Section 1

ALGEBRAIC THINKING AND APPLICATIONS

Objective 1: Simplify Algebraic Expressions Involving One or Two Variables

Students have great difficulty recognizing the differences among linear, quadratic, and constant terms in algebraic form. Exponents seem insignificant to them. Viewing each type of term as an area helps students visualize the role each term plays in an expression. The following activities provide experience with such visualization in the combining of like terms. It is assumed that students have already mastered the four operations with integers.

Activity 1

Manipulative Stage

Materials

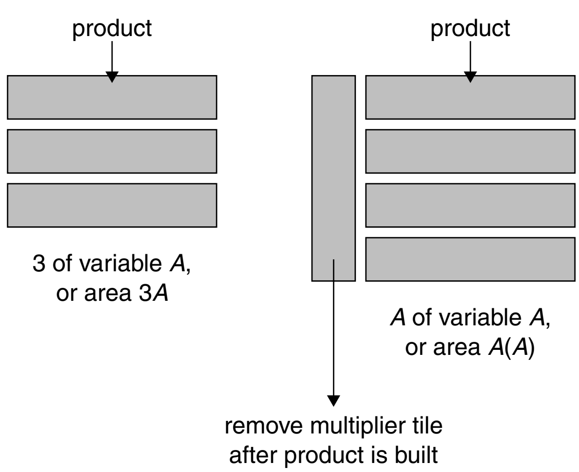
Packets of variable and unit tiles (described in step 1 below)
Worksheet 1–1a
Legal-sized plain paper or light tagboard (for building mats)
Regular paper and pencils

Procedure

1. Give each pair of students a packet of tiles, two copies of Worksheet 1–1a, and a sheet of plain paper or tagboard (approximately 8.5 inches by 14 inches) for a building mat. If preferred, laminate the mats to make them more durable. Mats define a specific space on which to represent a problem being solved. If teacher-made tiles are used, each packet should contain the following in different colors of laminated tagboard: 8 square (quadratic) variable tiles, each 3 inches by 3 inches (color #1); 8 square variable tiles, each 3.25 inches by 3.25 inches (color #2); 12 rectangular (linear) variable tiles, 0.75 inches by 3 inches (color #1); 12 rectangular variable tiles, 0.75 inch by 3.25 inches (color #2); and 20 unit tiles, 0.75 inch by 0.75 inch (color #3). Each tile should have a large X drawn on one side to show the inverse of that tile. Use tagboard that is thick enough so that the X will not show through to the other side. Commercial tiles are also available for two different variables, but a large X must be drawn on one of the largest faces of each tile in order to represent the inverse of that tile when the X faces up.

2. The meaning of a large square tile needs to be connected to a long rectangular tile of the same color. Have students place a rectangular variable tile of color #1 (call it variable A) horizontally on the mat. Then have them place two more variable tiles below and parallel to the first tile on the mat. Ask: "If a single variable tile A is considered to cover an area of 1 by A, or A, how can we describe the arrangement indicated by these tiles on the mat?" ("3 rows of A.") "What product or area is this?" ("3A.") Ask: "How can we show A rows of A on the mat if we do not know what the value of A is?" Show students how to build several rows of one variable tile each, using one variable tile A as the multiplier, or "ruler," that indicates when to stop putting tiles in the product on the mat (see the illustration below). When the product is finished, the multiplier tile should be removed from the mat. Depending on the dimensions used to make the tiles, whether commercial or teacher-made, the width across several rectangular tiles placed with their longer sides touching may or may not match the length of the longer side of the same type of tile. Such a match is not important and should be deemphasized since the variable tile A is not considered to have a specific length or value in unit tiles. Therefore, although the width of 4 of the variable tile A may appear to match to one variable tile length as shown on the mat below, do not allow students to say that 4 rows of A equal 4A.

3. Ask: "Is there another single block that will cover the same surface area on the mat that the product A of A, or $A(A)$, covers?" ("Yes. The large square tile in color #1; its side length equals the length of the variable tile A.") Again, discuss the idea that the large square tile in color #1 may or may not fit perfectly on top of the "A rows of A" tile arrangement; it will be close enough. Since both the square and rectangular tiles in color #1 are representing variables without known values, we want to maintain their variable nature as much as possible. Physical models like the tiles naturally have specific dimensions that affect or limit areas being built with the tiles, but for our purpose, we will assume that *only the unit tiles may be used to represent exact amounts of area.* We will now assign the large square tile in color #1 the name of A-squared, or A^2. Hence, A rows of A equal A^2. From now on, whenever A rows of A are needed, the large square tile will be used to show that amount of area on the mat.



4. Similarly the areas of the square and long rectangular variable tiles in color #2 might be described as B-squared, or B^2, and B, respectively. If an X appears on the top side of a variable tile, the <u>inverse</u> or <u>opposite</u> of the tile's area will be indicated. For

example, a B-squared tile with X on top will be called "the opposite or inverse of B-squared" and written as $(-B^2)$. Each small square tile in color #3 represents an area of 1 by 1, or 1 square unit of area. If a given set of unit tiles all have an X showing—for example, 5 tiles with X—then the tile value will be the "negative of 5 square units of area" and written as (–5). Note that area itself is an absolute measure, neither positive nor negative. Area, however, can be assigned a direction of movement in real applications; hence, we can consider the opposite or negative of a given area.

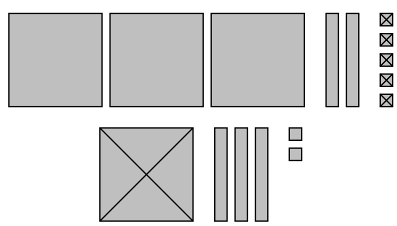
5. After the area of each type of tile is identified, have students do the exercises on Worksheet 1–1a. For each exercise, they should place a set of tiles on the building mat to show the first expression. Then they will either add more tiles to this initial set or remove some tiles from the set according to the second expression of the exercise.

6. After combining tiles that have the same amount of area, students should record an expression for the total or remaining area on the worksheet.

7. Discuss an addition exercise and a subtraction exercise with the class before allowing students to work the other exercises independently.

Consider Exercise 1 on Worksheet 1–1a for addition: $(3A^2+2A-5)+(-A^2+3A+2)$: Use color #1 variable tiles with the color #3 unit tiles.

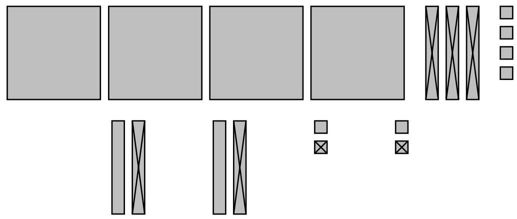
Have students place 3 large square (quadratic) variable tiles, 2 long rectangular (linear) variable tiles, and 5 negative unit tiles on the building mat to represent the first expression. Any such group of tiles is called a *polynomial*, that is, a combination of variable tiles and/or unit tiles. Leaving this set of tiles on the mat, have students place additional tiles on the mat below the initial tiles to represent the second expression. The second set should contain a quadratic variable tile with X showing, 3 linear variable tiles, and 2 unit tiles:



Ask: "Can any 0-pairs be made through joining, then removed from the mat?" (One 0-pair of the large quadratic tiles and two 0-pairs of the small unit tiles should be formed and removed from the building mat.) "Can you now describe the total in tiles still on the mat?" Since tiles for two of A-squared, 5 of A, and –3 remain on the mat, students should complete the recording of Exercise 1 on Worksheet 1–1a: $(3A^2+2A-5)+(-A^2+3A+2)=2A^2+5A-3$.

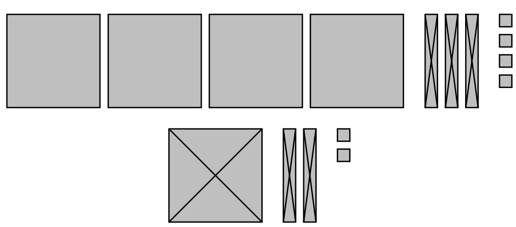
Now consider Exercise 2 on Worksheet 1–1a for subtraction: $(4A^2-3A+4)-(A^2+2A-2)$. Again, use color #1 variable tiles with the unit tiles.

Have students place tiles on their mats to show the first group. There should be 4 of the quadratic variable tile, 3 of the linear variable tile with the X-side showing for the inverse variable, and 4 positive unit tiles on the building mat. Discuss the idea that the subtraction symbol between the two polynomial groups means to *remove* each term in the second group from the first group. Ask: "Can we remove one quadratic variable tile from the original four? ("Yes; 3 quadratic variable tiles, or $3A^2$, will remain.") "Can $2A$ be removed from –3*A*?" Since only inverse variable tiles are present initially, 0-pairs of *A* and –*A* tiles will need to be added to the mat until two of the variable *A* are seen. Then 2*A* can be removed, leaving 5 of –*A* on the mat. Similarly, –2 will be removed from +4 by first adding two 0-pairs of +1 and –1 to the mat. Then –2 can be removed from the mat, leaving +6. The mat arrangement of the initial tiles and the extra 0-pairs of tiles is shown here <u>before</u> any tile removal occurs. Have students complete Exercise 2 on Worksheet 1–1a by writing an expression for the tiles left on the mat: $(4A^2 - 3A + 4) - (A^2 + 2A - 2) = 3A^2 - 5A + 6$.



Remind students that when they use 0-pairs of a tile and remove one form of the tile (for example, positive), then the other form (for example, negative) remains to be added to the other tiles on the mat. Show students that when they needed to remove 2*A* from the mat earlier, two 0-pairs of *A* and –*A* were placed on the mat. After 2*A* was removed to show subtraction, the two inverse variable tiles, –2*A*, still remained on the mat to be <u>combined</u> with the other tiles for the final answer. Hence, a <u>removal</u> of a tile from the mat is equivalent to <u>adding the inverse or opposite</u> of that tile to the mat.

To confirm this, have students place the original group of tiles $(4A^2 - 3A + 4)$ on the mat again. The opposites needed ($-A^2$, –2*A*, and +2) should then be placed on the mat and <u>combined</u> with the original tiles. See the illustration below. Remove any 0-pairs formed, leaving tiles for $3A^2$, –5*A*, and +6 on the mat as the answer. Finally, have students write another equation below Exercise 2 on Worksheet 1–1a, this time showing the alternate method that uses addition: $(4A^2 - 3A + 4) + (-A^2 - 2A + 2) = 3A^2 - 5A + 6$. Encourage students to use whichever of these two methods seems comfortable to them.



In the answer key for Worksheet 1–1a, when the coefficient of a final variable is 1, the number 1 will be written with the variable. This approach seems to be helpful to many students. Nevertheless, discuss the idea with the class that the 1 in such cases is often not recorded but simply understood as being there.

Answer Key for Worksheet 1–1a

1. $2A^2 + 5A - 3$
2. $3A^2 - 5A + 6$; alternate: $(4A^2 - 3A + 4) + (-A^2 - 2A + 2) = 3A^2 - 5A + 6$
3. $3B^2 + 2$; alternate: $(5B^2 + 3) + (-2B^2 - 1) = 3B^2 + 2$
4. $2A^2 + 1A - 4$
5. $1B^2 + 2A - 8$
6. $3A + 1A^2 + 5$; alternate: $3A + (A^2 + 5) = 3A + 1A^2 + 5$
7. $1A^2$; alternate: $(5B^2 - 4) + (-5B^2 + 4 + A^2) = 1A^2$
8. $3A + 1A^2 - 13 + 1B$

Worksheet 1–1a

Building Sums and Differences with Tiles

Name ______________________

Date ______________________

Build each polynomial exercise with tiles. Different variables require different tiles. Record the result beside the exercise. For each subtraction exercise, also write the alternate addition equation below the subtraction equation.

1. $(3A^2 + 2A - 5) + (-A^2 + 3A + 2) =$

2. $(4A^2 - 3A + 4) - (A^2 + 2A - 2) =$

3. $(5B^2 + 3) - (2B^2 + 1) =$

4. $(2A^2 - 3A + 1) + (4A - 5) =$

5. $(4A - 2 + B^2) + (-6 - 2A) =$

6. $3A - (-A^2 - 5) =$

7. $(5B^2 - 4) - (5B^2 - 4 - A^2) =$

8. $3A - 2A^2 - 5 + B - 8 + 3A^2 =$

Activity 2
Pictorial Stage

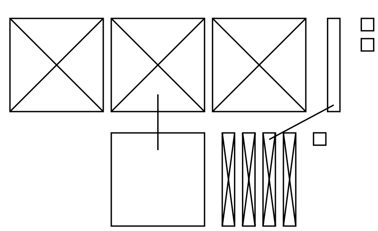
Materials

Worksheet 1–1b
Regular paper and pencil

Procedure

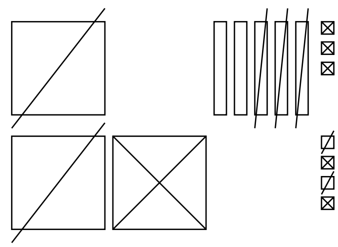
1. Give each student a copy of Worksheet 1–1b. Have students work in pairs, but they should draw the diagrams separately on their own worksheets. Large squares will be drawn for the quadratic variable, a long rectangle whose length equals an edge length of the large square will be drawn for the linear variable, and a small square will represent the integral unit. A large X should be drawn in the interior of a shape to show the inverse of that shape. If an exercise involves two different variables, letters need to be written on the drawn shapes to identify the different variables. The product of two different variables, for example, A and B, should be shown as a large rectangle similar in size to the quadratic squares and labeled as AB. The notation AB simply means A rows of B, or the area AB.
2. For addition exercises, students should draw the required shapes and connect any two shapes that represent a 0-pair. The remaining shapes will be recorded in symbols to show the sum.
3. For subtraction exercises, students will be asked to use either the removal method or the alternate method, which involves addition of inverses. To remove a shape, students should mark out the shape. When needed, two shapes should be drawn together as a 0-pair. For the alternate method, inverses of the subtrahend expression should be drawn and combined with the first expression to produce a sum. The result will be recorded symbolically.
4. When checking students' work after all are finished, allow time for students to explain their steps; do not just check for answers. Students need to practice expressing their ideas mathematically. Such verbal sharing is also very beneficial to *auditory learners*.
5. Discuss Exercises 1 and 2 on Worksheet 1–1b with the class before allowing partners to work together on their own.

Consider Exercise 1: $(-3B^2 + B + 2) + (B^2 - 4B + 1)$. Students should draw the necessary shapes on their papers to represent each polynomial group. The shapes for the first polynomial group may be drawn in a row from left to right following the order of the given terms. The shapes for the second polynomial group should be drawn as a second row below the first row, but students may rearrange the shapes and draw them below other like shapes in the first row. Since only one variable is involved, no labeling is needed for the shapes. Any 0-pairs should be connected. Remaining shapes will then be counted and recorded as the answer. A sample drawing is shown here:



The final equation will be as follows and should be recorded on Worksheet 1–1b: $(-3B^2+B+2)+(B^2-4B+1)=-2B^2-3B+3$. At this point, begin to encourage students to record the terms of a polynomial with their exponents in decreasing order.

Now consider Exercise 2: $(A^2+5A-3)-(2A^2+3A+2)$. Since the removal process is required for this exercise, students should draw shapes for the first polynomial group and then draw any 0-pairs below that group, which will be needed in order to mark out the shapes shown in the second group. The shapes remaining or not marked out in the finished diagram will be the difference. Here is the completed diagram:



The final equation should be recorded on Worksheet 1–1b as follows: $(A^2+5A-3)-(2A^2+3A+2)=-A^2+2A-5$. It may be helpful for some students to write $-1A^2$ instead of $-A^2$. This is acceptable notation.

Answer Key for Worksheet 1–1b

Only symbolic answers may be given.

1. $-2B^2-3B+3$ (see sample diagram in text)

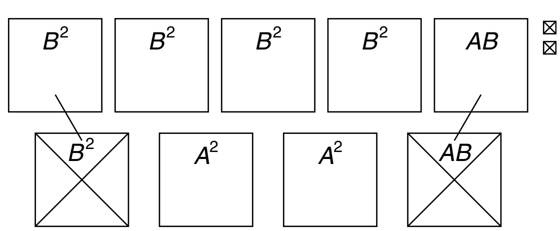
2. $-A^2+2A-5$ (see sample diagram in text)

3. $5A^2-2A+6$

4. $-3A-2B+1$

5. $2A^2+3B^2-2$

   Sample diagram for Exercise 5:



6. $B^2-3AB-A+1$

Worksheet 1–1b
Drawing Sums and Differences

Name ____________________
Date ____________________

Use shapes to simplify each polynomial exercise according to the directions. Label shapes to identify different variables when necessary. Record the algebraic result beside the exercise.

1. $\left(-3B^2+B+2\right)+\left(B^2-4B+1\right)=$

2. $\left(A^2+5A-3\right)-\left(2A^2+3A+2\right)=$

   [use removal]

3. $\left(6A^2-A+3\right)-\left(A^2+A-3\right)=$

   [use addition of inverses]

4. $(A-5B+6)+(3B-4A-5)=$

5. $\left(4B^2+AB-2\right)+\left(2A^2-B^2-AB\right)=$

6. $\left(2B^2-3AB+A\right)-\left(B^2+2A-1\right)=$

   [use either method]

Activity 3
Independent Practice

Materials
Worksheet 1–1c
Regular paper and pencil

Procedure
Give each student a copy of Worksheet 1–1c. After all students have completed the worksheet, ask various students to show their solutions or any illustrations they might have used to the entire class. In particular, select students to share their work who have solved the same problem in different ways.

Answer Key for Worksheet 1–1c

1. C
2. D
3. A
4. A
5. C

Possible Testing Errors That May Occur for This Objective
- When combining polynomials, students fail to recognize 0-pairs among the terms; for example, they write the sum $(+3x) + (-3x)$ as $(+6x)$ instead of 0.
- When finding differences by the alternate method of adding inverses of the subtrahend group to the original minuend group, students do not exchange all the terms for their inverse forms; hence, they add the wrong terms together. For example, in $(2N + 5) - (N + K - 3)$, they actually add $(2N + 5)$ to $(-N + K + 3)$ instead of to $(-\mathrm{N} - K + 3)$.
- Students make computational errors when combining like terms. For example, $(+4y) + (-7y)$ is incorrectly written as $(+11y)$ instead of $(-3y)$.

