1

Basic Fuzzy Mathematics for Fuzzy Control and Modeling

1.1. INTRODUCTION

Fuzzy control and modeling use only a small portion of the fuzzy mathematics that is available; this portion is also mathematically quite simple and conceptually easy to understand. In this chapter, we introduce some essential concepts, terminology, notations, and arithmetic of fuzzy sets and fuzzy logic. We include only a minimum though adequate amount of fuzzy mathematics necessary for understanding fuzzy control and modeling. To facilitate easy reading, these background materials are presented in plain English and in a rather informal manner with simple and clear notation as well as explanation. Whenever possible, excessively rigorous mathematics is avoided. The materials covered in this chapter are intended to serve as an introductory foundation for the reader to understand not only the fuzzy controllers and models in this book but also many others in the literature.

1.2. CLASSICAL SETS, FUZZY SETS, AND FUZZY LOGIC

1.2.1. Limitation of Classical Sets

In traditional set theory, membership of an object belonging to a set can only be one of two values: 0 or 1. An object either belongs to a set completely or it does not belong at all. No partial membership is allowed. Crisp sets handle black-and-white concepts well, such as "chairs," "ships," and "trees," where little ambiguity exists. They are not sufficient, however, to realistically describe vague concepts.

In our daily lives, there are countless vague concepts that we humans can easily describe, understand, and communicate with each other but that traditional mathematics, including the set theory, fails to handle in a rational way. The concept "young" is an example. For any specific person, his or her age is precise. However, relating a particular age to "young" involves fuzziness and is sometimes confusing and difficult. What age is young and what age is not? The nature of such questions is deterministic and has nothing to do with stochastic concepts such as probability or possibility.

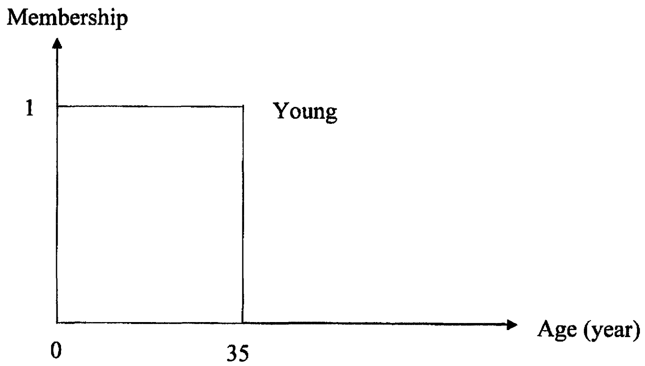

Figure 1.1 A possible description of the vague concept "young" by a crisp set.

A hypothetical crisp set "young" is given in Fig. 1.1. This set is unreasonable because of the abrupt change of the membership value from 1 to 0 at 35. Although a different cutoff age at which membership value changes from 1 to 0 may be used, a fundamental problem exists. Why is it that a 34.9-year-old person is completely "young," while a 35.1-year-old person is not "young" at all? No crisp set can realistically capture, quantitatively or even qualitatively, the essence of the vague concept "young" to reasonably match what "young" means to human beings. This simple example is not meant to discredit the traditional set theory. Rather, the intention is to demonstrate that crisp sets and fuzzy sets are two different and complementary tools, with each having its own strengths, limitations, and most effective application domains.

1.2.2. Fuzzy Sets

Fuzzy set theory was proposed by Professor L. A. Zadeh at the University of California at Berkeley in 1965 to quantitatively and effectively handle problems of this nature [277]. The theory has laid the foundation for computing with words [285][287]. Fuzzy sets theory generalizes 0 and 1 membership values of a crisp set to a membership function of a fuzzy set. Using the theory, one relates an age to "young" with a membership value ranging from 0 to 1; 0 means no association at all, and 1 indicates complete association. For instance, one might think that age 10 is "young" with membership value 1, age 30 with membership value 0.75, age 50 with membership value 0.1, and so on. That is, every age/person is "young" to a certain degree. By plotting membership values versus ages, like the one shown in Fig. 1.2, we generate a fuzzy set "young." The curve in the figure is called the membership function of the fuzzy set "young." All possible ages, say 0 to 130, form a universe of discourse. From this example, a definition of fuzzy sets naturally follows.

Fuzzy set: A fuzzy set consists of a universe of discourse and a membership function that maps every element in the universe of discourse to a membership value between 0 and 1.

