Section Four

Algebra

LIST 134

How to Express Operations Algebraically

Changing verbal phrases into algebraic expressions is necessary to solve problems. The list below contains key words that will help you in your study of algebra.

Addition	*Subtraction*	*Multiplication*	*Division*
add	decreased by	cubed	average
augment	deduct	double	(a) fourth
combine	depreciate	factor	divided by
enlarge	difference	multiple of	equally
exceeds	diminish	multiply	half
gain	drop	quadruple	per
greater than	fewer	squared	quotient
grow	left	times	ratio
in all	less than	triple	shared
increased by	lose	twice	split
larger than	loss		(a) third
longer than	lower		
more than	minus		
plus	remain		
rise	remove		
sum	shorten		
total	smaller than		
	subtract		
	take away		

LIST 135

Algebraic Grouping Symbols

The following symbols are important to grouping numbers, variables, and operations in algebra. Operations to be done first are enclosed in grouping symbols.

Parentheses	( )
Brackets	[ ]
Braces	{ }
Fraction Bar	——
Absolute Value	| |

LIST 136

Properties of Real Numbers

Real numbers include positive numbers, negative numbers, and zero. Since integers are a subset of the real numbers, all properties of integers are also properties of real numbers. However, some properties of real numbers are not the properties of integers.

In the chart below, *a, b,* and *c* are real numbers.

Property	*Addition*	*Multiplication*
Closure Property	$a + b$ is a unique real number	ab is a unique real number
Commutative Property	$a + b = b + a$	$ab = ba$
Associative Property	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Identity Property	$a + 0 = a$	$1(a) = a$
Inverse Property	$a + (-a) = 0$	$a \times \frac{1}{a} = 1 \quad a \neq 0$
Property of Zero		$a(0) = 0$
Property of –1		$-1 \times a = -a$
Property of Opposites		$-(-a) = a$ $-(a + b) = -a + (-b)$ $-(ab) = (-a)b = a(-b)$
Zero Product Property		$ab = 0$ if and only if $a = 0$, $b = 0$, or both a and $b = 0$
Distributive Property	$a(b + c) = ab + ac$	
Completeness Property	Every real number can be paired with a point on the number line.	
Density Property	Between any two real numbers, there is another real number.	

LIST 137

Summary of Properties of Sets of Numbers

Each set of numbers—natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers—has specific properties. The chart below summarizes the properties of each. A check means the property applies all the time. A blank means that a property is not always applicable.

Property	*Natural*	*Whole*	*Integer*	*Rational*	*Irrational*	*Real*
Closure (Add.)	✓	✓	✓	✓		✓
Closure (Sub.)			✓	✓		✓
Closure (Mult.)	✓	✓	✓	✓		✓
Closure (Div.)				✓		✓
Additive Identity		✓	✓	✓	✓	✓
Multiplicative Identity	✓	✓	✓	✓	✓	✓
Additive Inverse			✓	✓	✓	✓
Multiplicative Inverse				✓	✓	✓

LIST 138

Relating Operations on the Real Numbers

From addition and multiplication of real numbers (see List 136, "Properties of Real Numbers"), we can define subtraction and division. Along with each definition, other equations relating the operations of addition, subtraction, multiplication, and division follow. In the equations, *a, b, c,* and *d* are real numbers.

- Definition of subtraction: $a - b = a + (-b)$
- Definition of division: $\frac{a}{b} = ab^{-1} = a\left(\frac{1}{b}\right)$ $\quad b \neq$
- $\frac{ac}{bc} = \frac{a}{b}$ $\quad b \neq 0,\ c \neq 0$
- $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ $\quad c \neq 0$
- $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$ $\quad c \neq 0$
- $\frac{a}{c} + \frac{b}{d} = \frac{ad+bc}{cd}$ $\quad c \neq 0,\ d \neq 0$
- $\frac{a}{c} - \frac{b}{d} = \frac{ad-bc}{cd}$ $\quad c \neq 0,\ d \neq 0$
- $\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$ $\quad c \neq 0,\ d \neq 0$
- $\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{b} = \frac{ad}{cb}$ $\quad b \neq 0,\ c \neq 0,\ d \neq 0$
- If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$
- $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$ $\quad b \neq 0$
- $\frac{-a}{-b} = \frac{a}{b}$ $\quad b \neq 0$ $\quad b \neq 0$

LIST 139

Axioms of Equality

An *axiom* is a self-evident principle. In algebra (and also geometry), the four following statements about equality are true for all real numbers a, b, and c.

Reflexive Property: $a = a$. Any number is equal to itself.

Symmetric Property: If $a = b$, then $b = a$.

Transitive Property: If $a = b$ and $b = c$, then $a = c$.

Substitution Property: If $a = b$, then a may replace b or b may replace a.

LIST 140

Axioms of Order

Just as there are general statements about equality in mathematics, there are statements about inequality. These are called *axioms of order,* also known as *axioms of inequality.*

Trichotomy Property: For all real numbers a and b, one and only one of the following statements is true: $a > b$, $a = b$, or $a < b$.

Transitive Property: For all real numbers a, b, and c: If $a > b$ and $b > c$, then $a > c$. If $a < b$ and $b < c$, then $a < c$.

LIST 141

Properties of Equality

Pretend you are watching a basketball game, and the score is tied. In the next two plays, each team scores a basket. Do you agree that the score is tied once again? This is an example of the Addition Property of Equality. Try to find some examples of the other properties of equality listed below.

The following hold true when a, b, and c are real numbers.

Addition Property: If $a = b$, then $a + c = b + c$ and $c + a = c + b$. (If the same number is added to equal numbers, the sums are equal.)

Subtraction Property: If $a = b$, then $a - c = b - c$. (If the same number is subtracted from equal numbers, the differences are equal.)

Multiplication Property: If $a = b$ and $c \neq 0$, then $ac = bc$. (If equal numbers are multiplied by the same nonzero number, the products are equal.)

Division Property: If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$. (If equal numbers are divided by the same nonzero number, the quotients are equal.)

LIST 142

Properties of Inequalities

Inequalities are mathematical sentences that show a relationship between two or more variables. The following signs are used to show inequalities:

less than: $<$

less than or equal to: $\leq$

greater than: $>$

greater than or equal to: $\geq$

is not equal to: $\neq$

The following properties of inequalities hold true for all real numbers a, b, c, and d.

Addition	If $a > b$, then $a + c > b + c$. If $a < b$, then $a + c < b + c$. If $a < b$ and $c < d$, then $a + c < b + d$. If $a > b$ and $c > d$, then $a + c > b + d$.
Subtraction	If $a > b$, then $a - c > b - c$. If $a < b$, then $a - c < b - c$.
Multiplication*	If $a > b$ and $c > 0$, then $ac > bc$. If $a > b$ and $c < 0$, then $ac < bc$. If $0 < a < b$ and $0 < c < d$, then $ac < bd$.
Division*	If $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$. If $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$. If $a < b$ and $ab > 0$, then $\frac{1}{a} > \frac{1}{b}$.

*For multiplication and division, the above properties are not valid if $c = 0$.

LIST 143

Powers of Real Numbers

Some numbers may be written as the product of numbers that have identical factors—for example, $100 = 10 \times 10$ or 10^2. In this case, 10 is called the *base,* and 2 is the *exponent.* The exponent shows the number of times the base is a factor.

Powers of a number can be written in factored form or in exponential form:

- Factored form indicates the products of the factors as in $b \cdot b \cdot b \cdot b \cdot b$.
- Exponential form indicates the base and exponents as in b^5.

Examples of some of the powers of the real number b follow.

	Factored Form	***Exponential Form***	***Read***
Zero power of b	$1 (b \neq 0)$	b^0	1
First power of b	b	b^1 or b	b to the first power
Second power of b	$b \cdot b$	b^2	b to the second power, b squared, or the square of b
Third power of b	$b \cdot b \cdot b$	b^3	b to the third power, b cubed, or the cube of b
Fourth power of b	$b \cdot b \cdot b \cdot b$	b^4	b to the fourth power, or b to the fourth
nth power of b (n is a positive integer that represents the number of times b is multiplied)	$b \cdot b \cdot b \cdot b \ldots b$	b^n	b to the nth power, or b to the nth
$-n$th power of b (n is a positive integer that represents the number of times b is multiplied. $b \neq 0$)	$\frac{1}{b \cdot b \cdot b \cdot b \ldots b}$	b^{-n}	b to the $-n$th power, or b to the $-n$th

LIST 144

Rules for Exponents

The following rules for exponents hold for real numbers a and b. m and n are rational numbers.

