Chapter 1

UNCERTAINTY MANAGEMENT IN MEDICAL APPLICATIONS

Bernadette Bouchon-Meunier

1. INTRODUCTION

In biology and medicine, as well as in many other domains, imperfect knowledge cannot be avoided. It is difficult to construct automatic systems to provide classification or pattern recognition tools or help specialists make a decision. There exist two kinds of difficulties: (1) those related to the type of imperfection we have to consider (partial information, uncertainties, inaccuracies) and (2) those due to the type of problem we have to solve (e.g., images to process, expert rules, databases).

Which mathematical model are we supposed to choose to manage this imperfect knowledge? What is the best knowledge representation for a given problem? The answers to such questions are not obvious, and our purpose is to present several frameworks available to represent and manage imperfect knowledge, particularly in biological and medical domains. We indicate principles, interest and limits of these frameworks. We give more details about numerical approaches that have given rise to more practical applications than about symbolic approaches, which will be mentioned only briefly.

2. IMPERFECT KNOWLEDGE

2.1. Types of Imperfections

Imperfections may have several forms, which we present briefly.

2.1.1. Uncertainties

Imperfections are called *uncertainties* when there is doubt about the validity of a piece of information. This means that we are not certain that a statement is true or false because of

- The random behavior of a phenomenon (for instance, the factors of transmission of genetic features) related to probabilistic uncertainty.
- The reliability or limited soundness of an observer of the phenomenon who expresses the statement, or of the sensor used for a measurement. The uncertainty is then nonprobabilistic.

Uncertainties can be represented either by numbers, such as probabilities or confidence degrees indicating the extent to which we are certain of the validity of a statement, or by phrases such as "I believe that..." or "it is possible that...."

2.1.2. Imprecisions

The second type of imperfection is *imprecision*, when some characteristics of a phenomenon cannot be described accurately. Imprecisions have two main forms: approximate values (for instance, the limits of the normal glycemia level at a given age are not sensitive to a variation of 1‰) or vague descriptions using terms of natural language (for instance, "a high temperature" or "frequent attacks").

2.1.3. Incompleteness

Incomplete knowledge is the last kind of imperfection, in which there is a lack of information about some variables or criteria or elements of a given situation. Such incompleteness can appear because of defaults in knowledge acquisition (for instance, the age of a patient has not been recorded) or because of general rules or facts that are usually true but admit a few exceptions, the list of which is impossible to give (for instance, generally, the medication X does not cause any drowsiness).

2.1.4. Causes of Imperfect Knowledge

These imperfections may have various causes:

- They can be related to conditions of observation that are insufficient to obtain the necessary accuracy (for instance, in the case of radiographic images).
- They can be inherent in the phenomenon itself. This is often the case in biology or medicine, because natural factors often have no precise value or precise limit available for all patients. Conditions or values of criteria vary in a given situation (e.g., the size and shape of malignant microcalcifications in breast cancer).

It happens that several forms of imprecision cannot be managed independently. For instance, uncertainties are generally present at the same time as inaccuracies, and incompleteness entails uncertainties. It is then necessary to find the knowledge representation suitable for all the existing imperfections.

2.2. Choice of a Method

The choice of a method to process data is linked to the choice of knowledge representation, which can be numerical, symbolic, logical, or semantic, and it depends on the nature of the problem to be solved: classification, automatic diagnosis, or decision support, for instance. The available knowledge can consist of images or databases containing factual information or expert knowledge provided by specialists in the domain. They are, in some cases, directly managed by an appropriate tool, such as an expert system or pattern recognition method if the object to identify on images is not too variable, for instance. In other cases, learning is necessary as a preliminary step in the construction of an automatic system. This means that examples of well-known situations are given and assigned to a class, a diagnosis, a decision, or more generally a label by a specialist. On the basis of these examples, a general method is constructed to perform a similar assignment in new situations, for instance, by inductive learning, case-based reasoning, or neural networks.

It is also possible that explanations are required for the reasons leading the system to a given diagnosis or choice of a label. It is, for instance, interesting if the conceived automatic system has training purposes. Such problems of human–machine communication are studied in artificial intelligence.

We indicate briefly in Table 1 the main knowledge representation and management methods corresponding to the three kinds of imperfection we have mentioned. In the following, we will focus on the numerical methods listed in the bold frame in Table 1.

There are other kinds of methods that are not directly dedicated to one of the imperfections we have mentioned but provide numerical approaches to data management, such as chaos, fractals, wavelets, neural networks, and genetics-based programming, which are also intensively used, especially in medicine.

All these tools have their own advantages as well as some disadvantages. It is therefore interesting to use several of them as complementary elements of a general data processing system, taking advantage of synergy between them such that qualities of one method compensate for disadvantages of another one. For instance, fuzzy logic is used for its ability to manage imprecise knowledge, but it can take advantage of the ability of neural networks to learn coefficients or functions. Such an association of methods is typical of so-called *soft computing*, which was initiated by L.A. Zadeh in the 1990s and provides interesting results in many real-world applications. In the next sections, we present the fundamentals of the main numerical methods mentioned in Table 1. For more details, see the books or basic papers indicated at the end of this chapter [1–10].

TABLE 1 Classification of Methods for the Management of Imperfect Knowledge

Type of imperfection	Representation method	Management method
		Modal logic
	Symbolic beliefs	Truth maintenance systems
		Autoepistemic logic
		Probabilistic logic
Uncertainties	Probabilities	Bayesian Induction
		Belief networks
	Confidence degrees	Propagation of degrees
	Belief, plausibility measures	Evidence theory
	Possibility, necessity degrees	Possibilistic logic
		Fuzzy logic
Imprecisions	Fuzzy sets	Fuzzy set–based techniques
	Error intervals	Interval analysis
	Validity frequencies	Numerical quantifiers
General laws, exceptions	Hypotheses	Hypothetical reasoning
	Default rules	Default reasoning

3. FUZZY SET THEORY

3.1. Introduction to Fuzzy Set Theory

Fuzzy set theory, introduced in 1965 by Zadeh [11] provides knowledge representation suitable for biological and medical problems because it enables us to work with imprecise information as well as some type of uncertainty. We present such a representation using an example. Let us think of the glycemia level of patients. We can use a threshold of 1.4 g/l, providing two classes of levels: those at most equal to the threshold, labeled "normal," and those greater than the threshold, labeled "abnormal."

The transition from one label to the other appears too abrupt because a level of 1.39 g/l is considered normal and a level of 1.41 g/l is considered abnormal. Instead, we can establish a progressive passage from the class of normal levels to the class of abnormal ones and consider that the level is normal up to 1.3 g/l; that the greater the level between 1.3 and 1.5 g/l, the less normal this level; and finally that the level is considered really abnormal when greater than 1.5 g/l. We then define a fuzzy set A of the set X of possible values of the glycemia level by means of a membership function f_A, which associates a coefficient $f_A(x)$ in [0, 1] to every element x of X. This coefficient indicates the extent to which x belongs to A (see Figure 1).

The main novelty of fuzzy set theory compared with classical set theory is the concept of partial membership of an element in a class or a category. This corresponds to the idea that a level can be "somewhat abnormal." The possibility of representing gradual knowledge stems from this concept, such as "the more the value increases between given limits, the more abnormal the level," and of allowing progressive passage from one class (the class of normal levels) to another one (the class of abnormal levels). This possibility justifies the use of such a knowledge representation for modeling biological phenomena, in which there is generally no strict boundary between neighboring situations.

Such a representation is also interesting because it can be adjusted to the environment. If the observed patients are elderly, the membership function of the class of abnormal glycemia levels indicated in Figure 1 must be shifted 0.4 g/l to the right. Another advantage of this approach is that one can set up an interface between numerical values (1.3 g/l) and symbolic ones expressed in natural language (normal level). For instance, a young patient with a glycemia level of 1.7 g/l (numerical value) is associated with the symbolic value "abnormal." Conversely, a new patient with no record in a hospital can indicate that he had an abnormal glycemia level in the past; this symbolic

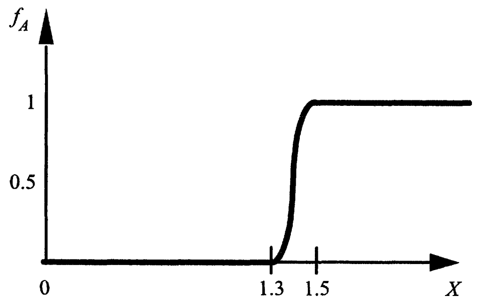

Figure 1 Fuzzy set A representing the category "abnormal" of the glycemia rate.

information will be taken into account together with measured (numerical) levels obtained in the future.