Unless otherwise stated, we always use a capital letter and tilde (e.g., $\tilde{A}$) to represent a fuzzy set in this book. If an element is denoted by $x \in X$, where X is a universe of discourse, the membership function of fuzzy set $\tilde{A}$ is mathematically expressed as $\mu_{\tilde{A}}(x)$, $\mu_{\tilde{A}}$, or simply μ. We will use all three representations in the book; the decision of which one to use depends

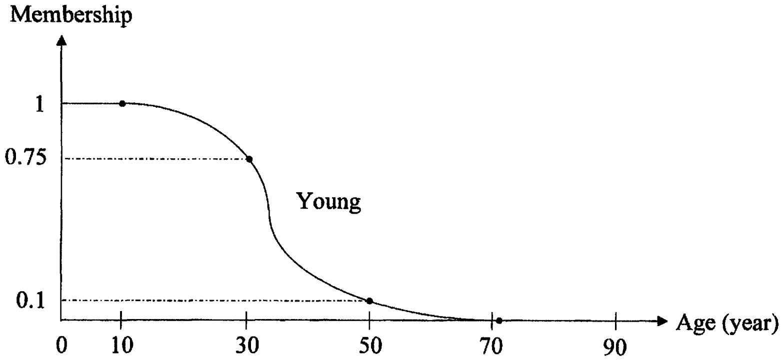

Figure 1.2 A possible description of the vague concept "young" by a fuzzy set.

on the circumstance. For the above age example, $X = [0, 130]$. Letting $\tilde{A}$ denote fuzzy set "young," we can represent its membership function by $\mu_{\tilde{A}}(x)$, where $x \in X$.

People have different views on the same (vague) concept. Fuzzy sets can be used to easily accommodate this reality. Continue the age example. Some people might think age 50 is "young" with membership value as high as 0.9, whereas others might consider that 20 is "young" with membership value merely 0.2. Different membership functions can be used to represent these different versions of "young." Figure 1.3 shows two more possible definitions of the fuzzy set "young." Not only do different people have different membership functions for the same concept, but even for the same person, the membership function for "young" can be different when the context in which age is addressed varies. For instance, a 40-year-old president of a country would likely be regarded as young, whereas a 40-year-old athlete would not. Two different fuzzy sets "young" are needed to effectively deal with the two situations.

These examples show that (1) fuzzy sets can practically and quantitatively represent vague concepts; and (2) people can use different membership functions to describe the same vague concept. We now introduce some definitions needed to describe fuzzy controllers and models.

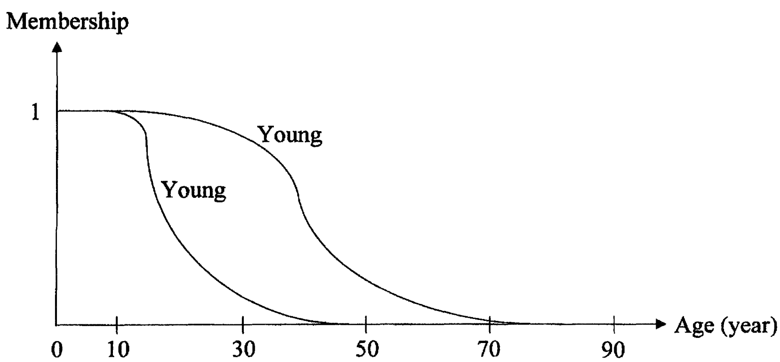

Figure 1.3 Two more possible descriptions of the vague concept "young" by fuzzy sets.

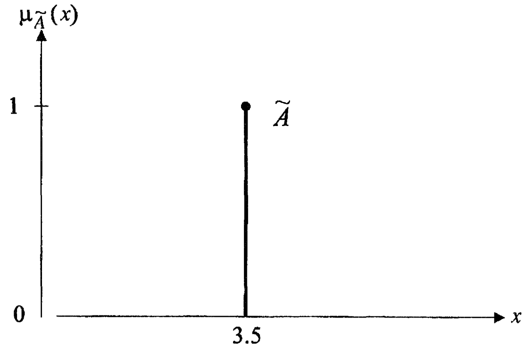

Figure 1.4 An example of the membership function of a singleton fuzzy set.

Continuous fuzzy sets: A fuzzy set is said to be continuous if its membership function is continuous.

Most fuzzy controllers and models nowadays use continuous fuzzy sets.

Singleton fuzzy sets: A fuzzy set that has nonzero membership value for only one element of the universe of discourse is called a singleton fuzzy set. Figure 1.4 exhibits a singleton fuzzy set whose membership value is 0 everywhere except at $x = 3.5$ where the membership value is 1.

The majority of typical fuzzy controllers and models employ singleton fuzzy sets in the consequent of fuzzy rules, as will be shown later in this book.

Support of a fuzzy set: For a fuzzy set whose universe of discourse is X, all the elements in X that have nonzero membership values form the support of the fuzzy set.

As an illustrative example, the support for the fuzzy set "young," shown in Fig. 1.2, is [0,70].

Height of a fuzzy set: The largest membership value of a fuzzy set is called the height of the fuzzy set.

For instance, the height of the fuzzy set "young" in Fig. 1.2 is 1. The height of the fuzzy sets used in fuzzy controllers and models is almost always 1.

Normal fuzzy set and subnormal fuzzy set: A fuzzy set is called normal if its height is 1. If the height of a fuzzy set is not 1, the fuzzy set is said to be subnormal.

The fuzzy sets in Figs. 1.2 and 1.3 are normal fuzzy sets, whereas the fuzzy set in Fig. 1.5 is a subnormal one. Subnormal fuzzy sets are rarely used in fuzzy controllers and models.