For Multiplication: $a^m \cdot a^n = a^{m+n}$

For Division: $\frac{a^m}{a^n} = a^{m-n}$ $\quad a \neq 0$

For a Power of a Power: $\left(a^m\right)^n = a^{mn} = \left(a^n\right)^m$

For a Power of a Product: $(ab)^m = a^m b^m$

For a Power of a Quotient: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $\quad b \neq 0$

For a Zero Exponent: $a^0 = 1$ $\quad a \neq 0$

For an Exponent of 1: $a^1 = a$

For a Negative Exponent: $a^{-n} = \frac{1}{a^n}$ $\quad a \neq 0$

For a Base of 1: $1^n = 1$

LIST 145

Order of Operations

In algebra, finding the right answer often depends on the way you go about solving a problem. There is a specific order of operations you must follow.

1. Simplify expressions within grouping symbols. If several grouping symbols are used, simplify the innermost group first, and continue simplifying to the outermost group. As you do, be sure to follow steps 2, 3, and 4.
2. Simplify powers.
3. Perform all multiplicative operations (multiplication and division) from left to right.
4. Perform all additive operations (addition and subtraction) from left to right.

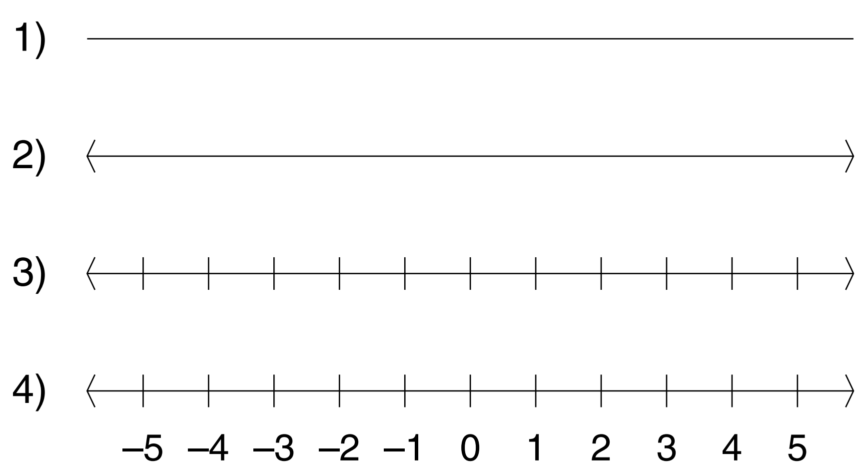
LIST 146

How to Construct a Number Line

Use the following steps to construct number lines accurately.

1. Use a ruler to draw a line segment.
2. Show continuity by drawing arrows on both ends.
3. Using a ruler, divide the line into equal segments.
4. Pair the endpoints or successive endpoints with integers listed in chronological order. Be sure to label numbers to the left of zero with a negative sign.

Steps



LIST 147

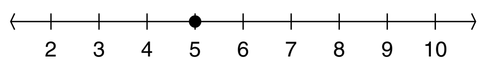
Steps for Graphing on a Number Line

The following list provides simple procedures for graphing points and inequalities on number lines.

To Graph a Point

1. Locate the coordinate (number) on the number line.
2. Place a shaded circle on the number line above the coordinate.

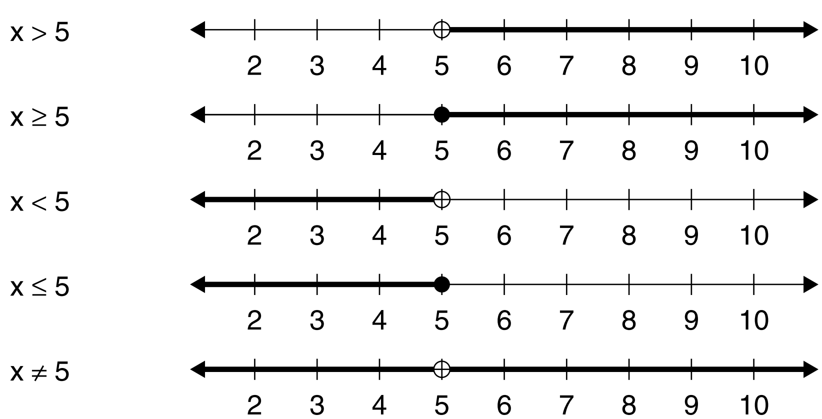
   Example: $x = 5$.



To Graph an Inequality

1. Locate the coordinate (number) on the number line.
2. If $x >$ a number, circle that number on the number line and shade the number line to the right.
3. If $x \geq$ a number, place a shaded circle on the number line and shade the number line to the right.
4. If $x <$ a number, circle that number on the number line and shade the number line to the left.
5. If $x \leq$ a number, place a shaded circle on the number line and shade the number line to the left.
6. If $x \neq$ a number, circle that number on the number line and shade to left and right.

Examples:



LIST 148

The Absolute Facts on Absolute Value

Absolute value is the distance a number is from zero on the number line.

$|a| = a$ if $a \geq 0$

$|a| = -a$ if $a < 0$

The distance between two real numbers a and b on the number line can be found by $|a - b|$ or $|b - a|$.

To evaluate an expression using absolute value, evaluate the expression within the absolute value symbols first. The symbols have the same priority as parentheses in the order of operations.

The absolute value of a sum is less than or equal to the sum of the absolute values.

$|a + b| \leq |a| + |b|$

The absolute value of a product is the product of the absolute values.

$|ab| = |a||b|$

If the absolute value of an expression is greater than a positive number, a *disjunction* results.

For a real number $c > 0$,

$|a| > c$ is equivalent to $a < -c$ or $a > c$

If the absolute value of an expression is less than a positive number, a *conjunction* results.

For a real number $c > 0$,

$|a| < c$ is equivalent to $-c < a < c$ or $-c < a$ and $a < c$

If the absolute value of an expression is less than a negative number, there is no solution. The absolute value is always greater than or equal to zero and therefore is greater than any negative number—for example: $|n| < -4$ $\quad n = \varnothing$ (no solution).

If the absolute value of an expression is greater than a negative number, the solution set is all real numbers. The absolute value is always greater than or equal to zero and therefore is always greater than a negative number—for example: $|n| > -4$ $\quad n =$ all numbers.

LIST 149

Steps to Solve an Equation in One Variable

Equations that have the same solution are called *equivalent equations.* The following is a step-by-step list for rewriting and transforming the original equation into an equivalent equation that has the same solution or root. If a step does not apply to a specific problem, go on to the next step.

1. Simplify each side of the equation. This may include:
   - Combining similar terms within grouping symbols
   - Using the Distributive Property
   - Removing any unnecessary parentheses
   - Combining similar terms

2. Add (or subtract) the same real number to (or from) each side of the equation. (If you add or subtract zero, an equivalent equation will result. It will be the same as the previous equations and will not be easier to solve. Although you can add or subtract zero, it is an unnecessary step and should be avoided.)

3. Multiply (or divide) each side of the equation by the same nonzero real number. (If you multiply each side of the equation by zero, the result will always be $0 = 0$, and the equation would not be solved. You cannot divide each side of an equation by zero because division by zero is undefined.)

4. In most cases, there is only one solution. The final transformation will result with the variable equaling a real number.

5. If the final transformation is equivalent to a false statement such as $3 = 7$ or $0 = 8$, the equation has no solution or root. It is written as $\varnothing$.

6. If the final transformation is equivalent to a statement that is always true such as $x = x$ *or* $3 = 3$, the equation is called an *identity* and is true for all real numbers.

LIST 150

Steps to Solve an Inequality in One Variable

Just as equivalent equations have the same solution set, equivalent inequalities have the same solutions. To solve an inequality, try to rewrite it, and transform it into an equivalent inequality using many of the same steps you use to solve equations. Be careful, however, when you multiply or divide both sides of the inequality by the same negative number, because that reverses the direction of the inequality. To solve an inequality in one variable, follow the steps below. If a step does not apply to a specific problem, go on to the next step.

1. Simplify each side of the inequality. This may include:
   - Combining similar terms within grouping symbols
   - Using the Distributive Property
   - Removing any unnecessary parentheses
   - Combining similar terms
2. Add (or subtract) the same real number to (or from) each side of the inequality. Adding or subtracting zero should be avoided.
3. Multiply (or divide) each side of the inequality by the same positive real number.
4. Multiply (or divide) each side of the inequality by the same negative real number, and reverse the direction of the inequality.
5. In most cases, the final transformation will be a comparison of a variable and real number, such as $x > 7$.
6. If the final transformation is equivalent to a false statement, such as $2 > 7$, $6 > 9$, or $x < x$, the inequality has no solution or root.
7. If the final transformation is equivalent to a statement that is always true, such as $x \geq x$ or $4 < 5$, the inequality is true for all real numbers.