Worksheet 1–1c

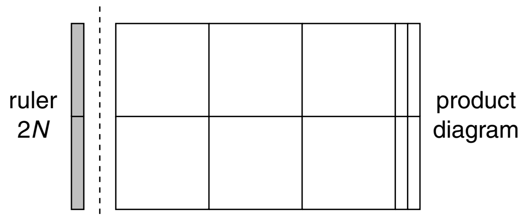
Finding Sums and Differences of Polynomials

Name ______________________

Date ______________________

Solve the exercises provided and be ready to discuss your methods and answers with the entire class.

1. The following diagram represents the product of $2N$ rows of $(3N + 2)$. Which expression is equivalent to the total area, $2N(3N + 2)$, of the product diagram?



A. $6N^2+2$ B. $6N+2$ C. $6N^2+4N$ D. $6N+4$

2. Which expression is equivalent to $\left(\frac{2}{3}\right)(6x-3y)+(4y-7x)$?

A. $-x+y$ B. $-3x+y$ C. $4x-2y$ D. $-3x+2y$

3. Which expression is equivalent to $(5k-2)(3k)-(5k-2)(k-1)$?

A. $10k^2+k-2$ B. $10k^2-4k-2$ C. $10k^2-2$ D. $5k^2+2$

4. Simplify the expression and evaluate for $T = -5$: $(2T^2+3T)+(5T-2T^2+3) = ?$

A. –37 B. 38 C. –43 D. Not here

5. Which expression is equivalent to $(5y^2-3xy-4)-(5y^2-y-3xy-4)$?

A. 0 B. $-6xy - 4$ C. y D. $-y$

Objective 2: Solve a Linear Equation Involving One Variable with a Fractional Coefficient

Fraction operations are difficult for most students to comprehend. Extending fraction multiplication to partial sets of a variable is even more complicated. Rote methods are often taught and students seemingly master them, yet they do not attain a deeper understanding of the method. Students need experiences with partial sets that will lead to discovering what the whole set will contain. The following activities provide such experiences.

Activity 1

Manipulative Stage

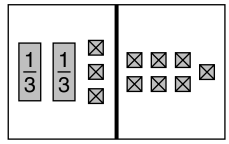
Materials

Tile sets (minimal set: 1 variable tile with its inverse tile, 30 positive unit tiles with their inverse tiles)
Building Mat 1–2a
Pieces of colored yarn (approximately 12 inches long) or flat coffee stirrers
Extra construction paper (use colors that match the variable tiles in the sets)
Scissors
Regular paper and pencils

Procedure

1. Give each pair of students a set of tiles, a copy of Building Mat 1–2a, a piece of yarn or coffee stirrer, scissors, and a sheet of construction paper (the same color as their variable tiles).
2. To make fractional variable tiles, have students cut out 6 rectangular strips from their sheet of construction paper. The paper strips should be the same size and color as one of their variable tiles. Show them how to fold the paper strips, mark the creases, and label the parts with fractional names. Two strips should be folded and labeled for halves, yielding 4 half-variables total. Two more strips should be folded and labeled for thirds, and another 2 strips for fourths. Use the ratio format for labeling the fractional parts, for example, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. On one side of each fractional part made, have students mark a thin, large X to represent the *inverse* of the fractional part. Be sure that the X does not show through on the other side of the paper. Additional paper strips may be cut out as needed.
3. Have students place tiles on Building Mat 1–2a to build each equation shown on Worksheet 1–2a. Below each equation on Worksheet 1–2a, they should record the symbolic steps they used to solve the equation.
4. For each equation solved, students should confirm their solution. Have them rebuild the original equation with tiles and then substitute the variable's discovered value in unit tiles for the variable tile itself to show the true equality. Students should write a check mark beside the solution on the worksheet after the correct value has been confirmed.
5. Discuss Exercises 1 and 2 on Worksheet 1–2a with the class before allowing students to work the rest of the exercises independently.

Consider Exercise 1 on Worksheet 1–2a: $\frac{2}{3}K-(+3)=-7$. Have students build this equation with tiles. A paper variable folded into thirds should be cut apart and 2 of the thirds placed on the left side of Building Mat 1–2a. The subtraction sign on the left side of the equation indicates that +3 must be removed from the mat. Since there are not 3 positive unit tiles on the left side, three 0-pairs of positive and negative unit tiles should be placed on the left side of the mat, followed by removal of the 3 positive unit tiles. Some students may realize that subtracting +3 is equivalent to adding –3; if so, they may simply place –3 on the left side of the mat with the fractional variable tiles instead of working with the 0-pairs first. Finally, 7 negative unit tiles should be placed on the right side of the mat.



Ask: "What changes can we make to the mat that will leave the variable tiles by themselves on the left?" ("Either take away –3 or bring in +3 on the left.") The students must then repeat whichever action they choose on the right side of the mat and record their steps with symbolic notation below Exercise 1 on Worksheet 1–2a.

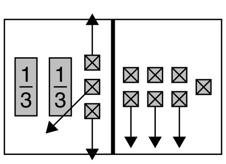
(a) Take-Away Model

$$\frac{2}{3}K-(+3)=-7$$

$$\frac{2}{3}K+(-3)=-7$$

$$\underline{-(-3)} \quad \underline{-(-3)}$$

$$\frac{2}{3}K=-4$$



remove -3 from both sides of mat

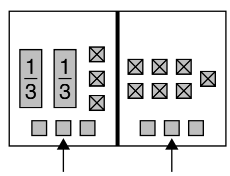
(b) Add-On Model

$$\frac{2}{3}K-(+3)=-7$$

$$\frac{2}{3}K+(-3)=-7$$

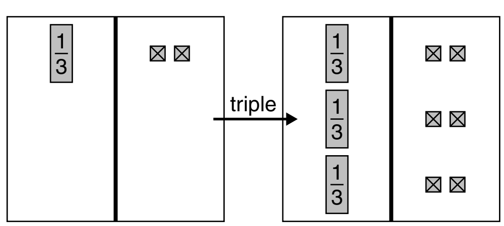
$$\underline{+(+3)} \quad \underline{+(+3)}$$

$$\frac{2}{3}K=-4$$



add +3 to both sides of mat

Ask: "Two of the thirds of a variable remain on the left by themselves. What do we need to do to find a whole variable tile now?" ("First find a single third of the variable by separating the 2 variable parts or by finding 'half' of the present variable group; 3 of this single third will make a whole variable tile.") Remember that you need to use *multiplication* language here, not *addition* language. That is, you want *3 of* the third-variable in order to make a whole variable, not *2 more of* the third-variable.



After students perform these two actions, halving and then tripling, on both sides of the mat, have them record their new actions as shown below:

(a) Take-Away Model

$$\frac{2}{3}K-(+3)=-7$$

$$\frac{2}{3}K+(-3)=-7$$

$$\underline{-(-3)} \quad \underline{-(-3)}$$

$$\frac{2}{3}K=-4$$

$$\frac{1}{2}\left(\frac{2}{3}K\right)=\frac{1}{2}(-4)$$

$$\frac{1}{3}K=-2$$

$$3\left(\frac{1}{3}K\right)=3(-2)$$

$$K=-6$$

(b) Add-On Model

$$\frac{2}{3}K-(+3)=-7$$

$$\frac{2}{3}K+(-3)=-7$$

$$\underline{+(+3)} \quad \underline{+(+3)}$$

$$\frac{2}{3}K=-4$$

$$\frac{1}{2}\left(\frac{2}{3}K\right)=\frac{1}{2}(-4)$$

$$\frac{1}{3}K=-2$$

$$3\left(\frac{1}{3}K\right)=3(-2)$$

$$K=-6$$

Note: The hope is that most students will realize that the halving-tripling process used above can be combined and shown in the recording as follows:

$$\frac{2}{3}K=-4$$

$$\frac{3}{2}\left(\frac{2}{3}K\right)=\frac{3}{2}(-4)$$

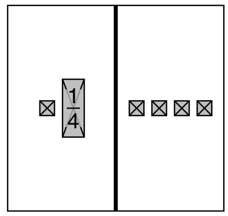
$$K=-6$$

If the students do not easily accept this combination of steps, allow them to continue recording the two separate steps of division and multiplication. Also notice that when we previously needed "half of the 2-thirds of the variable *K*," the factor $\frac{1}{2}$ was used with multiplication, rather than using 2 as a divisor, or dividing by 2. This helps students connect to the reciprocal method more easily.

The solution, $K = -6$, should be confirmed on the mat by exchanging variable parts for units. Have students rebuild the original equation on their mats and place 6 negative unit tiles on the mat just above the variable tiles. Since thirds are involved, –6 should be separated into 3 equal groups of –2 each. Each variable part on the mat should be replaced with one of the groups of –2. The unused –2 should be removed from the top of the mat. Now 2 groups of –2, along with another group of –3, can be seen on the left side, and –7 appears on the right. Since the two sides have the same total value, $K = -6$ is the correct solution. A check mark should be written beside the solution equation on the worksheet.

Now discuss the equation for Exercise 2 on Worksheet 1–2a: $(-1)-\frac{1}{4}p=-4$.

Have students build the equation on Building Mat 1–2a, using paper fourths of a variable. Each fourth-variable should have an X marked on one side. First have students place –4 in unit tiles on the right side of the mat and –1 in unit tiles on the left side. Discuss the idea that just as with unit tiles earlier, subtraction of a variable group is equivalent to the addition of the inverse variable group. Hence, after showing a 0-pair of fourths of a variable (one plain fourth and one fourth with an X-side facing up) on the left side of the mat with –1 and then removing $+\frac{1}{4}p$, students will have $-\frac{1}{4}p$ and –1 still remaining on the left side.



Students should now isolate the fourth of an inverse variable by either the take-away or the add-on method. The tiles on each side of the mat should then be quadrupled to yield 4 variable parts that are equivalent to a whole inverse variable, $-p$. If students prefer, they may replace the 4 variable parts on the mat with a whole inverse variable tile.

Since a solution usually involves a value for the regular variable p, not its inverse, $-p$, a coffee stirrer or piece of yarn should be placed below the last row of tiles on the mat: $-p = -12$. The inverse of $-p$ is p and the inverse of –12 is +12; therefore, the regular, whole variable tile should be placed on the mat below the coffee stirrer or yarn and the 4 fourths of the inverse variable, and 12 positive units should be placed below the negative units and coffee stirrer or yarn on the right. The building and recording are as follows:

(a) Add-On Method

$$(-1) - \frac{1}{4}p = -4$$

$$(-1) + \left(-\frac{1}{4}p\right) = -4$$

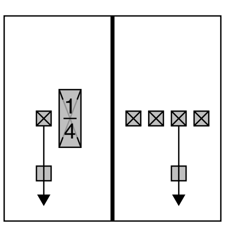
$$\underline{+(+1)} \quad \underline{+(+1)}$$

$$-\frac{1}{4}p = -3$$

$$4\left(-\frac{1}{4}p\right) = 4(-3)$$

$$-p = -12$$

$$\text{So, } p = +12$$



remove 0-pairs

(b) Take-Away Method

$$(-1) - \frac{1}{4}p = -4$$

$$(-1) + \left(-\frac{1}{4}p\right) = -4$$

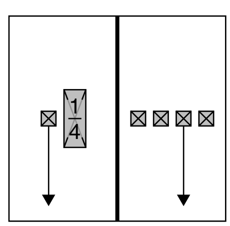
$$\underline{-(-1)} \quad \underline{-(-1)}$$

$$-\frac{1}{4}p = -3$$

$$4\left(-\frac{1}{4}p\right) = 4(-3)$$

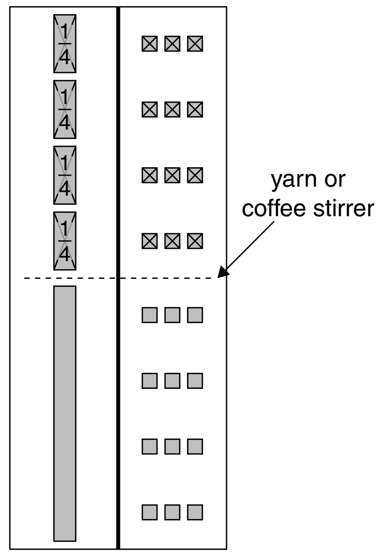
$$-p = -12$$

$$\text{So, } p = +12$$



remove -1 from both sides of mat

Here are the final steps for both methods:



Finally, the solution needs to be confirmed. The original equation should be rebuilt on the mat, including one of the fourths of the inverse variable tile. If $p = +12$, then $-p$ must equal –12; hence, –12 should be placed on the mat above the tiles on the left side. Since a fourth of the variable is needed, –12 should be separated into four equal groups of –3 each. One group of –3 should be exchanged for the variable part and the other three groups of –3 removed from the mat. A single, negative unit tile, as well as the group of –3, are now seen on the left side of the mat, and a group of –4 is seen on the right. Both sides of the mat have the same total value, so the solution, $p = +12$, is correct. A check mark should be written beside the solution equation on the worksheet.

Answer Key for Worksheet 1–2a

Only solutions are provided; no mats are shown.

1. $K = -6$

2. $p = +12$

3. $C = +10$

4. $m = -4$

5. $B = +4$ [use 3 half-variable tiles; then isolate the variable tiles on the left side of the mat; separate each side of the mat into 3 equal groups to find the value for one half-variable tile]

6. $w = -12$

Building Mat 1–2a

Worksheet 1–2a

Solving Linear Equations with Tiles

Name ____________________

Date ____________________

Solve the equations with tiles on a building mat. Below each equation, record the steps used with symbolic notation. Confirm each solution found by exchanging the appropriate amount of unit tiles for the variable tiles given in the original equation. Write a check mark beside any solution equation shown to be correct with tiles.

1. $\frac{2}{3}K-(+3)=-7$

4. $+4=-\frac{3}{4}m-(-1)$

2. $(-1)-\frac{1}{4}p=-4$

5. $\frac{3}{2}B+6=+12$

3. $7=2+\frac{1}{2}C$

6. $(-5)+\left(-\frac{2}{4}w\right)=+1$

Activity 2
Pictorial Stage

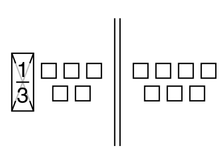
Materials

Worksheet 1–2b
Regular pencils
Red pencils

Procedure

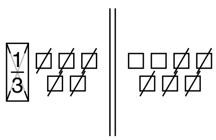
1. Give each student a copy of Worksheet 1–2b and a red pencil. Have students work in pairs.
2. Have students draw a small square to represent each unit integer. For a negative unit, they should draw diagonals inside the square. Tall, narrow rectangles will represent whole variable bars. To show a fractional amount of a variable, have students draw a short rectangle (but slightly taller than the squares drawn for integral units) and write the fractional label inside the rectangle. A large but light X should be drawn inside the rectangle to show the inverse form of the variable.
3. For each equation on Worksheet 1–2b, students should draw a diagram for the equation on the worksheet. They should transform the diagram in order to find the solution to the equation.
4. After each new step has been performed on the diagram, have students record that step in symbols beside the diagram on Worksheet 1–2b.
5. After a solution is found, students should confirm the solution by drawing in red pencil the appropriate number of small squares on each variable shape in the diagram. The total number of small squares on the left side of the frame should equal the total number of small squares on the right side. Also have them confirm their solutions by writing a number sentence below the symbolic steps to show the substitutions used.
6. Discuss Exercise 1 on Worksheet 1–2b with the class before allowing students to work independently.

Consider the equation for Exercise 1: $-\frac{1}{3}M + 5 = +7$. To make a diagram of this equation, ask students to draw a short rectangle (but slightly taller than the squares drawn for integral units) on the left side of a pair of parallel line segments (the equal sign). Have them draw a light but large X (to show the inverse) in the interior of the rectangle and write the fraction, $\frac{1}{3}$, over the X. This will represent $-\frac{1}{3}M$, read as "one-third of the inverse of the variable M." Avoid the language "negative one-third M" until later. Also draw 5 small, plain squares on the left side with the variable rectangle. Seven small, plain squares should then be drawn on the right side to represent +7. The diagram should be drawn on the left half of Worksheet 1–2b below the equation.

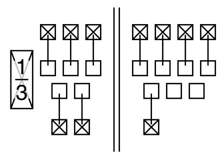


To isolate the variable by itself, students have the usual two methods available: remove +5 from both sides of the diagram, or bring in –5 to both sides of the diagram to form 0-pairs of the units. The transformed diagrams will appear as follows:

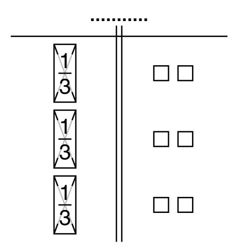
(a) Removal Method



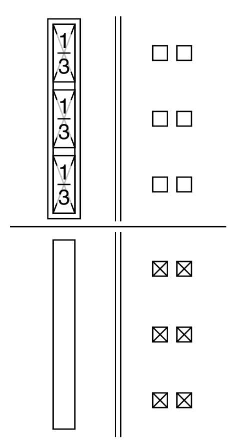
(b) Add-On Method



A horizontal bar should be drawn below the initial diagram that shows the removal or add-on step, and the shapes remaining for $-\frac{1}{3}M$ on the left and +2 on the right should be redrawn below that bar. Students now need to form a whole variable. This is done by drawing more rows of shapes on the second diagram until there are <u>3 rows in all</u>; each row shows a $-\frac{1}{3}M$ on the left of the vertical bars and +2 on the right. Remind students that they now have *3 times as many* $-\frac{1}{3}M$*'s* and *3 times as many groups of +2* as they did before they drew the extra amounts on the diagram. “3 times as many” is multiplicative language, which is needed for this type of equation.



By this time, students should recognize the three inverse thirds of a variable, one drawn above another, as a whole variable, and they should not need to draw a new, longer rectangle to show the whole inverse variable. It might be helpful, however, to have students draw a larger rectangle around the three smaller rectangles to show them grouped together. Since the diagram shows –*M* = +6, a horizontal bar should be drawn below this diagram and a plain rectangle for *M* drawn below the bar on the left side and 3 rows of –2 drawn below the bar on the right. That is, inverses have to be taken of –*M* and +6 in order to solve for the regular variable, *M*.



