1 Introduction

1.1 Introduction

1. In the following examples we briefly illustrate the problems considered in the book and the results obtained.

A. *A cylindrical mirror* is given; its director is an arc of a circle. The mirror is illuminated by a field with the polarization parallel to the cylinder's axis. The induced current has the same direction. For simplicity, we assume that both the field and the current do not depend on the coordinate in this direction. Can this current generate (at a suitable illumination) the pattern $1 + \cos\varphi$, or a pattern close to it?

The answer to this question depends on the value of the circle radius a. It is impossible to generate this pattern or even the one close to it if $J_0(ka) = 0$, where J_0 is the Bessel function and $k = \omega/c$ is the wave number (ω is the frequency). The closest pattern to the given one is $\cos\varphi$. This result does not depend on the mirror width.

The pattern can bear the stamp of the region in which the current is located. This stamp cannot be erased by changing the current distribution. If the condition $J_0(ka) = 0$ holds, then it is impossible for the pattern generated by the current, located on such a mirror, to be equal (even approximately) to a pattern, the Fourier series of which contains the constant term. If ka is not equal to a zero of the function J_0, then it is possible to approximate such a pattern or even realize it, but if ka is close to this zero, then the current generating the pattern must be very large.

B. *A metal screen* is a part of a sphere. It is illuminated by an electromagnetic impulse. Is it possible to verify whether the screen is really a part of a sphere, and to find its radius by the measured pattern of the scattered field? Both the spatial and time structures of the impulse as well as the shape of the mirror contour are unknown.

The answer is positive. To do it, the vector pattern generated (at any illumination) by the current on the screen should be multiplied by a certain weight vector function, and the product should be integrated over the solid angle 4π and Fourier-transformed over time. If the screen is a part of a sphere, then at some frequencies the Fourier transform will be equal to zero. The sphere radius can be calculated by values of these frequencies.

The above examples concerning the parts of cylindrical or spherical surfaces are not exotic. There are many such surfaces, moreover, in their neighborhood infinite numbers of other surfaces of this kind exist. This fact makes the study of the problem reasonable.

The shape of the surface, where the currents are located, can be decisive for the approximation not only of any pattern by the patterns of these currents, but it is also relevant to the fields in the near zone.

C. *A plane is illuminated by a beam* incoming from the antenna located in another plane parallel to the given one. Can some given field be created on it? The answer is positive only if the field fulfils the condition that some expression, containing this field, equals zero. The problem is a two-dimensional generalization of the problem on the existence condition for the field, located on the finite straight-line segment, which creates a given pattern. In both cases the problem is reduced to a first-kind integral equation. The above condition means that the *pseudo-solution* to this equation solves it, that is, the solution exists. In the case of the one-dimensional problem on the pattern of a linear current, the above condition means that the given pattern belongs to the class of functions defined by the Paley–Wiener theorem.

2. Using simple reasoning, one can easily explain the impossibility of approximating some class of patterns from the first example. Let us give such an explanation, emphasizing that it is not universal – in general, the physical explanation is more complicated.

As in most of the book, we will consider here the two-dimensional scalar formulation, because it is shorter and more demonstrative. In essence, the three-dimensional vector problems are not more complicated, but they are much more cumbersome. These problems will be considered in Chapter 5.

First, we prove that if $J_0(ka) = 0$, then no current on the whole circle $r = a$ generates a pattern having a constant term in the Fourier series. At this frequency there exists the solution to the homogeneous Helmholtz equation for the electric field, which equals zero on the circle and has no singularity inside it. This solution is $\widehat{u}(r, \varphi) = J_0(kr)$. In other words, the circle is a *resonant* one.

Consider an auxiliary interior problem on the field $\widehat{u}(r, \varphi)$ in a hollow metal cylinder of given radius $r = a$. At the resonant frequency the eigenoscillation $\widehat{u}(r, \varphi) = J_0(kr)$ can exist in such a volume. The current on the walls is proportional to $\partial u/\partial r$ and does not depend on the angle φ. This current generates a field equal to zero outside the cylinder. Below we will use only the fact that the current, independent of the angle φ, is not radiating at this frequency.

We return to the problem of the field generated by an arbitrary current on the circle arc. Starting with the case of the whole circle, expand the current in the Fourier series. Every term of the series is proportional to $\cos n\varphi$ or $\sin n\varphi$ $(n = 0, 1, ...)$ and generates the field with the same angular dependence. But the zero-order term does not generate a field outside the circle, therefore for $r > a$ such a term is absent in the field expansion of any current as well as in the pattern.

This result is also valid for any arc of the circle, in spite of the fact that the arc is not a closed line and, therefore, nonradiating current cannot be induced on it. Assume that some current on the arc generates a pattern with zero-order term in the Fourier series. Then we can supply the arc to the whole circle and set the current to be zero on the supplementary arc. In this way we have constructed the current on the whole circle, generating the pattern with nonzero constant term. But, as it follows from the above, it is impossible.