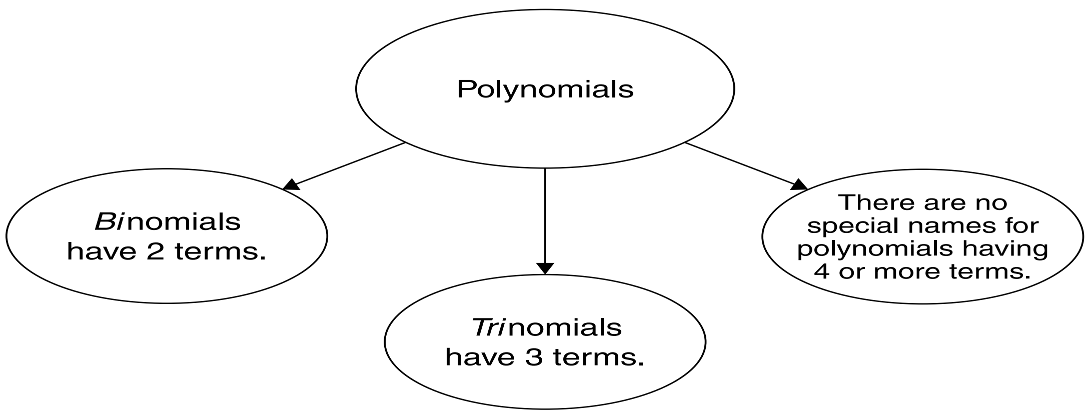
LIST 151

Polynomials

A *monomial* is an expression that is either (1) a real number, (2) a variable, or (3) the product of a real number and one or more variables. Remember that a variable cannot be in the denominator. The degree of a monomial is the sum of the exponents of its variables.

A *polynomial* is the sum or difference of monomials. Monomials that make up a polynomial are called its *terms.* Polynomials of two or three terms have special names, as shown below.

- *Binomial:* a polynomial with two terms.
- *Trinomial:* a polynomial with three terms.



To simplify a polynomial, combine similar terms. Similar terms contain the same variables and same exponents. Simplified polynomials can be arranged in the following orders:

- Descending order in which the degree of each term decreases in successive terms.
- Ascending order in which the degree of each term increases in successive terms.

The degree of a polynomial is the highest degree of any of its terms after it has been simplified.

LIST 152
Multiplication with Monomials and Polynomials

To multiply a polynomial by a monomial, use the distributive property.

$$a(b + c + d) = ab + ac + ad$$

To multiply two binomials, use the FOIL method.

$$(a + b)(c + d) = ac + ad + bc + bd$$

FOIL is an acronym for:

F—product of the FIRST terms	ac
O—product of the OUTERMOST terms	ad
I—product of the INNERMOST terms	bc
L—product of the LAST terms	bd

If the outermost and innermost products can be simplified, do so.

Special cases:

$$c(a + b) = ca + cb$$
$$c(a - b) = ca - cb$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a - b)(a + b) = a^2 - b^2$$
$$(a + c)(a + d) = a^2 + (c + d)a + cd$$

LIST 153

Guidelines for Factoring Polynomials of Degree 2

A polynomial is factored completely when it is written as the product of a prime polynomial and monomial, or it is the product of prime polynomials. A *prime polynomial* is a polynomial that cannot be factored. Use the following suggestions for factoring polynomials completely.

- Factor out the greatest monomial factor (GMF). The GMF is the largest monomial that is a factor of each term in the polynomial.
- If the polynomial has two terms, look for the difference of squares.
- If the polynomial has three terms, look for a perfect square trinomial or a pair of binomial factors.
- If the polynomial has four or more terms, group the terms, if possible, in ways that can be factored. Factor out common polynomials.
- Be sure each polynomial is prime.
- Check by multiplying all factors.
- Remember that not all polynomials can be factored.

For additional information, see List 154, "Common Factoring Formulas."

LIST 154

Common Factoring Formulas

While there are many ways to factor polynomials, the use of formulas can be very helpful. The list below contains formulas for factoring polynomials of degree 2 or higher.

Factoring the Greatest Common Factor

$ca + cb = c(a + b)$

$ca - cb = c(a - b)$

Difference of Squares

$a^2 - b^2 = (a - b)(a + b)$

Sum of Squares

$a^2 + b^2$ prime polynomial (cannot be factored over real numbers)

Perfect Square Trinomials

$a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2$

Other Polynomials

$a^2 + (c + d)a + cd = (a + c)(a + d)$

$a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$

$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$

$a^4 - 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = (a - b)^4$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$1 - a^n = (1 - a)(1 + a + a^2 + \ldots + a^{n-1})$

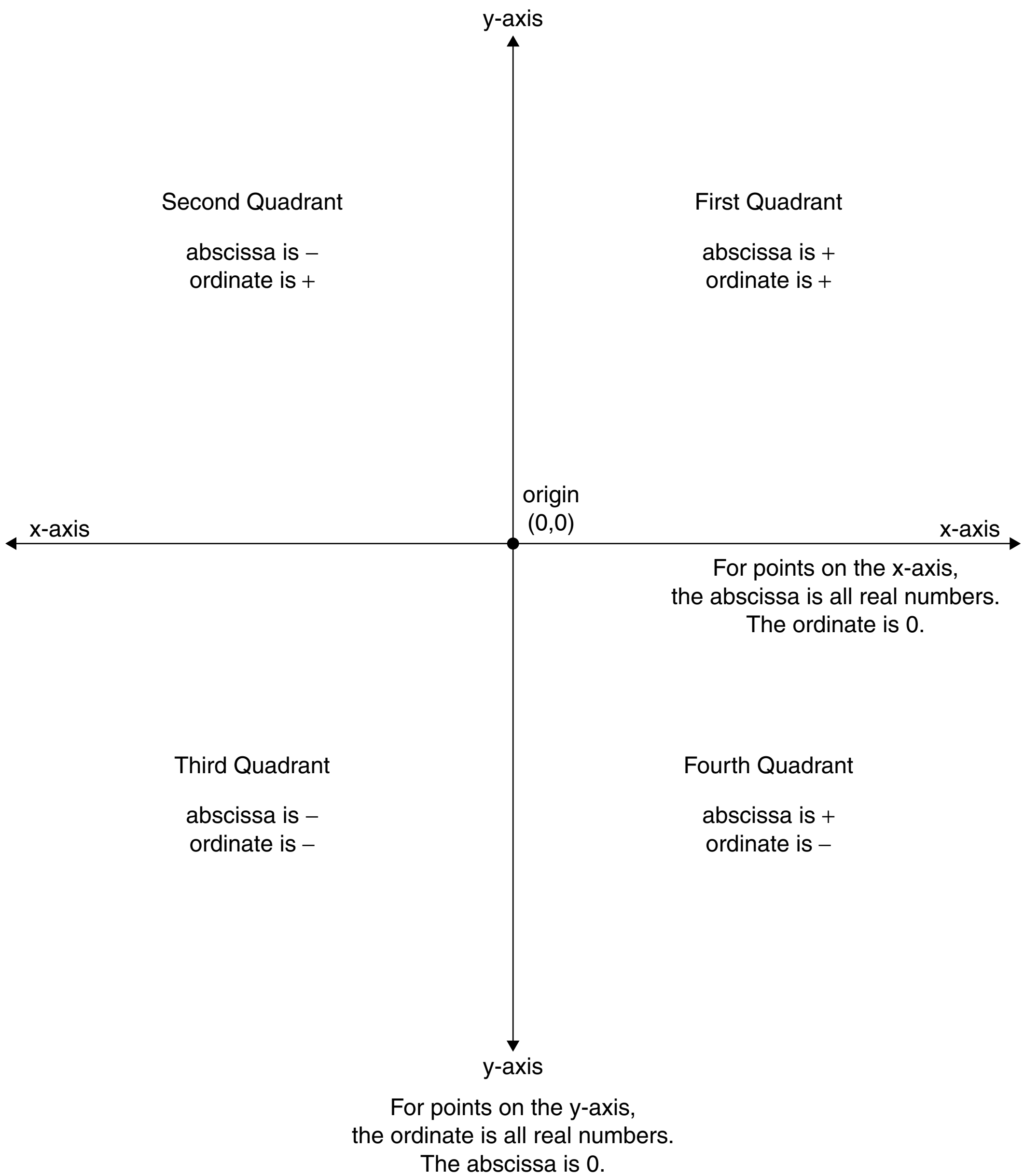
$a^n - b^n = (a - b)\left(a^{n-1} + ba^{n-2} + b^2a^{n-3} + \ldots + b^{n-2}a + b^{n-1}\right)$ for n positive and even

$a^n + b^n = (a + b)\left(a^{n-1} - ba^{n-2} + b^2a^{n-3} - \ldots - b^{n-2}a + b^{n-1}\right)$ for n positive and odd

LIST 155

Characteristics of the Coordinate Plane

The *coordinate plane* may be thought of as a flat surface divided into four parts, or quadrants, by the intersection of a vertical number line (called the *y*-axis) and a horizontal number line (called the *x*-axis). The coordinate plane is used to graph ordered pairs of the form (x,y), straight lines, and other functions or relations. The *x*-coordinate of the ordered pair is called the *abscissa.* The *y*-coordinate of the ordered pair is called the *ordinate.* A coordinate plane and some of its most important characteristics are shown below.