Center of a fuzzy set: We need to define this concept for four different situations. If the membership function of a fuzzy set reaches its maximum at only one element of the universe of discourse, the element is called center of the fuzzy set (Fig. 1.6a). If the membership function of a fuzzy set achieves its maximum at more than one element of the universe of discourse and all these elements are bounded, the middle point of the element is the center (Fig. 1.6b). If the membership function of a fuzzy set attains its maximum at more than one element of the universe of discourse and not all of the elements are bounded, the largest element is the center if it is bounded (Fig. 1.6d); otherwise, the smallest element is the center (Fig. 1.6c).

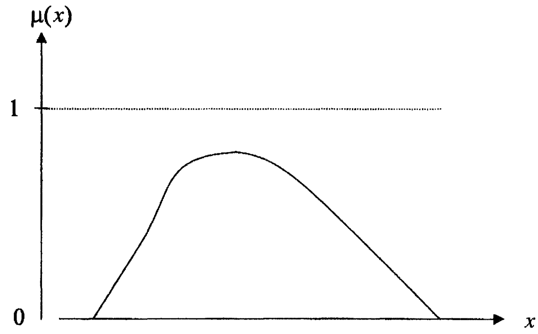

Figure 1.5 An example of a subnormal fuzzy set.

Convex fuzzy sets: Fuzzy set $\tilde{A}$, whose universe of discourse is [a, b], is convex if and only if

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)], \quad \forall x_1, x_2 \in [a,b] \quad \text{and} \quad \forall \lambda \in [0,1],$$

where min() denotes the minimum operator that uses the smaller membership value of the two memberships as the operation result.

The fuzzy set illustrated in Fig. 1.7 is convex, whereas the one shown in Fig. 1.8 is not. To avoid possible confusion, it is important to note that the definition of convex fuzzy sets does not necessarily imply that the membership functions of convex fuzzy sets are convex functions. Nevertheless, the definition requires membership functions to be concave. Of course, according to the definition of convex fuzzy sets, if the membership function of a fuzzy set is convex, the fuzzy set is convex. Typical fuzzy controllers and models employ convex fuzzy sets.

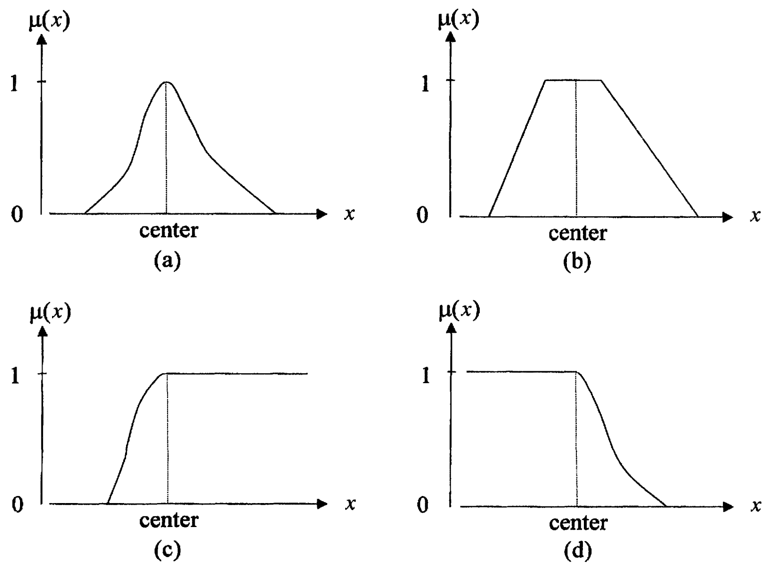

Figure 1.6 A definition of the center of a fuzzy set for four different cases.

According to the definition of fuzzy sets, any function, continuous or discrete, can be a membership function as long as its value falls in [0,1]. The discrete type is uncommon,

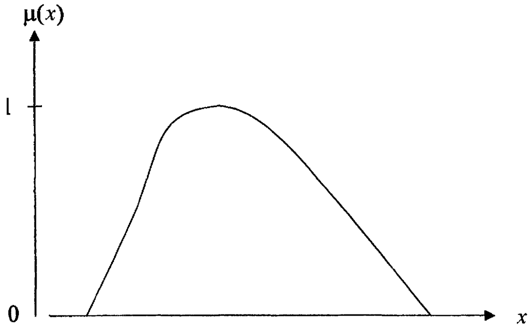

Figure 1.7 An example of a convex fuzzy set.

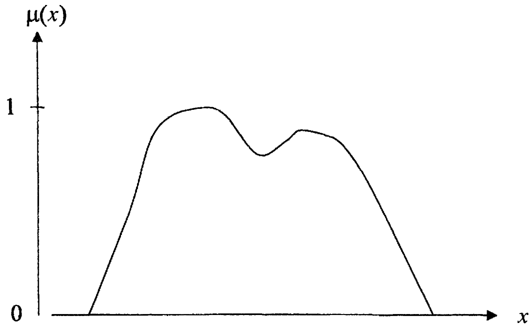

Figure 1.8 An example of a nonconvex fuzzy set.

however. Indeed, one of the key issues in the theory and practice of fuzzy sets is how to define the proper membership functions of fuzzy sets. Fuzzy control and modeling are no exception.

