometimes attributed to Ampere.)

The magnitude of (1.12) is

$$dH = \frac{I dl \sin\theta}{4\pi R^2} \quad \text{A/m.} \tag{1.13}$$

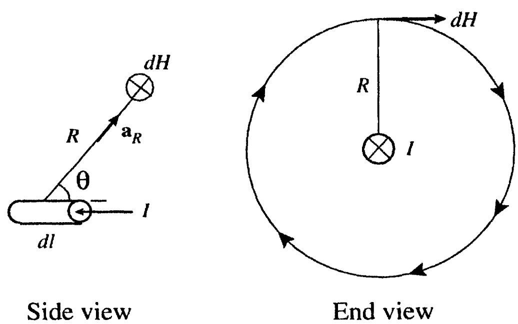


Figure 1.6 Illustration of the Biot-Savart law.

1.9 MAGNETIC FIELD FROM A LINE CURRENT

The magnetic field strength H_ϕ from a line current of infinite extent (Figure 1.7) is obtained directly from the Biot-Savart law by integrating (1.12) over the length of the conductor. The result is

$$H_\phi = \frac{I}{2\pi R} \quad \text{A/m.}$$

The magnetic field strength from an infinite line current falls off as $1/R$. This solution has applications in transmission line analysis, including overhead powerlines.

The direction of the H-field vector in relation to the direction of current flow is given by the "right-hand rule." See Figs. 1.6 and 1.7. If the thumb points in the direction of the current flow, the fingers indicate the positive direction of the magnetic field strength.

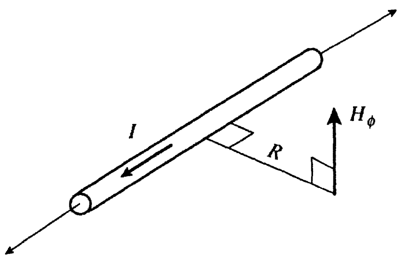


Figure 1.7 Magnetic field from an infinite line current.

1.10 MAGNETIC FLUX DENSITY AND MAGNETIC FLUX

Magnetic field strength **H** has dimensions of amperes per meter and is a measure of the *intensity* of the field, analogous to the electric field strength **E**.

Magnetic flux density **B** is defined as

$$\mathbf{B} = \mu \mathbf{H} \tag{1.14}$$

where **B** magnetic flux density, webers/m^2 (Wb/m^2) or tesla (T)
$\mu = \mu_r \mu_o$ permeability of the medium, henrys/m (H/m)
μ_r relative permeability of the medium
μ_o permeability of free space, $4\pi \times 10^{-7}$ henrys/meter
H magnetic field strength, A/m.

B has dimensions of weber/m^2, or flux per unit area, and is therefore called flux *density*. **B** is analogous to electric flux density **D**.

Lines of magnetic flux are conceptually similar to lines of electric flux (except that lines of magnetic flux close on themselves while electric flux lines terminate on charges). The magnetic flux Φ passing through a surface **S** is defined as the product of the normal magnetic flux density B_n and the surface area S. This assumes that **B** is uniform over **S**. More generally, when **B** is not uniform over the surface (see Fig. 1.8), the magnetic flux is given by the integral over the surface of the scalar product of **B** and $d\mathbf{S}$, that is,

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \qquad \text{webers (Wb).} \tag{1.15}$$

(Note that the symbol used for magnetic flux is the uppercase Greek Φ to distinguish it from the spherical coordinate denoted by the lowercase Greek ϕ.) The unit of webers is equivalent to volt-seconds.

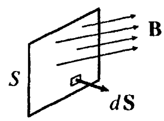


Figure 1.8 Magnetic flux density.

1.11 AMPERE'S LAW

> Andre Marie Ampere (1775–1836) was a French mathematician and physicist. His experiments led to the law of force between current carrying conductors and to the invention of the galvanometer. He postulated that magnetism was due to circulating currents on an atomic scale, showing the equivalence of magnetic fields produced by currents and those produced by magnets.

Ampere's law states that the line integral of the tangential magnetic field strength around any closed path is equal to the net current enclosed by that path. Mathematically,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}}. \tag{1.16}$$

For example, the closed integral of the tangential H-field around the circular path in Fig. 1.6 yields the current I. However, Ampere's law is more general in that it applies to any closed path and it applies to magnetic fields arising from both conduction and convection currents. (Convection currents are charges or charge densities moving with velocity v, i.e., qv and ρv.) For time-varying fields, the right-hand side of (1.16) contains an additional displacement current term.

Ampere's law is the magnetic field analog of Gauss's law, except that it involves a closed contour rather than a closed surface. Comparing (1.16) with (1.6) reveals that while the static electric field is a conservative field, the magnetic field is not.

1.12 LORENTZ FORCE

Hendrick Antoon Lorentz (1853–1928) was a Dutch physicist who became famous for his electron theory of matter. In addition to the formulation of the *Lorentz force* equation, he developed the *Lorentz transformations*, which show how bodies are deformed by motion, and the *Lorentz condition*, which has special significance in relativistic field theory. He shared the 1902 Nobel prize for physics with Pieter Zeeman for discovering the Zeeman effect of magnetism on light.