Students should record their pictorial steps in symbolic notation on the right half of the worksheet below the equation. Depending on the method used, their recordings should appear as follows:

(a) Removal-Multiplication Method

$$-\frac{1}{3}M+(+5)=+7$$

$$\underline{-(+5)}\quad\underline{-(+5)}$$

$$-\frac{1}{3}M=+2$$

$$3\left(-\frac{1}{3}M\right)=3(+2)$$

$$-M=+6$$

$$\text{So, } M=-6$$

(b) Add-On/Multiplication Method

$$-\frac{1}{3}M+(+5)=+7$$

$$\underline{+(-5)}\quad\underline{+(-5)}$$

$$-\frac{1}{3}M=+2$$

$$3\left(-\frac{1}{3}M\right)=3(+2)$$

$$-M=+6$$

$$\text{So, } M=-6$$

To confirm the solution as M = –6, have students first draw in red pencil two plain unit squares above the small rectangle for $-\frac{1}{3}M$ in the initial diagram; that is, if M = –6, the fractional amount must equal +2. The left side of the initial diagram now contains +7 in small plain squares and the right side contains +7, which confirms the solution. Below the symbolic steps, students should write the number sentence that shows their substitution: $-\frac{1}{3}M+(+5)=\frac{1}{3}(+6)+(+5)=(+2)+(+5)=+7$ [viewing $-\frac{1}{3}M$ as "a third of the inverse variable, –M," or $\frac{1}{3}$ of +6]; or $-\frac{1}{3}M+(+5)=-\frac{1}{3}(-6)+(+5)=(+2)+(+5)=+7$ [viewing $-\frac{1}{3}M$ as "the opposite of $\frac{1}{3}$ of the variable, M," or $-\frac{1}{3}$ of –6]. The +7, found after substitution to be the value of the left side of the original equation, agrees with the +7 given for the right side of the equation. Thus, M = –6 is confirmed again to be the correct solution.

Answer Key for Worksheet 1–2b

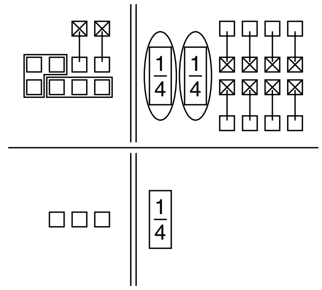
Only solutions and possible substitution number sentences are provided here. No diagrams are shown except for a partial diagram from Exercise 2.

1. M = –6; $-\frac{1}{3}M+(+5)=\frac{1}{3}(+6)+(+5)=(+2)+(+5)=+7$

   [The complete diagram and recordings are shown in the text.]

2. A = +12; $\frac{2}{4}A-8=\frac{2}{4}(+12)-8=(+6)-8=-2$

   [When multiple fractional parts of a variable are present in an equation and the variable parts are isolated by removing appropriate unit squares, rings should be drawn around the remaining shapes to form the same number of equal groups on each side of the diagram. Each fractional variable represents one group, so in this exercise, there will be two groups on the right side; this requires two groups of +3 to be formed on the left side. One group from each side is then redrawn and will be repeated to form the equivalent of a whole variable amount. Part of the diagram is shown here.]



3. $c = +8$; $\frac{1}{4}c - 4 = \frac{1}{4}(+8) - 4 = (+2) - 4 = -2$

4. $A = -10$; $(+5) + \left(-\frac{3}{5}A\right) = (+5) + \frac{3}{5}(+10) = (+5) + (+6) = +11$

Worksheet 1–2b

Solving Linear Equations by Drawing Diagrams

Name ______________________

Date ______________________

Solve each equation by drawing a diagram below the equation. Beside each diagram, record the steps used with symbolic notation. Confirm each solution found by drawing in red pencil the appropriate number of unit squares for each fractional variable above that variable shape in the initial diagram. Also, below the symbolic steps, write a number sentence that shows the substitutions made in the original equation.

1. $-\frac{1}{3}M + (+5) = +7$

3. $-2 = \frac{1}{4}c - 4$

2. $-2 = \frac{2}{4}A - 8$

4. $(+5) + \left(-\frac{3}{5}A\right) = +11$

Activity 3
Independent Practice

Materials

Worksheet 1–2c
Regular pencils

Procedure

Give each student a copy of Worksheet 1–2c to complete independently. After all have finished, have them share their methods and answers with the class.

Answer Key for Worksheet 1–2c

1. C
2. A
3. $n = +15$
4. $X = +16$
5. $c = -30$

Possible Testing Errors That May Occur for This Objective

- Students do not correctly interpret the subtraction sign in an equation and use addition instead; for example, for the expression $N-(-8)$ they will use $N+(-8)$. Other sign errors also occur.
- When an equation involves a fractional coefficient with the variable, students attempt to use the reciprocal method, but do not multiply by the correct fraction. For example, when trying to solve $\frac{3}{4}K = +6$, they will use either $\frac{1}{3}\left(\frac{3}{4}K\right)$ or $4\left(\frac{3}{4}K\right)$ instead of the combination step, $\frac{4}{3}\left(\frac{3}{4}K\right)$.
- When applying the reciprocal method to solve an equation, students will multiply the variable group by the reciprocal of the variable's coefficient, but fail to multiply the equivalent constant by the same value. As an example, for $\frac{2}{5}A = -4$ students will compute $\frac{5}{2}\left(\frac{2}{5}A\right)$, but will continue to use –4 instead of $\frac{5}{2}(-4)$.

Worksheet 1–2c
Solving Linear Equations

Name ____________________
Date ____________________

Complete each exercise provided. Be ready to explain to other students the steps or reasoning you used to work each exercise.

1. If $A = +22$ is a solution for the equation $-\frac{1}{2}A + 9 = -2$, which expression may be used to confirm the solution?

   A. $-\frac{1}{2}(+11) + 9$

   B. $-\frac{1}{2}(-22) + 9$

   C. $\frac{1}{2}(-22) + 9$

   D. $\frac{1}{2}(+22) + 9$

2. Which expression is not a correct interpretation of the expression $-\frac{2}{3}p$?

   A. $-3\left(\frac{1}{2}p\right)$ B. $-2\left(\frac{1}{3}p\right)$ C. $\frac{2}{3}(-p)$ D. $2\left(-\frac{1}{3}p\right)$

3. Solve for n: $+7 = \frac{1}{3}n - (-2)$.

4. Solve for X: $-18 = (-6) - \frac{3}{4}X$.

5. Solve for c: $8 + \frac{2}{6}c = -2$.

Objective 3: State an Equation with One or More Variables That Represent a Linear Relationship in a Given Situation; Apply the Equation to Solve the Problem, If Appropriate

Practical applications of mathematics require that students be able to translate the actions of a situation into an equation. Students may know how to solve a linear equation written in symbolic language, but this in no way guarantees that they can translate a word problem into an equation. This lesson focuses on the translation, but some actual solving may be required for additional practice. It is assumed that students already have a basic knowledge of how to solve simple linear equations involving integers. Objective 2 provides specific training with fractional coefficients if a review is necessary.

Activity 1

Manipulative Stage

Materials

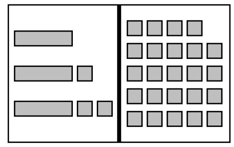
Sets of tiles (minimum set: 8 linear variable tiles of equal length, 30 unit tiles, and the fractional variable tiles from Objective 2; inverse tiles should be included for each type of tile)
Building Mat 1–3a
Worksheet 1–3a
Regular pencils

Procedure

1. Give each pair of students a set of tiles, a copy of Building Mat 1–3a, and two copies of Worksheet 1–3a.
2. Using the tiles, students should build equations on the building mat to represent the situations described in each exercise on Worksheet 1–3a.
3. For each exercise modeled with tiles, have students write the initial equations in symbols below the word problem. Then have them solve the equations with the tiles. The solutions should also be recorded beside the equations on Worksheet 1–3a.
4. Discuss Exercises 1 and 2 on Worksheet 1–3a with the class before allowing students to work independently with their partners.

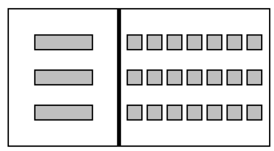
Consider Exercise 1 on Worksheet 1–3a: "Three consecutive positive integers have a sum of 24. Find the three consecutive integers."

Discuss examples of consecutive positive integers like 2, 3, and 4, or 25, 26, and 27. Guide students to recognize that each new number is one more than the previous number. Since the actual numbers are not yet known, have students place one variable tile on the left side of Building Mat 1–3a to represent the first number. The second number is one more than the first number, so students should place another variable tile (same color or tile length as the first variable tile), along with 1 unit tile, on the left side of the mat. Finally, they should place an additional variable tile and 2 unit tiles on the left side to represent the third number. The collective set of tiles on the left is the sum of the 3 numbers. Since the sum equals 24, students should place 24 unit tiles on the right side of Building Mat 1–3a. Here is the initial appearance of the building mat:



Have students record the following unsimplified equation below Exercise 1, using N for the variable tile: $N+(N+1)+(N+2)=24$.

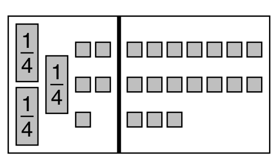
Students should then solve the equation with the tiles. Remind them that the variable tiles need to be isolated, which requires that the 3 unit tiles be removed from the left side of the mat. To keep the building mat balanced, 3 unit tiles must also be removed from the right side. Continuing, the remaining variable tiles must be separated into 3 groups (each variable tile forms a "group"), which forces the 21 unit tiles on the right side to be separated into 3 groups as well. The groups on the mat should now have this appearance:



Each variable tile determines a row on the right side of the mat. Hence, the value of one variable tile equals +7, which becomes the first number of the three consecutive numbers being sought. The next two numbers are represented by $N + 1$ and $N + 2$, so their values are $7 + 1 = 8$ and $7 + 2 = 9$. Students should record the following three equations beside their initial equation for Exercise 1 on Worksheet 1–3a: $N = 7$, $N + 1 = 7 + 1 = 8$, and $N + 2 = 7 + 2 = 9$.

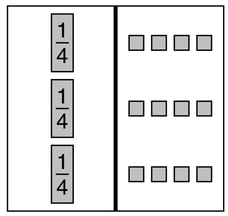
Now consider Exercise 2 on Worksheet 1–3a: "Three-fourths of the Math I class and five students from the Math II class are planning to go to the museum. Seventeen students in all will go on the field trip. How many students total attend the Math I class?"

Have students use their paper strips from Objective 2 to show fractional amounts of a variable tile. The whole variable tile equals the number of students in Math I, so 3 of the fourth-variable tiles should be placed on the left side of Building Mat 1–3a to represent the students from Math I going on the field trip. Five unit tiles should also be placed on the left side to show the 5 students from Math II who will be going. Since the combined groups equal 17, students should place 17 unit tiles on the right side of the building mat. The building mat will have the following initial appearance:



Using the variable M as the total number of students in Math I, have students record the following equation below Exercise 2 to represent the tiles shown on the building mat: $\frac{3}{4}M+5=17$.

Now have students use the tiles on the mat to solve for the value of the whole variable, M. At first they must remove 5 unit tiles from both sides of the mat to isolate the fourth-variable group on the left side. Then the fourth-variables should be separated into three equal groups (a fourth-variable forms a "group" this time), which forces the 12 remaining unit tiles on the right side to be separated into three equal groups as well. The new tile arrangement will be as shown:



Discuss the idea that one row on the building mat shows that a fourth-variable tile equals +4. To make a whole variable tile, four of the fourth-variable tiles will be needed on the left side of the mat. Similarly, four rows of +4 will be needed on the right side to keep the mat balanced. Have students place more tiles on the building mat in order to have the four rows needed. The solution to the initial equation is now found. Four fourth-variable tiles or 1 whole variable tile equals 4 rows of +4 each. Have students record the following statement beside the original equation below Exercise 2: M = 16 students in Math I.

Answer Key for Worksheet 1–3a

1. $N + (N+1) + (N+2) = 24;\ N = 7,\ N + 1 = 7 + 1 = 8,\ N + 2 = 7 + 2 = 9$
2. $\frac{3}{4}M + 5 = 17$; M = 16 students in Math I
3. $4p - 6 = 10;\ p = 4$ cents per piece of gum
4. $\frac{1}{2}R - 3 = 4;\ R = 14$ rings in a full box
5. $T + (T+4) + 3T = 19$; $T = 3$ movie passes for Toni
6. $\frac{1}{3}G = 8$; $G = 24$ gallons in a full drum

Building Mat 1–3a

Worksheet 1–3a

Modeling Linear Relationships with Tiles

Name ____________________

Date ____________________

For each exercise, build an equation with tiles on Building Mat 1–3a to represent the situation. Write the equation in symbols below the exercise. Solve the equation with tiles, and write the solution beside the symbolic equation.

1. Three consecutive positive integers have a sum of 24. Find the 3 consecutive integers.

2. Three-fourths of the Math I class and 5 students from the Math II class are planning to go to the museum. Seventeen students in all will go on the field trip. How many students are in the Math I class?

3. Kate bought 4 pieces of chewing gum at the school store. After a discount of 6 cents was applied to the total purchase, she paid 10 cents in all. What was the original price for each piece of gum?

4. Jorge won one-half of a box of rings at the carnival, but on the way home lost 3 of the rings. Later at home, he still had 4 rings left. How many rings were originally in a full box?

5. Maria has 4 more movie passes than Toni. Angela has 3 times as many passes as Toni. Together the three girls have a total of 19 movie passes. How many movie passes does Toni have?

6. Two-thirds of the oil in an oil drum has leaked out. Eight gallons of oil are left in the drum. How many gallons total can the drum hold?

Activity 2
Pictorial Stage

Materials

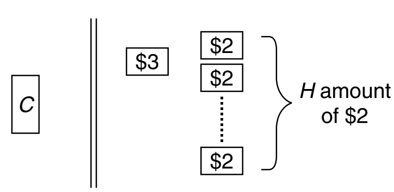
Worksheet 1–3b
Regular pencils
Regular paper

Procedure

1. Give each student a copy of Worksheet 1–3b. Have students work in pairs.
2. For each exercise on Worksheet 1–3b, have students draw on regular paper a diagram for each relationship described in the given situation. If more than one type of variable is needed, students will need to label each variable shape with a letter they select for that variable.
3. Beside each diagram, students should record an equation in symbolic language that is equivalent to the diagram.
4. For practice in solving equations, students might also be instructed to solve for the variables in the equations they find for Exercises 2 through 5 and Exercise 8.
5. Discuss Exercises 1 and 2 on Worksheet 1–3b before allowing students to work independently with a partner.

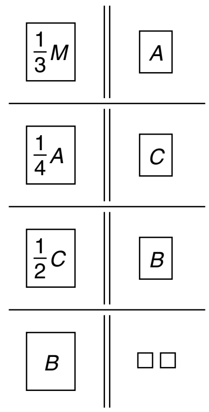
Consider Exercise 1 on Worksheet 1–3b: "Eddie's Dogwalking Service charges $3 for each walk plus $2 per hour for each hour the dog is walked. Find an equation that shows the relationship between the number of hours walked, *H*, and the total cost, *C*, for one walk."

Discuss the idea that the total cost is the combination of the single fee of $3 and the charges based on time. Each hour is worth $2, so *H* hours indicates how many of the $2 are needed. That is, the *H* hours serves as the multiplier, or counter of sets, and $2 is the multiplicand, or the set being repeated. Have students draw an equation frame that shows a variable *C* on the left side and shapes for the sum of the two kinds of charges on the right side. A pair of vertical parallel bars should be drawn to indicate equality of the two sides of the diagram. Because the multiplier in this case is a variable or an unknown amount, an exact amount of $2 sets cannot be shown; rather, a *countable amount* will be indicated in the diagram through special labeling. Here is a possible final diagram for the situation in Exercise 1. The following equation should be recorded beside the diagram: $C = \$3 + H(\$2)$. Since *H* serves as the multiplier, it is written in the multiplier position, which is the first factor of the product.



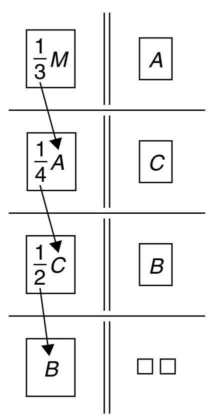
Now consider Exercise 2 on Worksheet 1–3b: "Marian took cookies to a party. She gave a third of her cookies to Adam. Adam then gave a fourth of his cookies to Charles. Charles gave half of his cookies to Barbara. If Barbara received two cookies in all, how many cookies did Marian have in the beginning? Show the initial equations. Then try to combine them into one equation that involves only the variable for Marian's amount of cookies."

Since several relationships are involved in this situation, have students draw a diagram for each one. A horizontal bar should be drawn between each touching pair of diagrams. Here is a possible sequence of diagrams to use, along with their recorded equations:



$$\frac{1}{3}M = A$$
$$\frac{1}{4}A = C$$
$$\frac{1}{2}C = B$$
$$B = 2$$

Exercise 2 asks students to find a single equation that relates Marian's amount of cookies to the 2 cookies that Barbara received. Guide students to apply backward thinking to their diagrams, beginning with the last diagram. Through a substitution process, the diagrams can be stacked on each other. Looking only at the left side of each diagram and moving upward, one-half C replaces the B in the last diagram, then one-fourth A replaces C, and finally one-third M replaces A. It might be helpful for students to draw arrows on their diagrams to show where the substitutions occur. Here is a possible example, along with the combined equation that results:



$$\frac{1}{3}M = A$$
$$\frac{1}{4}A = C$$
$$\frac{1}{2}C = B$$
$$B = 2$$

$$2 = \frac{1}{2}C = \frac{1}{2}\left(\frac{1}{4}A\right) = \frac{1}{2}\left(\frac{1}{4}\left[\frac{1}{3}M\right]\right) \qquad \text{So, } 2 = \frac{1}{2}\left(\frac{1}{4}\left[\frac{1}{3}M\right]\right) = \frac{1}{24}M.$$

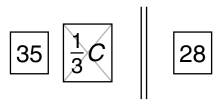
Answer Key for Worksheet 1–3b

Suggested diagrams and their equations are provided; other formats are possible.

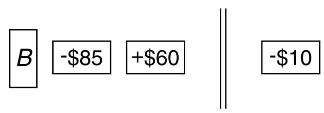
1. $C = \$3 + H\,(\$2)$ [The diagram is shown in the text.]

2. [See the diagrams and sequence of equations in the text.]

