I

The Magnetic Field

1. Historical

The writings of Thales, the Greek, establish that the power of loadstone, or magnetite, to attract iron was known at least as long ago as 600 B.C. It has been claimed that the Chinese used the compass sometime before 2500 B.C. That magnetite can induce iron to acquire attractive powers, or to become magnetic, was mentioned by Socrates. Thus permanent and induced magnetism represent two of man's earliest scientific discoveries. However, the only real interest in magnetism in antiquity appears to be concerned with its use in the construction of the compass. For example, it is illuminating that it was not until many centuries later that Gilbert (1540–1603) realized that the earth was a huge magnet, even though the operation of the compass depends on this very fact.

The discovery of two regions called magnetic poles, or sometimes just "poles," which attracted a piece of iron more strongly than the rest of the magnetite, was made by P. Peregrines about 1269 A.D. Coulomb (1736–1806), in accurate quantitative experiments with the torsion balance, investigated the forces between magnetic poles of long thin steel rods. His results form the starting point of this treatise on magnetism.

2. The Magnetic Field Vector H

Coulomb found that there were two types of poles, now called positive or north, and negative, or south. Like poles repel one another and unlike poles attract one another. This force of attraction or repulsion is proportional to the product of the strength of the poles and inversely proportional to the square of the distance between them. This is Coulomb's law, which can be stated mathematically as

$$\mathbf{F} = k\,\frac{m_1 m_2}{r^2}\,\mathbf{r}_0, \tag{1-2.1}$$

where $\mathbf{F}$ is the force,[1] m_1 and m_2 the pole strengths, r the distance between the poles, and $\mathbf{r}_0$ a unit vector directed along r. The constant of proportionality k that occurs permits a definition of pole strength. In the cgs system of units two like poles are of unit strength if they repel each other with a force of 1 dyne when they are 1 cm apart; that is, $k = 1$. Other systems of units and their relationship to the cgs system are discussed in Appendix I.

It is convenient to consider $\mathbf{F}$ as separated into two factors. One factor is just one of the poles, say m_2, usually called the test pole. The other factor depends on the other pole, called the source, and on the location with respect to it; it is called the magnetic field $\mathbf{H}$. This field is defined as the force the pole exerts on a unit positive pole, or

$$\mathbf{H} = \frac{m}{r^2}\,\mathbf{r}_0. \tag{1-2.2}$$

In addition to this use of $\mathbf{H}$ as the field at a point, we will employ the same symbol $\mathbf{H}$ as the set of values of the magnetic field at all points: no confusion should result, since the correct meaning will be clear from the context. The cgs unit of magnetic field is the oersted, although the term gauss is still frequently used. Should several poles be present, experiments show that the field is the vector sum of the forces on the test pole.

Instead of the vector field quantity $\mathbf{H}$, it is often convenient to use a scalar potential φ. The quantity φ is defined so that its negative gradient is the magnetic field

$$\mathbf{H} = -\nabla\varphi, \tag{1-2.3}$$

where the operator ∇ is

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}.$$

Here $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, are the unit vectors of a Cartesian coordinate system, and (x, y, z) are the coordinates at the point where the field or potential is under consideration.

[1] Boldface type indicates a vector quantity.

It follows immediately that for an isolated pole

$$\varphi = \frac{m}{r}. \tag{1-2.4}$$

The work done in bringing a unit test pole from infinity to the point (x, y, z) a distance r from m is found by integrating (1-2.2) along the path.[2] This turns out to be equal to φ and provides a simple physical meaning of the scalar potential.

Consider the work done in taking a unit pole from point 1 to 2 in a field **H** given by the line integral

$$\int_2^1 \mathbf{H} \cdot d\mathbf{s} = \int_2^1 H_s \, ds. \tag{1-2.5}$$

Here ds is a line element of the path from point 1 to point 2. Now, since $H_s = -\partial\varphi/\partial s$, we get

$$\int_2^1 \mathbf{H} \cdot d\mathbf{s} = \varphi_2 - \varphi_1. \tag{1-2.6}$$

This integral has the same value for any path having the same first and final point; that is, the work done is independent of the path. Mathematically this is stated as

$$\oint \mathbf{H} \cdot d\mathbf{s} = 0. \tag{1-2.7}$$

3. The Magnetization Vector M

Isolated magnetic poles have never been observed in nature, but occur instead in pairs, one pole being positive, the other negative. Such a pair is called a dipole. The magnetic moment of a dipole is defined as

$$\boldsymbol{\mu} = m\mathbf{d}, \tag{1-3.1}$$

where **d** is a vector pointing from the negative to the positive pole and equal in magnitude to the distance between the poles assumed to be points. If **d** approaches zero and m increases so that $\boldsymbol{\mu} = m\mathbf{d}$ is a constant, then in the limit in which $\mathbf{d} = 0$ the dipole is said to be ideal.

Atomic theory has shown that the magnetic dipole moments observed in bulk matter arise from one or two origins: one is the motion of electrons about their atomic nucleus (orbital angular momentum) and the other is the rotation of the electron about its own axis (spin angular momentum).

[2] This is the work done *against* the magnetic force; to compute the work done *by* it, the limits of the integration are reversed.

The nucleus itself has a magnetic moment. Except in special types of experiments, this moment is so small that it can be neglected in the consideration of the usual macroscopic magnetic properties of bulk matter.

It turns out that a magnetic field **H** interacts with the electrons of an atom in such a way that a magnetic moment is induced. This phenomenon is called *diamagnetism*. Since all matter contains electrons moving in orbits, diamagnetism occurs in all substances.

Depending on the electronic structure of an atom, it may or may not have a *permanent* magnetic moment. All magnetic effects other than diamagnetism result because of permanent atomic magnetic moments. If the coupling between the moments of different atoms is small or zero, the phenomenon of *paramagnetism* results. In the absence of an applied field **H** such materials will exhibit no net magnetic moment. If the

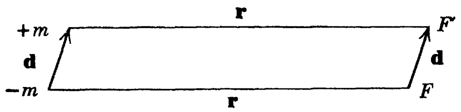

Fig. 1-3.1. F is the field point at which the potential produced by the dipole is calculated. F' is a point located a step **d** from F.

coupling between the atomic moments is very large, there are three important classifications. If the atomic moments are aligned parallel, the substance is said to be *ferromagnetic*. The magnetic moments may be aligned parallel within groups, usually two. If pairs of groups are aligned antiparallel and the atomic moments of the groups are equal, the substance is *antiferromagnetic*. However, the atomic moments of the groups may not be equal—for example, when two different elements are present; thus when they are aligned antiparallel there is a net moment. This phenomenon is called *ferrimagnetism*; some writers consider it to be just a special case of antiferromagnetism.

Because magnetic poles occur in pairs, it is of interest to calculate the magnetic field produced by such a combination. The dipole shown in Fig. 1-3.1 is considered to be almost but not quite ideal, so that $\mathbf{d} \ll \mathbf{r}$. The potential it produces at the field point F, φ_F, is due to both the positive and negative poles, that is,

$$\varphi_F = \varphi_F{}^+ + \varphi_F{}^-.$$

Now let the point a step **d** from F be called F'. Except for sign, the potential $-m$ produces at F is the same $+m$ produces at F'. Hence

$$\begin{aligned}\varphi_F &= \varphi_F{}^+ - \varphi_{F'}{}^+ \\ &= -\mathbf{d} \cdot \nabla_F \varphi_F{}^+.\end{aligned}$$

Here ∇_F indicates differentiation with respect to the field coordinate (x, y, z) and not the source coordinate (x_i, y_i, z_i) since

$$\nabla_F \frac{1}{r} = -\nabla_s \frac{1}{r},$$

where

$$r = [(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{1/2}.$$

Now, the point source $+m$ produces a potential at F given by

$$\varphi_F^{+} = \frac{m}{r}.$$

Therefore

$$\varphi_F = -m\mathbf{d} \cdot \nabla \frac{1}{r}.$$

Now suppose the dipole to be ideal, that is, $\mathbf{d} = 0$. In the limit we have

$$\varphi_F = -\boldsymbol{\mu} \cdot \nabla \frac{1}{r}.$$

Since

$$\nabla \frac{1}{r} = -\frac{1}{r^2}\left[\frac{(x - x_i)}{r}\mathbf{i} + \frac{(y - y_i)}{r}\mathbf{j} + \frac{(z - z_i)}{r}\mathbf{k}\right],$$

we get

$$\varphi_F = -\boldsymbol{\mu} \cdot \nabla \frac{1}{r} = \frac{1}{r^3}\boldsymbol{\mu} \cdot \mathbf{r} = \frac{1}{r^2}|\boldsymbol{\mu}| \cos \theta. \tag{1-3.2}$$

The magnetic field $\mathbf{H}_F$ is then

$$\mathbf{H}_F = -\nabla\left(-\boldsymbol{\mu} \cdot \nabla \frac{1}{r}\right)$$

$$= -\frac{\boldsymbol{\mu}}{r^3} + \frac{(3\boldsymbol{\mu} \cdot \mathbf{r})\mathbf{r}}{r^5}. \tag{1-3.3}$$

For substances that have a net magnetic moment it is usual to define a magnetization vector **M** as the ratio of the magnetic moment of a small volume at some point to that volume. The size of the volume chosen must be large enough so that a somewhat larger volume will still yield the same result for **M**; in this way we ensure that atomic fluctuations are negligible. If **M** is constant for the specimen, the material is said to be uniformly magnetized. From the definition of **M** it is clear that it is also the pole strength for a unit area perpendicular to **M**, that is

$$\sigma = \mathbf{M} \cdot \mathbf{n}, \tag{1-3.4}$$

where σ is the pole strength per unit area and $\mathbf{n}$ is a unit vector normal to the surface.