LIST 156

Plotting Points on the Coordinate Plane

Plotting, or graphing, points on the coordinate plane involves moving horizontally and vertically, depending on the values of x and y. The following "points" will help you to graph correctly.

1. In the ordered pair (x,y), the first coordinate (called the abscissa) is the value of x. Start from the origin. If the abscissa is:
   - 0, remain at the origin.
   - positive, move to the right the required number of spaces along the x-axis and stop.
   - negative, move to the left the required number of spaces along the x-axis and stop.
2. The second coordinate (called the ordinate) is the value of y. If the ordinate is:
   - 0, do not move up or down from the point where your pencil stopped after finding the abscissa. Graph the point by making a dot.
   - positive, move directly up the required number of spaces from where your pencil stopped after finding the abscissa. Graph the point by marking a dot.
   - negative, move directly down the required number of spaces from where your pencil stopped after finding the abscissa. Graph this point by marking a dot.
3. Label the point by writing the coordinates near the point.

   Following are some special points.
   - $(0,0)$ is the origin.
   - $(0,y)$ is a point on the y-axis, provided y is a real number.
   - $(x,0)$ is a point on the x-axis, provided x is a real number.
   - When $x > 0$ and $y > 0$, (x,y) is a point in the first quadrant.
   - When $x < 0$ and $y > 0$, (x,y) is a point in the second quadrant.
   - When $x < 0$ and $y < 0$, (x,y) is a point in the third quadrant.
   - When $x > 0$ and $y < 0$, (x,y) is a point in the fourth quadrant.

LIST 157

Common Forms of Linear Equations

A *linear equation* is an equation whose graph is a straight line. It is an equation of the first degree and is usually expressed in one of the following forms.

$ax + by = c$	**Standard Form** *a, b,* and c are integers. Both *a* and *b* cannot equal zero.
$x = k$	**Vertical Line** *k* is any real number. Vertical lines have no slope.
$y = k$	**Horizontal Line** *k* is any real number. Horizontal lines have a slope of zero.
$y = mx + b$	**Slope-Intercept Form** *m* stands for the slope. *b* stands for the *y*-intercept.
$y = mx$	**Slope-Intercept Form of a Line Passing Through the Origin** *m* stands for the slope.
$y - y_1 = m(x - x_1)$	**Point-Slope Form** *m* stands for the slope. (x_1, y_1) is a point on the line.

LIST 158

Formulas and the Coordinate Plane

Some equations are frequently used to graph lines or points on the coordinate plane. They are summarized below.

- Slope of a line given two points (x_1,y_1) and (x_2,y_2).
  - $m = \frac{y_2 - y_1}{x_2 - x_1}$
  - m stands for the slope.
  - Horizontal lines have a slope of 0.
  - Vertical lines have no slope.
- Slope-Intercept Equation: $y = mx + b$.
  - m stands for the slope.
  - b stands for the y-intercept.
- Standard form of a linear equation: $ax + by + c = 0$.
  - a, b, and c are integers.
  - Both a and b cannot equal 0.
- Point-Slope Form: $y - y_1 = m(x - x_1)$.
  - m stands for the slope.
  - (x_1,y_1) is a point on the line.
- Distance Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - d is the distance between two points, (x_1,y_1) and (x_2,y_2).
- Midpoint Formula: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
  - This formula gives the coordinate of the point halfway between (x_1,y_1) and (x_2,y_2).
- Parallel lines have the same slope.
  - $m_1 = m_2$ if and only if l_1 and l_2 are two nonvertical, noncollinear straight lines with slopes m_1 and m_2.
- The slopes of perpendicular lines are negative reciprocals of each other.
  - $m_1 \cdot m_2 = -1$, m_1, $m_2 \neq 0$ if and only if l_1 and l_2 are nonvertical, nonhorizontal straight lines with slopes m_1 and m_2.

LIST 159

Graphing Linear Equations in Two Variables on the Coordinate Plane

You can graph linear equations in two variables on the coordinate plane in three ways: plotting points, using intercepts, or using the slope-intercept method.

Plotting Points

1. Find two ordered pairs that satisfy the equation. This can be done by looking at the equation, or by choosing an x value (choose a value that will simplify the arithmetic), and then finding a corresponding y value.
2. Plot these points on the coordinate plane.
3. Draw a straight line through the points. This line is the graph of the equation.
4. Check the accuracy of your graph by plotting a third point. This point should be on the line. If it is not, go back and check your work and graph. All points should be collinear (lie on the same line).

Using Intercepts

1. Transform the equation into the form $Ax + By = C$.
2. Substitute 0 for y to find the x-intercept.
3. Substitute 0 for x to find the y-intercept.
4. Plot the intercept points.
5. Draw a straight line through the points. This is the graph of the line.

Using the Slope-Intercept Method

1. Transform the equation into the form of $y = mx + b$. If there is no y term, the graph is a vertical line. Solve for x. This is the x-intercept.
2. Using the equation $y = mx + b$, and, assuming there is a y term, graph the point $(0,b)$. This is the y-intercept.
3. Write the slope m as a fraction. Remember that $m = \frac{rise}{run}$.
4. From the y-intercept, count out the rise and run. Graph this point. If the rise is positive, count up. If the rise is negative, count down. If the run is positive, count to the right. If the run is negative, count to the left.
5. Draw a straight line through the y-intercept and the point plotted by using the slope. This is the graph of the line.

LIST 160

Graphing a Linear Inequality in Two Variables on the Coordinate Plane

The following steps and table will help you to graph linear inequalities in two variables on coordinate planes.

1. Graph the inequality as if it were an equation. This will enable you to find the boundary, which is the line that divides the coordinate plane into two half-planes. (You may find it helpful to refer to List 159, "Graphing Linear Equations in Two Variables on the Coordinate Plane.")
2. Graph the line. If the inequality symbol is ≥ or ≤, draw a solid line since these solutions are included. If the inequality symbol is > or <, draw a broken line since these solutions are not included.
3. Choose a point in the plane that is not on the line. Substitute the coordinates of this point in the inequality.
4. If the inequality is true, shade the half-plane that includes the point. If the inequality is false, shade the other half-plane.

The following table offers some guidelines.*

Equation	*Type of Line*	*Shaded*
$x \geq k$	Solid Vertical	Right
$x > k$	Broken Vertical	Right
$x \leq k$	Solid Vertical	Left
$x < k$	Broken Vertical	Left
$y \geq k$	Solid Horizontal	Above
$y > k$	Broken Horizontal	Above
$y \leq k$	Solid Horizontal	Below
$y < k$	Broken Horizontal	Below
$y \geq mx + b$	Solid	Above
$y > mx + b$	Broken	Above
$y \leq mx + b$	Solid	Below
$y < mx + b$	Broken	Below

*k stands for any real number, m stands for the slope, and b stands for the y-intercept.

LIST 161

Steps to Solve a System of Linear Equations in Two Variables

There are four methods to solve a system (more than one) of linear equations. Although any method can be used, some may be more efficient and direct for certain systems than others. A list of methods and how they can best be used follows.

Graphing Method

1. Draw the graph of each equation on the same coordinate plane. The lines will either intersect at one point, be parallel, or coincide.
2. If the lines intersect, the coordinates of the point of intersection are the solution to the two equations.
3. If the lines are parallel, there is no solution.
4. If the lines coincide (that is, the lines are identical), each point on the line is a solution. The number of solutions is infinite.

Use this method when you wish to approximate the solution. It is also most helpful when the solution is near the origin. This method is used the least for solving systems of linear equations.

Substitution Method

1. Solve one equation for one of the variables whose coefficient is one.
2. Substitute this expression in the equation you have not used. You should now have an equation in one variable. Solve this equation.
3. Substitute this expression in the equation you used in Step 1, and solve it.
4. Check your answers in both original equations.

Use this method when the coefficient of one of the variables is 1 or –1.

Addition-or-Subtraction Method

1. Add or subtract equations to eliminate one variable. Add the equations if the coefficients of one of the variables are opposites; subtract if the coefficients of one of the variables are the same.
2. Solve the equation resulting from Step 1.
3. Substitute this value in either of the original equations.
4. Check your answers in both of the original equations.

