1 Foreign Exchange Options

FX Structured Products are tailor-made linear combinations of FX Options including both vanilla and exotic options. We recommend the book by Shamah [1] as a source to learn about FX Markets with a focus on market conventions, spot, forward and swap contracts and vanilla options. For pricing and modeling of exotic FX options we suggest Hakala and Wystup [3] or Lipton [4] as useful companions to this book.

The market for structured products is restricted to the market of the necessary ingredients. Hence, typically there are mostly structured products traded in the currency pairs that can be formed between USD, JPY, EUR, CHF, GBP, CAD and AUD. In this chapter we start with a brief history of options, followed by a technical section on vanilla options and volatility, and deal with commonly used linear combinations of vanilla options. Then we will illustrate the most important ingredients for FX structured products: the first and second generation exotics.

1.1 A JOURNEY THROUGH THE HISTORY OF OPTIONS

The very first options and futures were traded in ancient Greece, when olives were sold before they had reached ripeness. Thereafter the market evolved in the following way:

16th century Ever since the 15th century tulips, which were popular because of their exotic appearance, were grown in Turkey. The head of the royal medical gardens in Vienna, Austria, was the first to cultivate Turkish tulips successfully in Europe. When he fled to Holland because of religious persecution, he took the bulbs along. As the new head of the botanical gardens of Leiden, Netherlands, he cultivated several new strains. It was from these gardens that avaricious traders stole the bulbs in order to commercialize them, because tulips were a great status symbol.

17th century The first futures on tulips were traded in 1630. From 1634, people could buy special tulip strains according to the weight of their bulbs, the same value was chosen for the bulbs as for gold. Along with regular trading, speculators entered the market and prices skyrocketed. A bulb of the strain "Semper Octavian" was worth two wagonloads of wheat, four loads of rye, four fat oxen, eight fat swine, twelve fat sheep, two hogsheads of wine, four barrels of beer, two barrels of butter, 1,000 pounds of cheese, one marriage bed with linen and one sizable wagon. People left their families, sold all their belongings, and even borrowed money to become tulip traders. When in 1637, this supposedly risk-free market crashed, traders as well as private individuals went bankrupt. The government prohibited speculative trading; this period became famous as Tulipmania.

18th century In 1728, the Royal West-Indian and Guinea Company, the monopolist in trading with the Caribbean Islands and the African coast issued the first stock options. These were options on the purchase of the French Island of Ste. Croix, on which sugar plantings were planned. The project was realized in 1733 and paper stocks were issued in 1734. Along with the stock, people purchased a relative share of the island and the possessions, as well as the privileges and the rights of the company.

19th century In 1848, 82 businessmen founded the Chicago Board of Trade (CBOT). Today it is the biggest and oldest futures market in the entire world. Most written documents were lost in the great fire of 1871, however, it is commonly believed that the first standardized futures were traded in 1860. CBOT now trades several futures and forwards, not only T-bonds and treasury bonds, but also options and gold.

In 1870, the New York Cotton Exchange was founded. In 1880, the gold standard was introduced.

20th century

- In 1914, the gold standard was abandoned because of the war.
- In 1919, the Chicago Produce Exchange, in charge of trading agricultural products was renamed to Chicago Mercantile Exchange. Today it is the most important futures market for Eurodollar, foreign exchange, and livestock.
- In 1944, the Bretton Woods System was implemented in an attempt to stabilize the currency system.
- In 1970, the Bretton Woods System was abandoned for several reasons.
- In 1971, the Smithsonian Agreement on fixed exchange rates was introduced.
- In 1972, the International Monetary Market (IMM) traded futures on coins, currencies and precious metal.
- In 1973, the CBOE (Chicago Board of Exchange) first traded call options; and four years later also put options. The Smithsonian Agreement was abandoned; the currencies followed managed floating.
- In 1975, the CBOT sold the first interest rate future, the first future with no "real" underlying asset.
- In 1978, the Dutch stock market traded the first standardized financial derivatives.
- In 1979, the European Currency System was implemented, and the European Currency Unit (ECU) was introduced.
- In 1991, the Maastricht Treaty on a common currency and economic policy in Europe was signed.
- In 1999, the Euro was introduced, but the countries still used their old currencies, while the exchange rates were kept fixed.