3. C = number of jelly beans in whole cup; $35 - \frac{1}{3}C = 28$

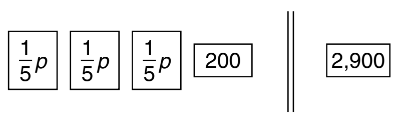


4. B = initial balance in bank account; $B - \$85 + \$60 = -\$10$

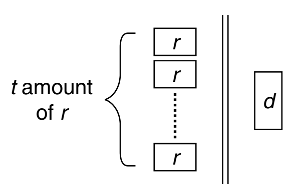


Alternate format:

5. p = size of parking lot in square feet; $\frac{3}{5}p + 200 = 2900$

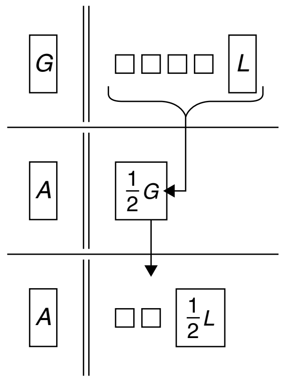


6. $d = t(r)$. [Note: The factor t in this case is the *multiplier*, so it is written first in the product; later, the commutative property might be applied to rewrite the equation in the more familiar form, $d = rt$.]

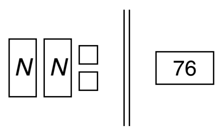


7. G, L, and A: Number of tokens each person has:

   $G = 4 + L$; $A = \frac{1}{2}G$; $A = \frac{1}{2}(4 + L) = 2 + \frac{1}{2}L$



8. N, $N + 2$: consecutive odd integers; $N + (N + 2) = 76$



Worksheet 1–3b

Drawing Diagrams for Linear Relationships

Name ______________________

Date ______________________

For each exercise, draw a diagram on another sheet of paper to represent each relationship in the situation. Write equations in symbols beside the diagrams.

1. Eddie's Dogwalking Service charges $3 for each walk plus $2 per hour for each hour the dog is walked. Find an equation that shows the relationship between the number of hours walked, H, and the total cost, C, for 1 walk.

2. Marian took cookies to a party. She gave a third of her cookies to Adam. Adam then gave a fourth of his cookies to Charles. Charles gave half of his cookies to Barbara. If Barbara received two cookies in all, how many cookies did Marian have in the beginning? Show the initial equations. Then try to combine them into one equation that involves only the variable for Marian's amount of cookies.

3. Jaime removed one-third of a cup of jelly beans from a jar that held 35 jelly beans at first. She recounted and found that there were still 28 jelly beans in the jar. Approximately how many jelly beans would fill a whole cup?

4. On Friday, Sam wrote a check for $85. The following Monday, he deposited $60 into his bank account. On Wednesday, he checked his bank's Web site and learned that he had overdrawn his account by $10. If Sam made no other transactions between Friday and Wednesday, what was his balance before he wrote the check on Friday?

5. Three-fifths of a parking lot is scheduled to be resurfaced with new asphalt. Another 200 square feet of driveway will also be resurfaced at that time. The contractor has agreed to repave 2,900 square feet total. What is the size of the parking lot in square feet? Hint: To show a large quantity in a diagram, write the number inside a rectangle.

6. Let r represent the average speed in miles per hour that a car traveled on a trip. Let d represent the distance in miles that the car had traveled t hours after the beginning of the trip. Find an equation that relates the distance traveled to the speed and the time traveled.

7. Gary has 4 more game tokens than Leo has. Angie has half as many tokens as Gary. Find an equation that relates Angie's tokens to Leo's tokens.

8. Two consecutive odd integers have a sum of 76. What are the two integers?

Activity 3
Independent Practice

Materials
Worksheet 1–3c
Regular pencils

Procedure

Give each student a copy of Worksheet 1–3c to complete independently. After all have finished, ask various students to explain the steps they used to get their answers.

Answer Key for Worksheet 1–3c

1. D
2. A
3. C
4. B
5. B

Possible Testing Errors That May Occur for This Objective

- When writing equations for several relationships in a word problem, students will omit one or more of the relationships.
- When one variable is described as twice as much as another variable, students will reverse the relationship. For example, if A should be twice as much as B, students will write the equation as $B = 2A$ instead of $A = 2B$.
- When one variable is described as some amount less than another variable, for example, X is 3 less than Y, students will write $X = 3 - Y$, instead of $X = Y - 3$.

Worksheet 1–3c

Name ____________________

Translating Situations into Linear Equations

Date ____________________

Solve the exercises provided. Be ready to share your answers and procedures with others in the class.

1. David's Grocery Mart sells 3 cans of soup for \$1.45 total. The total cost, C, of buying N cans of this same soup can be found by which of the following procedures?

   A. Multiplying N by \$1.45

   B. Dividing N by the cost of one can

   C. Dividing N by \$1.45

   D. Multiplying N by the cost of one can

2. A situation involves relationships that lead to the equation: $3m - 4 + 5m = 4m + 8$. Solve for the value of m.

   A. 3 B. 2 C. 1.5 D. 1

3. Susan has a third as many movie passes as Joe. Angie has 2 fewer passes than Joe. Together the 3 students have a total of 19 movie passes. Which equation can be used to find how many movie passes Joe has?

   A. $J + 3J + (J - 2) = 19$

   B. $\frac{1}{3}J + J + (J + 2) = 19$

   C. $\frac{1}{3}J + (J - 2) + J = 19$

   D. $3J + (J + 2) + J = 19$

4. A person of normal weight has a wrist circumference, w, equal to half of his or her neck circumference, n. Which equation best describes this relationship?

   A. $w = \frac{1}{2} + n$

   B. $w = \frac{1}{2}n$

   C. $w = n - \frac{1}{2}$

   D. $w = 2n$

5. A student needs to find three consecutive whole numbers whose sum is 72. He writes the equation: $(n - 1) + n + (n + 1) = 72$. What does the variable n represent in the equation?

   A. The greatest of the 3 numbers

   B. The middle of the 3 numbers

   C. The least of the 3 numbers

   D. None of the 3 numbers

Objective 4: Apply Number Properties to Solve Word Problems

There are various number properties that students need to master, such as common multiples and consecutive even or odd integers. They also need the logical reasoning and the language involved in working with Venn diagrams. These topics are covered in the following activities.

Activity 1

Manipulative Stage

Materials

Set of 30 small counters per pair of students
Building Mat 1–4a per pair of students
Worksheet 1–4a
Regular pencils

Procedure

1. Give each pair of students one set of counters, two copies of Worksheet 1–4a, and one copy of Building Mat 1–4a.
2. Have each pair of students use their counters to model each exercise on Worksheet 1–4a on Building Mat 1–4a. They should use the quantities found on the building mat to complete the table below the exercise.
3. For each exercise on Worksheet 1–4a, guide students to write several statements about their results on the back of the worksheet.
4. Discuss Exercise 1 with the class before allowing students to work Exercise 2 independently.

Consider Exercise 1 on Worksheet 1–4a: "Let the circles on Building Mat 1–4a represent these characteristics: Circle A, 'things that have four legs'; Circle B, 'things that eat meat'; and Circle C, 'things that climb trees.' Place counters in the mat's disjoint regions to locate the following groups: 2 rabbits, 4 squirrels, 2 cats, 1 dog, 3 caterpillars, 8-year-old Kate, Kate's hat, Grandpa who loves hot dogs, and Grandma the vegetarian. After placing the counters, find the region(s) described in each row of the table below, and record the total counters for that row. How many things were counted in all?"

Guide students to identify each group by the three characteristics in order to locate the appropriate region on Building Mat 1–4a for the necessary counters. For example, the 4 squirrels have 4 legs AND climb trees AND do NOT eat meat. So 4 counters should be placed in the region that belongs to Circle A and to Circle C, but not to Circle B. Since Grandma is a vegetarian and assuming she can no longer climb trees, a counter for her should be outside all three circles. One counter for Kate's hat will also be outside all three circles. Grandpa's counter will be in Circle B but not in Circle A or Circle C (too old to climb trees). We will assume that Kate is a good tree climber.

After all counters have been placed, students should complete the totals for the table as follows:

Four Legs	**Eats Meat**	**Climbs Trees**	**Total**
yes	yes	yes	2 cats
yes	yes	no	1 dog
yes	no	yes	4 squirrels
yes	no	no	2 rabbits
no	yes	yes	1 Kate
no	yes	no	1 Grandpa
no	no	yes	3 caterpillars
no	no	no	1 Grandma and 1 hat

Using the table or the counters on the building mat, students should find the total objects involved (without repeats). For Exercise 1, there are 16 people or things found in the disjoint regions of the building mat or the rows of the table.

Also on the back of Worksheet 1–4a, have students write sample descriptions for objects that share one characteristic, share two characteristics, or share three characteristics. Language can be confusing at this point. We are not consistent with when "having 4 legs" is intended to be exclusive (that is, "only having 4 legs" but not "eating meat" or "climbing trees"), or inclusive (that is, "having 4 legs" whether or not it "eats meat" or "climbs trees"). Generally with this type of logic problem, writers intend for the inclusive meaning to be used, and we will use that here. Here are sample descriptions and totals that students might use for the following combinations with A, B, and C:

Circle A: 2 rabbits, 4 squirrels, 2 cats, and 1 dog have four legs.
Circles A and B: 1 dog and 2 cats have 4 legs and eat meat.
Circles A, B, and C: 2 cats have 4 legs, eat meat, and climb trees.
Not Circles A, B, or C: Grandma and Kate's hat do not have 4 legs, do not eat meat, and do not climb trees.

Other combinations of characteristics are possible. For example, students might use B, and (B and C), along with (A, B, and C) and Not (A, B, or C). Remind students that Not (A, B, or C) is logically equivalent to (Not A, Not B, and Not C).

Answer Key for Worksheet 1–4a

1. Table, total, and sample descriptors are shown in the text.

2. Total = 5 + 2 + 2 + 3 + 4 + 1 + 5 = 22 students surveyed

Sample descriptors [others are possible]:

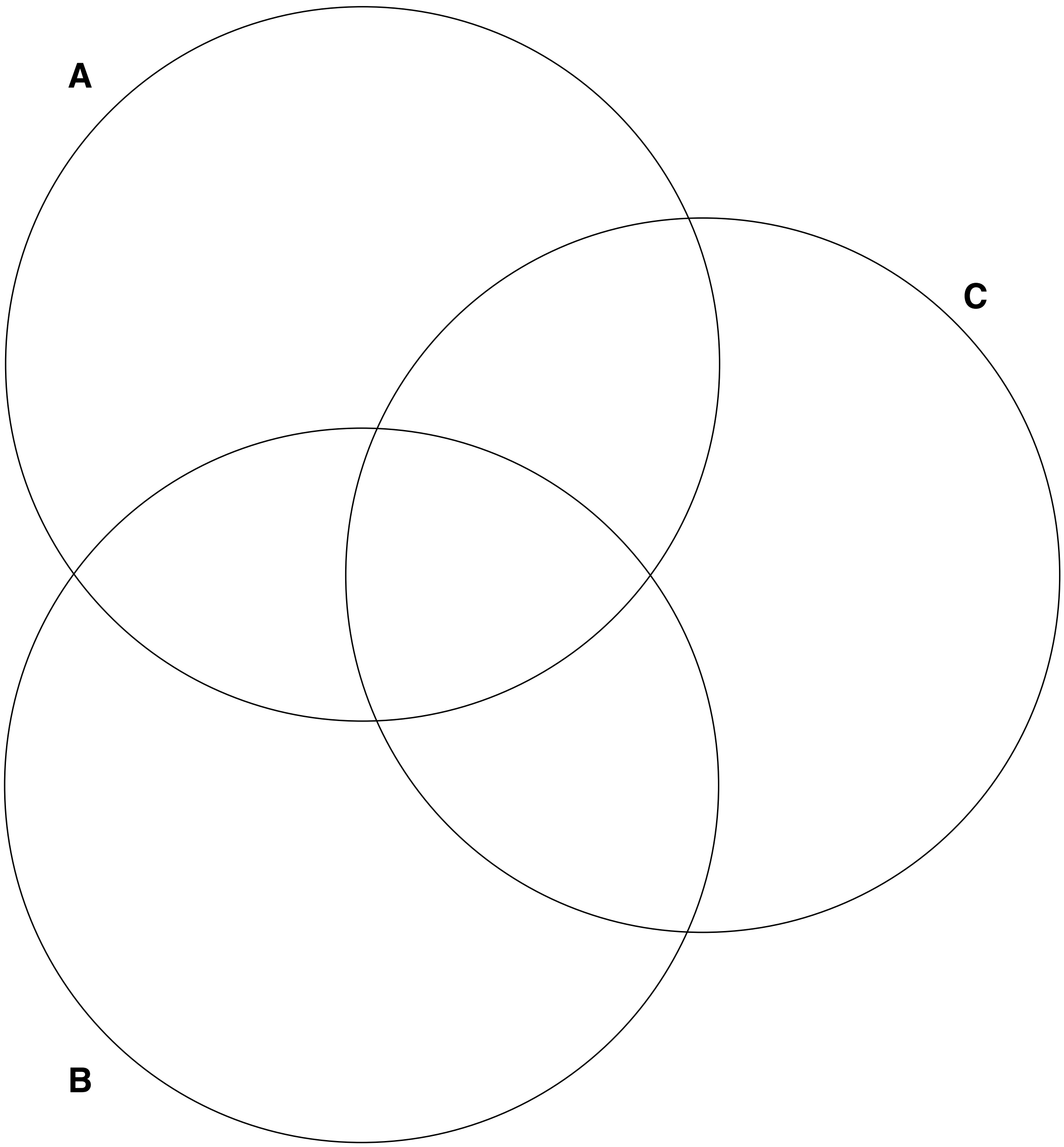
Circle C: (5 + 2 + 2 + 1) or 10 students are Service Club members.
Circles B and C: (2 + 1) or 3 students are Service Club members and music students.

Circles A, B, and C: 2 students are juniors, music students, and Service Club members.

Not Circles A, B, or C: 0 students are not juniors, not music students, and not Service Club members.

Junior Class	**Music Student**	**Service Club**	**Total**
yes	yes	yes	2
yes	yes	no	3
yes	no	yes	2
yes	no	no	5
no	yes	yes	1
no	yes	no	4
no	no	yes	5
no	no	no	0

Building Mat 1–4a



Worksheet 1–4a
Solving with Characteristics

Name ______________________
Date ______________________

Use counters on Building Mat 1–4a to solve the exercises provided.

1. Let the circles on Building Mat 1–4a represent these characteristics: Circle A, “things that have 4 legs”; Circle B, “things that eat meat”; and Circle C, “things that climb trees.” Place counters in the mat’s disjoint regions to locate the following groups: 2 rabbits, 4 squirrels, 2 cats, 1 dog, 3 caterpillars, 8-year-old Kate, Kate’s hat, Grandpa who loves hot dogs, and Grandma the vegetarian. After placing the counters, find the region(s) described in each row of the table below and record the total counters for that row. How many things were counted in all?

Four Legs	**Eats Meat**	**Climbs Trees**	**Total**
yes	yes	yes	
yes	yes	no	
yes	no	yes	
yes	no	no	
no	yes	yes	
no	yes	no	
no	no	yes	
no	no	no	

2. Repeat the steps of Exercise 1 with the following: Circle A, “junior class”; Circle B, “music students”; Circle C, “Service Club members.” The survey included 12 juniors, 10 music students, and 10 Service Club members. Use totals given in the table to place counters in the regions of Building Mat 1–4a. Find the missing totals in the table. How many students were actually in the survey?

Junior Class	**Music Student**	**Service Club**	**Total**
yes	yes	yes	2
yes	yes	no	3
yes	no	yes	
yes	no	no	
no	yes	yes	1
no	yes	no	
no	no	yes	5
no	no	no	0

Activity 2
Pictorial Stage

Materials

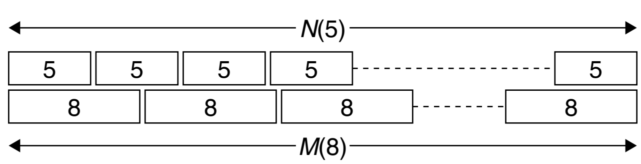
Worksheet 1–4b
Regular pencils

Procedure

1. Give each student a copy of Worksheet 1–4b. Have students work in pairs.
2. Below each exercise on Worksheet 1–4b, have students draw a diagram to represent the word problem. The diagram should help students decide which equations are needed. They should then write the equation(s) beside the diagram and solve them. It is assumed that students have the necessary skills to solve the simple equations involved in this activity.
3. When all have finished, have various students draw their diagrams on the board and discuss the different approaches used, along with the results found.
4. Discuss Exercise 1 on Worksheet 1–4b before allowing students to work the other exercises with their partners.

Consider Exercise 1 on Worksheet 1–4b: "A low brick wall is to be built along a portion of one edge of a terrace. The wall must be over 5 feet long. Five-inch-long tan bricks will form the top layer of the wall, and 8-inch-long brown bricks will form the second layer. What is the minimal length the wall must be so that the bricks in the top two layers align with each other at both ends of the wall without partial bricks being used?"

The length of the top layer of bricks must be a multiple of 5 inches since 5-inch bricks are used. The second layer must be a multiple of 8 inches since 8-inch bricks are used. For the two layers to match, the two multiples must equal each other. Have students draw the following diagram below Exercise 1 to represent the multiples involved:



The diagram leads to the equation: $N(5) = M(8)$. Students need to find values for N and M, so that $N(5) = M(8)$. One possible solution would be $N = 8$ and $M = 5$. These choices, however, would produce a wall 40 inches long, and the wall must be over 5 feet or 60 inches long. The next possible solution will be $N = 16$ and $M = 10$. Therefore, the minimal wall length will be 80 inches long. Have students record this minimal wall length beside the diagram, along with the equation and solution selected.

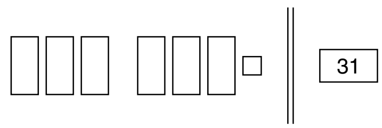
Answer Key for Worksheet 1–4b

Suggested diagrams are shown; other forms are possible.