Introduction of the vector $\mathbf{M}$ permits generalization of equation 1-3.2 for bulk material. Summing over the dipoles gives the total potential at a point external to the specimen as

$$\varphi_F = -\sum \boldsymbol{\mu} \cdot \nabla \frac{1}{r}$$

$$= -\int \mathbf{M} \cdot \nabla_F \frac{1}{r}\, dv_s. \tag{1-3.5}$$

A special form of Green's theorem gives

$$\varphi_F = \int \frac{1}{r} \mathbf{M} \cdot \mathbf{n}\, dS - \int \frac{1}{r} \nabla_s \cdot \mathbf{M}\, dv \tag{1-3.6}$$

or

$$= \int \frac{\sigma\, dS}{r} - \int \frac{\rho}{r}\, dv,$$

where dS is an element of area. This result permits an interesting physical interpretation to be made. The magnetic potential can be considered to

(a) (b)

Fig. 1-3.2. In (*a*) each arrow represents a dipole, each with the same magnetic moment. The uncompensated charges for this dipole distribution are shown in (*b*). This illustrates the origin of volume charges for a simple case.

be due to two causes. One, the surface charge (or pole) density σ, and, two, a volume charge density ρ. The first of these can be easily pictured as arising from the uncompensated ends of the dipoles that end on the surface. The volume density may be pictured as the uncompensated poles that arise from an inhomogenity of the distribution of the moments, as illustrated in Fig. 1-3.2.

4. Magnetic Induction, the Vector B

The magnetic forces that must exist inside a ferro- or ferrimagnetic medium pose some special problems. Such forces have meaning only if it is possible to specify a method of measuring them. The approach adopted by Maxwell was to consider the medium as a continuum and to make a cavity around the point at which the force on the test pole was to be determined. However, the force per unit pole depends on the shape of the cavity, since this force depends partly on the pole distribution around the cavity, so that there exist an infinite number of ways that the field could be defined. In fact, two particular cavity shapes are chosen.

The field **H** is defined as the field vector in a needle-shaped cavity with an infinitesimally small diameter. The reason for this choice is that the field defined in this way satisfies equation 1-2.7. The field vector obtained when the cavity is a disk of infinitesimally small height is called the magnetic induction **B**; the reason for this choice is that Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ is then satisfied. With the aid of Gauss's theorem, we can show that **B** and **H** are related by[3]

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}; \tag{1-4.1}$$

B is said to be in the units of gauss in the cgs system.

The magnetic flux Φ is defined as the flux of the vector **B** through a surface of area A; that is,

$$\Phi = \int_A \mathbf{B} \cdot \mathbf{n}\, dS. \tag{1-4.2}$$

The unit of flux is called the maxwell. Thus the induction in gauss, at some field point, is equal to the flux density, the number of maxwells per square centimeter. The foregoing definition of flux is possible only because $\nabla \cdot \mathbf{B} = 0$, one of Maxwell's equations. Often a graphical meaning is given to the flux. It can be represented as lines or tubes whose density is equal to B and direction is along **B**.

When the vectors **B**, **H**, and **M** are parallel, it is useful to define the permeability μ by

$$B = \mu H \tag{1-4.3}$$

and the susceptibility χ by

$$M = \chi H. \tag{1-4.4}$$

The susceptibility per unit mass χ_ρ is defined as χ/ρ, where ρ is the density. The atomic or molar susceptibility χ_A or χ_m then is found by multiplying χ_ρ by the atomic or molecular weight. From (1-4.1) it follows immediately that

$$\mu = 1 + 4\pi\chi. \tag{1-4.5}$$

[3] A good treatment of this problem for the analogous electrical case may be found in C. J. F. Böttcher, *Theory of Electric Polarization*, Elsevier Publishing Co., New York (1952), Ch. II.

If the magnetic vectors are not parallel, the components of **B** and **H** relative to an arbitrarily chosen Cartesian coordinate system can be related by the set of equations:

$$B_x = \mu_{11}H_x + \mu_{12}H_y + \mu_{13}H_z,$$

$$B_y = \mu_{21}H_x + \mu_{22}H_y + \mu_{23}H_z,$$

$$B_z = \mu_{31}H_x + \mu_{32}H_y + \mu_{33}H_z.$$

The quantities μ_{ij} are components of the permeability tensor $\tilde{\boldsymbol{\mu}}$. Similarly, the relationship between M and H can be expressed with the aid of a susceptibility tensor $\boldsymbol{\chi}$.

It is an experimental result that χ is negative for diamagnetic materials and positive for the other types of magnetism, being very large for ferri- and ferromagnetic substances.

The magnetization of dia-, para-, and antiferromagnetic substances disappears if the applied field is removed. This is in contrast to the behavior of ferro- and ferrimagnetic materials, which usually retain at least part of their induced magnetic moment in the absence of an applied field. For these materials the susceptibility is a function of the applied field, the temperature, and the history of the samples. Discussion of the temperature and field dependence of the susceptibility is left until later.

5. The Demagnetization Factor D

The field H' inside a specimen is different from the applied field H because of the magnetization or equivalently, the poles. Consider a ferromagnet with ellipsoidal shape in a uniform external field **H**. As discussed later, the magnetization of the ellipsoidal specimen will also be uniform. The poles that appear on the surface, indicated in Fig. 1-5.1, produce a uniform internal field, $H' - H$, opposite in direction to **H**. For specimens with an ellipsoidal shape it is usual to write

$$\mathbf{H}' = \mathbf{H} - D\mathbf{M}, \tag{1-5.1}$$

where D is called the demagnetization factor. D depends on the geometry of the specimen. For diamagnets $H' > H$; for all other magnets $H' < H$. The difference in the field H' and H can usually be neglected for dia- and paramagnets, but it can be very large for ferro- and ferrimagnets. From the reasoning of Section 1-4, it can be seen that for a disk $D = 4\pi$ for the direction perpendicular to the plane of the disk. In general the demagnetizing factor is a tensor **D**.

In the easiest general case to calculate **M** is uniform. In equation 1-3.6,

$$\varphi = \int \frac{1}{r} \mathbf{M} \cdot \mathbf{n} \, dS - \int \frac{1}{r} \nabla \cdot \mathbf{M} \, dv, \tag{1-5.2}$$

the second term is zero, and the potential, and therefore the field, is due only to the surface pole distribution. Also, in expression 1-3.5 for φ, **M** can be taken outside the integral sign, and we have

$$\varphi = -\mathbf{M} \cdot \int \nabla \frac{1}{r} \, dv. \tag{1-5.3}$$

Now $-\int \nabla(1/r)\, dv$ is the gravitational force due to a volume of uniform unit mass density (the gravitational constant $G = 1$ here) or the electric force due to a volume of unit charge density. We therefore have the

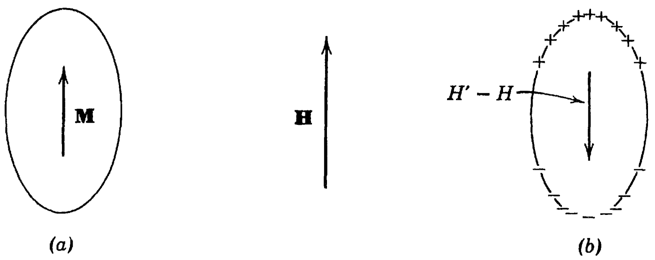

Fig. 1-5.1. (*a*) A uniformly magnetized ellipsoid and (*b*) the equivalent poles. The uncompensated surface poles produce a uniform internal field $H' - H$ and an external field that is identical to that of an equivalent dipole positioned at the ellipsoid's center. **H** is a uniform applied field.

important result that equations derived in potential theory may be used to find φ, next $H' - H$, and finally D.

