1 The Fourier Transform and the Helix

The Fourier transform is a common tool in physics, engineering, and statistics. Examples of Fourier transformations are

- A voltage or current that varies with time into its frequency spectrum;
- The illumination function of an antenna into its pattern in sine space;
- The probability density function in statistics into the characteristic function.

The use of the Fourier transform has been introduced into a number of disciplines independently and unfortunately each uses its own convention. Other than finding spectra, antenna patterns, and so on, the Fourier transforms of functions are added or multiplied and then undergo an inverse Fourier transform to produce the required results as in other forms of operational mathematics.

1.1 Fourier Transform Conventions

The conventions in physics, electrical engineering, and statistics are often different and vary from country to country [1].

1.1.1 Fourier Transforms in Physics

As the fundamental unit of angle is the radian and of frequency radians/second, the Fourier transform used in physics and given in mathematics programs (such as Maple) uses the minus omega convention, $-\omega$, given by

$$\mathrm{F}(\omega) = \int_{-\infty}^{+\infty} \mathrm{f}(t)\ \exp(-i\omega t)\ \mathrm{d}t \tag{1}$$

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Where F(ω) is the Fourier transform with variable ω;
f(t) is a time function with variable t;
and i is $\sqrt{-1}$.

The inverse transform used here is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\ \exp(+i\omega t)\ d\omega \tag{2}$$

This convention will not be used further in this book.

1.1.2
Fourier Transform in Electrical Engineering

Electrical engineers measure frequency in cycles or rotations per second (Hz) so that ω is replaced by $2\pi f$ to give Fourier transforms using the $-f$ convention. The Fourier transform which connects, among others, waveforms and their spectra can be given by [1, p. 381, 4, p. 27]

$$F(f) = \int_{-\infty}^{+\infty} f(t)\ \exp(-j2\pi ft)\ dt \tag{3}$$

where F(f) is the Fourier transform with variable f;
f(t) is a time function with variable t;
and j is $\sqrt{-1}$.

The inverse transform used here is given by

$$f(t) = \int_{-\infty}^{+\infty} F(f)\ \exp(+j2\pi ft)\ df \tag{4}$$

Notice that this convention that is used in all chapters, except for Chapter 6, has the convenient property the Fourier transform of the Fourier transform returns the original function. The $1/2\pi$ in (2) is the result of the fact that $df = d\omega/2\pi$.

1.1.3
Fourier Transform in Statistics

In statistics the following form for the Fourier transform to obtain the characteristic function is used to be consistent with other statistics texts in Chapter 6. This convention is called the +omega convention in this book.

$$C(\xi) = \int_{-\infty}^{+\infty} p(x)\ \exp(+i\xi x)\ dx \tag{5}$$

where C(ξ) is the characteristic function with variable ξ;
p(x) is a probability density function with variable x;
and i is $\sqrt{-1}$.

The inverse transform used here is given by

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} C(\xi)\ \exp(-i\xi x)\ d\xi \tag{6}$$

1.2
The Fourier Transform and the Helical Functions

The Fourier transform used by electrical engineers contains the helical function $\exp(-j\ 2\pi ft)$ that interacts with the f(t) function. The function $\exp(+j\ 2\pi ft)$ is shown in Figure 1.1.

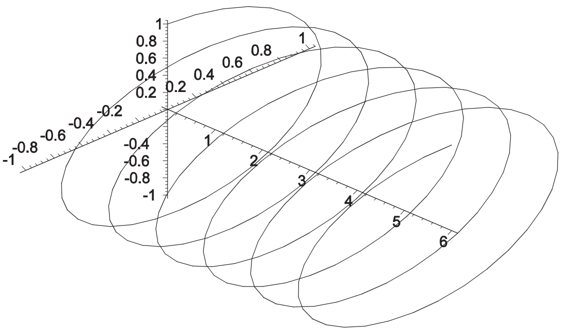


Figure 1.1 The helix $1\exp(j\ 2\pi ft)$.