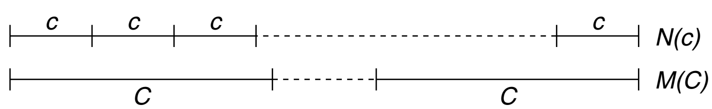
1. $N(5) = M(8)$; $N = 16$ and $M = 10$ [a possible diagram is shown in the text]

2. $3N + (3N + 1) = 31$; $N = 5$, so $3N = 15$, the fifth multiple of 3

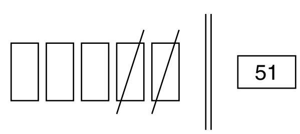
Possible diagram:



3. $N(c) = M(C)$, so $N(2)(3.14)(2) = M(2)(3.14)(7)$; $N = 7$ and $M = 2$ for the first possible solution, so the small gear will rotate 7 times while the large gear rotates 2 times before the 0-marks coincide again.



4. $5N - 2N = 51$; $N = 17$, the number needed.



Worksheet 1–4b
Solving with Diagrams

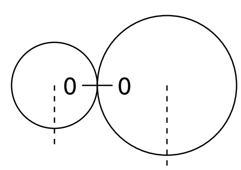
Name ______________________
Date ______________________

Draw diagrams to model the exercises provided. Then find their equations and solutions.

1. A low brick wall is to be built along a portion of one edge of a terrace. The wall must be over 5 feet long. Five-inch-long tan bricks will form the top layer of the wall, and 8-inch-long brown bricks will form the second layer. What is the minimal length the wall must be so that the bricks in the top two layers align with each other at both ends of the wall without partial bricks being used?

2. Two consecutive integers have a sum of 31. The first number is a multiple of 3. Is the first integer the fourth, fifth, or sixth multiple of 3?

3. In a machine, a small gear wheel of radius 2 centimeters turns along the rim of a large gear wheel of radius 7 centimeters. The gears are initially positioned with their 0-marks touching. How many complete revolutions will the small gear make and the large gear make before the 0-marks touch again?



4. Five times a number minus two times that same number equals 51. Find the number.

Activity 3
Independent Practice

Materials

Worksheet 1–4c
Regular pencils

Procedure

Give each student a copy of Worksheet 1–4c to complete independently. Encourage students to draw diagrams to help them set up any needed equations. When all have finished, have various students share the diagrams and equations they used to solve the different exercises.

Answer Key for Worksheet 1–4c

1. D
2. B
3. C
4. B
5. A

Possible Testing Errors That May Occur for This Objective

- When seeking the least common multiple of a set of numbers, students will find a common multiple by multiplying all the given numbers together, but will fail to find the least multiple possible.
- When using a Venn diagram to analyze the quantities resulting from a survey, students will use the quantity listed for a single characteristic in the exclusive sense rather than in the inclusive sense. That is, if 20 people are identified as "eating a hot dog," students will view them as "only eating a hot dog." The 20 people are excluded from "drinking a cola" or "eating ice cream," for example. This causes students to find the wrong total involved in the survey.
- When the sum of two consecutive even or two consecutive odd numbers is required, students will use $N + (N + 1)$ for the sum instead of $N + (N + 2)$.

Worksheet 1–4c
Solving Numerical Problems

Name ______________________
Date ______________________

Solve the exercises provided. Be ready to share your steps and your reasoning with others in the class.

1. A bag of wieners contains 10 wieners. A bag of hot dog buns contains 8 buns. If enough bags are bought so that there are equal numbers of wieners and buns, what is the least number of hot dogs that can be made that will use all the wieners and buns bought?

   A. 8 B. 10 C. 18 D. 40

2. Two positive integers have a sum of 57. The lesser number is an even number, and the difference between the two integers is 5. What is the even number?

   A. 5 B. 26 C. 31 D. 57

3. Henry and Georgia both leave campus at the same time to begin a driving marathon. Henry drives at 55 mph, and Georgia drives at 60 mph. If each driver records the total distance traveled only at full hour intervals, what is the least distance at which they both will record the same mileage?

   A. 115 mi. B. 132 mi. C. 660 mi. D. 3,300 mi.

4. Some vacationers were exiting a tour bus, so Mary and her friends decided to count them in various ways. Each looked for something different. Here are the results: 18 people wore jeans; 15 wore tennis shoes; 10 wore hats; 6, tennis shoes and hat; 9, jeans and tennis shoes; 7, hat and jeans; and 4, hat, jeans, and tennis shoes. Five people did not wear a hat, jeans, or tennis shoes. How many people actually were counted leaving the bus?

   A. 43 B. 30 C. 25 D. 18

5. Maude bought a bag of 5 dozen assorted cookies at the bakery. When she opened the bag, she found the following amounts: 10 cookies had only pecans; 10 had only raisins; 15, only cinnamon; 15, raisins and pecans; and 5, pecans and cinnamon without raisins. How many cookies had cinnamon and raisins without pecans?

   A. 5 B. 10 C. 30 D. 60

Objective 5: Apply Ratio and Proportion to Solve Numeric Problems or Problems Involving Variables

Proportional thinking allows several major ideas to be placed into a larger family of concepts. These ideas include probability, percents, and rates. For example, given the rate of 30 mph, students are typically taught to multiply the given rate by a new time to find a new corresponding distance in miles. Using the *proportion* approach, however, students may equate the ratio of 30 miles to 1 hour to the ratio of the new distance to the new time. Similarly, probability and percent problems may be set up as proportions. The concept of proportions and its applications will be developed in the following activities.

Activity 1

Manipulative Stage

Materials

Building Mat 1–5a per pair of students
Bag of small counters (2 colors, 20 counters per color) per pair of students
Worksheet 1–5a
Regular pencil

Procedure

1. Give each pair of students two copies of Worksheet 1–5a, a copy of Building Mat 1–5a, and a bag of small counters (2 colors, 20 counters per color). The top level on the mat will be for the *basic ratio* (a ratio that uses the smallest whole numbers possible), and the bottom level on the mat will be for the *secondary ratio* (the larger amounts formed by multiple amounts of the basic ratio's numbers, each amount being arranged as an array). One color of counter will be used to show the first amount in each ratio, and the second color of counter will show the second amount in each ratio.
2. Have students build basic ratios and their corresponding secondary ratios on Building Mat 1–5a with the counters, using the exercises on Worksheet 1–5a.
3. After students complete each exercise, have them write a word sentence below the exercise that describes their results.
4. Discuss Exercise 1 on Worksheet 1–5a before allowing students to work independently.

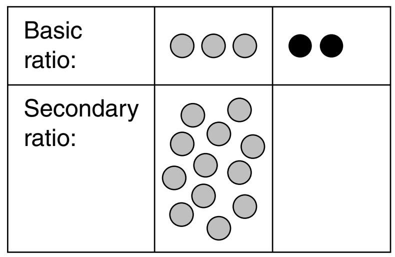
Consider Exercise 1 from Worksheet 1–5a: "At the store, 3 mechanical pencils cost $2. What will 12 pencils cost?"

Have students place 3 color #1 counters (the 3 pencils) in a row in the left empty region of the top level of Building Mat 1–5a and 2 color #2 counters (the $2 price for the 3 pencils) in the right empty region of the top level. The pencils will be represented in the left region since they were mentioned first, before the cost in the exercise. Now have the students randomly place 12 color #1 counters in the left empty region of the lower level of the mat.

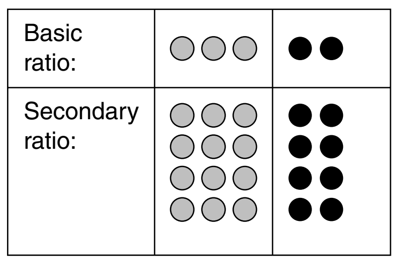
Students must find how many color #2 counters go in the lower right region of Building Mat 1–5a. Since the secondary ratio consists of repeats of the basic ratio, the color #1 counters in the lower left region must be rearranged into an array having 3 counters per row to match the row of color #1 counters in the top left region; 4 rows will be formed. The 4 rows indicate that 4 of the basic ratio have been used to make the secondary ratio. Thus, 4 rows of color #2 counters must be used in the lower right region of the mat.

Since two color #2 counters have been used in the basic ratio, two color #2 counters must be used in each of the four rows in the secondary ratio. The final mat arrangement yields 8 color #2 counters in the right region of the secondary ratio. This indicates that 12 pencils will cost $8. Have students record the results on Worksheet 1–5a below Exercise 1 as follows: "3 pencils compare to $2 like 12 pencils compare to $8. So 12 pencils cost $8." The initial and final stages of mat work are shown below. The final completed building mat represents a proportion (two equivalent ratios):

Initial Mat:



Final Mat:



There are four regions to fill on Building Mat 1–5a. The word problems in the exercises of Worksheet 1–5a will vary, so that different regions need to be filled. In Exercise 1, the lower right region was needed; that is, the value for that region was the unknown for the problem. Whenever one region is needed, the numbers for the other 3 regions must be given in the word problem. The placement in the left and right regions of the building mat should follow the initial order given for the ratios in each exercise. For example, if the exercise states "3 girls for every 5 boys," then "3 girls" should be shown in the upper left region and "5 boys" in the upper right region.

Answer Key for Worksheet 1–5a
Here are possible sentences to use.

1. 3 pencils compare to $2 like 12 pencils compare to $8. So 12 pencils cost $8.
2. 4 people compare to 3 dogs like 16 people compare to 12 dogs. So 16 people attended the dog show.
3. 12 girls compare to 15 boys like 4 girls compare to 5 boys. There are 5 boys in the class for every 4 girls.
4. 1 red marble compares to 3 blue marbles like 3 red marbles compare to 9 blue marbles. So there are 3 red and 9 blue marbles in 12 marbles total.
5. 12 cans compare to $16 like 3 cans compare to $4. The basic ratio is 3 cans for $4.
6. 5 miles compare to 1 hour like 15 miles compare to 3 hours. It will take 3 hours to ride 15 miles on the bike.

Building Mat 1–5a

Basic ratio:		
Secondary ratio:		

Worksheet 1–5a
Building Ratios

Name ____________________
Date ____________________

Solve the word problems provided by placing counters on Building Mat 1–5a. Below each exercise, write a word sentence that states as a proportion the results found for that exercise.

1. At the store, 3 mechanical pencils cost $2. What will 12 pencils cost?

2. At the dog show, there are 4 people for every 3 dogs. If there are 12 dogs in the show, how many people are present at the show?

3. Mr. Jordan's eighth-grade class has 12 girls and 15 boys. How many boys are in the class for every 4 girls?

4. There is 1 red marble for every 3 blue marbles in a box. If there are 12 marbles total in the box and each marble is either red or blue, how many marbles are red and how many are blue?

5. Jan bought 12 cans of stew for $16. What is the basic ratio of cans to dollars?

6. If George rides his bike at 5 mph, how many hours will it take him to ride 15 miles if his speed remains constant?

Activity 2
Pictorial Stage

Materials

Worksheet 1–5b
Red pencil and regular pencil

Procedure

1. Give each student a copy of Worksheet 1–5b and a red pencil.
2. Students should complete Exercises 1 to 4 by drawing small shapes in the ratio frames. The finished ratio frames should look much like the completed building mat illustrated in the Manipulative Stage. Because of larger numbers used in Exercise 4, students might draw rectangles and write a value inside each rectangle instead of using unit counters.
3. Beside each completed ratio frame, have students record equations that describe the proportion represented. A variable will be recorded for the missing part of the proportion. It is assumed that students can solve simple equations like $2N = 16$. The discussion of Exercise 1 will include the types of equations to use.
4. For Exercises 5 and 6, students will graph on a grid several ratio pairs that come from the same situation. On the grid, the points for the ratio pairs will then be connected in red pencil. In each case, a red line should be formed, which indicates that the proportional situations are linear relationships.
5. Discuss Exercise 1 on Worksheet 1–5b with the class before allowing students to finish the worksheet independently. Also review graphing techniques before having them work Exercises 5 and 6.

Consider Exercise 1: "There are 15 cats and 9 dogs at the pet shop. How many cats are there for every 3 dogs?"

Students should draw circles on the ratio frame to represent the cats and triangles to represent the dogs. At first, 3 triangles will be drawn in the upper right region of the ratio frame to represent the 3 dogs. The upper right region is used since "dogs" was stated second in the initial ratio of *cats* to *dogs*. Having "3 triangles in one row" then determines the number <u>per row</u> in the lower right region. This requires 3 <u>rows</u> of triangles to be drawn in order to show the 9 dogs total. Remind students that rows in an array are left to right, not up and down.

Students should now be aware that 3 <u>rows</u> of circles will have to be drawn in the lower left region of the frame when representing all the cats. Since 15 circles must be drawn to represent the 15 cats total and these circles must be drawn in 3 rows, this forces 5 circles to be in each row. In turn, this causes 5 circles to be drawn in the upper left region of the ratio frame. The initial and final stages of the ratio frame are shown here.

Initial Frame:

Basic ratio:		△△△
Secondary ratio:		△△△ △△△ △△△

Final Frame:

Basic ratio:	○○○○○	△△△
Secondary ratio:	○○○○○ ○○○○○ ○○○○○	△△△ △△△ △△△

Have students record their results beside the ratio frame that is now completed. Be sure that the two ratios keep the same order, that is, "cats to dogs." Have the students use the following formats where N represents the unknown amount of cats in the basic ratio:

15 cats to 9 dogs = N cats to 3 dogs

$\frac{N}{3} = \frac{3\,\text{rows} \times N}{3\,\text{rows} \times 3} = \frac{15}{9}$, so $3N = 15$ cats and $N = 5$ cats.

Notice that $\frac{15}{9}$ represents the secondary ratio, and $\frac{N}{3}$ represents the basic ratio, with N as the part to be found (read as "15 to 9" and "N to 3"). The left factor in the numerator and in the denominator of the expression, $\frac{3 \times N}{3 \times 3}$, indicates the number of <u>rows</u> that were drawn in each region of the secondary ratio. Since each of the denominators equals 9, students only need to compare the numerators, $15 = 3N$, in order to solve for N as 5 cats. Discuss the idea that finding equivalent ratios in a proportion is similar to finding equivalent fractions.

In preparation for Exercises 5 and 6 on Worksheet 1–5b, review the graphing of ordered pairs. Also discuss how to determine the step or interval sizes for the horizontal and vertical scales of a grid. In each of the two exercises, the three ordered pairs plotted will have collinear points. The red path connecting the three points will be straight. Equivalent ratios have a linear relationship. After students have completed Exercises 5 and 6, help them make these observations.

Answer Key for Worksheet 1–5b

Suggested equations are shown; other equivalent forms are possible.

1. 15 cats to 9 dogs = N cats to 3 dogs

   $\frac{N}{3} = \frac{3 \times N}{3 \times 3} = \frac{15}{9}$, so $3N = 15$ and $N = 5$ cats.

2. \$4 to 1 hour = \$20 to N hours

   $\frac{\$4}{1} = \frac{5 \times \$4}{5 \times 1} = \frac{\$20}{N}$ So, $N = 5 \times 1$, or 5 hours.

3. 2 blue to 5 total marbles = N blue to 20 total marbles

   $\frac{2}{5} = \frac{4 \times 2}{4 \times 5} = \frac{N \text{ blue marbles}}{20 \text{ total marbles}}$ So, $N = 4 \times 2$, or 8 blue marbles.

4. \$3 discount to N% discount = \$15 total price to 100% total

   $\frac{\$3}{N\%} = \frac{5 \times \$3}{5 \times N\%} = \frac{\$15}{100\%}$ So, $5 \times N\% = 100\%$ and $N\% = 20\%$ discount off total price.

Basic ratio:	\$3	20%
Secondary ratio:	\$3 \$3 \$3 \$3 \$3	20% 20% 20% 20% 20%

5. Ordered pairs graphed: (1 hour, \$4), (5 hours, \$20), (7 hours, \$28); connecting red path is straight

6. Ordered pairs graphed: (5 total, 2 blue), (15 total, 6 blue), (20 total, 8 blue); connecting red path is straight

Worksheet 1–5b
Drawing and Graphing Ratios

Name ______________________
Date ______________________

Solve the word problems by drawing shapes on a ratio frame provided. Beside Exercises 1 to 4, write equations that state as proportions the results found for each exercise. Solve for each variable or unknown part. For Exercises 5 and 6, graph ordered pairs for ratios on the grids as directed.

1. There are 15 cats and 9 dogs at the pet shop. How many cats are there for every 3 dogs?

Basic ratio:		
Secondary ratio:		

2. Lynn earns $4 per hour at her weekend job. How many hours must she work to earn $20?

Basic ratio:		
Secondary ratio:		

3. There are 2 blue marbles for every 5 marbles in a jar. If there are 20 marbles total, how many marbles are blue?

Basic ratio:		
Secondary ratio:		

Worksheet 1–5b Continued

Name ______________________

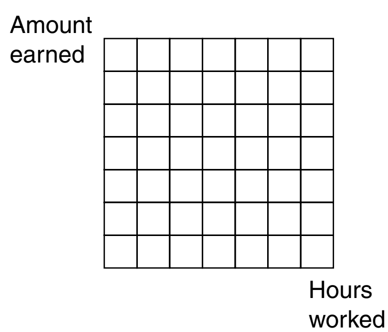
Date ______________________

4. Mario bought a music CD for $3 off the original price of $15. What was the percent of discount on the CD? Hint: Use "discount amount to discount %" for the basic ratio, and "total price to total %" for the secondary ratio.

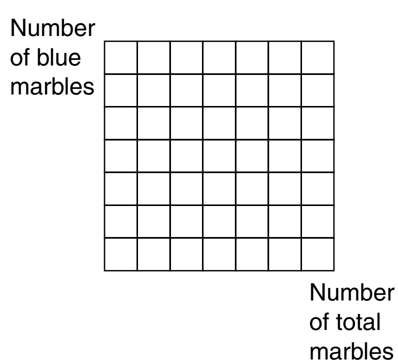
Basic ratio:		
Secondary ratio:		

On each grid for Exercises 5 and 6, plot points for the three ratios found, and draw a path in red pencil to connect the three points. What do you notice about the red path drawn on each grid?

5. Find another secondary ratio for Exercise 2, using 7 hours for the time. Then plot points for the 3 ratios found, using (hours worked, amount earned) as the ordered pair. Number the grid axes as needed.



6. Find another secondary ratio for Exercise 3, using 15 marbles total. Then plot points for the 3 ratios found, using (# total, # blue) as the ordered pair. Number the grid axes as needed.