Primary approaches include (1) asking the control/modeling expert to define them; (2) using data from the system to be controlled/modeled to generate them; and (3) making them in a trial-and-error manner. Each different approach has its benefits and drawbacks. In more than 25 years of practice, it has been found that the third approach, though *ad hoc*, works effectively and efficiently in many real-world applications.

Numerous applications have shown that only four types of membership functions are needed in most circumstances: trapezoidal, triangular (a special case of trapezoidal), Gaussian, and bell-shaped. Figure 1.9 shows an example of each type. All these fuzzy sets are continuous, normal, and convex. Among the four, the first two are more widely used. In the figure, we purposely use asymmetric membership functions to make the illustration more general. More often than not, however, symmetric functions are used.

1.2.3. Fuzzy Logic Operations

In classical set theory, there are binary logic operators AND (i.e., intersection), OR (i.e., union), NOT (i.e., complement), and so on. The corresponding fuzzy logic operators exist in fuzzy set theory. Fuzzy logic AND and OR operations are used in fuzzy controllers and models. Unlike the binary AND and OR operators whose operations are uniquely defined, their fuzzy counterparts are nonunique. Numerous fuzzy logic AND operators and OR operators have been proposed, some of them purely from the mathematics point of view. To a large extent, only the Zadeh fuzzy AND operator, product fuzzy AND operator, the Zadeh

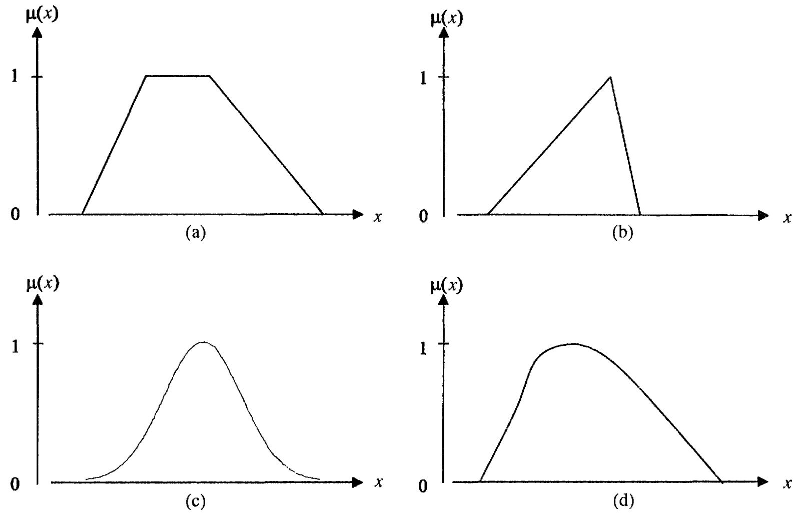

Figure 1.9 Examples of four commonly used input fuzzy sets in fuzzy control and modeling: (a) trapezoidal, (b) triangular, (c) Gaussian, and (d) bell-shaped. Note that they are all continuous, normal, and convex fuzzy sets.

OR operator, and the Lukasiewicz OR operator have been found to be most useful for fuzzy control and modeling [79]. Their definitions are as follows:

Zadeh fuzzy logic AND operator: $\mu_{\tilde{A}\cap\tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$

product fuzzy logic AND operator: $\mu_{\tilde{A}\cap\tilde{B}}(x) = \mu_{\tilde{A}}(x) \times \mu_{\tilde{B}}(x)$

Zadeh fuzzy logic OR operator: $\mu_{\tilde{A}\cup\tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$

Lukasiewicz fuzzy logic OR operator: $\mu_{\tilde{A}\cup\tilde{B}}(x) = \min(\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x), 1)$

where max() and min() are the maximum operator and minimum operator, respectively.

As a concrete demonstration, suppose that a specific age, say 30, is "young" (a fuzzy set) with a membership value of 0.8 and is "old" (another fuzzy set) with a membership value of 0.3. Then, the membership value for the age being "young and old" (a newly formed fuzzy set) is 0.3 if the Zadeh fuzzy AND operator is used or 0.24 if the product fuzzy AND operation is applied. By the same token, the membership value for the age being "young or old" (another newly formed fuzzy set) is 0.8 if the Zadeh fuzzy OR operator is utilized, or 1 if the Lukasiewicz fuzzy OR operation is involved.

1.3. FUZZIFICATION

Fuzzy control and modeling always involve a process called *fuzzification* at every sampling time. Fuzzification is a mathematical procedure for converting an element in the universe of discourse into the membership value of the fuzzy set. Suppose that fuzzy set $\tilde{A}$ is defined on [a,b]; that is, the universe of discourse is [a,b]; for any $x \in$ [a,b], the result of fuzzification is simply $\mu_{\tilde{A}}(x)$. Figure 1.10 shows an example in which the fuzzification result for $x = 7$ is 0.4.