It is easy to see that fuzzy sets are useful for representing imprecise knowledge with ill-defined boundaries, such as approximate values of vague characterizations (see Figure 2). Such a representation is also compatible with the representation of some kinds of uncertainty by means of possibility theory, which we will develop later.

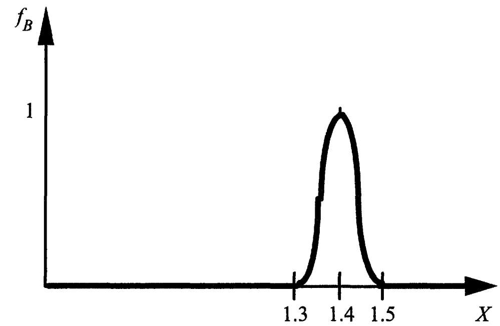

Figure 2 Fuzzy set B representing the approximate value "about 1.4 g/l" of the glycemia rate.

3.2. Main Basic Concepts of Fuzzy Set Theory

3.2.1. Definitions

For a given universe X, a classical subset C is defined by a characteristic function χ_C lying in {0, 1}, and a fuzzy set A is defined by a *membership function* $f_A : X \rightarrow [0, 1]$. A classical (or crisp) subset of X is then a particular case of a fuzzy set. We note that classical (or crisp) subsets of X are particular cases of fuzzy sets, corresponding to membership functions taking only the value 0 or 1.

Some particular elements are of interest in describing a fuzzy set:

Its support:

$$\text{supp}(A) = \{x \in X / f_A(x) \neq 0\} \tag{1}$$

Its height:

$$h(A) = \sup_{x \in X} f_A(x) \tag{2}$$

Its kernel or core:

$$\ker(A) = \{x \in X / f_A(x) = 1\} \tag{3}$$

Its cardinality:

$$|A| = \sum_{x \in X} f_A(x) \tag{4}$$

Fuzzy sets with a nonempty kernel and a height equal to 1 are called *normalized.*

In the example given in Figure 1, with a continuous membership function, we have $\mathrm{Ker}(A) = [1.5, +\infty[$, $\mathrm{Supp}(A) = [1.3, +\infty[$, $h(A) = 1$.

Let us remark that a fuzzy set can have several *interpretations*, depending on the situation:

- Partial membership ($f_A(x)$ is the membership degree of x in the class A)
- Preference ($f_A(x)$ is the degree of preference attached to x)
- Typicality ($f_A(x)$ is the degree of typicality of x in the class A)
- Possibility ($f_A(x)$ is the degree of possibility that x is the value of a variable defined on X)

Two fuzzy sets A and B of X are *equal* if and only if

$$\forall x \in X \quad f_A(x) = f_B(x) \tag{5}$$

3.2.2. Operations on Fuzzy Sets

We define the *inclusion* of fuzzy sets of X as a partial order such that A is included in B, and we note $A \subseteq B$, if and only if

$$\forall x \in X \quad f_A(x) \leq f_B(x) \tag{6}$$

with:

- The empty set ($\forall x \in X\ f_A(x) = 0$) as smallest element
- The universe itself ($\forall x \in X\ f_A(x) = 1$) as greatest element

It is then necessary to define operations on fuzzy sets extending the operations on crisp subsets of X.

The *intersection* of A and B (Figure 3) is defined as the fuzzy set $C = A \cap B$ of X with the following membership function:

$$\forall x \in X \quad \forall x \in X \quad (f_C(x) = \min f_A(x),\ f_B(x)) \tag{7}$$

The *union* of A and B (Figure 4) is defined as the fuzzy set $D = A \cup B$ of X with the following membership function:

$$\forall x \in X \quad f_D(x) = \max(f_A(x),\ f_B(x)) \tag{8}$$

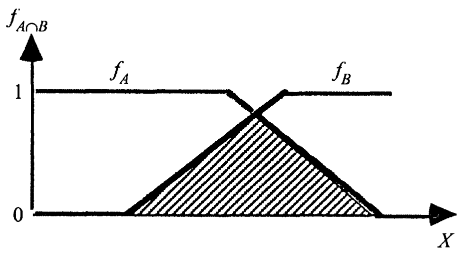

Figure 3 Intersection of fuzzy sets A and B.

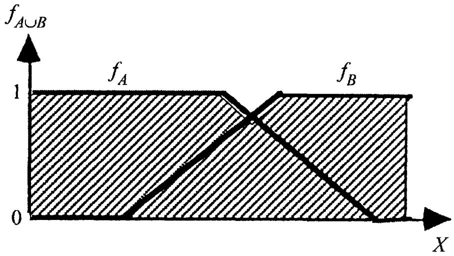

Figure 4 Union of fuzzy sets A and B.

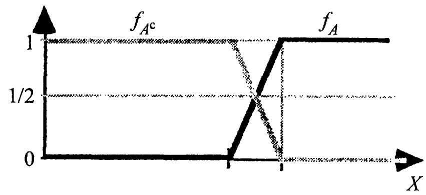

Figure 5 Complement of a fuzzy set A.

The properties of intersection and union of crisp subsets of X are preserved by these definitions: associativity of $\cap$ and $\cup$, commutativity of $\cap$ and $\cup$, $A \cup \emptyset = A$, $A \cup X = X$, $A \cap X = A$, $A \cap \emptyset = \emptyset$, $A \cup B \supseteq A \supseteq A \cap B$, and distributivity of $\cap$ over $\cup$ and, conversely, of $\cup$ over $\cap$.

We now define the *complement* of a fuzzy set A of X (Figure 5) as the fuzzy set A^c of X with the following membership function:

$$\forall x \in X \quad f_{A^c}(x) = 1 - f_A(x) \tag{9}$$

This definition preserves almost all the properties available in classical set theory, except the following ones:

- $A^c \cap A \neq \emptyset$
- $A^c \cup A \neq X$

which means that a class and its complement may overlap, in agreement with the basic idea of partial membership in fuzzy set theory.

In some cases, it can be interesting to lose some other properties and to use definitions of intersection and union with a slightly different behavior.

The most common alternative operators are *triangular norms* (t-norms) T: $[0, 1] \times [0, 1] \to [0, 1]$ to define the intersection and *triangular conorms* (t-conorms) $\perp$: $[0, 1] \times [0, 1] \to [0, 1]$ to define the union. These operators have been introduced in probabilistic metric spaces and they are

- Commutative
- Associative
- Monotonous
- Such that $T(x, 1) = x$, $\perp(x, 0) = x$ for any x in $[0, 1]$

It is easy to check that min is a t-norm and max a t-conorm, which are dual in the following sense:

- $1 - T(x, y) = \perp(1 - x,\ 1 - y)$
- $1 - \perp(x, y) = T(1 - x,\ 1 - y)$

The other widely used t-norms are the product $T(x, y) = xy$ and the so-called Lukasiewicz t-norm $T(x, y) = \max(x + y - 1, 0)$, respectively dual from the following t-conorms: $\perp(x, y) = x + y - xy$ and $\perp(x, y) = \min(x + y, 1)$ (Figures 6 and 7).

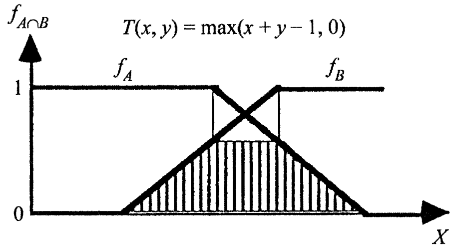

Figure 6 Intersection of A and B based on the Lukasiewicz t-norm.

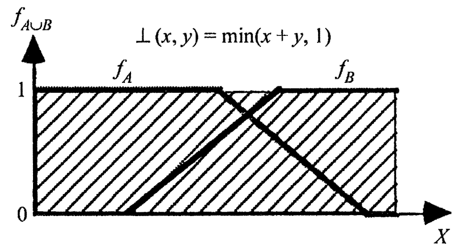

Figure 7 Union of A and B based on the Lukasiewicz t-conorm.

When several universes are considered simultaneously—for instance, several criteria to make a decision, attributes to describe an object, variables to control a system—it is necessary to define the *Cartesian product* of fuzzy sets of the various universes. This situation is very frequent, because a decision, a diagnosis, the recognition of a class, and so forth are generally based on the use of several factors involved simultaneously.