Activity 3
Independent Practice

Materials

Worksheet 1–5c
Regular pencil

Procedure

Give each student a copy of Worksheet 1–5c. Encourage students to set up proportions similar to those equations used in Exercises 1 to 4 on Worksheet 1–5b of Activity 2. For example, in Exercise 3 of Worksheet 1–5c, they might write the following:

$$\frac{13}{20} = \frac{5 \times 13}{5 \times 20} = \frac{N}{100}, \text{ where } N \text{ equals 65 shots made.}$$

Since N is compared to 100, finding N as 65 is equivalent to finding that 65% of the attempted shots were successful. Although Exercises 3 and 7 involve percents, students should focus on the proportion method to solve the problems and not on an alternative method often used for percents.

Also, in problems like Exercise 7, students may have difficulty finding the correct factor to use to form the new or equivalent ratio. In such cases, simple integral factors do not work. Show students how to construct the factor in the following way. The proportion needed is $\frac{\$24}{N} = \frac{40}{100}$, where the numerators are the dollars and the percent amount for the *discount*. The denominators N and 100 represent the dollars and percent amount for the *total* cost or regular price. The numerator and denominator of one ratio should multiply by the same factor to produce the numerator and denominator of the other ratio. To construct the factor, consider the two numerators with known values. How can 40 be changed to \$24? A factor can be constructed so that 40 is divided out and \$24 is brought in; that is, $\left(\frac{\$24}{40}\right)$ becomes the chosen factor. Then $40 \times \left(\frac{\$24}{40}\right)$ will equal \$24. Then the denominators must use the same factor: $100 \times \left(\frac{\$24}{40}\right)$ must equal N. Students should not *compute* $\frac{\$24}{40}$ until this last equation is set up. So they will have $100 \times \left(\frac{\$24}{40}\right) = \$60$ for N, which is the regular price (total cost) of the jacket.

Answer Key for Worksheet 1–5c

1. B
2. D
3. A
4. B
5. C
6. D
7. B
8. C

Possible Testing Errors That May Occur for This Objective

- Students multiply the numerator of the initial ratio by one factor and the denominator by another factor when changing the initial ratio to an equivalent ratio.
- The initial ratio's numerator and denominator are changed to the new ratio's numerator and denominator by *adding* a constant instead of *multiplying* by a constant. For example, to compare $\frac{3}{8}$ to $\frac{N}{12}$ to find N, students incorrectly use $\frac{(3+4)}{(8+4)} = \frac{7}{12}$ instead of $\frac{(1.5\times 3)}{(1.5\times 8)} = \frac{4.5}{12}$.
- Students do not maintain the same <u>order</u> in the two ratios being compared in a proportion. For example, they will equate cats/dogs to dogs/cats, instead of using cats/dogs = cats/dogs.

Worksheet 1–5c

Using Ratios to Solve Word Problems

Name ____________________

Date ____________________

Solve the word problems, using proportions.

1. On the first day of his vacation, Thomas counted the car license plates he saw from different states. Of the 80 plates he counted, 45 were from Ohio, 23 were from Illinois, and 12 were from other states. If he sees 160 license plates on his return trip home, how many of these could he expect to be from Illinois?

   A. 24 B. 46 C. 80 D. 90

2. The Disco Shop is selling CDs at $7.50 per package of 3 CDs. What will it cost to purchase 12 CDs?

   A. $7.50 B. $15 C. $22.50 D. $30

3. Carrie made 13 out of 20 shot attempts during a recent basketball game. To find the percent of shots that Carrie made, how many successful shots out of 100 attempts would be equivalent to her score?

   A. 65 B. 72 C. 78 D. 85

4. A car is traveling at an average speed of 62 miles per hour. At this rate of speed, which is the best estimate for how long it will take the car to travel 356 miles?

   A. 5 hours B. 6 hours C. 7 hours D. 8 hours

5. A box contains 24 colored cubes. There are 6 blue cubes, 3 red cubes, and 7 yellow cubes. The rest of the cubes are green. The ratio of green cubes to total cubes is equivalent to which probability for drawing out a green cube at random?

   A. $\frac{1}{8}$ B. $\frac{1}{4}$ C. $\frac{1}{3}$ D. $\frac{7}{24}$

Worksheet 1–5c Continued

Name ____________________

Date ____________________

6. Leo reported the following data on his time sheet at work: (2 hours, \$10), (5 hours, \$25), (8 hours, \$40), (7 hrs, \$30), and (4 hours, \$20). If his hourly rate was constant, which data pair was incorrect?

   A. (2, 10) B. (5, 25) C. (8, 40) D. (7, 30)

7. Myshondi saved \$24 when she bought a jacket on sale. If the discount was 40% of the regular price, what was the regular price of the jacket?

   A. \$52 B. \$60 C. \$75 D. \$80

8. A package of three Brand B batteries costs \$4.95, and a package of three Brand C batteries costs \$3.75. Which proportion can be used to determine the percent, *N*, of the cost of the Brand B batteries that will be saved if the Brand C batteries are purchased?

   A. $\frac{(4.95-3.75)}{3.75}=\frac{N}{100}$ C. $\frac{(4.95-3.75)}{4.95}=\frac{N}{100}$

   B. $\frac{3.75}{4.95}=\frac{N}{100}$ D. $\frac{4.95}{3.75}=\frac{N}{100}$

Objective 6: Identify Two Given Linear Equations, Given in Algebraic Form or Graphic Form, as Parallel, Perpendicular, or with the Same *x*- or *y*-Intercept

Students need experience with certain properties of lines and should be able to recognize these properties in the algebraic equations of the lines. Of particular interest are the parallel and perpendicular properties of a pair of lines, along with their possible sharing of an intercept point. For the following activities, it will be assumed that students are familiar with the $y = mx + b$ format for a line, even though their understanding may be somewhat limited.

Activity 1

Manipulative Stage

Materials

Worksheet 1–6a
Building Mat 1–6a
Rulers or straightedges
Index cards (3 inches by 5 inches; any color but white)
Regular pencils

Procedure

1. Give each pair of students two copies of Worksheet 1–6a, one copy of Building Mat 1–6a, a ruler, and one colored index card. The lower left corner of the building mat will be considered the origin of the grid.
2. For each exercise on Worksheet 1–6a, students must use the measuring edge of the ruler to locate line A on Building Mat 1–6a. The ruler's edge must pass through the points of the two ordered pairs given for line A. Students should identify the vertical change and the horizontal change as they move from one point to the other point given for line A, then record these changes as the ratio for line A's slope on Worksheet 1–6a. The ratio may be simplified, but should be kept in *a/b* format.
3. While holding the ruler in place on the grid, students should align the short edge of the index card with the ruler's edge and slide the card along the ruler's edge until the card's <u>long</u> edge touches the points given for line B. As with line A, students should find the vertical change and the horizontal change between the two points, then record the slope for line B on Worksheet 1–6a. Again, simplify the ratio if necessary, but keep the *a/b* format.
4. Have students estimate what the *y*-intercept might be for each line and write an equation for each line on the worksheet, using the $y = mx + b$ format.
5. Because of the alignment of the index card with the ruler, lines A and B in each exercise will be perpendicular to each other. Guide students to realize that line A's slope is the inverse or opposite of the reciprocal of line B's slope. Have students write a statement on the back of Worksheet 1–6a about two perpendicular lines and the relationship between their slopes.
6. Discuss Exercise 1 on Worksheet 1–6a with the class before allowing partners to work the exercises on their own.

Consider Exercise 1 on Worksheet 1–6a:

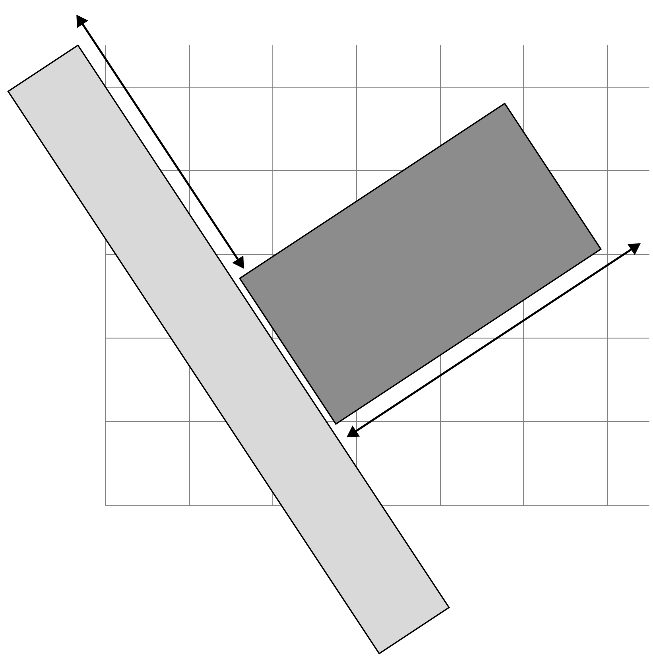
Line A: (0,5) and (2,2); slope = ___________; equation: ________________________________

Line B: (3,1) and (6,3); slope = ___________; equation: ________________________________

Have students position a ruler on Building Mat 1–6a so that its edge passes through the points for (0,5) and (2,2) to represent line A. There will be a vertical change of –3 and a horizontal change of +2, so students should record a slope of $\frac{-3}{+2}$ for line A on Worksheet 1–6a. Since (0,5) is the actual y-intercept, have students record the equation for line A as $y = \frac{-3}{+2}x + 5$, following the $y = mx + b$ format.

Holding the ruler in place on the grid, students should align the short edge of the index card with the ruler, then slide the card along the ruler's edge until the long edge of the card passes through the points for (3,1) and (6,3). This new position of the card's long edge will represent line B. The vertical change will be +2, and the horizontal change will be +3, so a slope of $\frac{+2}{+3}$ should be recorded for line B. Using the slope to extend to new points off the grid, the y-intercept is found to be at (0, –1). Students should now record the equation for line B on the worksheet as $y = \frac{+2}{+3}x + (-1)$.

Discuss the idea that the lines represented by the ruler's edge and the index card's long edge are perpendicular to each other (that is, they form a 90 degree angle with each other). Guide students to compare the slopes of line A and line B and to notice that each slope is the inverse of the reciprocal of the other slope. For this comparison, it is helpful to show the signs on the numerator and the denominator of each slope ratio. The ruler and the index card might be arranged as follows on the grid, with the bold arrows indicating the two lines being represented by the ruler's edge and the card's long edge:



When students have completed all four exercises, have them write a statement on the back of Worksheet 1–6a that describes two perpendicular lines and the special relationship between their slopes. Note that the y-intercepts of perpendicular lines may or may not be equal. The lines will share the same y-intercept if the two lines intersect on the y-axis.

Answer Key for Worksheet 1–6a

1. [The answer is given in the text.]
2. Line A: slope = $\frac{+4}{+2} = \frac{+2}{+1}$; $y = \frac{+2}{+1}x + 0$

   Line B: slope = $\frac{-2}{+4} = \frac{-1}{+2}$; $y = \frac{-1}{+2}x + 5$
3. Line A: slope = $\frac{-1}{+1}$; $y = \frac{-1}{+1}x + 5$

   Line B: slope = $\frac{+1}{+1}$; $y = \frac{+1}{+1}x + (-1)$
4. Line A: slope = $\frac{+3}{+1}$; $y = \frac{+3}{+1}x + 3$

   Line B: slope = $\frac{-1}{+3}$; $y = \frac{-1}{+3}x + 5\frac{1}{3}$

Building Mat 1–6a

Worksheet 1–6a

Name ______________________

Comparing Slopes of Perpendicular Lines

Date ______________________

For each exercise, use a ruler to locate line A on Building Mat 1–6a and use a colored index card to locate line B. Find their slopes and compare them. Estimate the y-intercepts of the two lines. Then write equations for the lines, using the $y = mx + b$ format.

1. Line A: (0, 5) and (2, 2); slope = ____________; equation: ____________

   Line B: (3, 1) and (6, 3); slope = ____________; equation: ____________

2. Line A: (1, 2) and (3, 6); slope = ____________; equation: ____________

   Line B: (2, 4) and (6, 2); slope = ____________; equation: ____________

3. Line A: (1, 4) and (2, 3); slope = ____________; equation: ____________

   Line B: (4, 3) and (5, 4); slope = ____________; equation: ____________

4. Line A: (0, 3) and (1, 6); slope = ____________; equation: ____________

   Line B: (1, 5) and (4, 4); slope = ____________; equation: ____________

Activity 2
Pictorial Stage

Materials

Worksheet 1–6b
Worksheet 1–6c
Rulers or straightedges
Colored pencils (2 bright colors; not yellow)
Regular pencils

Procedure

1. Give each pair of students 4 copies of Worksheet 1–6b, 2 copies of Worksheet 1–6c, a ruler or straightedge, and 2 colored pencils (2 different colors). On each grid, students should position the origin in the center of the grid and draw the horizontal and vertical axes.

2. Each exercise on Worksheet 1–6c should be drawn on its own copy of the grid on Worksheet 1–6b. For each exercise, have students draw line A through its two given points, using pencil color #1 on the grid. Then they should translate A's two points as directed to find two new points for line B. Line B should be drawn through its two points with pencil color #2 on the same grid.

3. After line A and line B are drawn, have students measure the perpendicular distance between the two lines at several different locations to confirm that the two lines are the same distance apart, no matter where the measurement is made. This indicates that the two lines are parallel to each other.

4. Then for each line of the same exercise, have students find the slope and y-intercept and write an equation for the line, using the $y = mx + b$ format. This information should be recorded below the exercise on Worksheet 1–6c.

5. After all exercises are completed, discuss the idea that in each exercise, line A and line B have the same slope but different y-intercepts. These are properties of parallel lines. Have students write a statement on the back of Worksheet 1–6c about parallel lines and the relationship between their slopes.

6. Discuss Exercise 1 on Worksheet 1–6c with the class before allowing partners to work the other exercises on their own.

Consider Exercise 1 on Worksheet 1–6c:

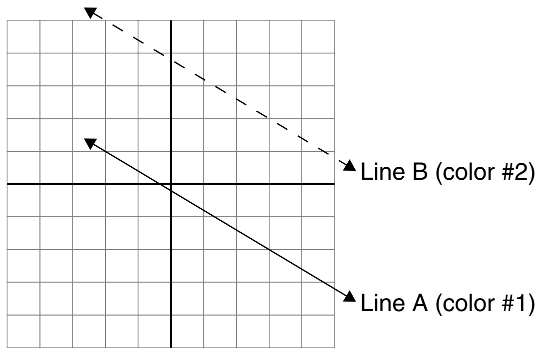
Line A: (–2, 1) and (3, –2); slope = ___________; equation: _____________________________

Translation: Move vertically by +4.

Line B: (___, ___) and (___, ___); slope = _________; equation: ___________________________

Have students number a copy of Worksheet 1–6b as grid #1 to correspond to Exercise 1, then draw and label a pair of coordinate axes, placing the origin in the center of the grid. In pencil color #1, have them locate the two points (–2, 1) and (3, –2) given for line A, then draw line A through the points on the grid.

The two points given for line A should be translated according to the given rule: Move vertically by +4. This means that the y-value of each ordered pair should be increased by 4. So (–2, 1) becomes (–2, 5), and (3, –2) becomes (3, 2). The two new points should be recorded on Worksheet 1–6c for line B. Students should use pencil color #2 to locate the two new points on grid #1 and draw a line through those points to show line B. Here is a graph of line A and line B shown on a partial grid #1:



Using the ruler, students should measure the perpendicular distance between line A and line B at various locations to confirm that the two lines remain the same distance apart at all points. The actual numerical measurement is not important here. In fact, in lieu of a ruler, the edge of an index card might be used to mark off the perpendicular distance between the two lines. Then the marked-off length might be used to test other positions along the two lines.

After graphing the two lines, students should find the vertical and horizontal changes between each line's two stated points and compute the slope, or estimate the changes directly from the graph of the line. For line A, the slope will be $\frac{(-2)-1}{3-(-2)}=\frac{-3}{+5}$, and for line B, the slope will be $\frac{2-5}{3-(-2)}=\frac{-3}{+5}$. Other equivalent forms are possible. For later comparisons, it is helpful to students to continue to show the signs in both the numerator and the denominator of the slope ratio. The slopes should be recorded in the appropriate blanks below Exercise 1 on Worksheet 1–6c.

In order to find the equation for each line, students need to estimate the y-intercept graphically or use substitution of the slope and one point of the line in $y = mx + b$ to find the value of b. Practice with both methods is valuable. For line A, a graphical estimate for b might be 0, and for line B, it might be +4. If substitution is used, a more accurate value for line A's y-intercept is $\frac{-1}{5}$, and for line B, it is $3\frac{4}{5}$. Students should record the equations for either or both methods below Exercise 1 on Worksheet 1–6c. The equations will be as follows: (A) $y=\frac{-3}{+5}x+\left(\frac{-1}{5}\right)$, or $y=\frac{-3}{+5}x+0$; and (B) $y=\frac{-3}{+5}x+3\frac{4}{5}$, or $y=\frac{-3}{+5}x+4$.

When students have completed all four exercises, guide them to notice that each pair of parallel lines has the same slope, but their y-intercepts differ. Have students write a statement on the back of Worksheet 1–6c that describes two parallel lines and the special relationship between their slopes. They should also note that the y-intercepts of distinct parallel lines will not be equal.

Answer Key for Worksheet 1–6c

Slope ratios and y-intercepts are shown as initially computed; equivalent forms are possible; estimates of y-intercepts may also be used.

1. Line A: (–2, 1) and (3, –2); $m = \frac{-3}{+5}$; $y = \frac{-3}{+5}x + \left(\frac{-1}{5}\right)$

   Line B: (–2, 5) and (3, 2); $m = \frac{-3}{+5}$; $y = \frac{-3}{+5}x + 3\frac{4}{5}$

2. Line A: (–3, –2) and (4, –2); $m = \frac{0}{+7} = 0$; $y = 0x + (-2)$

   Line B: (–3, –5) and (4, –5); $m = \frac{0}{+7} = 0$; $y = 0x + (-5)$

3. Line A: (0, 0) and (–5, –4); $m = \frac{-4}{-5}$; $y = \frac{-4}{-5}x + 0$

   Line B: (3, 0) and (–2, –4); $m = \frac{-4}{-5}$; $y = \frac{-4}{-5}x + \left(-2\frac{2}{5}\right)$

4. Line A: (5, 1) and (–5, –1); $m = \frac{-2}{-10} = \frac{-1}{-5}$; $y = \frac{-1}{-5}x + 0$

   Line B: (4, –1) and (–6, –3); $m = \frac{-2}{-10} = \frac{-1}{-5}$; $y = \frac{-1}{-5}x + \left(-1\frac{4}{5}\right)$

Worksheet 1–6b

Name ____________________

Date ____________________

Worksheet 1–6c Name ____________________

Comparing Slopes of Parallel Lines Date ____________________

For each exercise, use a ruler to draw line A in color #1 on a copy of Worksheet 1–6b. Translate A's points to find line B's points; then draw line B in color #2 on the same grid. Measure the perpendicular distance between the two lines at several locations. Find the slopes, and find or estimate the y-intercepts of the two lines. Then write equations for the lines, using the $y = mx + b$ format.