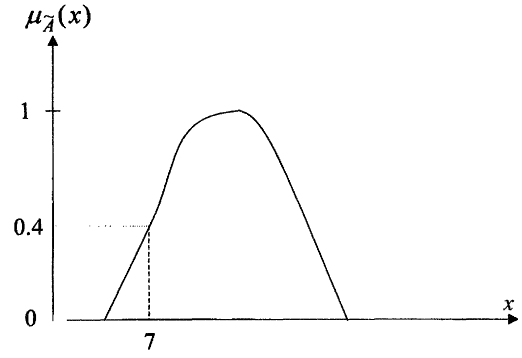

Figure 1.10 An example showing how fuzzification works.

1.4. FUZZY RULES

A fuzzy controller or model uses fuzzy rules, which are linguistic if-then statements involving fuzzy sets, fuzzy logic, and fuzzy inference. Fuzzy rules play a key role in representing expert control/modeling knowledge and experience and in linking the input variables of fuzzy controllers/models to output variable (or variables).

Two major types of fuzzy rules exist, namely, Mamdani fuzzy rules and Takagi-Sugeno (TS, for short) fuzzy rules [202].

1.4.1. Mamdani Fuzzy Rules

A simple but representative Mamdani fuzzy rule describing the movement of a car is:

IF Speed is High AND Acceleration is Small THEN Braking is (should be) Modest,

where Speed and Acceleration are input variables and Braking is an output variable. "High," "Small," and "Modest" are fuzzy sets, and the first two are called *input fuzzy sets* while the last one is named the *output fuzzy set.*

The variables as well as linguistic terms, such as High, can be represented by mathematical symbols. Thus, a Mamdani fuzzy rule for a fuzzy controller involving three input variables and two output variables can be described as follows:

$$\text{IF } x_1 \text{ is } \tilde{A} \text{ AND } x_2 \text{ is } \tilde{B} \text{ AND } x_3 \text{ is } \tilde{C} \text{ THEN } u_1 \text{ is } \tilde{D},\ u_2 \text{ is } \tilde{E}, \tag{1.1}$$

where x_1, x_2, and x_3 are input variables (e.g., error, its first derivative and its second derivative), and u_1 and u_2 are output variables (e.g., valve openness). In theory, these variables can be either continuous or discrete; practically speaking, however, they should be discrete because virtually all fuzzy controllers and models are implemented using digital computers. $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, and $\tilde{E}$ are fuzzy sets, and AND are fuzzy logic AND operators. "IF x_1 is $\tilde{A}$ AND x_2 is $\tilde{B}$ AND x_3 is $\tilde{C}$" is called the rule antecedent, whereas the remaining part is named the rule consequent.

The structure of Mamdani fuzzy rules for fuzzy modeling is the same. The variables involved, however, are different. An example of a Mamdani fuzzy rule for fuzzy modeling is

$$\begin{aligned}&\text{IF } y(n) \text{ is } \tilde{A} \text{ AND } y(n-1) \text{ is } \tilde{B} \text{ AND } y(n-2) \text{ is } \tilde{C} \text{ AND } u(n) \text{ is } \tilde{D}\\ &\text{AND } u(n-1) \text{ is } \tilde{E} \text{ THEN } y(n+1) \text{ is } \tilde{F},\end{aligned} \tag{1.2}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$, and $\tilde{F}$ are fuzzy sets, $y(n)$, $y(n-1)$, and $y(n-2)$ are the output of the system to be modeled at sampling time n, $n-1$ and $n-2$, respectively. And, $u(n)$ and $u(n-1)$ are system input at time n and $n-1$, respectively; $y(n+1)$ is system output at the next sampling time, $n+1$.

Obviously, a general Mamdani fuzzy rule, for either fuzzy control or fuzzy modeling, can be expressed as

$$\text{IF } v_1 \text{ is } \tilde{S}_1 \text{ AND} \ldots \text{AND } v_M \text{ is } \tilde{S}_M \text{ THEN } z_1 \text{ is } \tilde{W}_1, \ldots, \ z_P \text{ is } \tilde{W}_P \tag{1.3}$$

where v_i, $i = 1, \ldots, M$, is an input variable and z_j, $j = 1, \ldots, P$, is an output variable. $\tilde{S}_i$ is an input fuzzy set and $\tilde{W}_j$ an output fuzzy set.

As mentioned earlier, for most fuzzy controllers and models, input fuzzy sets are continuous, normal, and convex and are usually of the four common types. Output fuzzy sets are most often of the singleton type. Thus, the general Mamdani fuzzy rule (1.3) can be reduced to

$$\text{IF } v_1 \text{ is } \tilde{S}_1 \text{ AND} \ldots \text{AND } v_M \text{ is } \tilde{S}_M \text{ THEN } z_1 \text{ is } \beta_1, \ldots, \ z_P \text{ is } \beta_P, \tag{1.4}$$

where β_j represents singleton fuzzy set $\tilde{W}_j$ that is nonzero only at $z_j = \beta_j$.