The direct proof of the above statement is elementary for this example. If C is a circle arc of radius a and $j(\theta)$ is a current, located on C, then the pattern generated by $j(s)$ is (with accuracy to a nonessential factor)

$$f(\varphi) = \int\limits_C e^{ika\cos(\varphi-\theta)}\, j(\theta) d\theta. \tag{1.1}$$

The constant term in the Fourier series for the pattern is (with the same accuracy)

$$\int_0^{2\pi} f(\varphi)d\varphi = 2\pi J_0(ka)\int_C j(\vartheta)d\vartheta. \tag{1.2}$$

If $J_0(ka) = 0$, then the zeroth term in the series for $f(\varphi)$ is absent. This property does not depend on both the current and the length of arc C. If C is the whole circle, then this proves once more that, at the resonance, any current on the walls creates a field which does not have the zeroth Fourier term. However, the multiplier $J_0(ka)$ is factored out for the circle arc as well.

3. Let us state the problem in more specific terms, but still without aspiring to an exact formulation. Consider the current $j(s)$ located on the given line C (s is the coordinate on C). It generates the pattern

$$f(\varphi) = \int_C \mathcal{K}(s,\varphi)\, j(s)ds. \tag{1.3}$$

The form of a smooth kernel $\mathcal{K}$ in (1.3) is not important here. (In the three-dimensional vector case, the line C should be replaced by a surface, $\mathcal{K}$ – by a functional matrix, and so on.)

In the antenna synthesis theory, equation (1.3) is considered as the integral equation on the current $j(s)$. We are interested in the problem on the existence of a solution to this equation or to an equation in which $f(\varphi)$ is replaced by another function close to $f(\varphi)$ in the quadratic metric. The current $j(s)$ should have a finite norm. First of all, we will investigate how the *existence of a solution to this equation depends on the line C*.

For the validity of most of the results obtained below, it is not necessary for the norm of $j(s)$ to be finite. Moreover, it is acceptable that the current may have singularities and $|\,j(s)|^2$ may not be integrable. It is only significant that the current should be integrable itself, so that the integral on the right side of expression (2.6) exists (see below). This condition is fulfilled also for the current at the border of a semi-plane (for both polarizations), and for $j(s) \sim \delta(s - s_0)$, that is, for the approximation, usually used in the antenna array theory. However, for simplicity (particularly in Chapter 4), we will require $|j(s)|^2$ to be integrable.

There exist such lines, for which equation (1.3) has no solution, and (what is important) the equation, in which $f(\varphi)$ is replaced by a function close to it, has no solution, either. To obtain the solvable equation, we should change $f(\varphi)$ by a finite value. In terms of functional analysis this means that the complete set of currents $j(s)$ generates a noncomplete set of patterns $f(\varphi)$. It turns out, that line C possesses this property, if there exists a solution to the homogenous Helmholtz equation, equal to zero on C. In the above example “A” this solution is $J_0(kr)$. The investigation of such a solution turns out to be a very efficient method for solving the problem of approximability and the related ones.

The most important result is that there are “many” such lines and surfaces, and “many” patterns $f(\varphi)$, for which equation (1.3) has no solution even if $f(\varphi)$ is replaced by any function “close” to it. Therefore, the effect of nonapproximability and its consequences deserves a detailed analysis. We will explain below, what “many” and “close” mean .

1.2 Subject and Method of Investigation

1. In the first five chapters of the book, some properties of electromagnetic fields, it is three-dimensional vector problems are investigated. Almost all of the questions we are interested in are solved for these fields in much the same way as for the scalar ones. However, the equation itself as well as the material explanation and result formulations are, of course, much simpler in the scalar case. Therefore, the exposition will mainly be given for the scalar fields and, as a rule, for the two-dimensional ones. Transferring the methods and results to the three-dimensional case is almost always trivial; see Chapter 5.

Only one condition is nontrivial while transferring the results to the three-dimensional vector fields. It is connected with the fact that surfaces orthogonal to a given vector field $\mathbf{A}(x, y, z)$ do not always exist. Such surfaces exist only in the case, when

$$\mathbf{A} rot \mathbf{A} = 0. \tag{1.4}$$

There is no similar condition for existence of zero surfaces or zero lines in scalar fields. The methods developed in the book contain the construction procedure of such zero lines for some cases. While transferring the methods from $u(x, y)$ to $\mathbf{A}(x, y, z)$ (1.4) should be taken into account.

Thus, almost all the material in the book is related to properties of a scalar function of two variables. It will be denoted by $u(x, y)$ or $u(r, \varphi)$ depending on the coordinate system, in which the problem is investigated.

The monochromatic fields with time dependence $e^{i\omega t}$ are considered. Instead of the frequency ω, the wave number $k = \omega/c$ is involved in the formulas, and this number is called the frequency (c is the light velocity). We will consider the fields generated by the currents distributed on the surfaces.

This condition does not relate to the material of Chapter 6 where quasi-plane fields are investigated. They do not satisfy the condition (1.18) (see below). They are “generated” by some other fields (not by currents). We return to this question at the end of the subsection.

For the two-dimensional case, the line current (the current distributed along a line) is an analog of the surface one. It is more convenient to introduce the current, not as the right-hand side of the Helmholtz equation, but as the discontinuity of the function u or its derivative on the line.

The field $u(x, y)$ satisfies the homogenous Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1.5}$$

and one of two following conditions on a line C :

$$[u] = 0, \left[\frac{\partial u}{\partial N}\right] = j; \tag{1.6a}$$

$$[u] = j, \left[\frac{\partial u}{\partial N}\right] = 0, \tag{1.6b}$$

where Δ is the two-dimensional Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}, \tag{1.7}$$

$[u]$ is the difference between the values of u on the sides of C, from and to which the normal N is directed.