Let us consider universes $X_1, X_2, \ldots, X_r$ and their Cartesian product $X = X_1 \times X_2 \times \cdots \times X_r$, the elements of which are r-tuples $(x_1, x_2, \ldots, x_r)$, with $x_1 \in X_1, \ldots, x_r \in X_r$. From fuzzy sets $A_1, A_2, \ldots, A_r$, respectively defined on $X_1, X_2, \ldots, X_r$, we construct a fuzzy set of X denoted by $A = A_1 \times A_2 \times \cdots \times A_r$, considered as their *Cartesian product*, with membership function

$$\forall x = (x_1, x_2, \ldots, x_r) \in X \quad f_A(x) = \min(f_{A_1}(x_1), \ldots, f_{A_r}(x_r)) \tag{10}$$

3.2.3. The Zadeh Extension Principle

Another important concept of fuzzy set theory is the so-called *Zadeh extension principle*, enabling us to extend to fuzzy values the operations or tools used in classical set theory or mathematics. Let us explain how it works. Fuzzy sets of X are imperfect information about the elements of X. For instance, instead of observing x precisely, we can only perceive a fuzzy set of X with a high membership degree attached to x. The methods that would be available to manage the information regarding X in the case of precise information need to be adapted to be able to manage fuzzy sets.

We consider a mapping ϕ from a first universe X to a second one Y, which can be identical to X. The Zadeh extension principle defines a fuzzy set B of Y from a fuzzy set A of X, in agreement with the mapping ϕ, in the following way:

$$\forall y \in Y \quad f_B(y) = \sup_{x \in \phi^*(y)} f_A(x) \quad \text{if } \phi^*(y) \neq \emptyset \tag{11}$$

and $f_B(y) = 0$ otherwise, with $\phi^*(y) = \{x \in X/y = \phi(x)\}$ if $\phi\colon X \to Y$, and $\phi^*(y) = \{x \in X/y \in \phi(x)\}$ if $\phi\colon X \to P(Y)$ (i.e., ϕ is multivalued).

If A is a crisp subset of X reduced to a singleton $\{a\}$, the Zadeh extension principle constructs a fuzzy set B of Y reduced to $\phi(\{a\})$.

If ϕ is a one-to-one mapping, then

$$\forall y \in Y \quad f_B(y) = f_A\left(\phi^{-1}(y)\right) \tag{12}$$

If we consider the Cartesian product of universes $X = X_1 \times X_2 \times \cdots \times X_r$ and A the Cartesian product of fuzzy sets of these universes $A = A_1 \times A_2 \times \cdots \times A_r$, the Zadeh extension principle associates a fuzzy set B of Y with A as follows:

$$\forall y \in Y \quad f_B(y) = \sup_{x=(x_1,\ldots,x_r) \in \phi^*(y)} \min\left(f_{A_1}(x_1), \ldots, f_{A_r}(x_r)\right) \quad \text{if } \phi^*(y) \neq \emptyset \tag{13}$$

and $f_B(y) = 0$ otherwise

For example, let us consider the fuzzy set A representing "about 1.4" on the universe $X = [0, +\infty[$, as defined in Figure 2. If we know that the value of variable W defined on X is *greater* than the value of variable V and that the value of V is about 1.4, we can characterize the value of W by the fuzzy set B obtained by applying the extension principle to the order relation on $[0, +\infty[$. We have $Y = [0, +\infty[$ and $\phi(x) = \{y \in Y/y \geq x\}$. We get

$$\begin{gathered} \forall y \in Y \quad f_B(y) = \sup_{y > x} f_A(x) \\ f_B(y) = 0 \text{ if } x \leq 1.3, \quad f_B(y) = 1 \text{ if } y \geq 1.4 \end{gathered} \tag{14}$$

which corresponds to a representation of "greater than about 1.4."

Another example of application of application of the extension principle defines a *distance* between imprecise locations. Let us consider a set of points $Z = \{a, b, c, d\}$. The distance between any pair of points of Z is defined by a mapping ϕ: $Z \times Z \to [0, +\infty[$. If the points are observed imprecisely, we need to extend the notion of distance to fuzzy sets.

We use the extension principle with $X = Z \times Z$ and $Y = [0, +\infty[$, and we get a fuzzy set C of $[0, +\infty[$ with a membership function defined for any $d \in [0, +\infty[$ by

$$\begin{aligned} f_C(d) &= \sup_{\{(x,y) \in X, x \neq y, \phi(x,y) = d\}} \min(f_A(x), f_B(y)) \\ &\qquad \text{if } \left\{(x, y) \in X, x \neq y, \phi(x, y) = d\right\} \neq \emptyset \\ f_C(d) &= 0 \quad \text{otherwise} \end{aligned} \tag{15}$$

This kind of distance can be used, for instance, in image processing.

The Zadeh extension principle is fundamental for extending to fuzzy sets all the concepts we are familiar with in classical set theory, for instance, in reasoning or arithmetic.

3.3. Fuzzy Arithmetic

Arithmetic is precisely one of the domains where fuzzy sets are widely used. Many applications use the universe $\mathbb{R}$ of real numbers, with fuzzy sets representing imprecise measurements of real-valued variables (e.g., distance, weight). The membership functions are generally chosen as simple as possible, compatible with the intuitive representation of approximations. This simple form of functions corresponds to convex fuzzy sets.

A fuzzy set F on X is *convex* if it satisfies the following condition:

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R} \ \forall z \in [x, y] \quad f_F(z) \geq \min(f_F(x), f_F(y)) \tag{16}$$

- A *fuzzy quantity* Q is a normalized fuzzy set of $\mathbb{R}$.
- A *model value* of Q is an element m of $\mathbb{R}$ in the kernel of Q such that $f_Q(m) = 1$.
- A *fuzzy interval* I is a convex fuzzy quantity. It corresponds to an interval of $\mathbb{R}$ with imprecise boundaries.
- A *fuzzy number* M is a fuzzy interval with an upper semicontinuous membership function, a compact support, and a unique modal value. It corresponds to an imprecisely known real value.

It is often necessary to compute the addition or the product of imprecisely known real values. For instance, if a patient has lost approximately 5 pounds during the first week and 3 pounds during the second one, how much has he lost during these two weeks? Symbolically, we can conclude that he has lost approximately 8 pounds, but we need to formalize this operation to define automatic operations for more complex problems. We use the Zadeh extension principle to extend the classical arithmetic operations to fuzzy quantities. We do not go into detail with the general definition of fuzzy quantities. We focus on particular forms of membership functions for which the main operations are easily computable. They are called *L-R* fuzzy intervals.

An *L-R fuzzy interval I* is a fuzzy quantity with a membership function f_I defined by means of four real parameters (m, m', a, b) with a and b strictly positive, and two functions L and R, defined on $\mathbb{R}+$, lying in [0, 1], upper semicontinuous, nonincreasing, such that

$$\begin{aligned} &L(0) = R(0) = 1 \\ &L(1) = 0 \text{ or } L(x) > 0 \ \forall x \text{ with } \lim_{x \to \infty} L(x) = 0 \\ &R(1) = 0 \text{ or } R(x) > 0 \ \forall x \text{ with } \lim_{x \to \infty} R(x) = 0 \end{aligned} \tag{17}$$

The membership function of an *L-R* fuzzy interval defined by m, m', a, and b is then

$$
\begin{aligned}
f_I(x) &= L((m-x)/a) && \text{if } x \le m \\
f_I(x) &= 1 && \text{if } m < x < m' \\
f_I(x) &= R((x-m')/b) && \text{if } x \ge m'
\end{aligned} \tag{18}
$$

We note $I = (m, m', a, b)_{LR}$. It can be interpreted as "approximately between m and m'."

The particular case of an L-R fuzzy interval $(n, n, a, b)_{LR}$ is an *L-R fuzzy number* denoted by $M = (n, a, b)_{LR}$. It can be interpreted as "approximately n."

Fuzzy quantities often have trapezoidal or triangular membership functions. They are then L-R fuzzy intervals or numbers, with $R(x) = L(x) = \max(0, 1-x)$. It is also possible to use functions such as $\max(0, 1-x^2)$, $\max(0, 1-x)^2$ or $\exp(-x)$ to define R and L.

Given two L-R fuzzy intervals defined by the same functions L and R, respectively denoted by $I = (m, m', a, b)_{LR}$ and $J = (n, n', c, d)_{LR}$, the main arithmetic operations can be computed very simply, as follows:

- For the opposite of I: $-I = (-m', -m, b, a)_{LR}$
- For the addition: $I \oplus J = (m+n, m'+n', a+c, b+d)_{LR}$
- For the subtraction: $I \Theta J = (m-n', m'-n, a+d, b+c)_{LR}$ if $L = R$
- For the product: $I \otimes J$ is generally not an L-R fuzzy interval, but it is possible to approximate it by the following L-R fuzzy interval:

$$I \otimes J = (mn, m'n', mc + na, md + nb)_{LR} \tag{19}$$

These operations satisfy the classical properties of the analogous operations in classical mathematics except for some of them. For instance, $Q \oplus (-Q)$ is different from 0, but it accepts 0 as its modal value; it can be interpreted as "approximately null."