Use this method if the coefficients of one of the variables are the same or if the coefficients are opposites.

LIST 161
(Continued)

Multiplication with Addition-or-Subtraction Method

1. Multiply one or both equations so that the coefficients of one of the variables will be the same or opposite.
2. Follow steps 1 through 4 in the Addition-or-Subtraction Method.

Use this method for the following conditions:

- To clear the equations of fractions
- If the coefficients of a variable are relatively prime (have the greatest common factor of 1)
- If one of the coefficients of a variable is a factor (other than 1) of the other

LIST 162

Types of Functions

A *function* is a special type of relation in which every element in the domain is paired with exactly one element of the range. This is loosely translated as: "For each value of x, there is one and only one y." As a counterexample, if there are two or more y values for any value of x, the relation is not a function.

If a vertical line can be drawn anywhere on a graph and the vertical line intersects the graph at more than one point, then the graph is not the graph of a function. This is called the Vertical Line Test.

Following is a list of functions and their descriptions.

- Linear: $y = mx + b$ or $f(x) = mx + b$ where m and b are real numbers (m stands for the slope and b stands for the y-intercept). Special names for linear functions include:
  - Constant Function: $y = b$ or $f(x) = b$.

    The slope is zero.

    The graph is a horizontal line.
  - Identity Function: $y = x$ or $f(x) = x$.

    The slope is 1.

    The graph is a line passing through the origin.
  - Direct Variation: $y = mx$ or $f(x) = mx$.

    The slope is not equal to zero.

    The graph is a line passing through the origin.

- Absolute Value Function: $y = |x|$ or $f(x) = |x|$.
  - If $x \geq 0$, the graph is like the graph of $y = x$.
  - If $x < 0$, the graph is like the graph of $y = -x$.

- Greatest Integer Function: $y = [|x|]$ or $f(x) = [|x|]$.
  - This graph finds the greatest integer that is not greater than x.
  - The graph is a series of line segments with one open endpoint.

- Inverse Variation Function: $xy = k$ or $f(x) = \frac{k}{x}$; $k \neq 0, x \neq 0$.
  - The graph is a hyperbola.

- Quadratic Function: $y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$; $a \neq 0$.
  - The graph is a parabola.

- Cubic Function: $y = ax^3 + bx^2 + cx + d$ or $f(x) = ax^3 + bx^2 + cx + d$; $a \neq 0$.
  - The graph resembles a sideways S.

LIST 162

(Continued)

- Exponential Function: $y = b^x$ or $f(x) = b^x$; $b > 0$, $b \neq 1$, x is a real number.
  - The graph resembles part of a hyperbola.

- Logarithmic Function: $y = \log_a x$ if and only if $a^y = x$; $a > 0$, $a \neq 0$.
  - This function is the inverse of the Exponential Function.
  - The graph is the inverse of the graph of the Exponential Function.

Note: For Trigonometric Functions, see Lists 183 and 186 for definitions.

Some functions may be classified as odd or even. The properties of each are shown below:

- Even Function: $y = f(-x) = f(x)$ for all x in the domain.
  - The graph is symmetric with respect to the y-axis. [If (x,y) is on the graph, then so is $(-x,y)$.]
  - *Examples:* $y = |x|$, $y = x^2$

- Odd Function: $y = f(-x) = -f(x)$ for all x in the domain.
  - The graph is symmetric with respect to the origin. [If (x,y) is on the graph, then so is $(-x, -y)$.]
  - *Examples:* $y = x$, $y = x^3$

LIST 163

Direct Facts on Variation

Some functions are used so frequently in science and math that they have special names and general formulas. Types of "variations" fall into this category.

Common Types of Variations*

Formula	*Meaning*
$y = kx$ $\frac{y_1}{x_1} = \frac{y_2}{x_2}$	y varies directly as x or y is directly proportional to x.
$y = kx^2$ $\frac{y_1}{x_1^2} = \frac{y_2}{x_2^2}$	y varies directly as the square of x or y is directly proportional to the square of x.
$y = kx^3$ $\frac{y_1}{x_1^3} = \frac{y_2}{x_2^3}$	y varies directly as the cube of x or y is directly proportional to the cube of x.
$y = \frac{k}{x}$ $xy = k$ $x_1 y_1 = x_2 y_2$	y varies inversely as x or y is inversely proportional to x.
$y = \frac{k}{x_2}$ $x^2 y = k$ $x_1^2 y_1 = x_2^2 y_2$	y varies inversely as the square of x or y is inversely proportional to the square of x.
$z = kxy$ $\frac{z_1}{x_1 y_1} = \frac{z_2}{x_2 y_2}$	z varies jointly as x and y.
$z = \frac{kx}{y}$ $zy = kx$ $\frac{z_1 y_1}{x_1} = \frac{z_2 y_2}{x_2}$	z varies directly as x and inversely as y. This is a combined variation.

*k is a nonzero constant. It is called the *constant of variation* or the *constant of proportionality.*

LIST 164

Functional Facts About Functions

Functions are sometimes combined to form other sums, differences, products, quotients, and inverses. Following are some functional facts.

If f and g are any two functions with a common domain, then:

1. The sum of f and g, written $f + g$, is defined by $(f + g)(x) = f(x) + g(x)$.

2. The difference of f and g, written $f - g$, is defined by $(f - g)(x) = f(x) - g(x)$.

3. The product of f and g, written fg, is defined by $(fg)(x) = f(x)g(x)$.

4. The quotient of f and g, written $\frac{f}{g}$, is defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. $g(x) \neq 0$.

5. The composite of f and g, written $f \circ g$, is defined by $[f \circ g](x) = f[g(x)]$.

6. To find the inverse of a function, interchange the values for x and y and then solve for y. By doing this, the order of each pair in f is reversed. (Note that the inverse of a function is not always a function.)

7. Assume f and f^{-1} are inverse functions. Then $f(a) = b$ if and only if $f^{-1}(b) = a$.

8. f and f^{-1} are inverse functions if and only if their composites are the identity function,* meaning $[f \circ f^{-1}](x) = [f^{-1} \circ f](x) = x$.

*The identify function is $y = x$ or $f(x) = x$. See List 162, "Types of Functions."

LIST 165

Square Roots

Finding the square root of a number is the inverse of squaring a number. Since $5^2 = 25$ and $(-5)^2 = 25$, the square root of 25 is both 5 and –5. Square root notation and properties follow.

- $\sqrt{}$ is called the radical sign.
- The number written beneath the radical sign is called the radicand. *Example*: $\sqrt{a}$.
- $\sqrt{a}$ is used to denote the principal or nonnegative square root of a positive real number *a*.
- $-\sqrt{a}$ is used to denote the negative square root of a positive real number *a*.
- $\pm\sqrt{a}$ is used to denote the positive or negative square root of a positive real number *a*.
- The index of a root is the small number written above and to the left of the radical sign. It represents which root is to be taken. The index for square roots is 2. It is understood and therefore not included.
- $\left(\sqrt{a}\right)^2 = a$ where a is a positive real number.
- $\sqrt{a^2} = |a|$ where a is a real number.
- $\sqrt{a^{2m}} = |a|^m$ where a is any real number, and m is any positive integer.
- 0 has only one square root, $\sqrt{0} = 0$.
- Negative numbers do not have square roots in the set of real numbers. See List 170, "Imaginary Numbers and Their Powers."
- In accordance with the Product Property of Square Roots, $\sqrt{ab} = \sqrt{a}\sqrt{b}$, where a and b are nonnegative real numbers.
- In accordance with the Quotient Property of Square Roots, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, where a is any nonnegative real number and b is a positive real number.
- In accordance with the Property of Square Roots of Equal Numbers, $a^2 = b^2$ if and only if $a = b$ or $a = -b$, where a and b are real numbers.
- A radical is in simplest form when:
  - No radicand has a square root factor (other than 1).
  - The radicand is not a fraction.
  - No radicals are in the denominator.
- Only radicals with like radicands may be added or subtracted.

LIST 166

*N*th Roots

If the volume of a cube is 125 cubic units, then any side of the cube has a length of 5 units. You can write $\sqrt[3]{125} = 5$ because $5^3 = 125$. We can also say $\sqrt[3]{a} = b$, if $b^3 = a$.

The small number written in the upper left-hand corner of the radical is called the *index.* When finding the square root of a number, the index is generally not stated. It is understood to be 2.

For any integer $n \geq 2$, the nth root is defined as follows:

$\sqrt[n]{a} = b$, if and only if $b^n = a$

for $a \geq 0$ and $b \geq 0$ if n is even or

for any real number a if n is odd.