The function $u(r, \varphi)$ satisfies the Sommerfeld radiation condition at infinity (i.e. at $r \to \infty$)

$$u(r, \varphi) = f(\varphi)\frac{e^{-ikr}}{\sqrt{kr}} + O\left[(kr)^{-\frac{3}{2}}\right]. \tag{1.8}$$

Here $f(\varphi)$ is a radiation pattern. Field $u(r, \varphi)$ is completely defined by equation (1.5), the given current $j(s)$ of the form (1.6a) or (1.6b) and condition (1.8) at infinity. The function $f(\varphi)$ in (1.8) is not a given one, it should be determined after the field $u(r, \varphi)$ is found. The field $u(r, \varphi)$ is complex, because any real field cannot have asymptotics as in (1.8). In the general case the function $f(\varphi)$ is also complex.

2. Let us explain the physical sense of the currents in (1.6). The two-dimensional scalar problem of the electrodynamics appears, when either the electrical or magnetic field has only one component (more exactly, when $E_x = E_y = H_z \equiv 0$ or $H_x = H_y = E_z \equiv 0$), and the fields do not depend on the z-coordinate: $\partial/\partial z \equiv 0$.

In this book, the first of these cases is considered, namely, the case, when $E_x = E_y = H_z \equiv 0$ and the fields do not depend on z. *The scalar function* $u(x, y)$ *is* E_z, two other components of the electric field equal zero identically. The derivative $\partial u/\partial N|_C$ is the component of the magnetic field, tangential to the line C. The expression $[\partial u/\partial N]$ is the jump of the magnetic field, or, what is the same, the z-component of the electric current. The electric current located on the line C is denoted in (1.6a) by j. The jump $[u]$ of the function u on C is the tangent to C component of the magnetic current $j^{(m)}$. One can write (1.6) in the form

$$\left[\frac{\partial u}{\partial N}\right] = j; \quad [u] = j^{(m)}. \tag{1.9}$$

While referring to $j_z^{(e)}$ and $j_z^{(m)}$ we omit the lower and upper indices z and (e), respectively. Nonessential factors are omitted in (1.9), too: in fact, the currents j and $j^{(m)}$ are not equal, but only proportional to the jumps of u and $\partial u/\partial N$, correspondingly.

There is another way to introduce the scalar function. One can consider a three-dimensional vector field with $E_z \equiv 0$ and $\partial/\partial z \equiv 0$. Then H_z depends only on x and y and it can be used as the above scalar function u. We will not use this approach in the book.

3. No special or very complicated mathematical methods are used for investigation of the problems, considered. Properties of solutions to equation (1.5) are investigated by the usual methods applied in typical diffraction problems. No diffraction problems themselves are considered in the book.

The functions considered below may have singularities – discontinuities and poles. Currents are given in the form of discontinuity of the functions or their normal derivatives. The existence or absence of singularities in the whole domain or in any part of it is very important in all the constructions below. This importance is connected with the wide use of the Green formula

$$\int\limits_{\mathcal{L}} \left(u_1 \frac{\partial u_2}{\partial N} - u_2 \frac{\partial u_1}{\partial N} \right) ds = 0 \tag{1.10}$$

for two solutions $u_1(x, y)$ and $u_2(x, y)$ to equation (1.5). This formula is valid only if both functions have no singularity inside the closed line (contour) $\mathcal{L}$.

All the lines mentioned below (either closed or not) are smooth or have only a finite number of angular points. We also assume that all the contours satisfy the conditions, necessary for the existence of a solution to the interior Dirichlet (u is zero on the contour) or Neumann ($\partial u/\partial N$ is zero on the contour) problems. At some frequencies the homogeneous problem for equation (1.5) with the corresponding boundary condition on C is supposed to have nonzero solutions.

The variational technique is also widely applied. Two variational methods are used most often. The first one is the Lagrangian multipliers method. It reduces the problem on a conditional extremum to the problem on an unconditional one. The second one is the Ritz method reducing the problem on the extremum of the two bilinear functionals ratio to an algebraic equation, involving equating some determinant to zero.

In some sections, the terms of functional analysis, such as: operators, completeness of the function set and so on, are used. However, only two theorems are referred to in the book. The first is the theorem on completeness of eigenfunctions of the self-adjoint integral operators; the second is on the tendency of eigenvalues of such operators to zero as the order number increases. In the problems on fields, generated in the free space by currents on surfaces, the nonself-adjoint operators are paramount. Some simple properties of these operators are briefly formulated as they are used in the book. The theory of analytical properties of solutions to the Helmholtz equation (or the Maxwell system) and functional analysis, are used simultaneously. This makes it possible to obtain some results in a simple and clear way.

The functional analysis formalism is used a little (mainly for shortness) in Chapter 4. The pseudo-solutions to the first kind equations, that is, the functions which minimize the mean square difference between the given function and the calculated one, are investigated. The solution to such an equation exists if and only if this minimal difference is zero.

All the material below is within the capacity of the layman in mathematics. In view of this the intermediate derivations are replaced by their descriptions where possible. As a rule, the technique used is trivial for mathematicians. However, the fact that an interesting branch of mathematical physics with unsolved problems, simple by formulation but still complicated by nature exists, may be of interest to them.