For example, if I is a triangular fuzzy number with modal value 4 and support $]3, 5[$ and J a triangular fuzzy number with modal value 8 and support $]6, 10[$, we represent them as L-R fuzzy numbers $I = (4, 4-3, 5-4)_{LR} = (4, 1, 1)_{LR}$, $J = (8, 8-6, 10-8)_{LR} = (8, 2, 2)$.

Then we obtain the following results:

- $-I = (-4, 1, 1)_{LR}$ is a triangular fuzzy number with modal value -4 and with support $]-5, -3[$.
- $I \oplus J = (12, 3, 3)_{LR}$ is a fuzzy number with modal value 12 and support $]9, 15[$.
- $J \Theta I = (8-4, 2+1, 2+1)_{LR} = (4, 3, 3)_{LR}$ is a triangular fuzzy number with modal value 4 and support $]1, 7[$.

3.4. Fuzzy Relations

Because fuzzy set theory represents a generalization of classical set theory, we need to generalize all the classical tools available to manage crisp data. Fuzzy relations are among the most important concepts in fuzzy set theory.

A *fuzzy relation* R between X and Y is defined as a fuzzy set of $X \times Y$.

An example of fuzzy relation can be defined on $X = Y = \mathbb{R}$ to represent approximate equality between real values, for instance, with the following membership function:

$$\forall x \in X \; \forall y \in Y \quad f_{\mathrm{Re}}(x, y) = \frac{1}{1 + (x - y)^2} \tag{20}$$

Another example, also defined on $X = Y = \mathbb{R}$, is a representation of the relation "y is really greater than x" with the following membership function:

$$\forall (x, y) \in \mathbb{R}^2 \quad f_R(x, y) = \begin{cases} \min\left(1, \frac{y - x}{\beta}\right) & \text{if } y \geq x \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

for a parameter $\beta \geq 0$ indicating the range of difference between x and y we accept.

If we have three universes X, Y, and Z, it is useful to combine fuzzy relations between them. The *max–min composition* of two fuzzy relations R_1 on $X \times Y$ and R_2 on $Y \times Z$ defines a fuzzy relation $R = R_1 \circ R_2$ on $X \times Z$, with membership function:

$$\forall (x, z) \in X \times Z \quad f_R(x, z) = \sup_{y \in Y} \min\left(f_{R_1}(x, y), f_{R_2}(y, z)\right) \tag{22}$$

The main *utilizations* of fuzzy relations concern the representation of resemblances ("almost equal") or orders ("really smaller"). We need to define general classes of fuzzy relations suitable for such representations, based on particular properties of fuzzy relations: symmetry, reflexivity, transitivity, antisymmetry, extending the analogous properties of classical binary relations.

A *similarity relation* is a symmetrical, reflexive and max–min transitive fuzzy relation. It corresponds to the idea of resemblance and it can be used in classification, clustering, and analogical reasoning, for instance.

A *fuzzy preorder* is a reflexive and transitive fuzzy relation R. If R is also antisymmetrical, R is a fuzzy order relation. It corresponds to the idea of ordering or anteriority and it is useful in decision making, for instance, for the analysis of preferences or for temporal ordering of events.

4. POSSIBILITY THEORY

4.1. Possibility Measures

Fuzzy set theory provides a representation of imprecise knowledge. It does not present any immediate representation of uncertain knowledge, which is nevertheless necessary to reason with imprecise knowledge. Let us consider the precise and certain rule "if the patient is at least 40 years old, then require a mammography." Imprecise information such as "the patient is approximately 40 years old" leads to an uncertain conclusion, "we are not certain that the mammography is required." This simple example proves that imprecision and uncertainty are closely related.

Possibility theory was introduced in 1978 by Zadeh [12] to represent nonprobabilistic uncertainty linked with imprecise information in order to enable reasoning on imperfect knowledge. It is based on two measures defined for any subset of a given universe X, the possibility and the necessity measure. Let $P(X)$ denote the set of subsets of the universe X.

A *possibility measure* is a mapping Π: $P(X) \to [0, 1]$, such that

i. $\Pi(\emptyset) = 0$, $\Pi(X) = 1$, (23)
ii. $\forall A_1 \in P(X)$, $A_2 \in P(X) \ldots$ $\Pi(\cup_{i=1,2\ldots} A_i) = \sup_{i=1,2\ldots} \Pi(A_i)$. (24)

In the case of a finite universe X, we can reduce ii to ii′, which is a particular case of ii for any X:

ii′. $\forall A \in P(X), B \in P(X) \quad \Pi(A \cup B) = \max(\Pi(A), \Pi(B))$ (25)

We can interpret this measure as follows: $\Pi(A)$ represents the extent to which it is possible that the subset (or event) A of X occurs. If $\Pi(A) = 0$, A is impossible; if $\Pi(A) = 1$, A is absolutely possible.

We remark that the possibility measure of the intersection of two subsets of X is not determined from the possibility measure of these subsets. The only information we obtain from i and ii is the following:

$$\forall A \in P(X), \quad B \in P(X) \quad \Pi(A \cap B) \leq \min(\Pi(A), \Pi(B)) \tag{26}$$

Let us remark that two subsets can be individually possible ($\Pi(A) \neq 0$, $\Pi(B) \neq 0$) but jointly impossible ($\Pi(A \cap B) = 0$).

Let us consider the example of identification of a disease in a universe $X = \{d_1, d_2, , d_3, d_4\}$. We suppose that it is absolutely possible to be in the presence of disease d_1 or disease d_2, disease d_3 is relatively possible, and disease d_4 is impossible, and we represent this information as follows:

$$\Pi(\{d_1, d_2\}) = 1, \quad \Pi(\{d_3\}) = 0.8, \quad \Pi(\{d_4\}) = 0 \tag{27}$$

We deduce that it is absolutely possible that the disease is one of $\{d_1, d_2, d_4\}$, since

$$\Pi(\{d_1, d_2, d_4\}) = \max(1, 0) = 1 \tag{28}$$

It is relatively possible that the disease is one of d_3, d_4 since

$$\Pi(\{d_3, d_4\}) = \max(0.8, 0) = 0.8 \tag{29}$$

but the intersection $\{d_4\}$ of these two subsets $\{d_1, d_2, d_4\}$ and $\{d_3, d_4\}$ of X corresponds to a possibility measure equal to 0.

We deduce from conditions i and ii that

Π is monotonous with respect to the inclusion of subsets of X:

$$\text{If } A \supseteq B \text{ then } \Pi(A) \geq \Pi(B) \tag{30}$$

If we consider any subset A of X and its complement A^c, at least one of them is absolutely possible. This means that either an event or its complement is absolutely possible:

$$\forall A \in P(X) \quad \max(\Pi(A), \Pi(A^c)) = 1$$
$$\Pi(A) + \Pi(A^c) \geq 1 \qquad (31)$$

It is easy to see that possibility measures are less restricting than probability measures, because the possibility degree of an event is not necessarily determined by the possibility degree of its complement.

4.2. Possibility Distributions

A possibility measure Π is completely defined if we assign a coefficient in [0, 1] to any subset of X. In the example of four diseases, we need 16 coefficients to determine Π. It is easier to define possibility degrees if we restrict ourselves to the elements (and not to the subsets) of X and we use condition ii to deduce the other coefficients.

A *possibility distribution* is a mapping π: $X \to [0, 1]$ satisfying the normalization condition:

$$\sup_{x \in X} \pi(x) = 1 \qquad (32)$$

A possibility distribution assigns a coefficient between 0 and 1 to every element of X, for instance, to each of the four diseases d_1, d_2, d_3, d_4. Furthermore, at least one element of X is absolutely possible, for instance, one disease in $\{d_1, d_2, d_3, d_4\}$ is absolutely possible. This does not mean that this disease is identified, because several of them can be absolutely possible and other information is necessary to make a choice between them.

Possibility measure and distribution can be associated. From a possibility distribution π, assigning a coefficient to any element of X, we construct a possibility measure assigning a coefficient to any subset of X as follows:

$$\forall A \in P(X) \quad \Pi(A) = \sup_{x \in A} \pi(x) \qquad (33)$$

Conversely, from any possibility measure Π, we construct a possibility distribution as follows:

$$\forall x \in X \quad \pi(x) = \Pi(\{x\}) \qquad (34)$$

For instance, a possibility distribution such as

$$\pi(d_1) = 1, \; \pi(d_2) = 0.4, \; \pi(d_3) = 0.8, \; \pi(d_4) = 0 \qquad (35)$$

is compatible with the preceding possibility measure, which is not given completely as only 3 of the 16 coefficients are indicated.