The coordinates in Figure 1.1 have been chosen to give positive upwards and to the right when looking from the side (in this case a cosine wave) or from the top (in this case a sine wave). Notice that this waveform is balanced, the vector traces a circle, and rotates in the direction called in electrical engineering *positive phase sequence*. The helix in the Fourier transform in (3) has a *negative phase sequence*.

The function f(t) can be also a spatial spiral, so that the Fourier transform in mathematics, engineering, and statistics becomes much easier to understand. The Fourier transform is described in engineering form in Chapter 3, and later chapters, and its use in statistics in Chapter 6.

The helix of radius A can be given by $A\ \exp(j\ 2\pi ft)$, in polar coordinates as $A\angle 2\pi ft$, or may be expressed in Cartesian coordinates as

$$\begin{aligned} x &= A\sin 2\pi ft \\ y &= A\cos 2\pi ft \end{aligned} \tag{7}$$

where A is the amplitude (voltage or current);
x is the horizontal component;
y is the vertical component;
f is the frequency, Hz;
t is time in seconds;
and j is $\sqrt{-1}$.

Pure helices occur in spiral springs and the position of a point on a rotating shaft when plotted in three dimensions against time. This form of illustration may be extended to other phenomena that occur in the distribution of electrical power or signals where the cross-section is not purely circular, examples are

- The most common occurrence of a helical waveform is in the three-phase power distribution used to distribute electricity throughout the world. The voltages and currents in balanced three-phase electrical power circuits at a fixed frequency, typically 50, 60, or 400 Hz may be represented by a single rotating vector and is also true for any electrical system with more than two phases. The phase angle, ϕ, between the voltage and current vectors allows the load factor, cos ϕ to be calculated. The load factor is the ratio of the real power used to the product of voltage and current. When the voltages and currents are not equal the rotating vector no longer traces a circle and symmetrical component theory is used (see Section 2.2). This is also true for radar and sonar echo signals (Section 1.3).
- The modulation of (notional) carrier waves by two-phase quadrature amplitude modulation (QAM) waveforms in communications and echo signals received by radar and sonar from moving objects (Sections 1.3 to 1.5). The vector representation of the waveform does not have a constant radius the form is called in this book a *spatial spiral*.
- Circularly polarised waves (Section 1.7)
- Noise, discussed in Section 1.8, has the form of a *random spatial spiral*.

These waveforms are not able to be displayed on a normal oscilloscope, though those from the individual phases may be. The use of helices to represent these and alternating waveforms gives a physical explanation to the helical function in the Fourier transform and allows their explanation to technicians.

The spatial spiral forms, their appearance in the physical world, calculation methods, and the relationships to alternating functions are described in Chapter 2.

1.3 Radar and Sonar Echo Signals

The waves from radio-frequency transmitters are scattered or reflected by objects they meet. The phase of the waves entering a receiver contain fine distance information for the length of the path between transmitter and receiver. In a radar, a coherent oscillator (COHO) is used as a common phase reference for both transmitter and receiver so that differences in phase may be measured. If the scatterer moves towards the radar, the phase angle will decrease with time so that vector detection will give rise to a vector rotating at the Doppler frequency. For a receding scatterer the phase of the vector rotates in the opposite sense, the negative phase sequence. The filtering of such signals is the basis of the rejection of ground echoes that clutter up the radar displays and would overload the equipment that extracts aircraft echoes, for example. Radar (and sonar) echo signals contain a combination of background echo signals and echo signals from approaching and receding objects, that is a direct voltage component and components with both phase sequences. The method of symmetrical components is discussed in Chapter 2.

1.4
Colour Television Signals

The NTSC and PAL colour television systems use quadrature or Cartesian modulation to modulate the two colour difference signals (R–Y and B–Y) onto a common subcarrier. The reference phase is given by the colour burst signal that occurs after the end of each line synchronisation pulse and this is used to keep the subcarrier oscillator at the correct phase. The quadrature outputs of this oscillator are used in the two synchronous detectors to recover the two colour difference signals.