4. There are not many references to publications in the text. The general statement of the problem, denoted in the book title, and some essential results, were given in the papers published in 1988–2003. There are not references to all these papers in the book. The formulas and methods of diffraction theory used below are described, for instance, in [2]. Some formulations of functional analysis are in [3]. The formulas for special functions can be found in the reference book [4].

1.3 Realizability, Approximability, Amplitude Approximability

1. In the book some properties of a function $f(\varphi)$ are investigated. This function is determined by the current $j(s)$ and line C using (1.3). It is the radiation pattern of the field generated by this current. For brevity we will often use a term "the pattern generated by the line C" for the function $f(\varphi)$. Current $j(s)$ is an arbitrary function of the coordinate s on C. As a rule, we only assume that it has a finite norm N defined by

$$N^2 = \int_C |j(s)|^2 \, ds. \tag{1.11}$$

The typical formulation of our problem is the following. Given a function $F(\varphi)$ (in general, complex) with finite norm; for definiteness, we will usually put

$$\int_0^{2\pi} |F(\varphi)|^2 \, d\varphi = 1. \tag{1.12}$$

Can $F(\varphi)$ be a radiation pattern generated by the line C? If not, does a function, close to $F(\varphi)$, exist, which can be generated by C? Otherwise, does a current $j(s)$ on C exist, connected with the function $F(\varphi)$ or with that, close to it, by formula (1.3)?

The following four cases can occur.

1. *Realizability:* equality (1.3) (replacing $f(\varphi)$ by $F(\varphi)$), treated as an integral equation on $j(s)$, has a solution with finite norm. It means there exists a current $j(s)$ generating the pattern $f(\varphi)$ equal to $F(\varphi)$; more specifically,

$$\int_0^{2\pi} |F(\varphi) - f(\varphi)|^2 \, d\varphi = 0. \tag{1.13}$$

2. *Approximability:* for any given $\delta > 0$, a current $j(s)$ exists, such that the pattern $f(\varphi)$, generated by it, fulfils the condition :

$$\int_0^{2\pi} |F(\varphi) - f(\varphi)|^2 \, d\varphi \le \delta^2. \tag{1.14}$$

The realizability can be considered as a special case of approximability. It occurs when the above condition is valid for $\delta = 0$, too. Any function realizable by a line C is also approximable by it. If the function is not approximable, then it is not realizable, either.

3. *Amplitude approximability:* for any given $\delta > 0$, a current $j(s)$ and a real function (a phase) $\psi(\varphi)$ exist, such that

$$\int_0^{2\pi} \left| F(\varphi) e^{-i\psi(\varphi)} - f(\varphi) \right|^2 d\varphi \le \delta^2. \tag{1.15}$$

The approximability is a special case of the amplitude approximability, when (1.15) is valid for $\psi(\varphi) \equiv 0$.

4. *Amplitude nonapproximability:* the amplitude of any pattern generated by the line C is not close to $|F(\varphi)|$.

The nonapproximability of a given function $F(\varphi)$ means that the distance between any pattern generated by the line C and the given function $F(\varphi)$ (see below (1.32)) is finite. The amplitude nonapproximability means that the finite distance is between any pattern generated by the line C and an arbitrary function with amplitude $|F(\varphi)|$.

If the function $F(\varphi)$ is realizable, then the integral equation

$$\int_C \mathcal{K}(\varphi, s) j(s) ds = F(\varphi) \tag{1.16}$$

has a solution with a finite norm. We do not mention the last statement. If the function $F(\varphi)$ is approximable, then for any $\delta > 0$ there exists the function $\widetilde{F}(\varphi)$ satisfying the condition

$$\int_0^{2\pi} \left| F(\varphi) - \widetilde{F}(\varphi) \right|^2 d\varphi \le \delta^2, \tag{1.17}$$

so that the equation

$$\int_C \mathcal{K}(\varphi, s) j(s) = \widetilde{F}(\varphi) \tag{1.18}$$

has a solution. If the function $F(\varphi)$ is amplitude approximable, then for any $\delta > 0$ there exist the phase $\psi(\varphi)$ and the function $\widetilde{F}(\varphi)$ satisfying the condition

$$\int_0^{2\pi} \left| F(\varphi) e^{-i\psi(\varphi)} - \widetilde{F}(\varphi) \right|^2 d\varphi \le \delta^2, \tag{1.19}$$

so that equation (1.18) has a solution.

Below we will use the shortened expressions: “the function is realizable by the line C”, “...is approximable by C”, “...is amplitude approximable by C”, omitting the words: “...by the patterns of currents with finite norm, located on along the line C”.

2. The problem of realizability is not the subject of this book. Only a few results of this topic will be needed below. We will use the technique [5], based on the analysis of the convergence rate (with respect to n) of the Fourier series

$$F(\varphi) = \sum_{n=0}^{\infty} A_n \cos(n\varphi) \tag{1.20}$$

of the function $F(\varphi)$, the realizability of which is investigated, (see also Section 3.3, Subsection 7). In a similar way to many other cases, the terms with $\sin(n\varphi)$ are omitted in (1.20) for simplicity. The realizability depends on the existence and value of the limit

$$a_0 = \frac{2}{ek} \lim_{n\to\infty} (n \sqrt[n]{|A_n|}), \quad e = 2.718... \tag{1.21}$$

Strictly speaking, the term $\overline{\lim}$ should be used in (1.21) instead of $\lim$, but this fact is not essential here. The value of a_0 does not depend on the norm of function $F(\varphi)$, because $\lim\limits_{n\to\infty} \sqrt[n]{|c|} = 1$ for any given $c \neq 0$.