In the case of two universes X and Y, we need to define the extent to which a pair (x, y) is possible, with $x \in X$ and $y \in Y$.

The joint possibility distribution $\pi(x, y)$ on the Cartesian product $X \times Y$ is defined for any $x \in X$ and $y \in Y$ and it expresses the extent to which x and y can occur simultaneously.

The global knowledge of $X \times Y$ through the joint possibility distribution $\pi(x, y)$ provides marginal information on X and Y by means of the marginal possibility distributions, for instance on Y:

$$\forall y \in Y \quad \pi_Y(y) = \sup_{x \in X} \pi(x, y) \tag{36}$$

which satisfy:

$$\forall x \in X \quad \forall y \in Y \quad \pi(x, y) \leq \min(\pi_X(x), \pi_Y(y)) \tag{37}$$

We remark that a joint possibility distribution provides uniquely determined marginal distributions, but the converse is false. Determining a joint possibility distribution π on $X \times Y$ from possibility distributions π_X on X and π_Y on Y requires information about the relationship between events on X and Y. If we have no information, π cannot be known exactly.

The universes X and Y are *noninteractive* if

$$\forall x \in X \quad \forall y \in Y \quad \pi(x, y) = \min(\pi_X(x), \pi_Y(y)) \tag{38}$$

This possibility distribution $\pi(x, y)$ is the greatest among all those compatible with π_X and π_Y. Two variables respectively defined on these universes are also called noninteractive.

The effect of X on Y can also be represented by means of a *conditional possibility distribution* $\pi_{Y/X}$ such that

$$\forall x \in X \; \forall y \in Y \quad \pi(x, y) = \pi_{Y/X}(x, y) * \pi_X(x) \tag{39}$$

for a combination operator $*$, generally the minimum or the product.

For example, if we consider again the universe $X = \{d_1, d_2, d_3, d_4\}$ of diseases and we add a universe $Y = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ of symptoms, $\pi_Y(s_i)$ is the possibility degree that a patient presents symptom s_i and $\pi_X(d_j)$ is the possibility degree that a patient suffers from disease d_j. For a disease d_j and a symptom s_i, we define the possibility degree $\pi(d_j, s_i)$ that the pair (d_j, s_i) is possible. X clearly has an influence on Y and the universes are interactive. We can have $\pi_X(d_j) = 1$, $\pi_Y(s_i) = 1$, but $\pi(d_j, s_i) = 0.05$. We can also define the conditional possibility degree $\pi_{Y/X}(d_j, s_i)$ that the symptom is s_i given that the disease is d_j. For instance, if the available information provides the values $\pi_X(d_3) = 0.8$ and $\pi_{Y/X}(d_3, s_i) = 1$, then $\pi(d_3, s_i) = 0.8$, since $\pi(d_3, s_i) = \pi_{Y/X}(d_3, s_i) * \pi_X$ $(d_3) = 1 * 0.8 = 0.8$, when we choose the minimum or the product for the operator $*$. This means that if disease d_3 is relatively possible for a given patient and if symptom s_i is completely possible when disease d_3 is present, then it is relatively possible that the given patient presents both disease d_3 and symptom s_i.

4.3. Necessity Measures

In this example, we see that a possibility measure provides an information on the fact that an event can occur, but it is not sufficient to describe the uncertainty about this event and to obtain a conclusion from available data. For instance, if $\Pi(A) = 1$, the event A is absolutely possible, but we can have $\Pi(A^c) = 1$, which proves that we have an absolute uncertainty about A. A solution to this problem is to complete the information on A by means of a measure of necessity on X.

A *necessity measure* is a mapping N: $P(X) \to [0, 1]$, such that

iii. $N(\emptyset) = 0$, $N(X) = 1$, (40)
iv. $\forall A_1 \in P(X), A_2 \in P(X) \ldots \quad N(\cap_{i=1,2\ldots} A_i) = \inf_{i=1,2\ldots} N(A_i)$, (41)

In the case of a finite universe X, we can reduce iv to iv′ which is a particular case of iv for any X:

iv′. $\forall A \in P(X), B \in P(X) \quad N(A \cap B) = \min(N(A), N(B))$ (42)

We can interpret this measure as follows: $N(A)$ represents the extent to which it is certain that the subset (or event)A of X occurs. If $N(A) = 0$, we have no certainty about the occurrence of the event A; if $N(A) = 1$, we are absolutely certain that A occurs.

Necessity measures are monotonous with regard to set inclusion:

$$\text{if } A \supseteq B, \text{ then } N(A) \geq N(B) \tag{43}$$

The necessity degree of the union of subsets of X is not known precisely, but we know a lower bound:

$$\forall A \in P(X), \ B \in P(X) \quad N(A \cup B) \geq \max(N(A), \ N(B)) \tag{44}$$

We deduce also from iii and iv a link between the necessity measure of an event A and its complement A^c:

$$\begin{aligned} &\forall A \in P(X) \quad \min(N(A), N(A^c)) = 0 \\ &N(A) + N(A^c) \leq 1 \end{aligned} \tag{45}$$

We see that the information provided by a possibility measure and that provided by a necessity measure are complementary and their properties show that they are linked together. Furthermore, we can point out a *duality between possibility and necessity measures*, as follows.

For a given universe X and a possibility measure Π on X, the measure defined by

$$\forall A \in P(X) \quad N(A) = 1 - \Pi(A^c) \tag{46}$$

is a necessity measure on X if A^c denotes the complement of A in X. We are certain that A occurs ($N(A) = 1$) if and only if A^c is impossible ($\Pi(A^c) = 0$) and then $\Pi(A) = 1$.

If Π is defined from a possibility distribution π, we can define its dual necessity measure by

$$\forall A \in P(X) \quad N(A) = \inf_{x \in X}(1 - \pi(x)) \tag{47}$$

which means we need only one collection of coefficients between 0 and 1 associated with the elements of the universe X (the values of $\pi(x)$) to determine both possibility and necessity measures.

With the previous example, the certainty on the fact that the patient suffers from disease d_1 is measured by $N(\{d_1\})$ and it can be deduced from the greatest possibility that the patient suffers from one of the three other diseases:

$$N(\{d_1\}) = 1 - \Pi(\{d_2, d_3, d_4\}) = 1 - \max(\pi(d_2), \pi(d_3), \pi(d_4)) \tag{48}$$

The duality between Π and N also appears in the following relations, satisfied $\forall A \in P(X)$:

- $\Pi(A) \geq N(A)$,
- $\max(\Pi(A), 1 - N(A)) = 1$,
- If $N(A) \neq 0$, then $\Pi(A) = 1$,
- If $\Pi(A) \neq 1$, then $N(A) = 0$.

These properties are important if we elicit the possibility and necessity measures from a physician. For instance, if the physician provides first possibility degrees $\Pi(A)$ for events A, we should not ask the physician to give necessity degrees for events with possibility degrees strictly smaller than 1, because $N(A) = 0$ in this case. If the physician provides first degrees of certainty, corresponding to values of a necessity measure, we should not ask for possibility degrees for events with necessity degrees different from 0, as $\Pi(A) = 1$ in this case.

4.4. Relative Possibility and Necessity of Fuzzy Sets

Possibility and necessity measures have been defined for crisp subsets of X, not for fuzzy sets. In the case in which fuzzy sets are observed, analogous measures are defined with a somewhat different purpose, which is to compare an observed fuzzy set F to a reference fuzzy set A of X.

The *possibility of F relative to A* is defined as

$$\Pi(f; A) = \sup_{x \in X} \min(f_F(x), f_A(x)) \tag{49}$$

We remark that $\Pi(F; A) = 0$ indicates that $F \cap A = \emptyset$, and $\Pi(F; A) = 1$ indicates that $F \cap A \neq \emptyset$.

The dual quantity defined by $N(F; A) = 1 - \Pi(F^c; A)$ is the *necessity of F with regard to A*, defined as

$$N(F; A) = \inf_{x \in X} \max(f_F(x), 1 - f_A(x)) \tag{50}$$

These coefficients are used, among other things, to measure the extent to which F is suitable with A. For example, with the universe X of real numbers, we can evaluate the compatibility of the glycemia level F of a patient, described as "about 1.4 g/l" (Figure 2), with a reference description of the glycemia level as "abnormal" (Figure 1), by means of $\Pi(F; A)$ and $N(F; A)$, and this information will express the extent to which the glycemia level can be considered abnormal.