Note that when n is an even integer, $\sqrt[n]{a}$ is defined for nonnegative values of a.

Properties of *N*th Roots

a and b are real numbers. m and n are positive integers. Each property is valid for all values of a and b for which the equation is defined.

- $\left(\sqrt[n]{a}\right)^n = a$
- $\sqrt[n]{a^n} = |a|$ if n is even or
  $= a$ if n is odd
- $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
- $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad b \neq 0$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$

LIST 167

Powers and Roots

The following table shows the squares, square roots, cubes, and cubic roots of the numbers from 1 to 50. Where necessary, the values have been rounded to the nearest thousandth.

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$
1	1	1.000	1	1.000
2	4	1.414	8	1.260
3	9	1.732	27	1.442
4	16	2.000	64	1.587
5	25	2.236	125	1.710
6	36	2.449	216	1.817
7	49	2.646	343	1.913
8	64	2.828	512	2.000
9	81	3.000	729	2.080
10	100	3.162	1,000	2.154
11	121	3.317	1,331	2.224
12	144	3.464	1,728	2.289
13	169	3.606	2,197	2.351
14	196	3.742	2,744	2.410
15	225	3.873	3,375	2.466
16	256	4.000	4,096	2.520
17	289	4.123	4,913	2.571
18	324	4.243	5,832	2.621
19	361	4.359	6,859	2.668
20	400	4.472	8,000	2.714
21	441	4.583	9,261	2.759
22	484	4.690	10,648	2.802
23	529	4.796	12,167	2.844
24	576	4.899	13,824	2.884
25	625	5.000	15,625	2.924
26	676	5.099	17,576	2.962
27	729	5.196	19,683	3.000
28	784	5.292	21,952	3.037
29	841	5.385	24,389	3.072
30	900	5.477	27,000	3.107
31	961	5.568	29,791	3.141
32	1,024	5.657	32,768	3.175
33	1,089	5.745	35,937	3.208
34	1,156	5.831	39,304	3.240
35	1,225	5.916	42,875	3.271
36	1,296	6.000	46,656	3.302
37	1,369	6.083	50,653	3.332
38	1,444	6.164	54,872	3.362
39	1,521	6.245	59,319	3.391
40	1,600	6.325	64,000	3.420
41	1,681	6.403	68,921	3.448
42	1,764	6.481	74,088	3.476
43	1,849	6.557	79,507	3.503
44	1,936	6.633	85,184	3.530
45	2,025	6.708	91,125	3.557
46	2,116	6.782	97,336	3.583
47	2,209	6.856	103,823	3.609
48	2,304	6.928	110,592	3.634
49	2,401	7.000	117,649	3.659
50	2,500	7.071	125,000	3.684

LIST 168

Conditions for Simplifying Radical Expressions

An expression having a square root radical is in its simplest form when:

- No radicand has a square factor other than 1.
- The radicand is not a fraction.
- No radicals are in the denominator.

This can be extended for all radical expressions (not only square root radicals) by adding an additional stipulation.

A radical expression is in simplest form when:

- None of the factors of the radicand can be written as powers greater than or equal to the index. (No perfect squares, except 1, can be factors of the quantity under the square root sign; no perfect cubes, except 1, can be factors of the quantity under the cube root sign; and so on.)
- The radicand is not a fraction.
- No radicals are in the denominator.

LIST 169

Steps for Solving a Quadratic Equation

There are four methods to solving quadratic equations: (1) factoring, (2) using the square root property, (3) completing the square, and (4) using the quadratic formula. Factoring and using the square root property may be used only under certain conditions; completing the square and using the quadratic formula can always be used. A description of each method and when it may be best used follows.

Any quadratic equation has at most two solutions. Some may have the same solution twice. Others may have no real solutions. Some may have only one solution.

Factoring

Use this method if $ax^2 + bx + c$ can be factored.

1. Write the equation in the form $ax^2 + bx + c$. a, b, and c are real numbers, $a \neq 0$.	Solve $x^2 + 3x = 4$. $x^2 + 3x - 4 = 0$
2. Factor the polynomial.	$(x + 4)(x - 1) = 0$
3. Use the Zero-Product Property to set each factor equal to zero.	$(x + 4) = 0 \qquad (x - 1) = 0$
4. Solve each linear equation that results.	$x = -4 \qquad x = 1$

Using the Square Root Property

Use this method if $x^2 = k$ or $(ax + b)^2 = k$, $k \neq 0$.

1. Transform the equation so that a perfect square is on one side of the equation and a constant greater than or equal to zero is on the other.	Solve $x^2 - 16 = 0$. $x^2 = 16$
2. Use the Square Root Property to find the square root of each number. Remember that finding the square root of a constant yields positive and negative values.	$\sqrt{x^2} = \pm\sqrt{16}$
3. Solve each resulting equation. (If you are finding the square root of a negative number, there is no real solution.)	$x = \pm 4$

LIST 169

(Continued)

Completing the Square

This method may always be used to solve quadratic equations. It is best used, however, if the coefficient of the linear term is even.

1. Transform the equation so that the quadratic term plus the linear term equals a constant.
   Solve $x^2 + 12x + 2 = 0$.
   $x^2 + 12x = -2$
2. Divide each term by the coefficient of the quadratic term if it does not equal 1.
3. Complete the square:
   - Multiply the coefficient of x by $\frac{1}{2}$. $\quad 12 \cdot \frac{1}{2} = 6$
   - Square this value. $\quad 6^2 = 36$
   - Add the result to both sides of the equation. $\quad x^2 + 12x + 36 = -2 + 36$
   - Express one side of the equation as the square of a binomial and the other as a constant. $\quad (x + 6)^2 = 34$
4. Follow steps 2 and 3 of Using the Square Root Property.

$$\sqrt{(x+6)^2} = \pm 34$$
$$x + 6 = \pm 34$$
$$x = -6 + \sqrt{34}$$
$$x = -6 - \sqrt{34}$$

Using the Quadratic Formula

This method may always be used for any equation of the form $ax^2 + bx + c = 0$.

1. Write the equation in the form $ax^2 + bx + c = 0$. a, b, and c are real numbers, $a \neq 0$.
   Solve $3x^2 = 8x - 2$
   $3x^2 - 8x + 2 = 0$
2. The two roots (if they exist) are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$x = \frac{8 \pm \sqrt{64 - 24}}{6}$$
$$x = \frac{8 \pm \sqrt{40}}{6}$$
$$x = \frac{8 \pm 2\sqrt{10}}{6}$$
$$x = \frac{4 \pm \sqrt{10}}{3}$$

LIST 170

Imaginary Numbers and Their Powers

Equations such as $x^2 = -25$ have no solution in the set of real numbers. Equations like this have solutions that fall in the realm of imaginary numbers. Imaginary numbers use the imaginary unit i, which is defined as the square root of -1.

$$i = \sqrt{-1} \qquad i^2 = -1$$

The solution to the equation $x^2 = -25$ in the set of imaginary numbers is $x = \pm 5i$.

Generally, if $r > 0$, then $\sqrt{-r} = i\sqrt{r}$.

Some interesting patterns emerge for the powers of i as the exponent increases, as shown below.

$i = \sqrt{-1}$ (by definition)

$i^2 = -1$ (by definition)

$i^3 = i^2 \cdot i = -1 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i^6 = i^3 \cdot i^3 = -i \cdot -i = -1$

$i^7 = i^3 \cdot i^4 = -i \cdot 1 = -i$

$\vdots$

Any positive integer power of i equals 1, –1, or i or $-i$.

LIST 171

Discriminant and Coefficients

A quadratic equation of the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers, $a \neq 0$, has a discriminant equal to $b^2 - 4ac$, which determines the number and the kinds of solutions to the equation.

If $b^2 - 4ac$ is	*then the equation will have*
negative	two (conjugate) imaginary numbers
zero	one real root (double real root)
positive	two different real roots

For any rational numbers a, b, and c, $a \neq 0$:

If $b^2 - 4ac$ is	*then the two roots are*
positive and the square of a rational number	rational
positive and is not the square of a rational number	irrational

For any quadratic equation of the form $ax^2 + bx + c = 0$, and a, b, and c are real numbers, $a \neq 0$, the sum of the roots equals $-\frac{b}{a}$ and the product of the roots equals $\frac{c}{a}$.

LIST 172

Quadratic Functions

The function f given by $f(x) = ax^2 + bx + c$ is a quadratic function, provided $a \neq 0$. If its domain is all the real numbers, then:

- Its graph is a parabola.
- Its vertex is the highest or lowest point on the graph, depending on the value of a.
- If $a > 0$, the vertex is the lowest point and the parabola opens upward.
- If $a < 0$, the vertex is the highest point and the parabola opens downward.
- The vertex is the point $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.
- The axis of symmetry is the line $x = -\frac{b}{2a}$.
- The x-intercepts (if any) can be found by solving $f(x) = 0$ for x by factoring or by using the quadratic formula.
- The y-intercept is c.