If the limit (1.21) does not exist (i.e., $a_0 = \infty$), then the function $F(\varphi)$ is not realizable by patterns of any currents, wherever they are located. The example is the Π-function, its Fourier coefficients A_n diminish as slowly as $1/n$ and it is not realizable by any line.

If ka_0 is a finite number, then the realizability of $F(\varphi)$ by the line C depends on the mutual location of the line C and the circle of radius a_0 with its centre at the coordinate origin. The function $F(\varphi)$ is not realizable by any line lying wholly inside this circle, but it is realizable by any closed nonresonant (with respect to the Dirichlet boundary condition) contour, encircling the circle. If $a_0 = 0$, then $F(\varphi)$ is realizable by any closed nonresonant contour containing the coordinate origin, and it is not realizable by any unclosed line. For the pattern of any current located on a nonclosed line, $a_0 \neq 0$ (see [5]). We do not analyze all the possible mutual locations of the line C and the circle here. Notice only that the closed contours realize a much wider class of patterns, than do the unclosed ones. As it will be shown later in the book, this is valid for the approximability problem, too.

As an example, consider the function (not normalized by (1.12))

$$F(\varphi) = e^{-A\sin^2(\varphi/2)}. \tag{1.22}$$

Its Fourier coefficients decrease as $I_n\,(A/2)$, where I_n is the modified Bessel function. It is asymptotically (at $n \to \infty$) equal to $(eA/2)^n\, n^{-n}$, so that (1.21) yields: $a_0 = A/k$. Any circle of the radius greater than A/k, or any contour, encircling this circle can realize the function (1.22). Any line, located inside the circle does not realize it. The narrower a pattern, the larger must be the contour which can realize it. The width of the pattern (1.22) is about $A^{-1/2}$.

It is significant for comparison of the notions of realizability and approximability, that the first one is defined by the behavior of A_n at large n according to (1.21). Therefore, the number ka_0 may be altered in an arbitrary way, by altering the function $F(\varphi)$ as little as desired. For example, if we truncate the series (1.20) at large n, we almost do not affect the function $F(\varphi)$, but the value ka_0 is zero. Otherwise, if we add the function $\sum\limits_{n=N}^{\infty} \left(1/n^2\right)\cos(n\varphi)$ to $F(\varphi)$, then, at a large N, it will be altered by an arbitrary small value, but the Fourier series will converge so slowly that the limit (1.21) will not exist ($a_0 = \infty$), that means the new function will not be realizable by any line.

The realizability is a “gentle” property, the approximability is a more “rough” one. The nonapproximable function should be finitely altered to become an approximable one and vice versa. This statement will be justified below. It will be shown that if this alteration is small (although it is finite according to the approximability definition), then the difference between the presence and absence of the approximability will also be small. This corresponds to physical intuition.

3. Below, we often deal with the situation when all the patterns $f(\varphi)$, realizable by the line C, possess the property:

$$I = 0, \tag{1.23}$$

where

$$I = \int_0^{2\pi} f(\varphi)\hat{F}^*(\varphi)d\varphi. \tag{1.24}$$

Here $\hat{F}(\varphi)$ is a function, *determined by the line* C only. It has a finite norm and, by default, is normalized by the condition

$$\int_0^{2\pi} \left|\hat{F}(\varphi)\right|^2 d\varphi = 1. \tag{1.25}$$

The function $\hat{F}(\varphi)$ and its connection with the line C are very significant for the problems considered in the book.

If the condition (1.23) is fulfilled, then the functions $F(\varphi)$ exist, which cannot be approximated by the line C. We often use this relation between the property (1.23) of the line C and the nonapproximability.

Let us give an elementary proof of this known fact. Suppose that for some function $F(\varphi)$, the integral

$$b = \int_0^{2\pi} F(\varphi)\hat{F}^*(\varphi)d\varphi \tag{1.26}$$

does not equal zero. Consider the integral

$$\int_0^{2\pi} \left[F(\varphi) - f(\varphi)\right] \hat{F}^*(\varphi)d\varphi. \tag{1.27}$$

According to (1.26) and (1.23), the first summand in (1.27) equals b, and the second one equals zero, so that the integral equals b. Applying the Cauchy inequality

$$\int_a^b |\alpha(x)|^2 dx \int_a^b |\beta(x)|^2 dx \geq \left| \int_a^b \alpha(x)\beta(x)dx \right|^2, \tag{1.28}$$

which is valid for any square integrable functions $\alpha(x)$, $\beta(x)$, to (1.27) gives

$$\int_0^{2\pi} \left|F(\varphi) - f(\varphi)\right|^2 d\varphi \cdot \int_0^{2\pi} \left|\hat{F}(\varphi)\right|^2 d\varphi \geq |b|^2. \tag{1.29}$$

The second factor in (1.29) equals one by (1.25), and for any function $f(\varphi)$ generated by C we have

$$\int_0^{2\pi} \left|F(\varphi) - f(\varphi)\right|^2 d\varphi \geq |b|^2 . \tag{1.30}$$

That is the condition of nonapproximability of the function $F(\varphi)$ by the line C.