A point charge q moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ experiences a force, called the Lorentz force, which is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \qquad \text{newtons (N).} \tag{1.17}$$

Equation (1.17) assumes that there is no electric field acting on the point charge. If an electric field is present, the total force acting on the point charge is the sum of the Lorentz force and the $q\mathbf{E}$ force from (1.2), that is,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \text{newtons (N).} \tag{1.18}$$

The Lorentz force on an infinitesimal current element $I d\mathbf{l}$ immersed in a magnetic field $\mathbf{B}$ follows from (1.17), the definition of current $I = dq/dt$, and the velocity $\mathbf{v} = d\mathbf{l}/dt$. We have

$$dq\,\mathbf{v} = dq\,\frac{d\mathbf{l}}{dt} = \frac{dq}{dt}\,d\mathbf{l} = I\,d\mathbf{l}.$$

The instantaneous force on the infinitisimal current element is

$$d\mathbf{F} = I\,d\mathbf{l} \times \mathbf{B} \qquad \text{newtons (N).} \tag{1.19}$$

For the special case of a linear conductor of length $\mathbf{L}$ carrying a current I in a static uniform magnetic field $\mathbf{B}$ (Fig. 1.9), the Lorentz force is

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B} \quad \text{newtons (N)} \tag{1.20}$$

and the magnitude is

$$F = ILB \sin\theta. \tag{1.21}$$

The Lorentz force is proportional to the magnitude of the current, the length of the conductor, the strength of the magnetic field and the sine of the angle

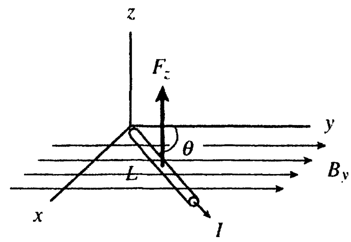


Figure 1.9 Lorentz force on a conductor in a uniform magnetic field.

between the current and the field. The direction of the force is perpendicular to the plane containing the current and the magnetic field.

The Lorentz force is utilized in Hall effect devices, in the focusing and deflection of electron beams in cathode-ray tubes, and in galvanometer movements, to mention a few applications. The Lorentz force has also been suggested as a possible factor in the biological effects of electromagnetic fields. In particular, the movement of charged electrolytes in body fluids in a magnetic field (for example, the earth's magnetic field) may cause changes in body chemistry.

Equation (1.20) can also be interpreted as the defining relation for the magnetic field **B**. If $I d\mathbf{l}$ and **B** are perpendicular

$$B = \frac{F}{I\,dl}. \tag{1.22}$$

That is, the magnetic flux density can be defined as a force per unit current element, analogous to the definition of the electric field as a force per unit charge in equation (1.2). The units of B are the Tesla, or equivalently, webers/m^2 or newtons/ampere-meter.

1.13 MAGNETIC FIELD UNITS AND CONVERSIONS

While SI units are preferred for magnetic field quantities, many applications still use centimeter-gram-second (cgs) units as a matter of custom or tradition. Table 1.3 is a summary of SI and cgs magnetic field units and the conversion factors from one system of units to the other.

ELF magnetic fields from appliances, video display terminals, and power distribution and transmission lines are commonly measured in milligauss. DC magnetic field measurements of magnets and magnetized objects (for complying with FAA regulations for air shipments, for example) are also commonly expressed in milligauss. In fact, most of the instruments used for DC and ELF magnetic field measurements are calibrated in milligauss.

Table 1.3 Magnetic Field Units

	Quantity			
	H	**B**	$\boldsymbol{\Phi}$	μ_o
SI units	Amps/meter A/m	Tesla T	Weber Wb	$4\pi \times 10^{-7}$ H/m
cgs units	Oersted Oe	Gauss Gs	Maxwell Mx	1
Conversion	1 Oe = 79.6 A/m	1 Gs = 10^{-4}T	1 Mx = 10^{-8}Wb	

1.14 STATIC MAGNETIC FIELD SUMMARY

A summary of important static magnetic field relations is provided in Table 1.4.

Table 1.4 Summary of Static Magnetic Field Relations

	Definition	Units
Biot–Savart law	$d\mathbf{H} = \dfrac{Id\mathbf{l} \times \mathbf{a}_R}{4\pi R^2}$	A/m
Flux density	$\mathbf{B} = \mu \mathbf{H}$	T
Magnetic flux	$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$	Wb
Ampere's law	$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}}$	A
Lorentz force	$\mathbf{F} = I\mathbf{L} \times \mathbf{B}$	N

References

[1] S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communications Electronics*, third edition, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1994.

[2] J. D. Kraus and K. R. Carver, *Electromagnetics*, second edition, McGraw-Hill, New York, 1973.

[3] R. Plonsey and R. E. Collin, *Principles and Applications of Electromagnetic Fields*, second edition, McGraw-Hill, New York, 1982.

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