1.4.2. TS Fuzzy Rules

Now, let us look at the so-called TS fuzzy rules. Unlike Mamdani fuzzy rules, TS rules use functions of input variables as the rule consequent. For fuzzy control, a TS rule corresponding to the Mamdani rule (1.1) is

$$\text{IF } x_1 \text{ is } \tilde{A} \text{ AND } x_2 \text{ is } \tilde{B} \text{ AND } x_3 \text{ is } \tilde{C} \text{ THEN } u_1 = f(x_1, x_2, x_3), \ \ u_2 = g(x_1, x_2, x_3),$$

where $f()$ and $g()$ are two real functions of any type. Similarly, for fuzzy modeling, a TS rule analogous to the Mamdani rule (1.2) is in the following form:

$$\begin{aligned}&\text{IF } y(n) \text{ is } \tilde{A} \text{ AND } y(n-1) \text{ is } \tilde{B} \text{ AND } y(n-2) \text{ is } \tilde{C} \text{ AND } u(n) \text{ is } \tilde{D}\\ &\text{AND } u(n-1) \text{ is } \tilde{E} \text{ THEN } y(n+1) = F(y(n), y(n-1), y(n-2), u(n), u(n-1)),\end{aligned}$$

where $F()$ is an arbitrary function. In parallel to the general Mamdani fuzzy rule (1.3), a general TS rule for both fuzzy control and fuzzy modeling is

$$\begin{aligned}&\text{IF } v_1 \text{ is } \tilde{S}_1 \text{ AND} \ldots \text{AND } v_M \text{ is } \tilde{S}_M\\ &\text{THEN } z_1 = f_1(v_1, \ldots, v_M), \ldots, \ z_P = f_P(v_1, \ldots, v_M).\end{aligned} \tag{1.5}$$

In theory, $f_j()$ can be any real function, linear or nonlinear. It seems to be appealing to use nonlinear functions for all the rules or to use a combination of linear and nonlinear functions as rule consequent (i.e., linear functions for some rules and nonlinear ones for the remaining). In this way, rules are more general and can potentially be more powerful. Unfortunately, this idea is impractical, for properly choosing or determining the mathematical formalism of nonlinear functions for every fuzzy rule is extremely difficult, if not impossible. This difficulty is fundamentally the same as those encountered in classical nonlinear control and modeling theory. It is well known that there is no general nonlinear control or modeling theory because general nonlinear system theory has not been, and most likely will not be, established. For these reasons, linear functions have been employed exclusively in theoretical research and practical development of TS fuzzy controllers and models. We call a TS rule employing a linear (nonlinear) function TS fuzzy rule with linear (nonlinear) rule consequent.

In this book, we focus only on fuzzy controllers and models that use the linear TS rule consequent.

1.5. FUZZY INFERENCE

Fuzzy inference is sometimes called fuzzy reasoning or approximate reasoning. It is used in a fuzzy rule to determine the rule outcome from the given rule input information. Fuzzy rules represent control strategy or modeling knowledge/experience. When specific information is assigned to input variables in the rule antecedent, fuzzy inference is needed to calculate the outcome for output variable(s) in the rule consequent.

Mamdani fuzzy rules and TS fuzzy rules use different fuzzy inference methods.

For the general Mamdani fuzzy rule (1.3), the question about fuzzy inference is the following: Given $v_i = \alpha_i$, for all i, where α_i are real numbers, what should z_j be? For fuzzy control and modeling, after fuzzifying v_i at α_i and applying fuzzy logic AND operations on the resulting membership values in the fuzzy rule, we attain a combined membership value, μ, which is the outcome for the rule antecedent. Then, the question is how to compute "THEN" in the rule. Calculating "THEN" is called fuzzy inference. Specifically, the question is: Given μ, how should z_j be computed? Since, mathematically, the computation is the same for different output variables, we use z and $\tilde{W}$ to represent, respectively, z_j and $\tilde{W}_j$ in the following discussion on fuzzy inference methods.

A number of fuzzy inference methods can be used to accomplish this task (e.g., [163]), but only four of them are popular in fuzzy control and modeling and we will use them only in this book [157]). They are the Mamdani minimum inference method, the Larsen product inference method, the drastic product inference method, and the bounded product inference method. We denote them by R_M, R_L, R_{DP}, and R_{BP}, respectively. The definitions of these methods are given in Table 1.1, where $\mu_{\tilde{W}}(z)$ is the membership function of fuzzy set $\tilde{W}$ in fuzzy rule (1.3) and μ is the combined membership in the rule antecedent.