The inverse statement is also valid. If the inequality (1.30) holds for any function $F(\varphi)$ and any pattern $f(\varphi)$, then it is also valid for $F(\varphi) = f(\varphi)$, where $f(\varphi)$ is a function generated by C, and (1.30) yields $b = 0$, that is, the condition (1.23).

4. Sometimes for the sake of brevity we will use the terms of functional analysis. However, we really do not need its technique. In these terms, condition (1.23) appears as the following: the set of functions $f(\varphi)$ is not complete, although it is generated by the integral operator (1.3) acting over the complete set of currents. There exist functions of the orthogonal complement. The above function $\hat{F}(\varphi)$ is an element of the space of these functions. It is orthogonal to all the functions $f(\varphi)$ of the form (1.3) . The orthogonality means that the integral (1.23) (the *inner product* of functions $f(\varphi)$ and $\hat{F}(\varphi)$) equals zero. The inner product of two functions $A(\varphi)$ and $B(\varphi)$ is the integral:

$$(A, B) = \int_0^{2\pi} A(\varphi) B^*(\varphi) d\varphi. \tag{1.31}$$

In particular, $(A, A) = ||A||^2$ is the squared norm of A. The distance between $A(\varphi)$ and $B(\varphi)$ is introduced as

$$\left\{ \int_0^{2\pi} |A(\varphi) - B(\varphi)|^2 \, d\varphi \right\}^{1/2} . \tag{1.32}$$

According to (1.23) and (1.30), the function is nonapproximable, if it is not orthogonal to $\hat{F}(\varphi)$. For instance, the function $\hat{F}(\varphi)$ itself is nonapproximable as well. The orthogonality of some function $F(\varphi)$ to $\hat{F}(\varphi)$ is the necessary condition for $F(\varphi)$ to be approximable. It is clear that this condition is not fulfilled for an arbitrary function. If there is only one function of the orthogonal complement for the given line C, then the above condition is also sufficient.

Let us give an obvious geometric illustration modelling the space of functions by the space of three-dimensional vectors. Then (1.31) is the usual scalar product of two vectors $\vec{A}$ and $\vec{B}$, and (1.32) is the length of the straight-line segment connecting the end points of these vectors if their origins coincide.

The condition (1.23) means that all the vectors $f(\varphi)$ lie in the plane orthogonal to $\hat{F}(\varphi)$. Only vectors lying in this plane are approximable. If the vector $F(\varphi)$ does not lie in this plane, then from comparison of (1.32) with (1.30) it follows: the distance from the end point of the vector to the plane, that is, the distance between $F(\varphi)$ and the vector $f(\varphi)$, closest to it, is equal to $|b|$. The statement “if the condition (1.23) is fulfilled, then almost all functions are nonapproximable” in terms of this illustration only, means that the dimension of the plane is less than the dimension of the whole space. The function $f(\varphi)$, closest to $F(\varphi)$, lying in the plane, orthogonal to $\hat{F}(\varphi)$, is the projection of $F(\varphi)$ onto the plane and equals $F(\varphi) - b\hat{F}(\varphi)$. In Chapter 4 we will obtain this result without referring to the three-dimensional illustration.

5. The orthogonal complement of functions $f(\varphi)$ may contain not only one function $\hat{F}(\varphi)$, but also a set of linear independent functions $\hat{F}_p(\varphi)$. In this case the three-dimensional illustration cannot be used. As we will see in Section 2.2, the number of functions $\hat{F}_p(\varphi)$ can even be infinite.

Give an obvious generalization of the formula (1.30) for this case. Let all the patterns $f(\varphi)$ satisfy the conditions

$$I_p = 0, \quad p = 1, 2, ..., P, \tag{1.33}$$

where

$$I_p = (f, \hat{F}_p). \tag{1.34}$$

Orthogonalize the linear independent functions $\hat{F}_p(\varphi)$ and normalize them to unity, so that

$$(\hat{F}_p, \hat{F}_q) = \delta_{pq}, \tag{1.35}$$

where δ_{pq} is the Kronecker symbol ($\delta_{pq} = 0$ for $p \neq q$, $\delta_{pp} = 1$).

Combine the expression, analogous to (1.27)

$$\left(F - f, \sum_{p=1}^{P} b_p \hat{F}_p \right), \tag{1.36}$$

where

$$b_p = (F, \hat{F}_p). \tag{1.37}$$

Applying the Cauchy inequality (1.28) to (1.36) with $\alpha = F - f$, $\beta = \sum\limits_{p=1}^{P} b_p \hat{F}_p$ and using (1.35), we have

$$\left| \left(F - f, \sum_{p=1}^{P} b_p \hat{F}_p \right) \right|^2 \leq \|F - f\|^2 \sum_{p=1}^{P} |b_p|^2 . \tag{1.38}$$

On the other hand, using (1.33) and (1.37) yields

$$\left(F - f, \sum_{p=1}^{P} b_p \hat{F}_p \right) = \sum_{p=1}^{P} |b_p|^2 , \tag{1.39}$$

which together with (1.38) gives

$$\int\limits_0^{2\pi} |F(\varphi) - f(\varphi)|^2 \, d\varphi \geq \sum_{p=1}^{P} |b_p|^2 . \tag{1.40}$$

This is a generalization of the formula (1.30).

If there are only P linearly independent functions of the orthogonal complement, that is if there are not other functions to which all the patterns, generated by the line C, are orthogonal, then the equalities

$$b_p = 0, \quad p = 1, 2, ..., P \tag{1.41}$$

are not only the necessary conditions for the function $F(\varphi)$ to be approximable, but they are also the sufficient ones.