It is instructive to consider the derivation of the foregoing result by a simple physical argument.[4] Let Ψ be the potential due to gravitation, or the electric charge of the body assumed of uniform density ρ. Now, if the body is moved a distance $-\delta x$ in the direction of x, the change of the potential at any point will be $-(d\Psi/dx)\,\delta x$. Instead, if we consider the body to be moved δx, and its original density ρ changed to $-\rho$, then $-(d\Psi/dx)\,\delta x$ is the resultant potential due to the two bodies (Fig. 1-5.2).

To any element of volume, mass, or charge $\rho\, dv$, there will correspond an element of the shifted body of $-\rho\, dv$ a distance $-\delta x$ away. Hence the dipole moment of these two elements is $\rho\, dv\, \delta x$, and the magnetization

[4] J. C. Maxwell, *Treatise on Electricity and Magnetism*, 3rd ed., Oxford University Press, Oxford (1891), vol. ii, p. 66. [Reprinted Dover Publications, New York (1954).]

is $\rho\,\delta x$. Therefore if $-(d\Psi/dx)\,\delta x$ is the magnetic potential of the body of magnetization $\rho\,\delta x$, then $-d\Psi/dx$ is the potential for a body of magnetization $\rho(= M)$.

In the volume common to the two bodies the density is effectively zero. A shell of positive charge or density resides on one side of the matter and one of negative on the other, each of density $\rho\cos\epsilon$, ϵ being the angle between the outward normal and the axis x. This then corresponds exactly to the first term of equation 1-5.2 and gives an immediate physical picture of the mathematics.

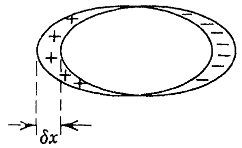

Fig. 1-5.2. Uncompensated charges or poles for two ellipsoids of opposite charge densities and a distance δx apart.

The x-component of $\mathbf{H}' - \mathbf{H}$ is $-(\partial\varphi/\partial x) = M_x(\partial^2\Psi/\partial x^2)$. Therefore, for $\mathbf{M}$ to be uniform, which implies that $\mathbf{H}'$ is uniform, Ψ must be a quadratic function of the coordinates. From potential theory[5] this occurs only when the body is bounded by a surface of second degree. The only physically possible body is then an ellipsoid.

The derivation of the expressions for Ψ is beyond the scope of this book and belongs to potential theory.[5] For completeness, and because of their practical usefulness, some of the important formulas for the demagnetization factor of ellipsoids of revolution are given.

We define a as the polar semiaxis and b as the equatorial semiaxis with $m = a/b$. Then for the prolate spheroid $(m > 1)$

$$D_a = \frac{4\pi}{(m^2-1)}\left\{\frac{m}{(m^2-1)^{1/2}}\ln\left[m + (m^2-1)^{1/2}\right] - 1\right\} \qquad (1\text{-}5.4)$$

and

$$D_b = \tfrac{1}{2}(4\pi - D_a), \qquad (1\text{-}5.5)$$

where D_a is the demagnetization factor for a and D_b along b.

In terms of the eccentricity $\epsilon^2 = 1 - (b/a)^2$

$$D_a = 4\pi\frac{(1-\epsilon^2)}{\epsilon^2}\left(\frac{1}{2\epsilon}\ln\frac{1+\epsilon}{1-\epsilon}\right) - 1. \qquad (1\text{-}5.6)$$

For the oblate spheroid $(m < 1)$

$$D_a = \frac{4\pi}{1-m^2}\left[1 - \frac{m}{(1-m^2)^{1/2}}\cos^{-1} m\right] \qquad (1\text{-}5.7)$$

or

$$D_a = \frac{4\pi}{\epsilon^2}\left[1 - \frac{(1-\epsilon^2)^{1/2}}{\epsilon}\sin^{-1}\epsilon\right]. \qquad (1\text{-}5.8)$$

[5] W. Thomson and P. G. Tait, *Treatise on Natural Philosophy*, 2nd ed., Vol. i, Part ii, Cambridge University Press, Cambridge (1883).

For the sphere ϵ approaches zero and

$$D = \frac{4\pi}{3}. \tag{1-5.9}$$

Appendix II lists some values of D calculated by Stoner,[6] and Osborn[7] has calculated the demagnetization factor for the general ellipsoid.

Practical specimens are seldom ellipsoidal in shape but may be in the form of bars or rods, in which case the second term of (1-3.6) may be comparable with the first. The solution of the problem requires approximations and is very laborious. Bozorth and Chapin[8] have studied the demagnetization factors of rods.

6. Energy of Interaction

The potential energy of a dipole in a pre-existing magnetic field, sometimes called the mutual or interaction energy, is equal to the work done in bringing the dipole from infinity. To calculate this energy, consider the dipole as two poles, $-m$ and $+m$, distance d apart, brought to the position shown in Fig. 1-6.1. If the potential at the poles is φ_{-m} and φ_{+m} respectively, the work done is $-m\varphi_{-m}$ and $+m\varphi_{+m}$. The potential energy is then

$$W = m(\varphi_{+m} - \varphi_{-m}) = md\frac{\partial \varphi}{\partial s},$$

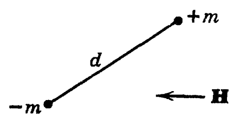

Fig. 1-6.1. Dipole in an applied magnetic field **H**.

where s denotes the direction between $-m$ and $+m$. Then

$$W = \boldsymbol{\mu} \cdot \nabla\varphi = -\boldsymbol{\mu} \cdot \mathbf{H} \tag{1-6.1}$$

when **H** is the pre-existing magnetic field. The interaction energy of a permanent magnet is found by summing (1-6.1) over all the dipoles, giving

$$W = \int \mathbf{M} \cdot \nabla\varphi \, dv = -\int \mathbf{M} \cdot \mathbf{H} \, dv. \tag{1-6.2}$$

In addition, a permanent magnet will have potential energy because of its own field. To calculate this self-energy, consider the work done in bringing up a sequence of dipoles, $a, b, c \cdots n$, whose sum makes up the permanent magnet. No work is done in bringing a into position. Let us denote by $W_a(b)$, the work done in bringing dipole b into position in a field due to dipole a. Further, the work done in positioning dipole c due

[6] E. C. Stoner, *Phil. Mag.* **36**, 816 (1946).
[7] J. A. Osborn, *Phys. Rev.* **67**, 351 (1945).
[8] R. M. Bozorth and D. M. Chapin, *J. Appl. Phys.* **13**, 320 (1942).

to fields of a and b will be $W_a(c) + W_b(c)$. For the n dipoles the total work done W is

$$\begin{aligned} W = \quad & W_a(b) \\ & + W_a(c) + W_b(c) \\ & + \cdots\cdots\cdots\cdots \\ & + W_a(n) + W_b(n) + W_c(n) + \cdots + W_{n-1}(n). \end{aligned}$$

On the other hand, if the dipoles had been brought up in the reverse order, we would get

$$\begin{aligned} W = \quad & W_b(a) + W_c(a) + \cdots\cdots\cdots\cdots + W_n(a) \\ & \qquad\quad + W_c(b) + \cdots\cdots\cdots\cdots + W_n(b) \\ & \qquad\qquad\qquad + \cdots\cdots \\ & \qquad\qquad\qquad\qquad\qquad + W_n(n-1) \end{aligned}$$

Adding, we get

$$\begin{aligned} 2W = \quad & \qquad\quad W_b(a) + W_c(a) + \cdots\cdots\cdots + W_n(a) \\ & + W_a(b) \qquad + W_c(b) + \cdots\cdots\cdots + W_n(b) \\ & + W_a(c) + W_b(c) \qquad + W_d(c) + \cdots\cdots + W_n(c) \\ & + \cdots\cdots\cdots\cdots \\ & + W_a(n) + W_b(n) + \cdots\cdots + W_{n-1}(n). \end{aligned}$$

Therefore

$$W = \tfrac{1}{2} \sum_{n=a}^{n} W(n), \tag{1-6.3}$$

where $W(a)$ is the energy of a dipole placed in a field due to the assembly of dipoles [$W_a(a) = 0$, etc.]. If the dipoles have a moment $\boldsymbol{\mu}$, then by (1-6.1)

$$W(n) = \boldsymbol{\mu} \cdot \nabla\varphi$$

and

$$\begin{aligned} W &= \tfrac{1}{2} \sum \boldsymbol{\mu} \cdot \nabla\varphi \\ &= -\tfrac{1}{2} \sum \boldsymbol{\mu} \cdot \mathbf{H}, \end{aligned} \tag{1-6.4}$$

where **H** is understood not to include the field of the dipole being summed.