1.5
Modulation and Demodulation

Signal vectors must be represented as two separate signals (normally considered to be voltages) representing the two coordinate systems: Cartesian or polar (each branch of electrical engineering uses different terminology that has changed with time, hence the use of basic English here [2]). Both vector modulation and demodulation may be carried out in terms of these coordinates and are shown in Figures 1.2 and 1.3 [3].

The two phase and amplitude modulation stages in Figure 1.2 may take place in any order since the two processes are linear without any suppression of signals.

1.6
Communications

Digital communications over normal telephone lines is limited to approximately 1200 signal changes per second. To increase the amount of information transferred in each element, a number of types of element must be used that carry more than one bit. Examples are

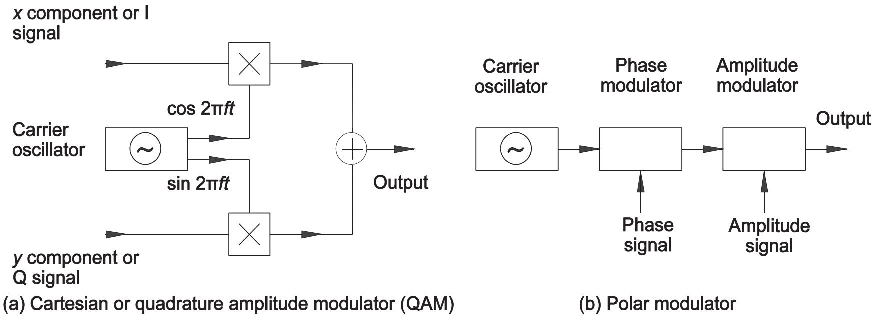


Figure 1.2 Vector modulators.

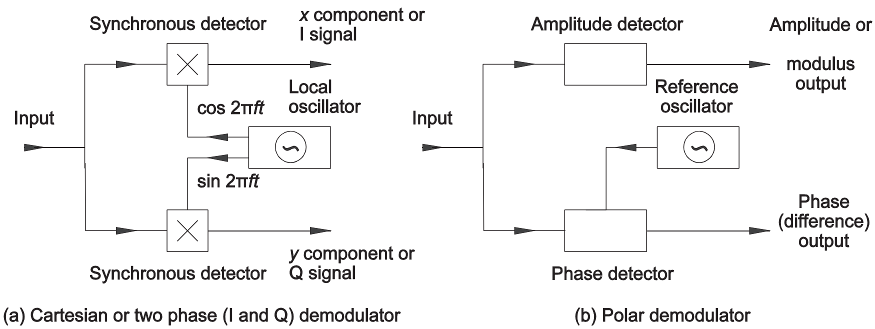


Figure 1.3 Vector demodulators or detectors.

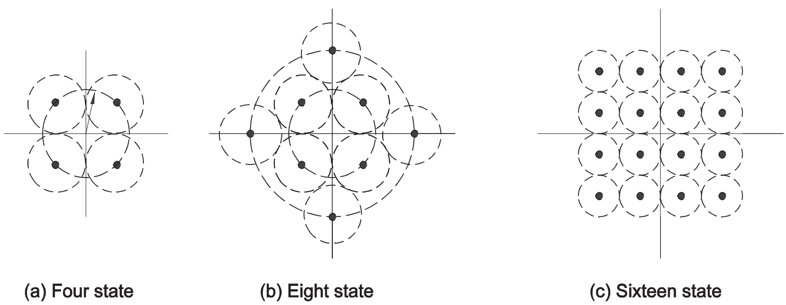
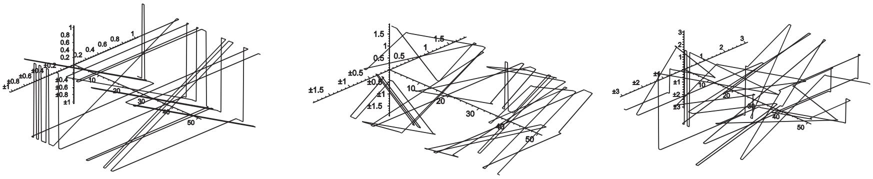


Figure 1.4 Signal space or constellation diagrams.

- Phase modulation with more than two states
- Quadrature amplitude modulation (QAM).