1.4 Outline of the Book

1. One of the purposes of the book is to investigate some properties of the *fields generated by surface currents*. The simplest two-dimensional model is mainly considered, which is based on the scalar Helmholtz equation with currents, located on a line. The property consists in the following: there are "many" lines such that the fields generated by them cannot approximate "many" functions themselves, as well as all functions close to them. This property is widely investigated throughout the book.

Such *specific* lines are zero lines of a *real* solution to the Helmholtz equation. As a matter of fact, this property is elementary, but it allows us to find and effectively investigate the above lines (surfaces). Even if a line is close to a specific one, very large currents are required for the approximation of almost all fields. The antenna surface should not be close to a specific one (see, however, Subsection 2.2.6).

The second purpose is to investigate the *two-dimensional Fourier transformation*: the main instrument for consideration of the fields in the form of a long narrow beam.

2. In Chapter 2 the approximability of the patterns (i.e., the fields at infinity) is analyzed. Different real fields, their zero lines, as well as prohibitions to the approximation connected with them, are under analysis. For some lines the prohibitions are so strict that many functions with a given amplitude and free phase cannot even be approximated by the fields of currents on these lines.

The existence of prohibitions to the approximation can be used for obtaining some information about the shape of a metallic body, if the scattered pattern is known but the incident field is unknown. This information can have the form of a probable statement. It is assumed that the incident field is unknown. In particular, if the measured pattern cannot be approximated by the patterns of currents, located on a specific surface, then the surface of the scatterer cannot be close to it.

3. The prohibitions to approximation exist not only for the patterns; they relate also to fields in the near zone (*near fields*). The questions connected with this problem are investigated in Chapter 3. In general, the prohibitions to the near field are stronger than to the field at infinity. They grow slowly while going away from the current. Similarly to the previous case, the problem is reduced to construction of a real field and determination of its zero lines.

If the Helmholtz equation has a solution, which equals zero on the given line, then the prohibition relates only to the part of the plane where the solution has no singularities. If the solution is analytical in the whole plane, then the prohibition relates to both the near field and the pattern. If singularities exist, then prohibitions relate only to the near field; any pattern can be approximated by currents on the line. When singularities exist at infinity only, then the prohibitions relate to the near field.

For this reason the problem of the *analytical extension* of the eigenoscillation field in the resonant domain outwards from its boundary is very important. The field should be real and continuous together with its normal derivative on the boundary. One should find the conditions for such an extension to be realizable in the case when singularities are absent, as well as to develop the method to locate these singularities, when they exist. This problem is not solved with the practically acceptable efficiency even in the two-dimensional scalar case. In the chapter several simple examples are given, when the singularities appear in a finite part of the plane, at infinity, or when they do not appear at all.

4. The concept of approximability is a generalization of the concept of realizability. The *optimal current synthesis* is a generalization of approximability. Even in the case, when a given pattern is approximable, it can require a very large current. It is often possible to alter the pattern *finitely but a little*, so that the new pattern can be realized by an essentially smaller current. The problem of finding the patterns, close to the given one but generated by currents with small norm, is considered in Chapter 4.

This problem is solved by using the known technique of *generalized functions of the double orthogonality.* This technique deals with two orthogonal sets of functions, mutually connected: the set of currents and the set of patterns. A current described by a function of the first set, generates the pattern described by an appropriate function of the second one.

Some closed line can have the two following properties. The first is to be the specific line. That means, there exists a pattern, which cannot be generated by a current located on the line. The second property consists in the existence of a current on the line (closed), which does not generate a field outside it. The usage of the function sets mentioned allows us to understand the connection between these properties. It results that they are independent, but mutually symmetrical in some sense.

In the last section of the chapter we give an expression for a lower bound of the norm of a current approximating a given pattern. To determine this estimation, it is not necessary to calculate preliminary the current. This bound can be calculated exactly as a result of some passing to the limit.

Each specific line has an influence on the norm of currents located on the nonspecific ones close to it. The mentioned estimation gives a value of this influence. The current norm is inversely proportional to a value having a meaning of the distance between the given line and the specific one near to it. To approximate almost any pattern, the current located on the line nearby a specific one should have a large norm. This statement is still more related to the approximation of fields in the near zone. It is in accordance with our physical intuition.

5. In Chapter 5 the results of the previous chapters are transferred to the case of three-dimensional vector fields satisfying the Maxwell equations. The transference is trivial almost everywhere. The first and second electrodynamic problems are formulated instead of the Dirichlet and Neumann ones; the Lorentz lemma is used instead of the Green formula, and so on. Section 5.1, devoted to this transference, is very short. The detailed exposition of results and intermediate derivations in the vector form are cumbersome but in fact do not provide anything new in the understanding of the nonapproximability phenomenon and the related prohibitions.

Section 5.2 is devoted to the essential difference between the three-dimensional vector problems and the related scalar ones (two- or three-dimensional). The auxiliary real field connected with a specific surface in the vector case is restricted by some conditions, which have no analogy in the scalar case. Some results, which are obvious or easy to prove in the scalar case, are formulated in the vector one only as probable hypothesis. That is the first effect caused by this difference. The second one is the limitation of the practical use of results obtained, in the problems, in which the vector nature of the field is essential. However, the general behavior of the nonapproximability phenomenon and the related problem of the optimal current synthesis differs a little in the scalar and vector cases.