For reference, the quadratic formula is stated below:

If $ax^2 + bx + c = 0$, $a \neq 0$, and $b^2 - 4ac \geq 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

LIST 173

Graph of a Circle

A *circle* is one of the conic sections consisting of all points in a plane at a fixed distance from a fixed point.

Some important properties of the graph of a circle centered at the origin are shown below:

- The standard form is $x^2 + y^2 = r^2$. $r \geq 0$
  (If $r = 0$, then the circle is called a point-circle.)
- The center is (0,0).
- The extreme points are $(-r,0)$, $(r,0)$, $(0,r)$, and $(0, -r)$.
- Lines of symmetry are infinite in number and include $x = 0$ and $y = 0$.
- The x-intercepts are r and $-r$.
- The y-intercepts are r and $-r$.
- This circle is not a function.

The properties of the graph of a circle centered at (h,k) include:

- The standard form is $(x - h)^2 + (y - k)^2 = r^2$. $r \geq 0$
  (If $r = 0$, then the circle is called a point-circle.)
- The center is (h,k).
- The extreme points are $(h - r,k)$, $(h + r,k)$, $(h,k + r)$, and $(h,k - r)$.
- Lines of symmetry are infinite in number and include $x = h$ and $y = k$.
- Set $y = 0$ and solve for x to find the x-intercepts (if they exist).
- Set $x = 0$ and solve for y to find the y-intercepts (if they exist).
- The graph is not a function.

LIST 174

Graph of an Ellipse

An *ellipse* is an elongated circle. It is one of the four conic sections.

The properties of the graph of an ellipse centered at the origin are provided below:

- The standard form is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $a > 0$, $b > 0$.
- The center is (0,0).
- The extreme points are $(a,0)$, $(-a,0)$, $(0,b)$, and $(0,-b)$.
- The lines of symmetry are $x = 0$ and $y = 0$.
- The x-intercepts are a and $-a$.
- The y-intercepts are b and $-b$.
- This ellipse is not a function.
- If $a = b$, then the ellipse is a circle.
- If $a > b$, then the x-axis is the major axis.
- If $a < b$, then the y-axis is the major axis.

The properties of the graph of an ellipse centered at (h,k) include:

- The standard form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. $a > 0$, $b > 0$.
- The center is (h,k).
- The extreme points are $(h+a,k)$, $(h-a,k)$, $(h,k+b)$, and $(h,k-b)$.
- The lines of symmetry are $x = h$ and $y = k$.
- Set $y = 0$ and solve for x to find the x-intercepts (if they exist).
- Set $x = 0$ and solve for y to find the y-intercepts (if they exist).
- This ellipse is not a function.
- If $a = b$, then the ellipse is a circle.
- If $a > b$, then the line $y = k$ is the major axis.
- If $a < b$, then the line $x = h$ is the major axis.

LIST 175

Graph of a Parabola

A *parabola* is a conic section shaped like a fountain. It can open up, down, to the left, or to the right. If it opens up or down, it is a quadratic function. Four general forms are addressed below.

The properties of the graph of a parabola with its vertex at the origin and which opens up or down are as follows:

- The standard form is $y = ax^2$. $a \neq 0$.
- The vertex is (0,0).
- The line of symmetry is $x = 0$.
- The y-intercept is 0.
- The parabola opens up if $a > 0$.
- The parabola opens down if $a < 0$.
- This parabola is a function.

The properties of the graph of a parabola with its vertex not at the origin and which opens up or down are as follows:

- The standard form is $y = ax^2 + bx + c$. $a \neq 0$.

  This can be transformed to $y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$ by completing the square.
- The vertex is $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.
- The line of symmetry is $x = -\frac{b}{2a}$.
- Use standard form and set $y = 0$ to find the x-intercept (if it exists).
- The y-intercept is c (from the standard form).
- The parabola opens up if $a > 0$.
- The parabola opens down if $a < 0$.
- This parabola is a function.

LIST 175

(Continued)

The properties of the graph of a parabola with its vertex at the origin and which opens left or right are as follows:

- The standard form is $x = ay^2$. $a \neq 0$.
- The vertex is (0,0).
- The line of symmetry is $y = 0$.
- The x-intercept is 0.
- The parabola opens to the right if $a > 0$.
- The parabola opens to the left if $a < 0$.
- This parabola is not a function.

The properties of the graph of a parabola with its vertex not at the origin and which opens left or right are as follows:

- The standard form is $x = ay^2 + by + c$. $a \neq 0$.

  This can be transformed to $x = a\left(y + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$ by completing the square.
- The vertex is $\left(c - \frac{b^2}{4a}, \frac{b}{2a}\right)$.
- The line of symmetry is $y = -\frac{b}{2a}$.
- The x-intercept is c (from the standard form).
- Use the standard form and set $x = 0$ to find the y-intercept (if it exists).
- The parabola opens to the right if $a > 0$.
- The parabola opens to the left if $a < 0$.
- This parabola is not a function.

LIST 176

Graph of a Hyperbola

A *hyperbola,* the fourth and final conic section, is a curve with two branches, each of which approaches other lines called asymptotes. The asymptotes are not part of the graph, but they are helpful in drawing the graph. A hyperbola may open to the right and left or up and down.

The properties of the graph of a hyperbola with the center at the origin and which opens right and left are as follows:

- The standard form is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. $a \neq 0$, $b \neq 0$.
- The center is (0,0).
- The vertices are $(-a,0)$ and $(a,0)$.
- The lines of symmetry are $x = 0$ and $y = 0$.
- The asymptotes are $y = -\frac{b}{a}x$ and $y = \frac{b}{a}x$.
- The x-intercepts are $-a$ and a.
- The y-intercepts do not exist.
- The hyperbola opens to the right and left.
- This hyperbola is not a function.

The properties of the graph of a hyperbola that opens right and left and is centered at a point other than the origin are as follows:

- The standard form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. $a \neq 0$, $b \neq 0$.
- The center is (h,k).
- The vertices are $(h-a,k)$ and $(h+a,k)$.
- The lines of symmetry are $x = h$ and $y = k$.
- The asymptotes are $y - k = \frac{b}{a}(x-h)$ and $y - k = -\frac{b}{a}(x-h)$.
- The hyperbola opens to the right and left.
- This hyperbola is not a function.

LIST 176

(Continued)

The properties of the graph of a hyperbola that opens up and down and is centered at the origin are as follows:

- The standard form is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. $a \neq 0$, $b \neq 0$.
- The center is (0,0).
- The vertices are $(0,b)$ and $(0, -b)$.
- The lines of symmetry are $x = 0$ and $y = 0$.
- The asymptotes are $y = -\frac{b}{a}x$ and $y = \frac{b}{a}x$.
- The x-intercepts do not exist.
- The y-intercepts are b and $-b$.
- The hyperbola opens up and down.
- This hyperbola is not a function.

The properties of the graph of a hyperbola that opens up and down and is centered at a point other than the origin are as follows:

- The standard form is $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$. $a \neq 0$, $b \neq 0$.
- The center is (h,k).
- The vertices are $(h,k + b)$ and $(h,k - b)$.
- The lines of symmetry are $x = h$ and $y = k$.
- The asymptotes are $y - k = \frac{b}{a}(x - h)$ and $y - k = -\frac{b}{a}(x - h)$.
- The hyperbola opens up and down.
- This hyperbola is not a function.

LIST 177

Properties of Complex Numbers

A *complex number* is any number of the form $a + bi$ where a and b are real numbers and $i^2 = -1$. The set of all complex numbers $a + bi$ with $b = 0$ is the set of real numbers. In a complex number of the form $a + bi$, a is called the *real* part and b is called the *imaginary* part. If $a = 0$, then the complex number of the form $a + bi$ is called *pure imaginary.*

Equality is defined as follows:

$a + bi = c + di$ if and only if $a = c$ and $b = d$.

Many of the properties of real numbers are also properties of complex numbers. In the properties of complex numbers listed below, w, y, and z are complex numbers.

Closure	$w + y$ is a unique complex number. $w \cdot y$ is a unique complex number.
Commutative Laws	$w + y = y + w$ $w \cdot y = y \cdot w$
Associative Laws	$w + (y + z) = (w + y) + z$ $w(yz) = (wy)z$
Identity Laws	$w + 0 = w$ $w \cdot 1 = w$
Distributive Law	$w(y + z) = wy + wz$
Additive Inverse	$w + (-w) = 0$
Multiplicative Inverse	$w \cdot w^{-1} = 1 \qquad w \neq 0$

LIST 178

Operations with Complex Numbers

Operations with complex numbers can be compared to corresponding operations for polynomials. In the operations below, $a + bi$ and $c + di$ are complex numbers. c is any real number.