For a better understanding, we graphically illustrate the definitions in Fig. 1.11. The results of the four fuzzy inference methods are the fuzzy sets formed by the shaded areas. Obviously, the resulting fuzzy sets can be explicitly determined since the formulas describing the shaded areas can be derived mathematically. Among the four methods, the Mamdani method is used most widely in fuzzy control and modeling.

TABLE 1.1 Definitions of Four Popular Fuzzy Inference Methods for Fuzzy Control and Modeling: (a) Mamdani minimum inference, (b) Larsen product inference, (c) drastic product inference, and (d) bounded product inference.

Fuzzy Inference Method	Definition[a]
Mamdani minimum inference, R_M	$\min(\mu, \mu_{\tilde{W}}(z))$, for all z
Larsen product inference, R_L	$\mu \times \mu_{\tilde{W}}(z)$, for all z
Drastic product inference, R_{DP}	$\begin{cases} \mu, & \text{for } \mu_{\tilde{W}}(z) = 1 \\ \mu_{\tilde{W}}(z), & \text{for } \mu = 1 \\ 0, & \text{for } \mu < 1 \text{ and } \mu_{\tilde{W}}(z) < 1 \end{cases}$
Bounded product inference, R_{BP}	$\max(\mu + \mu_{\tilde{W}}(z) - 1, 0)$

[a] General Mamdani fuzzy rule (1.3) is utilized in the definitions. $\mu_{\tilde{W}}(z)$ is the membership function of fuzzy set $\tilde{W}$ representing $\tilde{W}_j$ in the rule consequent, whereas μ is the final membership yielded by fuzzy logic AND operators in the rule antecedent.

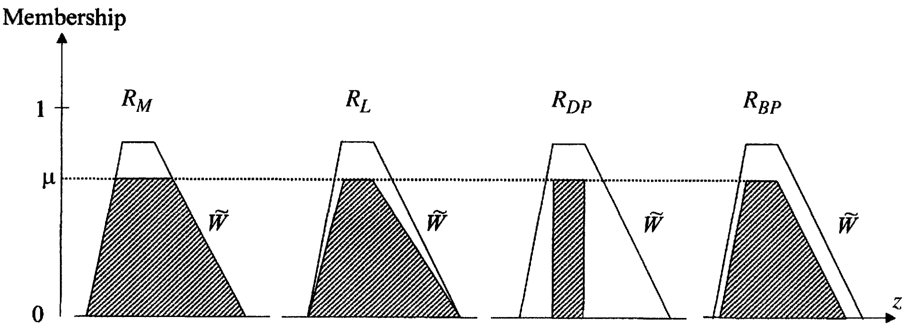

Figure 1.11 Graphical illustration of the definitions of the four popular fuzzy inference methods whose mathematical definitions are provided in Table 1.1: (a) the Mamdani minimum inference method, (b) the Larsen product inference method, (c) the drastic product inference method, and (d) the bounded product inference method.

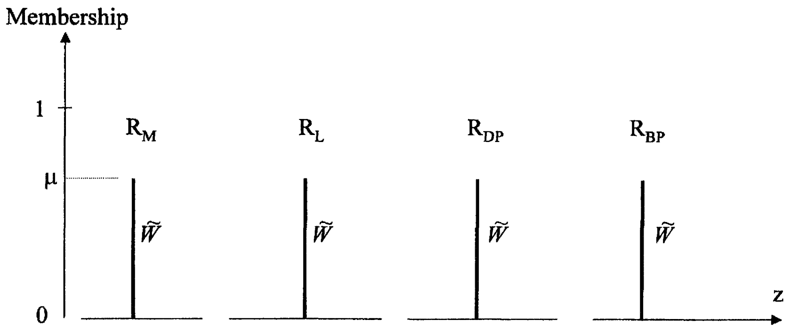

Figure 1.12 For Mamdani fuzzy controllers and models using singleton fuzzy sets in the rule consequent, the outcome of using the four different inference methods is identical.

As stated above, typical Mamdani fuzzy controllers and models employ singleton output fuzzy sets as the rule consequent (see rule (1.4). Under this condition, the four different inference methods produce the same inference result, as shown in Fig. 1.12.

For TS fuzzy rules, fuzzy inference is simpler and only one method exists. For general TS fuzzy rule (1.5), the result of the fuzzy inference is $\mu \times f_j(v_1, \ldots, v_M)$ for z_j. Instead of viewing this as a fuzzy inference result, one may also think of it as the rule consequent being weighted by the combined membership value from the rule antecedent.

1.6. DEFUZZIFICATION

Defuzzification is a mathematical process used to convert a fuzzy set or fuzzy sets to a real number. It is a necessary step because fuzzy sets generated by fuzzy inference in fuzzy rules must be somehow mathematically combined to come up with one single number as the output of a fuzzy controller or model. After all, actuators for control systems can accept only one value as their input signal, whereas measurement data from physical systems being modeled are always crisp.

Every fuzzy controller and model uses a defuzzifier, which is simply a mathematical formula, to achieve defuzzification. For fuzzy controllers and models with more than one output variable, defuzzification is carried out for each of them separately but in a very similar fashion. In most cases, only one defuzzifier is employed for all output variables, although it is theoretically possible to use different defuzzifiers for different output variables.