5. APPROXIMATE REASONING

Possibility theory, as presented in Section 4, is restricted to crisp subsets of a universe. The purpose of its introduction was to evaluate uncertainty related to inaccuracy. We need to establish a link between both approaches.

5.1. Linguistic Variables

A *linguistic variable* is a 3-tuple (V, X, T_V), defined from a variable V (e.g., distance, glycemia level, temperature) defined on a universe X and a set $T_V = \{A_1, A_2 \ldots\}$ of fuzzy characterizations of V. For instance, with $V =$ glycemia level, we can have

T_V = {normal, abnormal} (Figure 8). We use the same notation for a linguistic characterization and for its representation by a fuzzy set of X. The set T_V corresponds to basic characterizations of V.

We need to construct more characterizations of V to enable efficient reasoning from values of V.

A *linguistic modifier* is an operator m yielding a new characterization $m(A)$ from any characterization A of V in such a way that $f_{m(A)} = t_m(f_A)$ for a mathematical transformation t_m associated with m.

For a set M of modifiers, $M(T_V)$ denotes the set of fuzzy characterizations deduced from T_V. For example, with M = {almost, very}, we obtain $M(T_V)$ = {very abnormal, almost normal...} from T_V = {normal, abnormal} (Figure 9).

Examples of linguistic modifiers are defined by the following mathematical definitions, corresponding to translations or homotheties:

- $f_{m(A)}(x) = f_A(x)^2$ (very) introduced by Zadeh
- $f_{m(A)}(x) = f_A(x)^{1//2}$ (more or less) introduced by Zadeh
- $f_{m(A)}(x) = \min(1, \lambda f_A(x))$, for $\lambda > 1$ (approximately)
- $f_{m(A)}(x) = \max(0, \upsilon\phi(x) + 1 - \upsilon)$, for a parameter υ in [1/2, 1] (about)
- $f_{m(A)}(x) = \min(1, \max(0, \phi(x) + \beta))$, with $0 < \beta < 1$ (rather)
- $t_m(f_A(x) = f_A(x + \alpha)$, for a real parameter α (really or rather, depending on the sign of α) where ϕ is the function identical to f_A on its support and extending it out of the support.

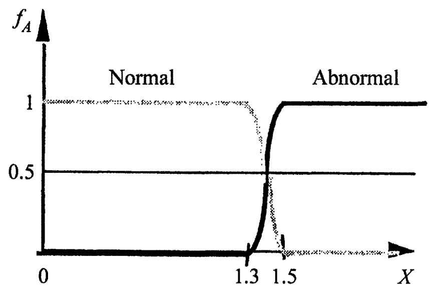

Figure 8 Set T_V of fuzzy characterizations associated with the variable V = glycemia rate.

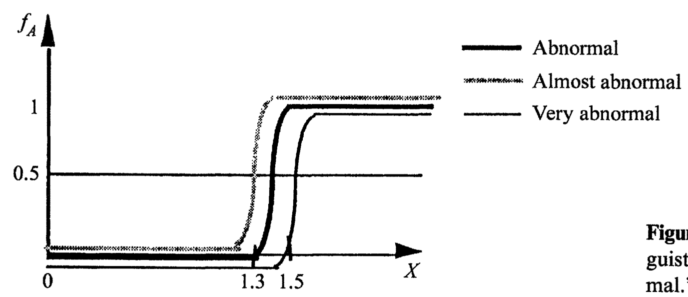

Figure 9 Representation of the effect of linguistic modifiers almost and very on "abnormal."

5.2. Fuzzy Propositions

We consider a set L of linguistic variables and a set M of linguistic modifiers.

For a linguistic variable (V, X, T_V) of L, and elementary proposition is defined as "V is A" ("the glycemia level is abnormal") by means of a normalized fuzzy set A of X in T_V or in $M(T_V)$.

The more suitable the precise value of V with A, the more true the proposition "V is A." The truth value of an *elementary fuzzy proposition* "V is A" is defined by the membership function f_A of A.

A *compound fuzzy proposition* is obtained by combining elementary propositions "V is A," "W is B," ... for noninteractive variables V.

The simplest fuzzy proposition is a conjunction of elementary fuzzy propositions "V is A and W is B" (for instance, "the glycemia level is abnormal and the cholesterol level is high"), for two variables V and W respectively defined on universes X and Y. It is associated with the Cartesian product $A \times B$ of fuzzy sets of X and Y, characterizing the pair (V, W) on $X \times Y$. Its truth value is defined by $\min(f_A(x), f_B(y))$ or more generally $T(f_A(x), f_B(y))$ for a t-norm T, in any (x, y) of $X \times Y$. Such a fuzzy proposition is very common in rules of knowledge-based systems and in fuzzy control.

Analogously, we can combine elementary propositions by a disjunction of the form "V is A or W is B" (for instance, "the glycemia level is abnormal and the cholesterol level is high"). The truth value of the fuzzy proposition is defined by $\max(f_A(x), f_B(y))$, or more generally $\perp(f_A(x), f_B(y))$ for a t-conorm $\perp$, in any (x, y) of $X \times Y$.

An *implication* between two elementary fuzzy propositions provides a fuzzy proposition of the form "if V is A then W is B" (for instance, "if the glycemia level is abnormal then the suggestion is sulfonylurea"), and we will study this form of fuzzy proposition carefully because of its importance in reasoning in a fuzzy framework.

More generally, we can construct fuzzy propositions by conjunction, disjunction, or implication on already compound fuzzy propositions.

A fuzzy proposition based on an implication between elementary or compound fuzzy propositions, for instance, of the form "if V is A and W is B then U is C" ("if the glycemia level is medium and the creatininemia level is smaller than k, then the suggestion is not sulfonylurea") is a *fuzzy rule*, "V is A and W is B" is its premise, and "U is C" is its conclusion.

5.3. Possibility Distribution Associated with a Fuzzy Proposition

The concepts of linguistic variable and fuzzy proposition are useful for the management of imprecise knowledge when we associate them with possibility distributions to represent uncertainty.

A fuzzy characterization A such as "abnormal" is prior information and its membership function f_A indicates to what extent each element x of X belongs to A. A fuzzy proposition such as "the glycemia level is abnormal" is posterior information, given after an observation, which describes to what extent it is possible that the exact value of the glycemia level is any element of X.

An elementary fuzzy proposition induces a possibility distribution $\pi_{V,A}$ on X, defined from the membership function of A by

$$\forall x \in X \quad \pi_{V,A}(x) = f_A(x) \tag{51}$$

From this possibility distribution, we define a possibility and a necessity measure for any crisp subset D of X, given the description of V by A:

$$\begin{aligned} \Pi_{V,A}(D) &= \sup_{x \in D} \pi_{V,A}(x) \\ N_{V,A}(D) &= 1 - \Pi_{V,A}(D^c) \end{aligned} \tag{52}$$

Analogously, a compound fuzzy proposition induces a possibility distribution on the Cartesian product of the universes. For instance, a fuzzy proposition such as "V is A and W is B," with V and W defined on universes X and Y, induces the following possibility distribution:

$$\forall x \in X, \ \forall y \in Y \quad \pi_{(V,W),A \times B}(x, y) = \min(f_A(x), f_B(y)) \tag{53}$$

Such a connection between membership functions and degrees of possibility, or equivalently between imprecision and uncertainty, appears clearly if we again use the example given in Figure 1. We see that a value of the glycemia level equal to 1.4 g/l belongs to the class of abnormal levels with a degree equal to 0.5. Conversely, if we know only that a given glycemia level is characterized as "abnormal," we deduce that

It is impossible that this level is less than 1.3 g/l, which means that the possibility degrees are equal to zero for the values of the glycemia level smaller than 1.3 g/l.

It is absolutely possible that this level is at least equal to 1.5 g/l, which means that the possibility distribution assigns a value equal to 1 to levels at least equal to 1.5 g/l.

It is relatively possible, with a possibility degree between 0 and 1, that the glycemia level is between 1.3 and 1.5 g/l.