Addition	Add the real parts and add the imaginary parts. $(a + bi) + (c + di) = (a + c) + (b + d)i$
Subtraction	Subtract the real parts and subtract the imaginary parts. $(a + bi) - (c + di) = (a - c) + (b - d)i$
Distributivity	Multiply the real part by c and multiply the imaginary part by c. $c(a + bi) = ca + cbi$
Multiplication	Carry out the multiplication as if the numbers were binomials and replace i^2 with -1. $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$
Division	Multiply both the numerator and denominator of the fraction by the conjugate of the denominator. See List 179, "Conjugate Complex Numbers." $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{\overline{c+di}}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$ $c + di \neq 0 + 0i$

LIST 179

Conjugate Complex Numbers

Conjugate complex numbers, which are also called *complex imaginaries,* are complex numbers that are identical except that the pure imaginary terms have opposite signs or are both zero.

$a + bi$ and $a - bi$ are conjugate complex numbers, where a and b are real numbers and $i^2 = -1$, and bi and $-bi$ are pure imaginary terms.

The conjugate of a complex number is denoted by a raised line ¯. $\overline{a+bi}$ is read "the conjugate of $a + bi$" and equals $a - bi$.

Important properties of conjugates follow:

- $\overline{a} = a$ where a is a real number because $a = a + 0i$ and $a = a - 0i = a$.
- $z \cdot \overline{z} = (a+bi)(a-bi) = a^2 - b^2$ where $z = a + bi$.
- $\overline{z+w} = \overline{z} + \overline{w}$. The conjugate of a sum is the sum of the conjugates. (z and w represent complex numbers.)
- $\overline{zw} = \overline{z} \cdot \overline{w}$. The conjugate of a product is the product of the conjugates. (z and w represent complex numbers.)
- $\overline{z^n} = \left(\overline{z}\right)^n$ where z is a complex number and n is a positive integer.
- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ where z and w represent complex numbers, $w \neq 0 + 0i$.

LIST 180

Vectors

A *vector* describes quantities that involve both magnitude (size) and direction. In geometry, a vector is a directed line segment, often described by a line segment with an arrow at one end. In algebra, a vector is described by the coordinates of the initial and terminal points of the directed line segments.

Definitions and notations about vectors are listed below.

- The ordered pair notation to describe a vector is $\langle a_1, a_2 \rangle$.
- The components of a vector are the numbers a_1 and a_2.
- $\overrightarrow{PQ}$ is the vector associated with the directed line segment with initial point $P = (x_0, y_0)$ and terminal point $Q = (x_1, y_1)$. This vector has components $x_1 - x_0$ and $y_1 - y_0$.
- Vectors such as $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ are equal if $a_1 = b_1$ and $a_2 = b_2$.
- Vectors are denoted by lowercase boldface letters, starting at the beginning of the alphabet. These letters are used to distinguish vectors from numbers that are sometimes called *scalars.* Since it is difficult to write a boldface letter by hand, a vector may be written as a lowercase letter with an arrow over it.
- A zero vector, $\langle 0, 0 \rangle$, is noted by **0**.
- The length (or norm) of a vector is $|a| = \sqrt{a_1^2 + a_2^2}$.
- A unit vector has a length of 1. $\boldsymbol{i} = \langle 1, 0 \rangle$ and $\boldsymbol{j} = \langle 0, 1 \rangle$ are unit vectors.
- A combination of vectors is $\boldsymbol{a} = \langle a_1, a_2 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2 \rangle$.
  Other examples include:
  $\boldsymbol{a} + \boldsymbol{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$
  $\boldsymbol{a} - \boldsymbol{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$
  $c\boldsymbol{a} = \langle ca_1, ca_2 \rangle$ where c is a number

LIST 180

(Continued)

- Properties of vectors include:

$\mathbf{0} + \boldsymbol{a} = \boldsymbol{a} + \mathbf{0} = \mathbf{a}$

$0\boldsymbol{a} = \mathbf{0}$

$c\mathbf{0} = \mathbf{0}$

$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$

$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-1)\boldsymbol{b}$

$\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$

$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$

$1\boldsymbol{a} = \boldsymbol{a}$

$c(\boldsymbol{a} + \boldsymbol{b}) = c\boldsymbol{a} + c\boldsymbol{b}$

$|c\boldsymbol{a}| = |c||\boldsymbol{a}|$

Any vector can be expressed $\boldsymbol{a} = \langle a_1, a_2 \rangle$ as a combination of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. Note the following:

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$$
$$|a_1\boldsymbol{i} + a_2\boldsymbol{j}| = \sqrt{a_1^2 + a_2^2}$$
$$(a_1\boldsymbol{i} + a_2\boldsymbol{j}) + (b_1\boldsymbol{i} + b_2\boldsymbol{j}) = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j}$$
$$(a_1\boldsymbol{i} + a_2\boldsymbol{j}) - (b_1\boldsymbol{i} + b_2\boldsymbol{j}) = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j}$$
$$c(a_1\boldsymbol{i} + a_2\boldsymbol{j}) = ca_1\boldsymbol{i} + ca_2\boldsymbol{j}$$

LIST 181

Matrices

A *matrix* is a set of quantities arranged in rows and columns to form a rectangular array, usually enclosed in parentheses. Matrices do not have a numerical value; they are used to represent relations between quantities. Matrices may also be used to represent and solve simultaneous equations.

Facts about matrices are shown below.

- If there are m rows and n columns, the matrix is an $m \times n$ matrix. This is called the order of the matrix.
- Matrices are named with capital letters.
- Individual members in a matrix are called elements (or entries) of the matrix.
- Particular elements may be identified by the horizontal row and vertical column to which they belong. a_{ij} denotes the element in the ith row and jth column of matrix A.
- Two matrices are equal if they are the same size and if all the corresponding elements are the same.
- To add matrices, add corresponding elements together to obtain another matrix of the same order.
- Only matrices of the same order may be added.
- To subtract matrices, subtract corresponding elements to obtain another matrix of the same order.
- Only matrices of the same order may be subtracted.
- To multiply a matrix by a number (also called a scalar), multiply each element by the scalar.
- To multiply matrices, multiply row by column and add.

For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 8 & 11 & 14 \\ 9 & 12 & 15 \\ 10 & 13 & 16 \end{bmatrix} = \begin{bmatrix} 56 & 72 & 92 \\ 137 & 182 & 227 \end{bmatrix}$$

$$1 \times 8 + 2 \times 9 + 3 \times 10 = 56$$
$$1 \times 11 + 2 \times 12 + 3 \times 13 = 72$$
$$1 \times 14 + 2 \times 15 + 3 \times 16 = 92$$
$$4 \times 8 + 5 \times 9 + 6 \times 10 = 137$$
$$4 \times 11 + 5 \times 12 + 6 \times 13 = 182$$
$$4 \times 14 + 5 \times 15 + 6 \times 16 = 227$$

LIST 181
(Continued)

- Two matrices A and B may be multiplied only if the number of columns in A is the same as the number of rows in B. Multiplication is not commutative.
- The determinant is a function of a square matrix derived by multiplying and adding the elements together to obtain a single number.
  - The determinant of a 1×1 matrix $[a_1]$ is its element.
  - The determinant of a 2×2 matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$
  - The determinant of a 3×3 matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} =$

    $a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$
- For any square matrix A, A has an inverse denoted A^{-1} if and only if the determinant of A does not equal zero.

LIST 182

Types of Matrices

Several specific types of matrices and their distinguishing characteristics are noted below.

Square Matrix—a matrix that has the same number of rows and columns. The diagonal from the top left to the bottom right is the leading diagonal (or principal diagonal). The sum of the elements in this diagonal is called the trace, or spur, of the matrix.

Row Matrix—a matrix with only one row.

Column Matrix—a matrix with only one column.

Zero Matrix (*Null Matrix*)—a matrix in which all the elements are equal to zero.

Unit Matrix (*Identity Matrix*)—a square matrix in which all the elements in the leading diagonal are one and the other elements are equal to zero.

Diagonal Matrix—a square matrix in which all the elements are zero except those in the leading diagonal.

Triangular Matrix—a square matrix in which either all the elements above the leading diagonal are zero or all the elements below the leading diagonal are zero.

Conformable Matrices—two matrices in which the number of columns in one is the same as the number of rows in the other.

Transpose of a Matrix—the matrix that results from interchanging the rows and columns.

Negative of a Matrix—the matrix whose elements are opposite each corresponding element of the original matrix.