We now wish to extend equation 1-6.4 for a permanent magnet. Direct passage from the sum to an integral over a volume element dv, in analogy to the step from equation 1-6.1 to equation 1-6.2, is not valid here.[9] The reason is that the field intensity which acts on a volume element dv, exclusive of its own field, depends on the shape of dv. One way out of this difficulty

[9] E. C. Stoner, *Phil. Mag.* **23,** 833 (1937); W. F. Brown, Jr., *Revs. Mod. Phys.* **25,** 131 (1953).

is to consider a cubic lattice of dipoles and to assume that the dipoles vary linearly (in direction) with position, to within negligible error, over distances containing many lattice spacings. It will also be assumed that there is no strain. The field acting on a dipole can then be evaluated by a method developed by Lorentz.

This method is as follows. A dipole on which the local field is to be calculated is considered as the center of a sphere of radius R that is large compared to the interdipole distance but small compared to macroscopic distances (Fig. 1-6.2). The difference between the field $\mathbf{H}_1$ of matter outside the sphere and the macroscopic field $\mathbf{H}$ is now computed. Both fields can be found from equation 1-3.6 by applying equation 1-2.3.

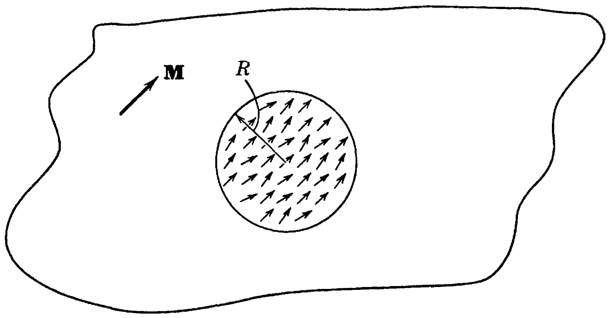

Fig. 1-6.2. The calculation of the Lorentz field.

However, $\mathbf{H}_1$ differs from $\mathbf{H}$ by omitting the volume integral over the interior of the sphere R and by including instead a surface integral over the surface of the sphere R. Now because of the assumption of the linear variation of μ, hence $\mathbf{M}$, with position $\nabla \cdot \mathbf{M} =$ constant, and the volume integral vanishes by symmetry. For the surface integral the expansion of $\mathbf{M}$ about the center contributes zero by symmetry, and $\mathbf{M}$ may be set equal to its value at the center of the sphere. Thus computation of the surface integral is elementary and gives for the difference $\mathbf{H}_1 - \mathbf{H}$ just $(4\pi/3)\mathbf{M}$. We must also compute the field $\mathbf{H}_2$ because of the dipoles inside the sphere. If the dipoles are parallel, it can be shown that $\mathbf{H}_2 = 0$; otherwise $\mathbf{H}_2 \neq 0$ in general. If $\mathbf{H}_2 = 0$, the total field $\mathbf{H}_1 + \mathbf{H}_2$ $(= \mathbf{H}_1)$ is known as the Lorentz field and is given by

$$\mathbf{H}_{\text{Lorentz}} = \mathbf{H} + \frac{4\pi}{3}\mathbf{M}. \tag{1-6.5}$$

If $\mathbf{H}_2 \neq 0$, the small energy term that results will not be considered as magnetic energy but will be lumped into other energy terms, such as anisotropy energy, to be discussed in Chapter 6.

In extending equation 1-6.4 to a permanent magnet, we make the following replacements:

$$\boldsymbol{\mu} \to \mathbf{M}\,dv \qquad \mathbf{H} \to \mathbf{H} + \frac{4\pi}{3}\mathbf{M} \qquad \sum \to \int$$

so that

$$W = -\tfrac{1}{2}\int \mathbf{M}\cdot\left(\mathbf{H} + \frac{4\pi}{3}\mathbf{M}\right) dv$$

$$= -\tfrac{1}{2}\int \mathbf{M}\cdot\mathbf{H}\,dv - \frac{2\pi}{3}\int M^2\,dv.$$

For a permanent magnet with ellipsoidal shape **M** is a constant, and the second term can be neglected by shifting the zero point of energy. Then we have

$$W = -\tfrac{1}{2}\int \mathbf{M}\cdot\mathbf{H}\,dv. \qquad (1\text{-}6.6)$$

The result also holds for a system of permanent magnets, the integration then being over all the magnetic specimens. The $\frac{1}{2}$ factor of equation 1-6.6 is the usual one found in all self-energy expressions.

For a permanent magnet its self-field is given by (1-5.1) as

$$\mathbf{H} = -D\mathbf{M}.$$

Substituting in (1-6.5) gives

$$W = \tfrac{1}{2}DM^2 \qquad (1\text{-}6.7)$$

per unit volume.

Expressions 1-6.5 and 1-6.6 indicate the energy to be localized at the magnetic particle. We now develop an equation which shows that the energy can be considered to be in the field, distributed throughout space. This result is in accordance with the ideas of Faraday and Maxwell.

First, we establish two relationships that will be useful in the derivation. By the divergence theorem

$$\int \nabla\cdot(\mathbf{H}\varphi)\,dv = \int \mathbf{n}\cdot\mathbf{H}\varphi\,dS.$$

Now consider the integration as being taken over the surface of an infinite sphere. Since $\varphi \propto 1/r^2$, $dS \propto r^2$, and $\mathbf{n}\cdot\mathbf{H} = 0$, we get

$$\int \nabla\cdot(\mathbf{H}\varphi)\,dv = 0. \qquad (1\text{-}6.8)$$

By similar reasoning

$$\int \nabla\cdot(\mathbf{M}\varphi)\,dv = 0. \qquad (1\text{-}6.9)$$

Now, returning to (1-6.6)

$$W = -\tfrac{1}{2}\int \mathbf{M}\cdot\mathbf{H}\,dv$$

$$= +\tfrac{1}{2}\int \mathbf{M}\cdot\nabla\varphi\,dv.$$

The integration can now be considered to be taken over all space, since **M** equals zero everywhere except in the permanent magnet. By virtue of equation 1-6.9

$$W = -\tfrac{1}{2}\int [\nabla\cdot(\mathbf{M}\varphi) - \mathbf{M}\cdot\nabla\varphi]\,dv$$

$$= -\tfrac{1}{2}\int (\nabla\cdot\mathbf{M})\varphi\,dv. \qquad (1\text{-}6.10)$$

Now, since

$$\mathbf{M} = \frac{1}{4\pi}(\mathbf{B} - \mathbf{H})$$

$$W = -\frac{1}{8\pi}\int [\nabla\cdot(\mathbf{B} - \mathbf{H})]\varphi\,dv.$$

It is one of Maxwell's fundamental relationships that $\nabla\cdot\mathbf{B} = 0$. By using equation 1-6.8,

$$W = -\frac{1}{8\pi}\int [\nabla\cdot(\mathbf{H}\varphi) - (\nabla\cdot\mathbf{H})\varphi]\,dv$$

$$= -\frac{1}{8\pi}\int \mathbf{H}\cdot\nabla\varphi\,dv$$

$$= \frac{1}{8\pi}\int H^2\,dv, \qquad (1\text{-}6.11)$$

and this is the equation we set out to develop.

7. Magnetic Effects of Currents. The Magnetic Shell. Faraday's Law

In 1820 Oersted discovered that an electric current produced a magnetic field. It was found by Ampère's experiments that the work per unit pole or the value of the line integral $\oint \mathbf{H}\cdot d\mathbf{s}$ (see equation 1-2.7) was no longer zero if the path taken enclosed the wire. Instead, it was directly proportional to the current i. Since the units of **H** have already been defined, this result permits a definition of the unit of current. For i to be in electromagnetic units, the constant of proportionality is taken as 4π so that

$$\oint \mathbf{H} \cdot d\mathbf{s} = 4\pi i. \tag{1-7.1}$$

The work done is no longer independent of the path. The use of the concept of the magnetostatic scalar potential to determine the magnetic field is thus limited to permanent magnet problems and situations in which the expression $\oint \mathbf{H} \cdot d\mathbf{s}$ is evaluated along paths that do not enclose current-carrying wires.