In the case of an *uncertain fuzzy proposition* such as "V is A, with an uncertainty ϵ," for $A \in T_V$, no element of the universe X can be rejected and every element x of X has a possibility degree at least equal to ϵ. Such a fuzzy proposition is associated with a possibility distribution:

$$\pi'(x) = \max(\pi_{V,A}(x), \epsilon) \tag{54}$$

For instance, a fuzzy proposition weighted by an uncertainty, such as "it is possible that the glycemia level is abnormal, with an uncertainty 0.4" or, equivalently, "it is possible that the glycemia level is abnormal, with a certainty 0.6," is represented by a possibility distribution π' as indicated in Figure 10 by using the possibility distribution

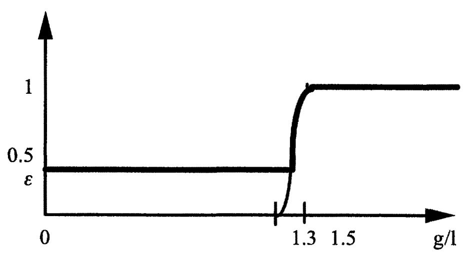

Figure 10 Possibility distribution of an uncertain fuzzy proposition.

$\pi_{V,A}$ deduced from the membership function of "abnormal" given in Figure 1 and the value 0.4 of ϵ.

5.4. Fuzzy Implications

The use of imprecise and/or uncertain knowledge leads to reasoning in a way close to human reasoning and different from classical logic. More particularly, we need:

To manipulate truth values intermediate between absolute truth and absolute falsity

To use soft forms of quantifiers, more gradual than the universal and existential quantifiers $\forall$ and $\exists$

To use deduction rules when the available information is imperfectly compatible with the premise of the rule.

For these reasons, fuzzy logic has been introduced with the following characteristics:

Propositions are fuzzy propositions constructed from sets L of linguistic variables and M of linguistic modifiers.

The truth value of a fuzzy proposition belongs to [0, 1] and is given by the membership function of the fuzzy set used in the proposition.

Fuzzy logic can be considered as an extension of classical logic and it is identical to classical logic when the propositions are based on crisp characterizations of the variables.

Let us consider a *fuzzy rule* "if V is A then W is B," based on two linguistic variables (V, X, T_V) and (W, Y, T_W).

A *fuzzy implication* associates with this fuzzy rule the membership function of a fuzzy relation R on $X \times Y$ defined as

$$\forall (x, y) \in X \times Y \quad f_R(x, y) = F(f_A(x), f_B(y)) \tag{55}$$

for a function F chosen in such a way that, if A and B are singletons, then the fuzzy implication is identical to the classical implication.

There exist many definitions of fuzzy implications. The most commonly used are the following:

$f_R(x, y) = 1 - f_A(x) + f_A(x) \cdot f_B(y)$	Reichenbach
$f_R(x, y) = \max(1 - f_A(x),\ \min(f_A(x), f_B(y))$	Willmott
$f_R(x, y) = \max(1 - f_A(x), f_B(y))$	Kleene-Dienes
$f_R(x, y) = \min(1 - f_A(x) + f_B(y), 1)$	Lukasiewicz
$f_R(x, y) = \min(f_B(y)/f_A(x), 1)$ if $f_A(x) \neq 0$ and 1 otherwise	Goguen
$f_R(x, y) = 1$ if $f_A(x) \leq f_B(y)$ and 0 otherwise	Rescher-Gaines
$f_R(x, y) = 1$ if $f_A(x) \leq f_B(y)$ and $f_B(y)$ otherwise	Brouwer-Gödel
$f_R(x, y) = \min(f_A(x), f_B(y))$	Mamdani*
$f_R(x, y) = f_A(x) \cdot f_B(y)$	Larsen*

The last two quantities (*) do not generalize the classical implication, but they are used in fuzzy control to manage fuzzy rules.

Generalized modus ponens is an extension of the scheme of reasoning called modus ponens in classical logic. For two propositions p and q such that $p \Rightarrow q$, if p is true, we deduce that q is true. In fuzzy logic, we use fuzzy propositions and, if p' is true, with p' approximately identical to p, we want to get a conclusion, even though it is not q itself.

Generalized modus ponens (g.m.p.) is based on the following propositions:

Rule	if V is A then W is B
Observed fact	V is A'
Conclusion	W is B'

The membership function $f_{B'}$ of the conclusion is computed from the available information: f_R to represent the rule, $f_{A'}$ to represent the observed fact, by means of the so-called combination-projection rule:

$$\forall y \in Y \quad f_{B'}(y) = \sup_{x \in X} T(f_{A'}(x),\ f_R(x, y)) \tag{56}$$

for a t-norm T called a generalized modus ponens operator.

The choice of T is determined by the compatibility of the generalized modus ponens with the classical modus ponens: if $A = A'$, then $B = B'$.

The most usual g.m.p. operators suitable with this condition are the following:

The Lukasiewicz t-norm $T(u, v) = \max(u + v - 1, 0)$ with any of the fuzzy implications mentioned above.

The product t-norm $T(u, v) = u.v$ with the five last fuzzy implications of our list

The min t-norm $T(u, v) = \min(u, v)$ with the four last ones

5.5. Fuzzy Inferences

The choice of a fuzzy implication is based on its behavior. Some fuzzy implications entail an uncertainty about the conclusion (Kleene–Dienes implication, for instance), whereas other provide imprecise conclusions (Reichenbach, Brouwer–Gödel, or Goguen implication, for instance). Some of them entail both types of imperfection (Lukasiewicz implication, for instance).

Let us consider the following example (see Figure 11):

Rule: "if the glycemia level is abnormal then sulfonylurea is suggested," with the universe of distances $X = \mathbb{R}^+$ and the universe of degrees of suggestion $Y = [0, 1]$.

Observation: the glycemia level is 1.4 g/l.

Conclusion:

- It is relatively certain that sulfonylurea is suggested (with the Kleene–Dienes implication).
- It is relatively certain that sulfonylurea is rather suggested (with the Reichenbach, Brouwer–Gödel, or Goguen implication).
- It is relatively certain that sulfonylurea is rather suggested (with the Lukasiewicz implication).

Fuzzy inferences are used in rule-based systems, when there exist imprecise data, when we need a flexible system, with representation of the linguistic descriptions

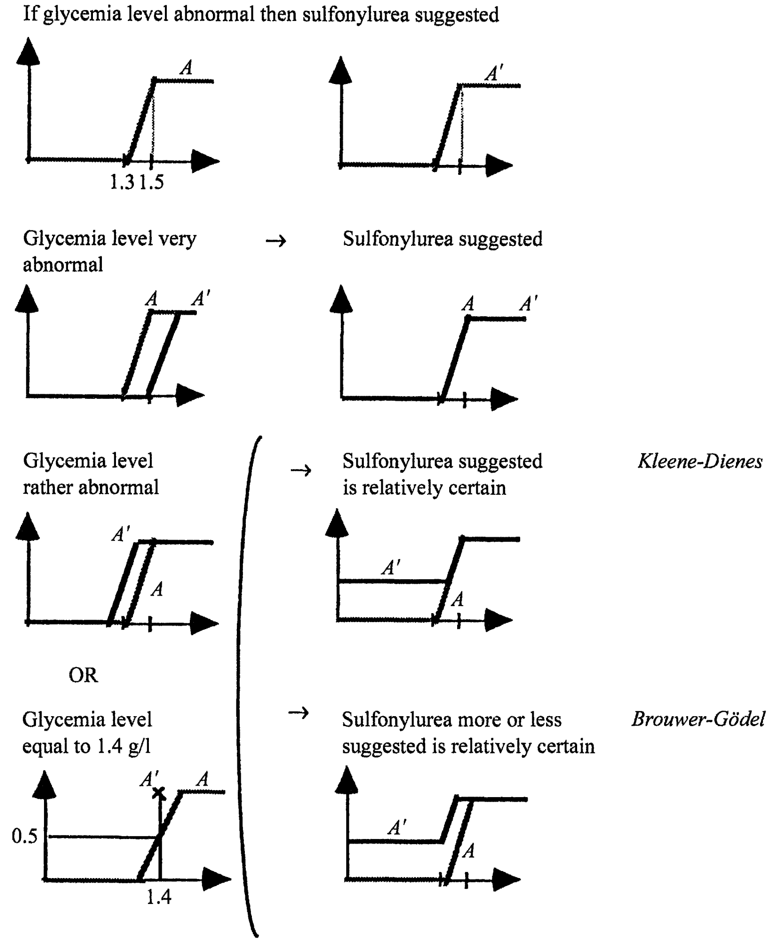

Figure 11 Example of a generalized modus ponens with various forms of observations A' and various fuzzy implications.

depending on the environment of the system or its conditions of utilization, when we cope with categories with imprecise boundaries, and when there exist subjective variables described by human agents.

6. EXAMPLES OF APPLICATIONS OF NUMERICAL METHODS IN BIOLOGY

There exist many knowledge-based systems using fuzzy logic. The treatment of glycemia, for instance, has given rise to several automatic systems supporting diagnosis or helping patients to take care of their glycemia level [13–15]. An example in other domains is a system supporting the prescription of antibiotics [16].