Different types of defuzzifiers are suitable for different circumstances; below, we present some of the more popular ones. Since most fuzzy controllers and models use singleton fuzzy sets in the fuzzy rule consequent, our presentation will concentrate on singleton output fuzzy sets. Nonetheless, extending the discussion to nonsingleton fuzzy sets is straightforward.

1.6.1. Generalized Defuzzifier

The generalized defuzzifier represents many different defuzzifiers in one simple mathematical formula [64].

Assume that the output variable of a fuzzy controller or model is z. Suppose that evaluating N Mamdani fuzzy rules using some fuzzy inference method produces N membership values, $\mu_1, \ldots, \mu_N$, for N singleton output fuzzy sets in the rules (one value for each rule). Let us say that these fuzzy sets are nonzero only at $z = \beta_1, \ldots, \beta_N$. The generalized defuzzifier produces the following defuzzification result:

$$z = \frac{\sum_{k=1}^{N} \mu_k^\alpha \cdot \beta_k}{\sum_{k=1}^{N} \mu_k^\alpha}, \tag{1.6}$$

where α is a design parameter.

Continue the above case, but assume that the fuzzy controller or model uses TS rules instead. Let us say that the rule consequents in the N fuzzy rules are $g_k(v_1, \ldots, v_M)$, $k = 1, \ldots, N$; then defuzzification outcome is achieved using the generalized defuzzifier

$$z = \frac{\sum_{k=1}^{N} \mu_k^\alpha \times g_k(v_1, \ldots, v_M)}{\sum_{k=1}^{N} \mu_k^\alpha}. \tag{1.7}$$

1.6.2. Centroid Defuzzifier, Mean of Maximum Defuzzifier, and Linear Defuzzifier

Different types of defuzzifiers are realized using different α values in the generalized defuzzifier, where $0 \leq \alpha + \infty$. When $\alpha = 1$, the most widely used centroid defuzzifier is obtained. The defuzzifier is of the centroid type because it computes, in a sense, the centroid of the singleton fuzzy sets from different rules. The occasionally used mean of maximum defuzzifier is realized when $\alpha = \infty$.

A few studies in the literature use a linear defuzzifier. When Mamdani fuzzy rules are involved, the defuzzification result is

$$z = \sum_{k=1}^{N} \mu_k^{\alpha} \times \beta_k. \tag{1.8}$$

On the other hand, for TS fuzzy rules, we get

$$z = \sum_{k=1}^{N} \mu_k^{\alpha} \times g_k(v_1, \ldots, v_M).$$

The difference is obvious: A linear defuzzifier does not have the denominator.

We will use the centroid defuzzifier and generalized defuzzifier only in this book because of their popularity.

1.7. SUMMARY

This chapter introduces the concept of fuzzy sets and their advantages over the classical sets. Also presented are concepts and notations of different types of fuzzy sets and fuzzy logic operations. The common building blocks of typical fuzzy controllers and models are described. They include fuzzification, fuzzy rules, fuzzy inference, and defuzzification.

1.8. NOTES AND REFERENCES

There are a number of introductory textbooks on fuzzy set theory and fuzzy systems (e.g., [101][102][242][293]). Fuzzification, fuzzy rules, fuzzy inference, and defuzzification are basic components of a typical fuzzy system, fuzzy controller, or fuzzy model. More information on these segments can be found in these books as well. A brief history of fuzzy sets, fuzzy logic, and fuzzy systems is given in [151].

EXERCISES

1. List some concepts in our daily lives that cannot be accurately described by conventional sets but can be by fuzzy sets.
2. Graphically draw your definitions of continuous fuzzy set "young" in some different circumstances. Can they be described by mathematical formulas? If not, can you approximate your definitions by formulas? Do your definitions belong to the four common types of fuzzy sets mentioned in this chapter?
3. Answer the same questions as in Problem 2 for continuous fuzzy set "middle age."
4. For the fuzzy sets that you defined in the above two problems, what are their supports, heights, and centers? Are they normal? Are they convex?
5. Derive two new fuzzy sets "young and middle age" and "young or middle age" from the fuzzy sets established in Problems 2 and 3. Use different fuzzy logic AND and OR operators discussed in this chapter. Do this exercise graphically and mathematically, if possible.
6. Describe a fuzzy integer 5 using the Gaussian fuzzy set (i.e., use the Gaussian formula in statistics). How do you use a singleton fuzzy set to represent integer 5?
7. What are the apparent similarities between fuzzy set and probability? What are the fundamental differences between them? What are the implications of the differences to application?

8. Make some Mamdani fuzzy rules and TS fuzzy rules of your own. Which type would you prefer? Why?

9. Is it meaningful to compare the effects of the different defuzzifiers? If yes, how can you compare them? If no, why?

10. If the same questions as Problem 9 are asked for the different fuzzy inference methods, what are your answers?