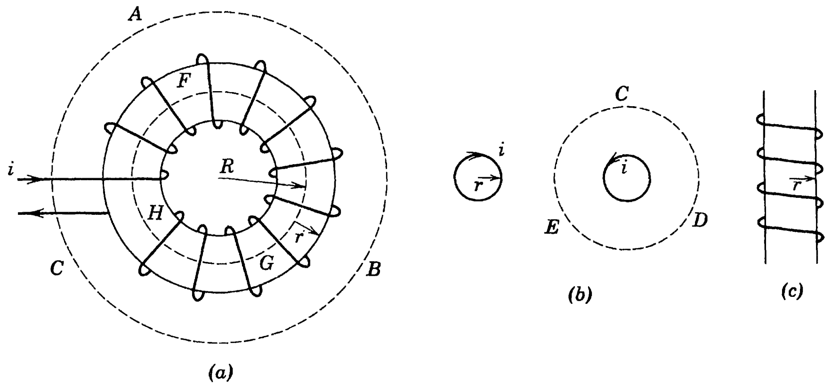

Fig. 1-7.1. The toroid is shown from above in (*a*) and in a vertical cross section in (*b*). The result when $R \to \infty$ is the solenoid shown in (*c*).

Application of (1-7.1) leads quickly to the calculation of **H** for a steady current if there is some symmetry to the problem. For example, if the path of integration for a straight wire of circular section is a concentric circle about the axis, and since by symmetry H is a constant, then $2\pi r H = 4\pi i$ and therefore $H = 2i/r$.

Consider now a toroid wound uniformly with n turns of wire per unit length. Figure 1-7.1*a* shows a view of the toroid from above a plane passing through its axis, whereas Fig. 1-7.1*b* shows a cross section for a plane perpendicular to the axis. Suppose the toroid is described by the radii R and r. We wish to determine the field when the toroid carries a current i. First we find the field outside the coil. On a circumferential path such as ABC of Fig. 1-7.1*a* no current is enclosed and $H = 0$. On a path such as CDE (Fig. 1-7.1*b*) one turn is crossed and $\oint \mathbf{H} \cdot d\mathbf{s} = 4\pi i$. Usually n is fairly large and i very small, so that $4\pi i$ can be neglected.

Hence the field outside the toroid is essentially zero. For a path inside the toroid and lying in a plane perpendicular to the axis no current is enclosed and therefore $H = 0$. For a path along the axis of the toroid (FGH) the field is

$$H = 4\pi ni. \tag{1-7.2}$$

Thus the only field is one inside the toroid and parallel to its axis. If now we let R approach infinity, in the limit the toroid becomes an infinitely long cylinder called a solenoid. The field inside the solenoid is parallel to the axis and uniform, equal to $4\pi ni$. If i is in amperes, the units are converted by dividing this result by 10; if it is in electrostatic units, the results are divided by $c = 3 \times 10^{10}$.

When the solenoid is of finite length and its length is appreciably larger than its radius, equation 1-7.2 is still accurate to a good degree at points not too close to the ends.

Often a solenoid is employed experimentally as a source of a uniform magnetic field. Practical difficulties limit the magnitude of the field that can be obtained when reasonable uniformity is required over a volume of several cubic centimeters. With only air cooling a rather husky coil is required to produce a continuous field in excess of 2000 oe. By passing cooling water through copper tubing, steady fields in the vicinity of 10,000 oe can be achieved.

Special winding arrangements, coupled with 2-Mw, high-current generators and large capacity cooling systems have permitted the attainment of static fields of over 100,000 oe[10]; the production of fields of 250,000 oe are planned for the future.[11] Besides water, organic fluids such as kerosene and orthodichlorobenzene, have been employed as coolants. Liquid N_2 and H_2 have also been used for cooling.[12] Low temperatures have the great advantage that the resistance of the electrical conductors is decreased by a factor of about 5. Hence the energy dissipated in the coil and the power source are also reduced by a factor of five.

[10] J. D. Cockroft, *Phil. Trans. Roy. Soc. (London)* **227**, 325 (1928); F. Bitter, *Rev. Sci. Instr.* **7**, 482 (1936), **10**, 373 (1939); J. M. Daniels, *Proc. Phys. Soc. (London)* **B-63**, 1028 (1950); F. Gaume, *J. Rech. Centre Natl. Rech. Sci.* **43**, 93 (1958); D. de Klerk, *Ned. Tijdschr. Natuurk.* **26**, 1 (1960), **26**, 345 (1960); S. Maeda, *High Magnetic Fields*, John Wiley and Sons, New York (1962), p. 406; R. S. Ingarden, *ibid.*, p. 427.

[11] B. Lax, *J. Appl. Phys.* **33**, 1025 (1962).

[12] T. W. Adair, C. F. Squire, and H. B. Utley, *Rev. Sci. Instr.* **31**, 416 (1960); C. E. Taylor and R. F. Post, *Advan. Cyrog. Eng.*, Plenum Press, New York (1960), Vol. 5; E. S. Borovik, F. I. Busel, and S. F. Grishin, *Zhur. Tekh. Fiz.* **31**, 459 (1961) [trans. *Sov. Phys.-Tech. Phys.* **6**, 331 (1961)]; J. R. Purcell, *High Magnetic Fields*, John Wiley and Sons, New York (1962) p. 166; H. L. Laquer, *ibid.*, p. 156.

Larger fields can be created by pulse techniques.[13] These methods consist of discharging a bank of condensers through a coil by means of some kind of switch. Fields of 150,000 oe are comparatively easy to make; with care, useful fields of close to 500,000 oe can be generated. Actually, fields in the neighborhood of 1 million oe have been produced, but the stresses on the coil are so great that it is usually destroyed. Pulsed fields have the disadvantage that special systems are required to detect the transient response.

Even larger fields have been produced by starting with a pulse field and then explosively reducing the volume of the flux container. The walls of the container act as perfect conductors; hence the total flux is conserved.[14] As a result the magnetic field is momentarily greatly increased. Fields in excess of 10 million oe lasting about 10^{-6} sec have been attained with this implosion technique.[15]

Solenoids constructed with hard superconductors are, at present, important sources of large continuous magnetic fields. They are likely, in the future, to be used even more widely. Hard superconductors are materials with zero electrical resistance that have the ability to remain superconducting in the presence of large magnetic fields. Hard superconductors are discussed at some length in Section 5-10. Finally, electromagnets, which are solenoids with soft iron cores, are also important sources of continuous magnetic fields; they will be considered in Section 1-9.

The magnetic shell. The magnetic effects of a current can be computed by using the concept of the equivalent magnetic shell. Such a shell is defined as a thin sheet of magnetic material, uniformly magnetized normal to its surface, the two surfaces having equal and opposite surface pole densities, $+\sigma$ and $-\sigma$. The boundary of the shell is taken to coincide with the current-carrying wire. If the surfaces of the shell are a small distance l apart, the moment or strength of the shell $\boldsymbol{\tau}$ is defined as

$$\boldsymbol{\tau} = \lim_{\substack{l \to 0 \\ \sigma \to \infty}} \sigma l \mathbf{n}$$

$$\sigma l = \text{constant},$$

[13] P. Kapitza, *Proc. Roy. Soc. (London)* **A-115**, 658 (1927); H. P. Furth and R. W. Waniek, *Rev. Sci. Instr.* **27**, 195 (1956); S. Foner and H. H. Kolm, *Rev. Sci. Instr.* **28**, 799 (1957); I. S. Jacobs and P. E. Lawrence, *Rev. Sci. Instr.* **29**, 713 (1958); D. H. Birdsall, *Rev. Sci. Instr.* **30**, 600 (1959); Y. B. Kim, *Rev. Sci. Instr.* **30**, 524 (1959); J. C. A. van der Sluijs, *High Magnetic Fields*, John Wiley and Sons, New York (1962), p. 290; L. W. Roeland and F. A. Muller, *ibid.*, p. 287; H. Zijlstra, *ibid.*, p. 281; M. A. Levine, *ibid.*, p. 277.

[14] Ia. P. Terletskii, *J. Exptl. Theoret. Phys. (USSR)* **32**, 387 (1957) [*trans. Soviet Phys.-JETP*, 301 (1957)].

[15] C. M. Fowler, W. B. Garn, and R. S. Caird, *J. Appl. Phys.* **31**, 588 (1960).

where **n** is a unit normal vector drawn outward from the surface with positive pole density. Thus the moment or strength of a shell is the magnetic moment per unit area. A uniform shell is one for which τ is everywhere constant; hence a uniform shell need not be plane. Obviously the total dipole moment of the shell is given by the product of $\boldsymbol{\tau}$ and the area of the shell A.

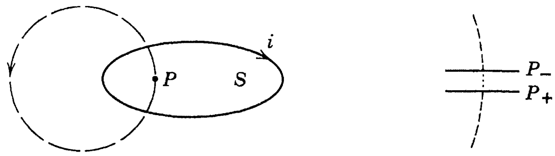

Fig. 1-7.2. S is the shell with wire-carrying current i as its boundary. For simplicity the shell is considered to be plane. The dotted curve is the path along which a unit pole is taken. (b) shows the points P_+, P_- at which the path crosses the shell of definite although infinitesimal thickness.