Some general systems, which are expert system engines using fuzzy logic, have been used to solve medical problems. MILORD is particularly interesting for its module of expert knowledge elicitation [17] and FLOPS takes into account fuzzy numbers and fuzzy relations and is used to process medical images in cardiology [18]. Also, CADIAG-2 provides a general diagnosis support system using fuzzy descriptions and also fuzzy quantifiers such as "frequently" or "rarely" [19].

The management of temporal knowledge in an imprecise framework can be solved by using fuzzy temporal constraints, and such an approach has been used for the management of data in cardiology [20], for instance.

It is also interesting to use fuzzy techniques for diagnosis support systems taking into account clinical indications that are difficult to describe precisely, such as the density, compacity, and texture of visual marks. Such systems have been proposed for the diagnosis of hormone disorders [21] or the analysis of symptoms of patients admitted to a hospital [22].

In medical image processing, problems of pattern identification are added to the difficulty in eliciting precise and certain rules from specialists, even though they are able to make a diagnosis from an image. A system for the analysis of microcalcifications in mammographic images has been proposed [23], a segmentation method based on fuzzy logic has been described [24], and the fusion of cranial magnetic resonance has been explained [25].

Databases can also be explored by means of imprecise queries, and an example of an approach to this problem using fuzzy concepts has been proposed [26].

In this section, we have listed the main directions in using fuzzy logic in the construction of automatic systems in medicine on the basis of existing practical applications. This list is obviously not exhaustive. More applications are discussed elsewhere [27].

7. CONCLUSION

We have presented the main problems concerning the management of uncertainty and imprecision in automatic systems, especially in medical applications. We have introduced methodologies that enable us to cope with these imperfections.

We have not developed evidence theory, also called Dempster–Shafer theory, which concerns the management of degrees of belief assigned to the occurrence of events. The main interest lies in the combination rule introduced by Dempster that provides a means of aggregating information obtained from several sources.

Another methodology used in medical applications is the construction of causal networks, generally regarded as graphs, the vertices of which are associated with situations or symptoms or diseases. The arcs forward probabilities of occurrence of events from one vertex to another and enable us to update probabilities of hypotheses when new information is received or to point out dependences between elements.

As we focused on methods for dealing with imprecisions, let us point out the reasons for their importance [1,2]: fuzzy set and possibility theory are of interest when at least one of the following problems occurs:

- We have to deal with imperfect knowledge.
- Precise modeling of a system is difficult.
- We have to cope with both uncertain and imprecise knowledge.

- We have to manage numerical knowledge (numerical values of variables "100 millimeters") and symbolic knowledge (descriptions of variables in natural language, "long") in a common framework.
- Human components are involved in the studied system (observers, users of the system, agents) and bring approximate or vague descriptions of variables, subjectivity (degree of risk, aggressiveness of the other participants), qualitative rules ("if the level is too high, reduce the level"), and gradual knowledge ("the greater, the more dangerous").
- We have to take into account imprecise classes and ill-defined categories ("painful position").
- We look for flexible management of knowledge, adaptable to the environment or to the situation we meet.
- The system is evolutionary, which makes it difficult to describe precisely each of its states.

The number of medical applications developed since the 1970s justifies the development we have presented.

REFERENCES

[1] B. Bouchon-Meunier, *La logique floue, Que Sais-Je*? 2nd ed. No. 2702. Paris: Presses Universitaires de France, 1994.

[2] B. Bouchon-Meunier, *La logique floue et ses applications*. Paris: Addison-Wesley, 1995.

[3] B. Bouchon-Meunier and H. T. Nguyen, *Les incertitudes dans les systèmes intelligents*. Paris: Presses Universitaires de France, 1996.

[4] D. Dubois and H. Prade, *Fuzzy Sets and Systems, Theory and Applications*. New York: Academic Press, 1980.

[5] D. Dubois and H. Prade, *Théorie des possibilités, applications à la représentation des connaissances en informatique*, 2nd ed. Paris: Masson, 1987.

[6] D. Dubois, H. Prade, and R. R. Yager, *Readings in Fuzzy Sets for Intelligent Systems*. San Mateo, CA: Morgan Kaufmann, 1993.

[7] G. Klir and T. Folger, *Fuzzy Sets, Uncertainty and Information*. Englewood Cliffs, NJ: Prentice Hall, 1988.

[8] L. Sombé, *Raisonnements sur des informations incomplètes en intelligence artificielle—Comparaison de formalismes à partir d'un exemple*. Toulouse: Editions Teknéa, 1989.

[9] R. E. Neapolitan, *Probabilistic Reasoning in Expert Systems*. New York: Wiley, 1994.

[10] R. R. Yager and D. P. Filev, *Essentials of Fuzzy Modeling and Control*. New York: Wiley, 1990.

[11] L. A. Zadeh, Fuzzy sets. *Information Control* 8: 338–353, 1965.

[12] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets Syst*. 1: 3–28, 1978.

[13] G. Soula, B. Vialettes, and J. L. San Marco, PROTIS, a fuzzy deduction rule system: Application to the treatment of diabetes. *Proceedings MEDINFO 83, IFIP-IMIA*, pp. 177–187, Amsterdam, 1983.

[14] P. Y. Glorennec, H. Pircher, and J. P. Hespel, Fuzzy logic control of blood glucose. *Proceedings International Conference IPMU*, pp. 916–920, Paris, 1994.

[15] J. C. Buisson, H. Farreny, and H. Prade, Dealing with imprecision and uncertainty in the expert system DIABETO-III. *Actes CIIAM-86*, pp. 705–721, Hermès, 1986.

[16] G. Palmer and B. Le Blanc, MENTA/MD: Moteur décisionnel dans le cadre de la logique possibiliste. *Actes 3èmes Journées Nationales sur les Applications des Ensembles Flous*, pp. 61–68, Nîmes, 1993.

[17] R. Lopez de Mantaras, J. Agusti, E. Plaza, and C. Sierra, Milord: A fuzzy expert system shell. In *Fuzzy Expert Systems*, A. Kandel, ed., pp. 213–223, Boca Raton, FL: CRC Press, 1992.

[18] J. J. Buckley, W. Siler, and D. Tucker, A fuzzy expert system. *Fuzzy Sets Syst* 20:1–16, 1986.

[19] K. P. Adlassnig and G. Kolarz, CADIAG-2: Computer-assisted medical diagnosis using fuzzy subsets. In *Approximate Reasoning in Decision Analysis*, M. M. Gupta and E. Sanchez, eds., pp. 219–247. Amsterdam: North Holland.

[20] S. Barro, A. Bugarin, P. Félix, R. Ruiz, R. Marin, and F. Palacios, Fuzzy logic applications in cardiology: Study of some cases. *Proceedings International Conference IPMU*, pp. 885–891, Paris, 1994.

[21] E. Binaghi, M. L. Cirla, and A. Rampini, A fuzzy logic based system for the quantification of visual inspection in clinical assessment. *Proceedings International Conference IPMU*, pp. 892–897, Paris, 1994.

[22] D. L. Hudson and M. E. Cohen, The role of approximate reasoning in a medical expert system. In *Fuzzy Expert Systems*, A. Kandel, ed. Boca Raton, FL: CRC Press, 1992.

[23] S. Bothorel, B. Bouchon-Meunier, and S. Muller, A fuzzy logic–based approach for semiological analysis of microcalcifications in mammographic images. *Int. J. Intell. Syst.*, 1997.

[24] P. C. Smits, M. Mari, A. Teschioni, S. Dellepine, and F. Fontana, Application of fuzzy methods to segmentation of medical images. *Proceedings International Conference IPMU*, pp. 910–915, Paris, 1994.

[25] I. Bloch and H. Maître, Fuzzy mathematical morphology. *Ann. Math. Artif. Intell.* 9:III–IV, 1993.

[26] M. C. Jaulent and A. Yang, Application of fuzzy pattern matching to the flexible interrogation of a digital angiographies database. *Proceedings International Conference IPMU*, pp. 904–909, Paris, 1994.

[27] M. Cohen and D. Hudson, eds., *Comparative Approaches to Medical Reasoning*. Singapore: World Scientific, 1995.

[28] R. R. Yager, S. Ovchinnikov, R. M. Tong, and H. T. Nguyen, eds., *Fuzzy Sets and Applications, Selected Papers by L. A. Zadeh*. New York: Wiley, 1987.

[29] H. J. Zimmermann, *Fuzzy Set Theory and Its Applications*. Dordrecht: Kluwer, 1985.