1. Line A: (–2, 1) and (3, –2); slope = ____________; equation: ____________________

   Translation: Move vertically by +4.

   Line B: (__, __) and (__, __); slope = _________; equation: ____________________

2. Line A: (–3, –2) and (4, –2); slope = ____________; equation: ____________________

   Translation: Move vertically by –3.

   Line B: (__, __) and (__, __); slope = _________; equation: ____________________

3. Line A: (0, 0) and (–5, –4); slope = ____________; equation: ____________________

   Translation: Move horizontally by +3.

   Line B: (__, __) and (__, __); slope = _________; equation: ____________________

4. Line A: (5, 1) and (–5, –1); slope = ____________; equation: ____________________

   Translation: Move vertically by –2 and horizontally by –1.

   Line B: (__, __) and (__, __); slope = _________; equation: ____________________

Activity 3
Independent Practice

Materials
Worksheet 1–6d
Regular pencils

Procedure
Give each student a copy of Worksheet 1–6d. After all have completed the worksheet, ask various students to explain their methods and their answers for the different exercises.

Answer Key for Worksheet 1–6d

1. B
2. C
3. A
4. B
5. D
6. C
7. A

Possible Testing Errors That May Occur for This Objective
- Students do not understand the role of the slope when identifying pairs of lines as parallel or perpendicular, so they use the same slope for both lines or randomly select slopes that are not related in any way when asked to find equations for such lines.
- When finding the slopes needed to produce two perpendicular lines, students will find reciprocals but fail to use inverses or opposites of the reciprocals. For example, they will use +3 and $\frac{+1}{3}$ for the required slopes, instead of using +3 and $\frac{-1}{3}$, or –3 and $\frac{+1}{3}$.
- When asked to match a pair of equations to their graphs, students incorrectly find or graph ordered pairs for the equations, thereby matching to the wrong pair of lines.

Worksheet 1–6d

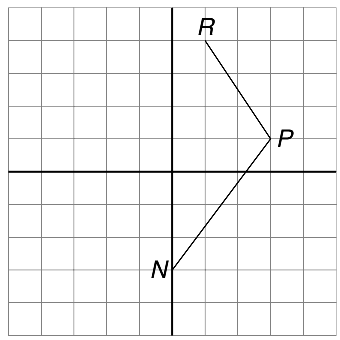
Applications of Parallel and Perpendicular Lines

Name ______________________

Date ______________________

Solve the exercises. Be ready to discuss your answers with the entire class.

1. A portion of trapezoid NPRT is shown on the grid. Through what coordinates should line RT be drawn to make side NP parallel to side RT in the trapezoid?



A. (–2, 3) C. (0, 2)

B. (–2, 0) D. (–1, –1)

2. Which equation describes the line that passes through the point (1,3) and is parallel to the line represented by the equation $-2x + y = -5$?

A. $y = -2x + 1$ B. $y = \frac{1}{2}x - 5$ C. $y = 2x + 1$ D. $y = -\frac{1}{2}x + 3$

3. What is the slope of a line that is perpendicular to the line having the equation $2x + 3y = 15$?

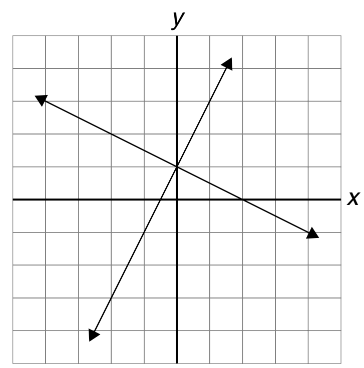
A. $\frac{+3}{+2}$ B. +5 C. $\frac{-2}{+3}$ D. –5

Worksheet 1–6d Continued

Name ______________________

Date ______________________

4. Which pair of equations represents the perpendicular lines shown on the graph?



A. $y = 4x + 1$ and $y = -4x + 1$

B. $y = -\frac{1}{2}x + 1$ and $y = 2x + 1$

C. $y = -x + 2$ and $y = x - \frac{1}{2}$

D. $y = 2x$ and $y = -\frac{1}{2}x$

5. Given the function $y = 2.6x - 35.4$, which statement best describes the graphical effect of increasing the y-intercept by 18.6, but making no other changes in the function?

A. The new line is perpendicular to the original.

B. The new line has a greater rate of change.

C. The x-intercept increases.

D. The new line is parallel to the original.

6. Which of the following best describes the graph of the two equations $4y = 2x - 4$ and $2y = -3x + 6$?

A. The lines have the same y-intercept.

B. The lines are perpendicular.

C. The lines have the same x-intercept.

D. The lines are parallel.

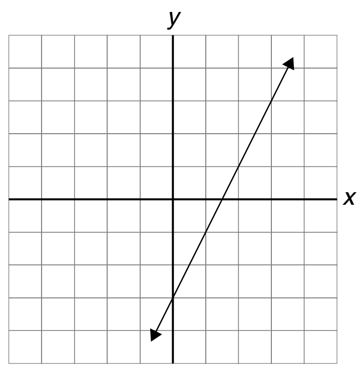
Worksheet 1–6d Continued

Name ____________________

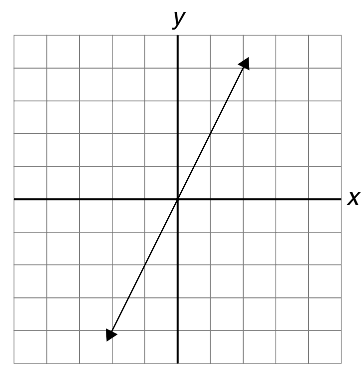
Date ____________________

7. Which graph best represents the line passing through the point (0, –3) and perpendicular to $y = -\frac{1}{2}x$?

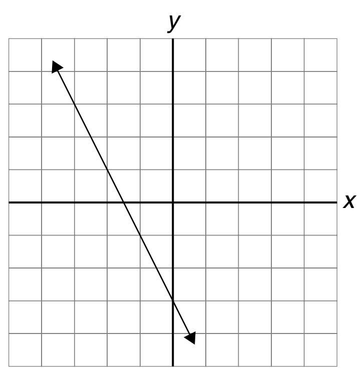
A.



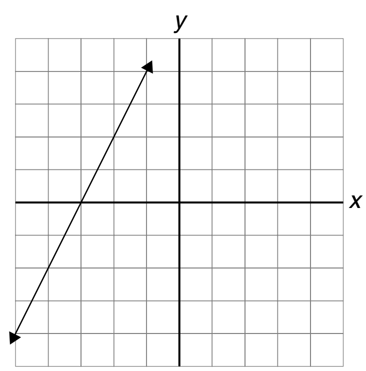
C.



B.



D.



Objective 7: Identify Algebraic Equations or Graphs Needed to Represent a System of Linear Equations Described in a Given Situation; Solve the System by a Substitution Method or a Graphical Method

Systems of equations are difficult for students. They do not understand that two different conditions or relationships are existing concurrently for the same situation. Several methods are available for solving such systems, but the following activities will focus on the substitution method and the graphical method, which seem to be more intuitive for students. It is assumed that students have already mastered the solving of linear equations in one variable, and that in equations having two variables they are able to solve for one variable in terms of the other variable.

Activity 1

Manipulative Stage

Materials

Sets of algebra tiles per pair of students (see step 1 under "Procedure" for details)
Building Mat 1–7a
Worksheet 1–7a
Regular pencils and paper

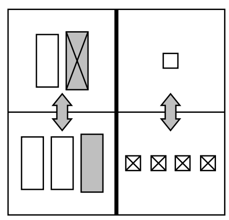
Procedure

1. Give each pair of students a set of algebra tiles, one copy of Building Mat 1–7a (system building mat), and two copies of Worksheet 1–7a. The set of tiles should contain 30 unit tiles, along with 12 variable tiles in each of two different sizes or colors. Consider the shorter variable tile as variable A and the longer variable tile as variable B for notation purposes. If teacher-made tiles are used, each packet should contain the following in different colors of laminated tagboard: 12 rectangular (linear) variable tiles, 0.75 inch by 3 inches (color #1); 12 rectangular variable tiles, 0.75 inch by 3.25 inch (color #2); and 30 unit tiles, 0.75 inch by 0.75 inch (color #3). Each tile should have a large X drawn on one side to show the inverse of that tile. Use tagboard that is thick enough so that the X will not show through to the other side. Commercial tiles are also available for two different variables, but a large X must be drawn on one of the largest faces of each tile in order to represent the inverse of that tile when the X faces up.
2. For each exercise on Worksheet 1–7a, have students represent the two equations with tiles on Building Mat 1–7a. Equation (a) should be shown in the upper half of the mat and equation (b) in the lower half of the mat. Discuss the idea that if two groups of tiles are considered equal in value, either one of the groups can replace the other on an equation mat. Also, since equations in a system have the same solutions, the replacement of equivalent groups can occur in any equation within the same system.
3. In one of the equations of an exercise, students will use the tiles to solve for one variable tile in terms of the other. The new group of tiles found for this first variable will then be substituted for that same variable in the other original equation. This results in a new equation involving only one variable. Students will solve this equation for its variable.
4. After the value of one variable has been found, its value will be substituted into the symbolic form of the first equation transformed, and that equation will be solved to find the value of the second variable. The substitution equation and the values of the two variables should then be recorded below the exercise on Worksheet 1–7a.

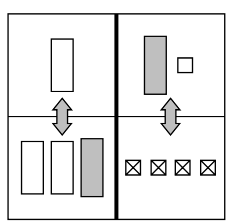
5. Now have students verify that their solutions for the two variables will satisfy both equation (a) and equation (b). Have them rebuild the original equations on the building mat, and then replace each variable tile with its value in unit tiles. The amounts on each side of an equation should be equal in value. Since the equality holds for both equations, this verifies that the values found for variable A and variable B form the solution pair for the given system of equations.

6. Discuss Exercise 1 on Worksheet 1–7a with the class before allowing partners to work on the other exercises on their own. Consider Exercise 1 on Worksheet 1–7a: "(a) $A - B = 1$ and (b) $2A + B = -4$."

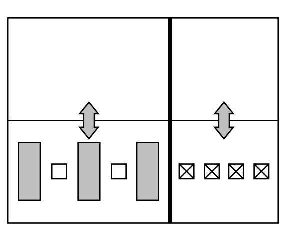
Have students show each equation with tiles on Building Mat 1–7a. Build equation (a) on the top half of the building mat, and build equation (b) on the bottom half of the mat. The initial mat and tiles should have the following appearance (variable A is represented by the plain rectangle and variable B by the shaded rectangle):



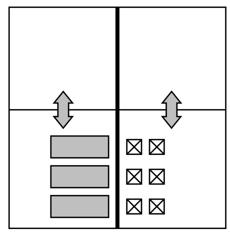
Students should now solve one of the equations for one of the variables in terms of the other variable. For discussion purposes, we will use equation (a) on the top half of the mat, since it is easy to solve for variable A. Have students add one variable B tile to each side of the equation mat and remove the 0-pair formed on the left side. The building mat will then have the following appearance:



Since A equals $B + 1$ in the top equation, have students replace each variable A in the bottom equation with tiles for $B + 1$. The top half of the mat should be cleared of any remaining tiles. After the substitution is completed, the building mat will appear as follows:

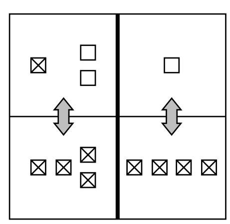


The two positive unit tiles on the left side of the equation should be removed by bringing in two negative unit tiles to both sides of the equation. Two 0-pairs will be formed on the left side and should be removed from the mat. A total of –6 will be on the right side of the equation. Students should separate the three variable *B* tiles on the left and form three rows. The same separation or division process should be shown with the –6 on the right side. Each complete row across the mat will show *B* = –2. The building mat will then appear as follows:



Have students remove two of the three rows of tiles from the building mat. The remaining row will show the solution for *B* in the original system of equations; that is, *B* = –2. This solution should be recorded below Exercise 1 on Worksheet 1–7a. Below Exercise 1, students should also rewrite equation (a), but substitute –2 for the variable *B: A* – (–2) = 1. They should solve for *A* in this equation, obtaining *A* = –1 and recording this result below Exercise 1.

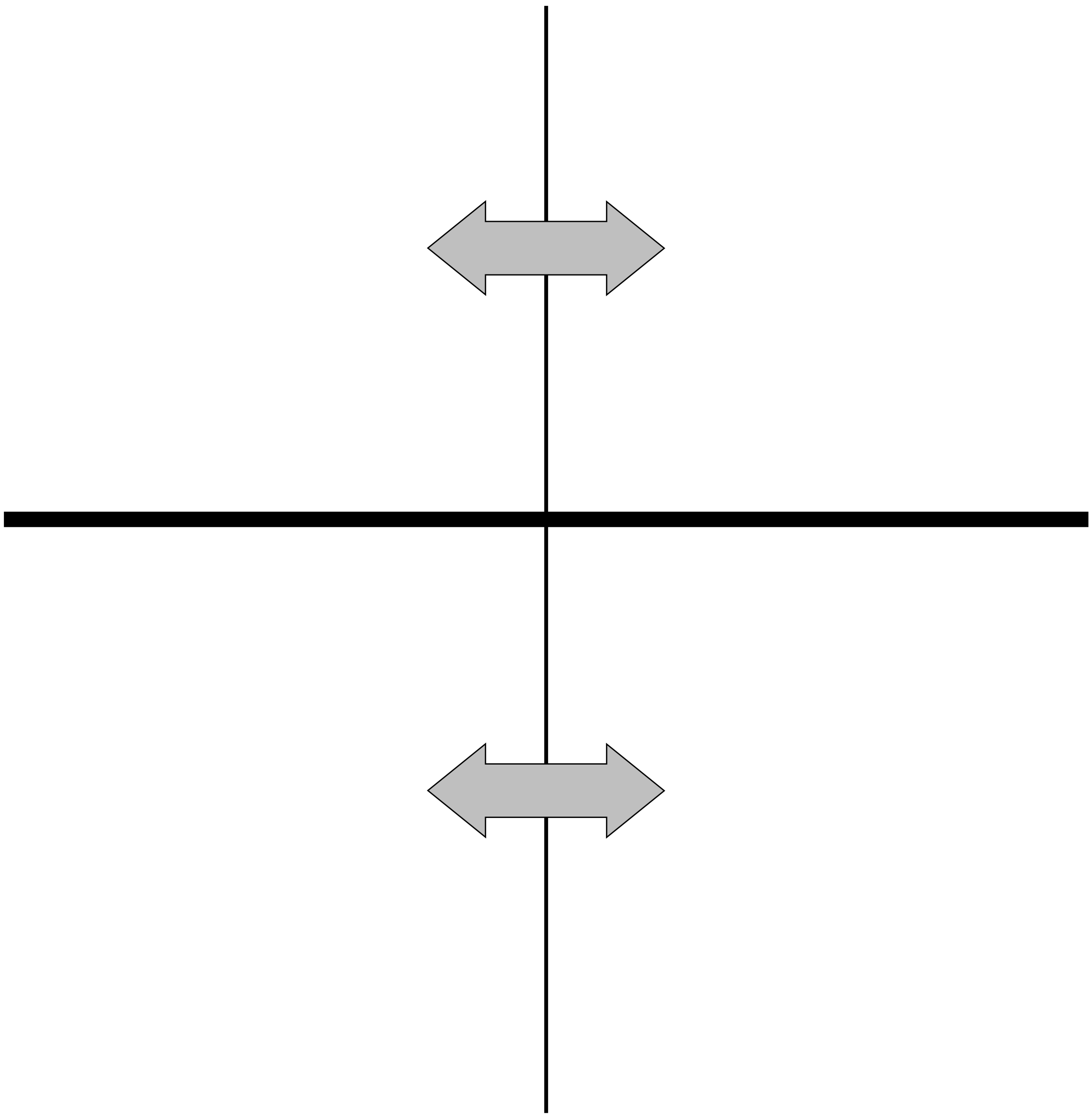
Now have students rebuild the original two equations on Building Mat 1–7a, then substitute the values found for variable *A* and variable *B* into the equations, using unit tiles. The solutions for *A* and *B* are verified when the left and right sides of each equation equal each other. This also reinforces the idea that in a system of equations, all equations have the same solution pair for (*A*,*B*). Here is a possible mat arrangement that shows the verification:



Answer Key for Worksheet 1–7a

1. *A* = –1, *B* = –2
2. *A* = –3, *B* = +5
3. *A* = +2, *B* = +3
4. *A* = –2, *B* = –1
5. *A* = –4, *B* = –2

Building Mat 1–7a



Worksheet 1–7a

Name ______________________

Solving Systems of Equations with Algebra Tiles

Date ______________________

For each exercise, build each equation with tiles on Building Mat 1–7a. Follow your teacher's instructions to solve the two equations for the two variables involved. Record the value of each variable below the exercise.