The potential at a point P due to a magnetic shell is easily calculated. By employing equation 1-3.5, we get

$$\varphi_P = \iint \boldsymbol{\tau}\, dS \cdot \nabla_s\left(\frac{1}{r}\right),$$

where dS is an element of area of the shell so that its moment (strength) is $\tau\, dS$ and $\mathbf{r}$ is the vector from P to dS. Now, since $\nabla_s(1/r) = -\mathbf{r}/r^3$

$$\varphi_P = \iint \tau \frac{\cos\theta}{r^2}\, dS,$$

where θ is the angle between $\boldsymbol{\tau}$ and $\mathbf{r}$. But $d\Omega = \pm(\cos\theta/r^2)\, dS$ is the solid angle of dS as seen at P. Therefore

$$\varphi_P = \iint \tau\, d\Omega.$$

Here $d\Omega$ is taken as positive if P faces the positive side of the shell. If τ is constant (uniform shell), then

$$\varphi_P = \tau\Omega. \tag{1-7.3}$$

Next we compute the work done on taking a unit pole around a path that crosses the magnetic shell once (Fig. 1-7.2).

By considering the potential, first of a plane surface distribution of poles, and then of a second plane with a surface distribution of poles of opposite sign, it may be shown[16] that the magnetic field between the layer

[16] M. Abraham and R. Becker, *Electricity and Magnetism*, Blackie and Son, London (1942), pp. 26–29.

is $4\pi\sigma$, which becomes infinite when $\sigma \to \infty$, plus a term that remains finite in the limit. The line integral of the field through the layer is $-4\pi\sigma l = -4\pi\tau$, a constant, plus a term that vanishes in the limit. The work done on the rest of the path is the difference in potential across the shell $\varphi_+ - \varphi_-$. Since the solid angle change is $\Omega = 4\pi$, we get $\varphi_+ - \varphi_- = 4\pi\tau$; hence the total work is zero. The field of a current differs from the field inside of the double layer by the singular term $4\pi\tau$; that is, the work done on traversing a circuit is $4\pi i$ (equation 1-7.1). Hence the field arising from a current-carrying circuit can be derived from a potential if complete traversals of the path are prevented by introducing a barrier surface. It then follows that a magnetic shell such as that defined above can be used to compute the field produced by a current if the magnetic moment τ is taken equal to the current i.

It is worth mentioning that other approaches are possible. A double layer can be defined as a surface distribution of dipoles, in which a dipole is defined as the limiting case of a pair of poles. Then, if a dipole is defined as the limiting case of a small circuit, we get a "double layer" with precisely the properties of a finite circuit. The little circuits side by side cancel each other everywhere except at the periphery.

Faraday's law. An effect, equally important as that investigated by Oersted and Ampère, was discovered by Faraday. He observed that a time-varying magnetic field caused a current to flow in a closed electrical circuit. Experimentally, it was found that

$$\oint_{\text{circuit}} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}\int_A \mathbf{B} \cdot \mathbf{n}\, dS, \tag{1-7.4}$$

where A is a surface which has the circuit as its boundary. Here $\mathbf{E}$ is the electrostatic field, or force per unit charge, so that the left-hand side of equation 1-7.4 is just the work done on taking a unit electric charge about the path of the circuit (compare with equation 1-2.6). It is usual to refer to this term as the induced emf (electromotive force) $\mathcal{E}$. By employing equation 1-4.2 we obtain

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{1-7.5}$$

The results of this section can be used to compute the energy required to change the magnetization of a material. Suppose that the material, of cross-sectional area A and length L, is placed in the solenoid, also of length L. If the solenoid is long and narrow, the magnetic field produced by a current i is given by equation 1-7.2,

$$H = 4\pi n i.$$

The induction in the solid is just (equation 1-4.1)

$$B = H + 4\pi M;$$

hence the total flux threading the solenoid is

$$\Phi = BnLA.$$

Suppose that this flux is changing with time; then an emf will be induced in the solenoid, given by equation 1-7.5 as

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

During this time work must be done to keep the current flowing; this energy is supplied by some power source such as a battery. If the amount of charge moved is dQ, the work that must be done is

$$\begin{aligned} dW &= \mathcal{E}\, dQ \\ &= -\frac{d\Phi}{dt}\, dQ \\ &= -i\, d\Phi \\ &= -\frac{H}{4\pi n}\, d[(H + 4\pi M)nLA] \\ &= -\frac{LA}{4\pi}(H\, dH + 4\pi H\, dM). \end{aligned}$$

Per unit volume of the material, the work done is

$$dW = -\frac{H\, dH}{4\pi} - H\, dM.$$

The first term is the work done to change the field H to $H + dH$; it is of no interest to us here. Let us take this term to the left-hand side of the equation and incorporate it into dW; for convenience this energy will still be called dW. This new dW is just the work done on increasing the magnetization of the material by an amount dM. Hence for a magnetic material we have[17]

$$dW = -H\, dM. \tag{1-7.6}$$

[17] This result can be obtained from a field, rather than a lumped parameter point of view. See, for example, E. A. Guggenheim, *Proc. Roy. Soc. (London)* **A-155,** 49, 70 (1936).

If the solenoid's current, hence its field, is increased or decreased slowly, the total amount of work done on the solid per unit volume is given by

$$W = -\int H\,dM, \tag{1-7.7}$$

where the limits of the integral are the initial and final values of M. If the material is isotropic so that equation 1-4.4 holds and the magnetic field is increased from 0 to H, the amount of work done per unit volume is

$$\begin{aligned} W &= -\int_0^H \chi H\,dH \\ &= -\tfrac{1}{2}\chi H^2. \end{aligned} \tag{1-7.8}$$

It can be shown that the form of equations 1-7.6 and 1-7.7 also holds if the body placed in the solenoid is ellipsoidal and so small that poles must be considered. By employing the equation for the self-field, $\mathbf{H} = -D\mathbf{M}$ (equation 1-5.1) we rederive equations 1-6.5 and 1-6.6.

8. Maxwell's and Lorentz's Equations

As a consequence of the experimental observations of Ampere and Faraday, Maxwell was led to the following equations:

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \tag{1-8.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1-8.2}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{1-8.3}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\left(4\pi\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}\right). \tag{1-8.4}$$

These equations are in gaussian units. Here $\mathbf{E}$ is the electric field, or force per unit charge, $\mathbf{D}$ is the displacement vector, $\mathbf{j}$ is the current density, and ρ is the electric charge volume density. $\mathbf{D}$ is related to $\mathbf{E}$ by the equation

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \tag{1-8.5}$$

where $\mathbf{P}$ is the polarization of the electric dipole moment per unit volume. This relation is analogous to equation 1-4.1. $\mathbf{D}$ and $\mathbf{E}$ can also be related by

$$\mathbf{D} = \epsilon\mathbf{E}, \tag{1-8.6}$$

where ϵ is the dielectric constant. Often Maxwell's equations are considered as postulates instead of being derived from the foregoing experiments. The ability of the equations to predict vast numbers of experimental results correctly is then considered proof of their validity.

Because of equation 1-8.2 and since $\nabla \cdot \nabla \times \mathbf{A} \equiv 0$, a vector potential $\mathbf{A}$ can be defined as

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1-8.7}$$

If the permeability μ is equal to 1, a dipole moment $\boldsymbol{\mu}$ will have a vector potential given by

$$\mathbf{A} = -\boldsymbol{\mu} \times \nabla\left(\frac{1}{r}\right) = \frac{1}{r^3}\boldsymbol{\mu} \times \mathbf{r}. \tag{1-8.8}$$

This follows since

$$\begin{aligned}\mathbf{H} = \mathbf{B} &= \nabla \times \mathbf{A} \\ &= \nabla \times \left[\boldsymbol{\mu} \times \nabla\left(\frac{1}{r}\right)\right] \\ &= (\boldsymbol{\mu} \cdot \nabla)\, \nabla\left(\frac{1}{r}\right) - \boldsymbol{\mu}\, \nabla^2\left(\frac{1}{r}\right).\end{aligned}$$

The term $\boldsymbol{\mu}\, \nabla^2(1/r) = 0$ except at $r = 0$. Therefore

$$\begin{aligned}\mathbf{H} &= \nabla\left(\boldsymbol{\mu} \cdot \nabla \frac{1}{r}\right) \\ &= -\nabla\varphi\end{aligned}$$

as in equation 1-2.3. The vector potential is used when currents that cannot be replaced by magnetic shells are present. It is also useful to introduce an electric scalar potential φ_E, defined by

$$\mathbf{E} = -\nabla\varphi_E - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{1-8.9}$$

Frequently the equation that expresses the force on a moving charge is required. Lorentz showed that this force is given by

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{H}, \tag{1-8.10}$$

where $\mathbf{v}$ is the velocity of the charge.