6. In Chapter 6 the theory of *long narrow beams* of electromagnetic waves is considered. Such beams are supposed to be applied for transmitting the power from the orbiting solar power stations to the Earth. The beams are created by large plane antennas; they preserve a quasi-plane structure at a long distance, that is, on the receiving antenna (rectenna). With an accuracy to simple phase factors, the field in the rectenna plane is described by two-dimensional Fourier transformation of the field on the antenna. The field distributions on the antenna are determined, which provide either maximum of the power transmission coefficient or the maximal closeness of the field in the rectenna plane to an “ideal” one. The first demand reduces the problem to a homogeneous equation for the field on the antenna; the second one is fulfilled if the field on the antenna is the pseudo-solution to some first-kind integral equation.

In this problem the pseudo-solution differs from the identical zero. Note that the pseudo-solution technique is not expedient to be applied in Chapters 2-5, because in the most interesting case, when the surface is specific, that is, when the orthogonal complement functions exist, the pseudo-solution to the appropriate integral equation can be equal to zero (“If an operator gives rise to a noncomplete set of functions, then the kernel of the adjoint operator is nonempty”).

The question of the antenna *shape* is of special interest in the beam problem. A hypothesis that the optimal shape is circular or elliptical is well founded (but not proved!).

7. The diffraction (scattering) problems on the nontransparent bodies (metallic ones or with impedance boundaries) as well as on the partially transparent obstacles or walls have been investigated for a long time. The key works in the mathematical theory of the patterns (*far fields*) completeness are [6], [7], [8]. This theory is explained in detail in the book [9]. In particular, it is shown there that some properties of the diffracted (scattered) field depend on the interior domain even for bodies with nontransparent boundaries, i.e. when the field does not penetrate into this domain. At its eigenfrequencies the space of the radiation patterns can be noncomplete, namely, at arbitrary incident field the pattern does not contain some elements of a complete set of the angle functions. This takes place in the case, when there exists an auxiliary field (the so-called *Herglotz wave function*) analytical in the whole space, which satisfies the Helmholtz equation (the Maxwell ones in the vector case) and equals zero on the scatterer boundary.

The material given in the book is closely related to the above problem. However, our aim is more to explain what physical sense these specific properties have and in which way they can show themselves in the practical problems, than to obtain some new mathematical properties of the scattered fields.

No diffraction problem is considered here. Common to the problems is the fact that the induced current is allocated on the surface and the volume currents do not arise. The *scattered field is a field of* the induced (*surface*) *currents*. The properties of the surface currents are investigated in the book without connection with the way in which these currents arise. This problem statement is based on the fact that the noncompleteness of the pattern set is not connected with any property of the surface currents.

The above noncompleteness is not obligatorily connected with the existence of the resonance of the interior domain. This property is also inherent in the fields scattered on nonclosed shells (screens), because these fields are also created by the surface currents at the diffraction on the screens; the auxiliary field mentioned above exists at all frequencies “almost always”. Just the *existence of* such an *auxiliary field is the necessary and sufficient condition* for the *noncompleteness* of the *scattered patterns* set (see [9, Theorem 6.32]). Of course, for the nontransparent bodies or closed shells the auxiliary field exists at resonant frequencies only. Great attention is paid in the book to the *structure of the orthogonal complement function set*. The connection of these functions with the auxiliary field is stated. The situation is investigated in detail, when there are many such functions, for instance, when the set of them is countable.

The analogous properties of the patterns created by the magnetic currents are stated. The currents of both types together create the complete set of patterns.

All the results related to the patterns are valid (even in a reinforced form) for the near fields. The noncompleteness of the set of scattered patterns can be used for the reconstruction of the scatterer shape and position.

As a consequence of the nonapproximability phenomenon, the problem of the optimal current synthesis arises. It is solved using the eigenfunctions of iterated operators (see[10], [11]). This technique allows to state the symmetry between two independent properties of the surface: the noncompleteness of the fields of surface currents and the existence of the nonradiating surface currents (i.e., those connected with the eigenoscillations of the interior volume).

The book is based on the material partially published by the author and his colleagues in a series of papers [5], [12]-[29].

8. In the Appendix some questions related to the optimal current synthesis at the given amplitude pattern are considered. In general, the pattern is supposed to be nonapproximable, and the problem is formulated in the variational form. The appropriate Lagrange-Euler equation is nonlinear, it has many solutions describing all stationary points of the functional. The solutions bifurcate as the antenna size (or the frequency) increases. In the particular case of the linear antenna, with an appropriate metric in the functional space of the patterns, expression (1.3) becomes the Fourier transformation. The problem is close to the so-called phase problem and can be considered as a modification of it.

In this case the solutions to the mentioned nonlinear equation can be expressed in an explicit form, with a finite number of unknown complex parameters. The parameters are determined from a system of transcendental equations. This allows to investigate the equation completely including the calculation of all its branching points. The case of the discrete Fourier transformation describing the equidistant antenna array, is also considered.

All theoretical results are reinforced by numerical ones.

The nonlinear equations considered in the Appendix are also interesting from the mathematical point of view, not only owing to their connection with the Fourier transformation theory, but also as representatives of an unstudied class of nonlinear integral equations of the Hammerstein type.