1. (a) $A - B = 1$, and (b) $2A + B = -4$

2. (a) $A - B = -8$, and (b) $B = 2 - A$

3. (a) $3B - 2A = 5$, and (b) $2A - B = 1$

4. (a) $A + B = -3$, and (b) $2B - 3A = 4$

5. (a) $A - 3B = 2$, and (b) $-2A + 2B = 4$

Activity 2
Pictorial Stage

Materials

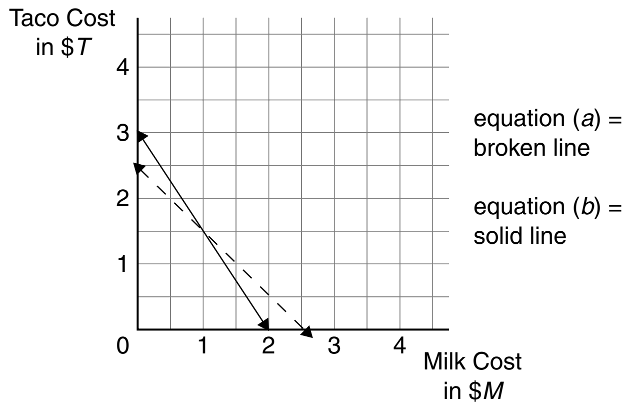
Worksheet 1–7b
Worksheet 1–7c
Rulers or straightedges
Regular pencils

Procedure

1. Give each pair of students two copies of Worksheet 1–7c, a ruler, and eight copies of Worksheet 1–7b (four copies per person).
2. For each exercise on Worksheet 1–7c, have students draw the necessary graphs on a copy of Worksheet 1–7b. It may be necessary in some exercises for students to identify the equations needed before they can graph them. Any equations found should be recorded below the exercise on Worksheet 1–7c. Some exercises will need the origin for the graphs to be in the lower left corner of the grid, while others will need the origin to be in the center of the grid. Students will need to decide how to number each scale on the grid, based on the data given in the exercise.
3. Students will use the pair of lines graphed for an exercise to find the solution to the system of equations involved. They will graphically estimate the coordinates of the intersection point of the two lines, then substitute the chosen coordinates into the system equations to verify the solution. If an intersection point lies between grid marks, the first estimate for a coordinate may not work in the given equations. Students will then need to adjust their estimate until it works. The numerical substitutions verifying the solution should be recorded below the exercise on Worksheet 1–7c.
4. After students have found the ordered pair that is the solution to the system of linear equations, they should use that information to answer any specific questions included in the exercise.
5. Discuss Exercise 1 with the class before having students work on other exercises with their partners.

Consider Exercise 1 on Worksheet 1–7c: "At a restaurant the cost for a breakfast taco and a carton of milk is \$2.50. The cost for 2 tacos and 3 cartons of milk is \$6.00. Write equations for the relationships described, then use graphs of the equations to find the cost of one taco and the cost of one carton of milk."

The two relationships include the 1 taco–1 milk combination and the 2 tacos–3 milks combination. Using T for the taco cost and M for the milk cost, guide students to write the following equations below Exercise 1 on Worksheet 1–7c: (a) $T + M = \$2.50$, and (b) $2T + 3M = \$6.00$. Have students graph each equation on the same copy of Worksheet 1–7b. Ordered pairs for each equation will need to be found by trial and error. Discuss the idea that each algebraic equation may represent a general set of ordered pairs that may or may not make sense in the given situation. For example, students might be able to purchase 2 cartons of milk for \$2, but they might not be able to buy 1 taco for \$0.50 at the same time, yet (\$2, \$0.50) is a solution for equation (a). So only specific points on each line might actually relate to the given situation. The graphs might appear as shown:



Have students identify the ordered pair for the intersection point of the two lines. It will be (M, T) = (\$1, \$1.50). The intersection point should be circled on the graphs and its ordered pair recorded below Exercise 1. This point belongs to both lines; therefore, it represents the solution to the system of the two equations. That is, it satisfies both relationships in the given situation. Note that it is the only solution shared by both equations.

Students should also verify the solution by substituting the values for M and T into the two equations. The substitutions should be shown on Worksheet 1–7c below Exercise 1 as follows:

(a) $T + M = \$1.50 + \$1 = \$2.50$
(b) $2T + 3M = 2(\$1.50) + 3(\$1) = \$6.00$

Answer Key for Worksheet 1–7c
Only solutions/answers and substitutions are provided; (a) and (b) assignments may vary; no graphs are shown.

1. $(M,T) = (\$1, \$1.50)$; (a) $T + M = \$1.50 + \$1 = \$2.50$, and (b) $2T + 3M = 2(\$1.50) + 3(\$1) = \$6.00$
2. $(x, y) = (3, 1)$; (a) $y = \frac{5}{3}x - 4 = \frac{5}{3}(3) - 4 = 1$, and (b) $y = \frac{-2}{3}x + 3 = \frac{-2}{3}(3) + 3 = 1$
3. $(t, a) = (16, 1200)$; (a) $a = -100t + 2800 = -100(16) + 2800 = 1200$, and (b) $a = 50t + 400 = 50(16) + 400 = 1200$; planes have same altitude 16 minutes after initial siting
4. $(a, c) = (50, 35)$; (a) $a + c = 50 + 35 = 85$ people, and (b) $a(\$5) + c(\$2) = 50(\$5) + 35(\$2) = \$320$; 50 adult tickets were sold

Worksheet 1–7b

Name ______________________

Date ______________________

Worksheet 1–7c

Solving Systems of Equations by Graphing

Name ______________________

Date ______________________

Use graphing to find the solution to each system of linear equations or the answer to the exercise. Verify that an ordered pair is a solution to a system by substituting its coordinate values into each equation of the system.

1. At a restaurant, the cost for a breakfast taco and a carton of milk is $2.50. The cost for 2 tacos and 3 cartons of milk is $6.00. Write equations for the relationships described, then use graphs of the equations to find the cost of one taco and the cost of one carton of milk.

2. Graph the line for the equation $y = \frac{5}{3}x - 4$. Graph $y = \frac{-2}{3}x + 3$ on the same grid. What is the solution to this system of equations?

3. At Miami International Airport, plane A is sited at an altitude of 2,800 feet as it descends toward runway 3 at a rate of 100 feet per minute. At the same moment, plane B is sited at 400 feet as it climbs at a rate of 50 feet per minute after taking off from runway 4. In about how many minutes after the initial siting will the two planes be at the same altitude?

4. At a spaghetti supper, band members served 85 people and raised $320. If each adult's ticket was $5 and each child's ticket was $2, how many adult tickets were sold to the supper?

Activity 3
Independent Practice

Materials

Worksheet 1–7d
Grid paper
Regular pencils and regular paper

Procedure

Give each student a copy of Worksheet 1–7d and a sheet of grid paper. Remind students that the solution to a system of equations may be found either by substitution or graphing. When all have finished the worksheet, ask various students to share their solutions and the methods they used to find the solutions.

Answer Key for Worksheet 1–7d

1. D
2. A
3. C
4. B
5. A

Possible Testing Errors That May Occur for This Objective

- When substitution is used to solve a system of equations, students fail to use the distributive property correctly. For example, if $y = 2x + 5$ is substituted in the equation $5x - 3y = 4$, $(-3)(2x + 5)$ is replaced with $-6x + 5$ instead of $-6x - 15$, thereby leading to the wrong solution.
- When the solution to a system of equations is to be found graphically, students incorrectly read the vertical or horizontal scale of the graph when finding the ordered pair for the intersection point of the two lines.
- When given a word problem, students set up linear equations that incorrectly reflect the conditions of the situation.

Worksheet 1–7d

Solving Systems of Equations by Graphing and Substitution

Name ____________________

Date ____________________

Solve each exercise provided. Be ready to share your reasoning and your answers with the entire class.

1. What is the x-coordinate of the solution to the system of linear equations $5x + 4y = 8$ and $-3x + 2y = -18$?

   A. –4 B. –3 C. +3 D. +4

2. In the system of equations $9x + 3y = 6$ and $7x + 3y = 10$, which expression can be correctly substituted for y in the equation $7x + 3y = 10$?

   A. $y = 2 - 3x$ B. $y = 2 + 3x$ C. $y = 6 - 3x$ D. $y = 6 + 3x$

3. In the movie theater parking lot, there are 57 cars and motorcycles altogether. If the wheels are counted on all the vehicles, there are 194 wheels total. How many cars and how many motorcycles are in the parking lot?

   A. 35 cars, 22 motorcycles C. 40 cars, 17 motorcycles

   B. 28 cars, 29 motorcycles D. 48 cars, 9 motorcycles

4. At the bakery, Carol bought 5 pieces of fudge and 3 chocolate chip cookies for a total of $5.70. Her friend Juan bought 2 pieces of fudge and 10 chocolate chip cookies, for a total of $3.60. Which system of equations could be used to determine the cost, f, of 1 piece of fudge, and the cost, c, of 1 chocolate chip cookie?

   A. $f + c = 20$
   $7f + 13c = \$9.30$

   C. $5f + 2f = \$3.60$
   $3c + 10c = \$5.70$

   B. $5f + 3c = \$5.70$
   $2f + 10c = \$3.60$

   D. $5f + 3c = \$3.60$
   $2f + 10c = \$5.70$

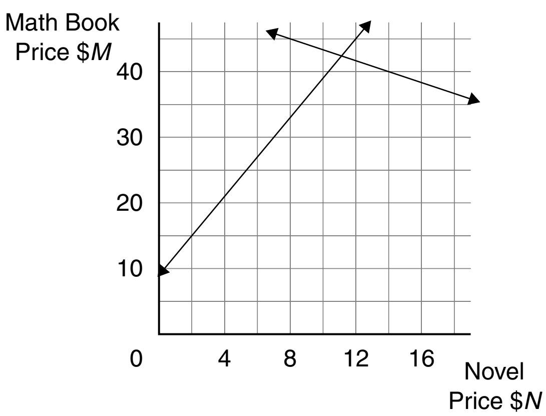
Worksheet 1–7d Continued

Name ____________________

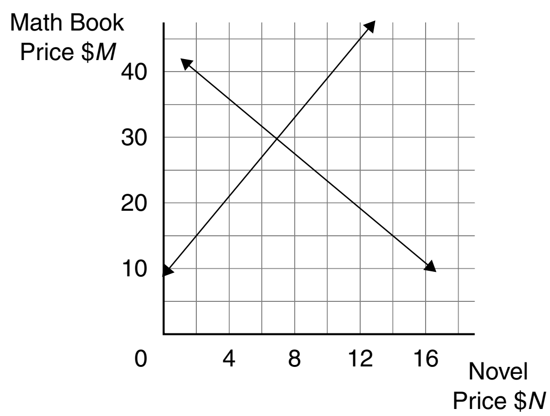
Date ____________________

5. Jimmy purchased a math book and a paperback novel for a total of $54 without tax. If the price, M, of the math book is $8 more than 3 times the price, N, of the novel, which graph of a system of linear equations could be used to determine the price of each book?

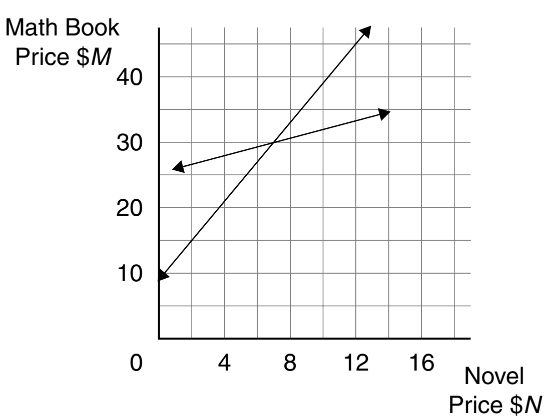
A.



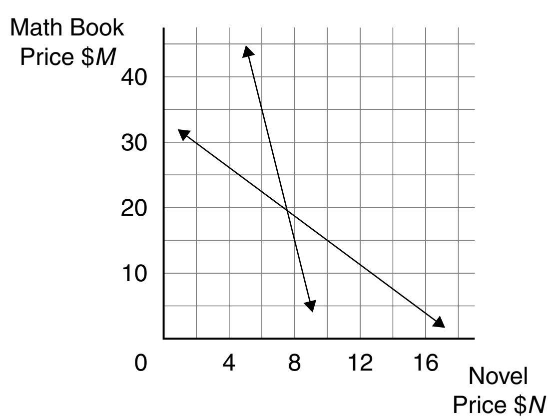
C.



B.



D.



Section 1

Name ______________________

Date ______________________

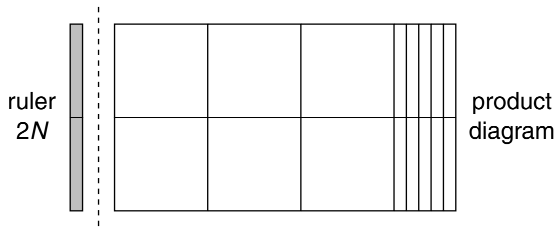
ALGEBRAIC THINKING AND APPLICATIONS: PRACTICE TEST ANSWER SHEET

Directions: Use the answer sheet to darken the letter of the choice that best answers each question.

1.	○ A	○ B	○ C	○ D	8.	○ A	○ B	○ C	○ D
2.	○ A	○ B	○ C	○ D	9.	○ A	○ B	○ C	○ D
3.	○ A	○ B	○ C	○ D	10.	○ A	○ B	○ C	○ D
4.	○ A	○ B	○ C	○ D	11.	○ A	○ B	○ C	○ D
5.	○ A	○ B	○ C	○ D	12.	○ A	○ B	○ C	○ D
6.	○ A	○ B	○ C	○ D	13.	○ A	○ B	○ C	○ D
7.	○ A	○ B	○ C	○ D	14.	○ A	○ B	○ C	○ D

Section 1: Algebraic Thinking and Applications: Practice Test

1. The following diagram represents the product of $2N$ rows of $(3N + 5)$. Which expression is equivalent to the total area, $2N(3N + 5)$, of the product diagram?



A. $6N^2+5$ B. $6N+5$ C. $6N^2+10N$ D. $6N+10$

2. Which expression is equivalent to $\left(\frac{2}{3}\right)(3x-6y)+(5y-8x)$?

A. $-6x+y$ B. $-2x-y$ C. $-6x-y$ D. $-5x+y$

3. If $A = -21$ is a solution for the equation $-\frac{1}{3}A-10=-3$, which expression may be used to confirm the solution?

A. $-\frac{1}{3}(+21)-10$ C. $\frac{1}{3}(+21)-10$

B. $-\frac{1}{3}(-7)-10$ D. $\frac{1}{3}(-21)-10$

4. Which expression is not a correct interpretation of the expression $-\frac{3}{5}p$?

A. $3\left(-\frac{1}{5}p\right)$ B. $-3\left(\frac{1}{5}p\right)$ C. $\frac{3}{5}(-p)$ D. $\frac{-3}{5}(-p)$

5. The Deli-Mart sells 3 cartons of chili for \$1.79 total. The total cost, C, of buying N cartons of this same chili can be found by which of the following procedures?

A. Multiplying N by \$1.79 C. Dividing N by \$1.79

B. Dividing N by the cost of one carton D. Multiplying N by the cost of one carton

6. Carrie has half as many movie passes as Jan. Anna has 5 fewer passes than Jan. Together, the three students have a total of 20 movie passes. Which equation can be used to find how many movie passes Jan (J) has?

A. $J+2J+(J-5)=20$ C. $\frac{1}{2}J+(J+5)+J=20$

B. $\frac{1}{2}J+J+(J-5)=20$ D. $2J+(J+5)+J=20$

Section 1: Practice Test (Continued)

7. Two positive integers have a sum of 73. The lesser number is an even number, and the difference between the two integers is 9. What is the even number?

   A. 32 B. 49 C. 26 D. 73

8. Some vacationers were exiting a tour bus, so Josh and his friends decided to count them in various ways. Each looked for something different. Here are the results: 21 people wore jeans; 18 wore sandals; 14 wore hats; 7, sandals and hat; 11, jeans and sandals; 8, hat and jeans; and 5, hat, jeans, and sandals. Five people did not wear a hat, jeans, or sandals. How many people did Josh and his friends count leaving the bus?

   A. 58 B. 37 C. 53 D. 21

9. A package of three Brand B batteries costs $5.85, and a package of three Brand C batteries costs $4.69. Which proportion can be used to determine the percent, *N*, of the cost of the Brand B batteries that will be saved if the Brand C batteries are purchased?

   A. $\frac{(5.85-4.69)}{4.69}=\frac{N}{100}$ C. $\frac{(5.85-4.69)}{5.85}=\frac{N}{100}$

   B. $\frac{4.69}{5.85}=\frac{N}{100}$ D. $\frac{5.85}{4.69}=\frac{N}{100}$

10. A box contains 18 colored cubes. There are 6 blue cubes, 5 red cubes, and 3 yellow cubes. The rest of the cubes are green. The ratio of green cubes to total cubes is equivalent to which probability for drawing out a green cube at random?

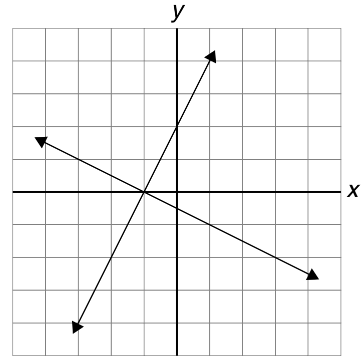
    A. $\frac{1}{3}$ B. $\frac{5}{18}$ C. $\frac{1}{6}$ D. $\frac{2}{9}$

11. Which equation describes the line that passes through the point (1, –5) and is parallel to the line represented by the equation $-3x + y = -4$?

    A. $y=3x-8$ B. $y=\frac{1}{3}x-4$ C. $y=3x+1$ D. $y=-\frac{1}{3}x+5$

Section 1: Practice Test (Continued)

12. Which pair of equations represents the perpendicular lines shown on the graph? (1 grid segment = 1 unit)



A. $y = 2x + 2$ and $y = -2x + 2$

B. $y = -0.5x - 1$ and $y = 2x - 1$

C. $y = 2x + 2$ and $y = -0.5x - 0.5$

D. $y = 2x + 2$ and $y = -2x - 1$

13. At the bakery, Carol bought 3 pieces of fudge and 8 chocolate chip cookies, for a total of \$4.35. Her friend Juan bought 2 pieces of fudge and 10 chocolate chip cookies for a total of \$3.60. Which system of equations could be used to determine the cost, f, of 1 piece of fudge, and the cost, c, of 1 chocolate chip cookie?

A. $f + c = 23$
$5f + 18c = \$7.95$

B. $3f + 8c = \$4.35$
$2f + 10c = \$3.60$

C. $3f + 2f = \$3.60$
$8c + 10c = \$4.35$

D. $3f + 8c = \$3.60$
$2f + 10c = \$4.35$

14. What is the x-coordinate of the solution to the system of linear equations $3x + 4y = -8$ and $-2x - 2y = 6$?

A. +1 B. +4 C. –1 D. –4

Section 1: Algebraic Thinking and Applications: Answer Key for Practice Test

The objective being tested is shown in brackets beside the answer.

1. C [1]
2. A [1]
3. C [2]
4. D [2]
5. D [3]
6. B [3]
7. A [4]
8. B [4]
9. C [5]
10. D [5]
11. A [6]
12. C [6]
13. B [7]
14. D [7]