It is of interest to consider the transformation of Maxwell's equations to coordinate systems moving linearly with respect to the original frame of reference. According to the special theory of relativity, the length element

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2\, dt^2$$

is an invariant quantity; that is $ds'^2 = dx'^2 + dy'^2 + dz'^2 - c^2\, dt'^2 = ds^2$. The linear transformations which keep ds^2 invariant are known as Lorentz transformations. If we let $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = ict$, the

Lorentz transformation may be written

$$x_i' = \sum_{j=1}^{4} a_{ij} x_j, \qquad i = 1, 2, 3, 4. \tag{1-8.11}$$

For a frame of reference Σ' moving with a velocity $\mathbf{v}$ parallel to the y-axis of the Σ frame the transformation coefficients are given by

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_r & 0 & i\gamma_r\beta \\ 0 & 0 & 1 & 0 \\ 0 & -i\gamma_r\beta & 0 & \gamma_r \end{pmatrix}, \tag{1-8.12}$$

where $\beta = v/c$ and $\gamma_r = 1/\sqrt{1 - (v/c)^2}$.

It is desirable to rewrite Maxwell's equations so that their terms, although not invariant, transform with the same properties; when this is done, the equations are said to be covariant in form. Covariance may be achieved by writing the current $\mathbf{j}$ and charge density ρ as the four-vector

$$j_i = (\mathbf{j}, ic\rho), \tag{1-8.13}$$

the vector potential $\mathbf{A}$ and the electric scalar potential φ_E as the four-vector

$$A_i = (\mathbf{A}, i\varphi_E), \tag{1-8.14}$$

and the field quantities $\mathbf{E}$ and $\mathbf{H}$ by the antisymmetric tensor of rank two,

$$\mathscr{F}_{ij} = \begin{pmatrix} 0 & H_3 & -H_2 & -iE_1 \\ -H_3 & 0 & H_1 & -iE_2 \\ H_2 & -H_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}. \tag{1-8.15}$$

Equations 1-8.1 and 1-8.4 then take the form

$$\sum_{j=1}^{4} \frac{\partial \mathscr{F}_{ij}}{\partial x_j} = \frac{4\pi}{c} j_i, \qquad i = 1, 2, 3, 4. \tag{1-8.16}$$

To be covariant, this equation must take the form

$$\sum_{l=1}^{4} \frac{\partial \mathscr{F}_{kl}'}{\partial x_l'} = \frac{4\pi}{c} j_k' \tag{1-8.17}$$

in the Σ' reference frame. That this is indeed true is easily seen by applying the transformation of equation 1-8.11; equation 1-8.17 then becomes

$$\sum_{i=1}^{4} a_{ki} \left(\sum_{j=1}^{4} \frac{\partial \mathscr{F}_{ij}}{\partial x_j} - \frac{4\pi}{c} j_i \right) = 0.$$

Similarily, equations 1-8.2 and 1-8.3 reduce to

$$\frac{\partial \mathscr{F}_{ij}}{\partial x_k} + \frac{\partial \mathscr{F}_{ki}}{\partial x_j} + \frac{\partial \mathscr{F}_{jk}}{\partial x_i} = 0. \tag{1-8.18}$$

Finally, the Lorentz force (equation 1-8.10) has the covariant form

$$\mathbf{F}_i = \frac{1}{c}\mathscr{F}_{ij}j_j. \tag{1-8.19}$$

It is instructive to see how the field quantities themselves transform. For a coordinate system Σ' moving with a velocity $\mathbf{v}$ relative to the reference frame Σ, it follows from the general transformation relation $\mathscr{F}_{ij}' = a_{ik}a_{jl}\mathscr{F}_{kl}$ and equation 1-8.12 that

$$E_{\parallel}' = E_{\parallel}, \qquad H_{\parallel}' = H_{\parallel}$$

and

$$\mathbf{E}_{\perp}' = \gamma_r\left(\mathbf{E}_{\perp} + \frac{\mathbf{v}}{c} \times \mathbf{H}\right), \qquad \mathbf{H}_{\perp}' = \gamma_r\left(\mathbf{H}_{\perp} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right), \tag{1-8.20}$$

where the subscripts $\parallel$ and $\perp$ refer to directions parallel and perpendicular, respectively, to $\mathbf{v}$. It is clear that $\mathbf{E}$ and $\mathbf{H}$ are not unique with respect to linear transformations. The physical origin of these results is closely related to the laws of Ampère and Faraday. Consider a charge or a permanent magnet at rest in the system Σ. In Σ' the charge or permanent magnet will be moving with respect to the axes. There will then be additional electric ($\mathbf{v}/c \times \mathbf{H}$) and magnetic ($\mathbf{v}/c \times \mathbf{E}$) fields present. Further details on the relativistic formulation of electrodynamics may be found in a number of standard textbooks.[18]

9. The Magnetic Circuit

The magnetic circuit, a concept analogous to the electric circuit, is of considerable use in solving many practical problems such as the design of electromagnets. We now develop this concept.

The work done on taking a unit pole around a closed path that goes through a solenoid of N turns is, by equation 1-7.1,

$$\oint \mathbf{H} \cdot d\mathbf{s} = \frac{4\pi N i}{10},$$

[18] P. G. Bergmann, *Introduction to the Theory of Relativity*, Prentice-Hall, Englewood Cliffs, N.J. (1942) Ch. 1–9; R. Becker, *Theorie der Elektrizität*, B. G. Teubner, Leipzig (1949), Sections 24–66; W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, Addison-Wesley, Cambridge, Mass. (1955), Ch. 14 and 15; J. D. Jackson, *Classical Electrodynamics*, John Wiley and Sons, New York (1962), Ch. 11.

where i is in amperes. The integral $\oint \mathbf{H} \cdot d\mathbf{s}$ is often called the magnetomotive force (mmf) since it is similar to the definition of the electromotive force (emf, equation 1-7.5). The mmf can be considered responsible for the production of the flux in a magnetic circuit, just as the emf causes current to flow in an electric circuit. In analogy to electrical resistance a magnetic reluctance is defined as

$$\mathcal{R} = \frac{\oint \mathbf{H} \cdot d\mathbf{s}}{\int \mathbf{B} \cdot \mathbf{n}\, dS} = \frac{\oint \mathbf{H} \cdot d\mathbf{s}}{\Phi}, \tag{1-9.1}$$

where Φ is the flux from equation 1-4.2. For the usual situation in which $\mathbf{B}$ is uniform,

$$\mathcal{R} = \frac{l}{\mu A}, \tag{1-9.2}$$

where l is the length of the circuit, A the cross-sectional area, and μ the permeability. When the magnetic circuit is made up of parts of length l_i, areas A_i, and permeabilities μ_i, equation 1-9.2 becomes

$$\mathcal{R} = \sum_i \frac{l_i}{\mu_i A_i}. \tag{1-9.3}$$

Usually a certain flux density in an air gap in the magnetic circuit is desired. Therefore for some particular system it is necessary to find the required value of the product Ni, or the ampere-turns. Since for air $\mu = 1$ and for a common ferromagnetic material such as iron $\mu \doteqdot 1000$, the reluctance in the short air gap is much greater than that in the iron part of the circuit. As stated in Section 1-4, μ is not a constant for iron but depends on the value of H or Ni. As a result, the solution of the magnetic circuit problem often involves a numerical successive approximation technique. This occurs most often in problems in which Ni is given and B is to be calculated.[19]

The magnetic flux in an air gap tends to spread out and have a larger value of cross-sectional area A_g than in the iron (A). A useful rule of thumb, based on experiment, is to increase each linear dimension of A by twice the length of the air gap in order to find A_g. If the air gap is very large, as, for example, in a "horseshoe" magnet, the reluctance can be

[19] R. A. Galbraith and D. W. Spence, *Fundamentals of Electrical Engineering*, Ronald Press, New York (1955), Ch. 9.

found by plotting the flux lines in the gap and drawing the corresponding magnetic equipotential lines.[20]

Examples of the design of some electromagnets are the following:

Uppsala University magnet. L. Dreyfus, *Arch. Electrotech.* **25**, 392 (1931); *ASEA J.* **12**, 8 (1935).

Leiden. G. Häden, *Siemens-Z.* **10**, 481 (1930).

Academie des Sciènces. S. Rosenblum, *Compt. rend. (Paris)* **188**, 1401 (1929).

Cambridge University. J. D. Cockcroft, *J. Sci. Instr.* **10**, 71 (1933).