In Figure 1.4(a), the four states may be obtained by phase modulation alone. Amplitude and phase modulation are needed in parts (b) and (c) in Figure 1.4. The dots in the state diagram represent the states and the circles around them represent the combined tolerances in the modulation, transmission, and demodulation processes.



(a) Waveform with 4 states (b) Waveform with 8 states (c) Waveform with 16 states

Figure 1.5 The complex waveforms for the four, eight, and sixteen states in Figure 1.4.

The complex waveforms for modulating signals using these states are shown in Figure 1.5. The four-state modulation uses simple phase modulation and the shape of the complex waveform is a box. With more states the complex modulation and when random characters are transmitted, the waveform becomes more like equally distributed noise.

1.7 Circularly Polarised Waves

Circularly polarised electromagnetic waves are generated when either the electric or magnetic component has its phase delayed by 90 degrees. These waves are used in microwave communications to avoid polarisation alignment and radar to reduce rain echoes.

1.8 Noise

The transformation from a recognisable modulation to a complex waveform similar to noise can be heard when logging onto an Internet provider as the protocol tries

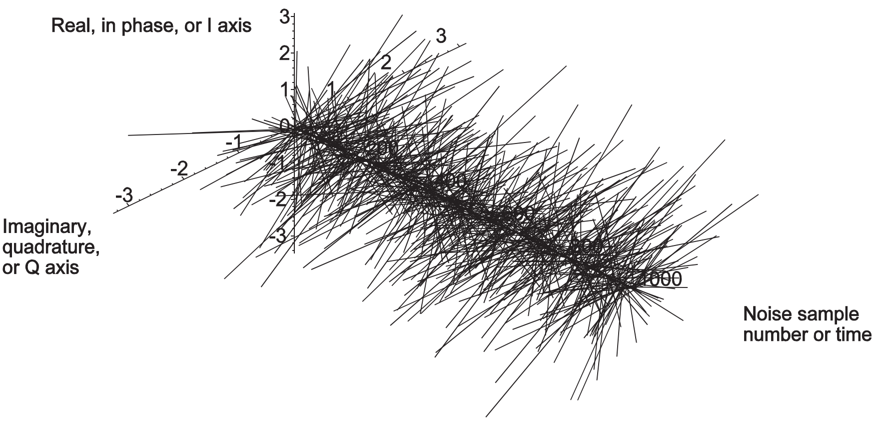


Figure 1.6 Gaussian noise samples demodulated from a (notional) carrier. [Source: Meikle, H.D., *Modern Radar Systems*, Artech House, Norwood, Massachusetts, 2001]

higher and higher transmission speeds. In contrast to communication systems where each state is equally likely, Gaussian noise is normally (Gaussian) distributed in the two Cartesian coordinates and looks like the shaggy bottlebrush in Figure 1.6. Complex noise waveforms are discussed in Chapter 7.

1.9 Other Forms of the Fourier Transform

The Fourier transform in t–f notation, discussed in Chapter 3, is the continuous Fourier transform with the limits of integration being from the beginning to the end of time. In real life time is limited and the finite transform is introduced in Chapter 4 together with the discrete Fourier transform for sampled waveforms that lend themselves to manipulation by digital logic and computers.

Erdélyi [4] and others describe the Fourier sine and cosine transforms that, being flat functions, are not treated further. The transforms are

$$\begin{aligned} \text{Fourier sine transform} &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{f}(x)\,\sin(xy)\,\mathrm{d}x \\ \text{Fourier cosine transform} &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{f}(x)\,\cos(xy)\,\mathrm{d}x \end{aligned} \tag{8}$$

Notice that both these transforms are one-sided, the limits for integration are from zero to infinity, in contrast to the exponential transforms elsewhere in this book.

The sampling of signals over finite times, shorter than the length of the signal, puts an unwanted modulation on the transform giving repetition and extra sidelobes and these effects are described in Chapter 4. Chapter 5 shows ways of reducing the sidelobes. Chapter 6 shows the Fourier transforms of probability distribution functions used in statistics in three dimensions. Statistical signals and noise are treated in Chapter 7.

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