Very large magnets for synchrocyclotrons, etc. For example, H. L. Anderson et al. *Rev. Sci. Instr.* **23**, 707 (1952).

10. Dipole in a Uniform Field

If a dipole of moment $\boldsymbol{\mu}$ is placed in a uniform field, it is subjected to a torque **L**, given by

$$L = -\frac{\partial W}{\partial \theta} = -\mu H \sin\theta, \tag{1-10.1}$$

where θ is the angle between $\boldsymbol{\mu}$ and **H** and W is given by equation 1-6.1. In vectors, remembering that the cross product anticommutes, this can be rewritten as

$$\mathbf{L} = \boldsymbol{\mu} \times \mathbf{H}. \tag{1-10.2}$$

The action of the uniform field **H** is thus to tend to rotate the dipole until it is parallel to **H**. This action, illustrated in Fig. 1-10.1, develops because we have assumed that the dipole acts like a conventional mechanical system, that is, that the dipole is a rigid body, capable only of rotation about an axis. In order to discuss the dynamical behavior of the dipole, we must consider how a dipole moment originates. For simplicity we

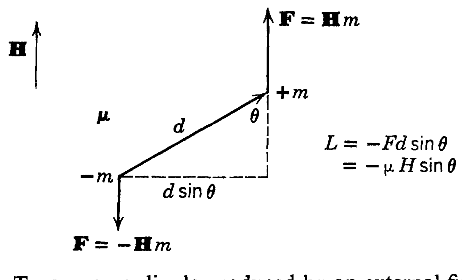

Fig. 1-10.1. Torque on a dipole produced by an external field H.

[20] For example, see W. B. Boast, *Principles of Electric and Magnetic Fields*, Harper and Bros, New York (1948), Ch. 14.

will assume that the dipole moment $\boldsymbol{\mu}$ arises from the motion of an electron in a circular orbit. Let r be the radius of the orbit, e its charge, and T the period of rotation. The moving electron can be considered essentially as a current flowing in a wire that coincides with the orbit. The magnetic effects can then be deduced by considering the equivalent magnetic shell. From Section 1-7 we have $\boldsymbol{\mu} = \boldsymbol{\tau} A$, in which $\boldsymbol{\mu}$ is the total dipole moment of the shell, $\boldsymbol{\tau}$ the strength of the shell, and A the shell area. Since $\tau = i$, and the current in this case is e/T, with e in gaussian units, we get

$$\boldsymbol{\mu} = \frac{eA}{Tc}\mathbf{n} \quad (A = \pi r^2). \qquad (1\text{-}10.3)$$

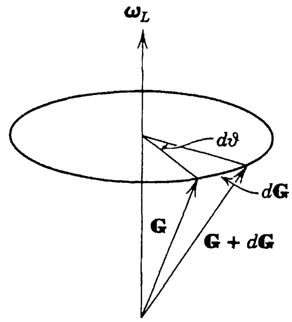

Fig. 1-10.2. Physical interpretation of equation 1-10.7.

Now, if m is the mass of the electron, the angular momentum is given by $\mathbf{G} = I\boldsymbol{\omega} = mr^2\boldsymbol{\omega} = mr^2(2\pi/T)\mathbf{n}$, in which I is the moment of inertia and ω ($= d\vartheta/dt$) is the angular frequency. Substituting this into (1-10.3) gives

$$\boldsymbol{\mu} = \frac{e}{2mc}\mathbf{G}. \qquad (1\text{-}10.4)$$

Because the electron is negatively charged, the value to be inserted into this equation is $e = -4.80 \times 10^{-10}$; that is, the vectors $\boldsymbol{\mu}$ and $\mathbf{G}$ are antiparallel. Equation 1-10.4, together with Newton's law for angular momentum, $d\mathbf{G}/dt = \mathbf{L}$, and equation 1-10.2 give finally

$$\frac{d\mathbf{G}}{dt} = -\frac{e}{2mc}\mathbf{H} \times \mathbf{G}. \qquad (1\text{-}10.5)$$

Let us set

$$\boldsymbol{\omega}_L = \frac{-e}{2mc}\mathbf{H}; \qquad (1\text{-}10.6)$$

then (1-10.5) can be written as

$$d\mathbf{G} = \boldsymbol{\omega}_L \times \mathbf{G}\, dt. \qquad (1\text{-}10.7)$$

Consideration of Fig. 1-10.2 shows that this equation represents the equation of motion of a vector $\mathbf{G}$ precessing with an angular velocity $\omega_L = d\vartheta/dt$.

The action, therefore, of a uniform field $\mathbf{H}$ on such a dipole is to cause the moment $\boldsymbol{\mu}$ to precess about $\mathbf{H}$ with an angular frequency $eH/2mc$. This is in addition to the motion possessed by the electron before the application of the field. The precession is called the Larmor precession frequency, after the physicist who first derived it. These results also hold for noncircular orbits, provided only that the forces acting on the electron are central. In this derivation we have assumed that the angular momentum is left unchanged by the application of **H**. This is true to a first-order term in **H** (see problem 1-8). The precession phenomenon is the origin of diamagnetism and is discussed further in Chapter 2.

Problems

1-1. Show that for a uniformly magnetized sphere of radius a, the magnetic potential at a point distance $r > a$,

$$\varphi = \frac{4\pi}{3} Ma^3 \frac{\cos\theta}{r^2},$$

where θ is the angle between **M** and **r**. Show that for $r < a$

$$\varphi = \frac{4\pi}{3} Mr\cos\theta.$$

1-2. From the result of problem 1-1, show that for a sphere $D = 4\pi/3$.

1-3. Show that the self-interaction energy of a permanent magnet is

$$\tfrac{1}{2}\int \rho\varphi\, dv + \tfrac{1}{2}\int \sigma\varphi\, dS,$$

where the symbols have the meaning used in the text.

1-4. Use the result of problem 1-3 to calculate the energy of an ellipsoidal specimen, volume v_1, which contains an ellipsoidal cavity, volume v_2. [*Hint.* Assume first an ellipsoid, volume v_1, with uniform magnetization M, then another volume v_2, with uniform magnetization $-M$, and consider the interaction between these ellipsoids. See W. F. Brown Jr., and A. H. Morrish, *Phys. Rev.* **105,** 1198 (1957).]

1-5. Show that a permanent magnet, permeability μ, permanent magnetic moment M_0, has the field energy

$$W = \frac{1}{8\pi}\int \mu H^2\, dv.$$

[*Hint.* $W = \frac{1}{2}DM_0^2$ and M_0 is defined by the equation $M = M_0 + (\mu - 1)/4\pi H$.]

1-6. An electromagnet whose magnetization M is uniform has truncated pole pieces of base radius b and gap face radius a. The center of the air gap coincides with the cone from which the truncated pieces are fashioned. Show that the field at the center of the air gap is given by

$$4\pi M\left(1 - \cos\theta + \sin^2\theta\cos\theta\log_e\frac{b}{a}\right),$$

where θ is the angle of the cone. If the poles on the flat face of the pole pieces are neglected, show that the maximum field is achieved for $\theta = 54° 44'$.

1-7. Show that the energy in a magnetic field caused (equation 1-6.11) by circuits carrying currents is given by $\frac{1}{2}\Sigma i\Phi$ where Φ is the flux that threads the current i.

1-8. Consider an electron moving in a circular orbit about a nucleus of charge e. Use the Lorentz equation to set up a force equation when a magnetic field **H** is applied and show $\omega = \pm [(eH/2mc)^2 + (e^2/mr^3)]^{1/2} - (eH/2mc)$. Consider the magnitudes of the terms and make a suitable approximation to obtain the Larmor frequency.

Bibliography

Abraham, M., and R. Becker, *Classical Theory of Electricity and Magnetism*, Blackie and Sons, London (1937).

Brown, W. F., *Magnetostatic Principles in Ferromagnetism*, North-Holland Publishing Co., Amsterdam (1962).

Jackson, J. D., *Classical Electrodynamics*, John Wiley and Sons, New York (1962).

Jeans, J. H., *Electricity and Magnetism*, Cambridge University Press, Cambridge (1925).

Kolm, H., B. Lax, F. Bitter, and R. Mills, editors, *High Magnetic Fields*, M.I.T. Press and John Wiley and Sons, New York (1962).

Panofsky, W. K., and M. Philips, *Classical Electricity and Magnetism*, Addison-Wesley Publishing Co., Cambridge, Mass. (1955).

Stoner, E. C., *Magnetism and Matter*, Methuen and Company, London (1934).

Stratton, J. A., *Electromagnetic Theory*, McGraw-Hill Book Co., New York (1941).