21st century In 2002, the Euro was introduced as new money in the form of cash.

1.2 TECHNICAL ISSUES FOR VANILLA OPTIONS

We consider the model *geometric Brownian motion*

$$dS_t = (r_d - r_f)S_t\,dt + \sigma S_t\,dW_t \tag{1.1}$$

for the underlying exchange rate quoted in FOR-DOM (foreign-domestic), which means that one unit of the foreign currency costs FOR-DOM units of the domestic currency. In the case of EUR-USD with a spot of 1.2000, this means that the price of one EUR is 1.2000 USD. The notion of *foreign* and *domestic* does not refer to the location of the trading entity, but only

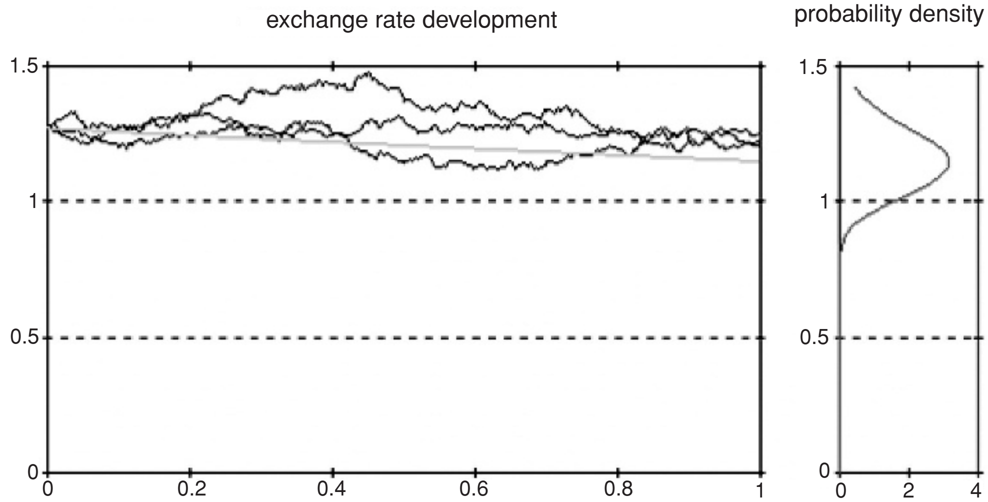


Figure 1.1 Simulated paths of a geometric Brownian motion. The distribution of the spot S_T at time T is log-normal. The light gray line reflects the average spot movement.

to this quotation convention. We denote the (continuous) foreign interest rate by r_f and the (continuous) domestic interest rate by r_d. In an equity scenario, r_f would represent a continuous dividend rate. The volatility is denoted by σ, and W_t is a standard Brownian motion. The sample paths are displayed in Figure 1.1.[1] We consider this standard model, not because it reflects the statistical properties of the exchange rate (in fact, it doesn't), but because it is widely used in practice and front office systems and mainly serves as a tool to communicate prices in FX options. These prices are generally quoted in terms of volatility in the sense of this model.

Applying Itô's rule to $\ln S_t$ yields the following solution for the process S_t

$$S_t = S_0 \exp\left\{\left(r_d - r_f - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \tag{1.2}$$

which shows that S_t is log-normally distributed, more precisely, $\ln S_t$ is normal with mean $\ln S_0 + (r_d - r_f - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. Further model assumptions are

1. There is no arbitrage
2. Trading is frictionless, no transaction costs
3. Any position can be taken at any time, short, long, arbitrary fraction, no liquidity constraints

The payoff for a vanilla option (European put or call) is given by

$$F = [\phi(S_T - K)]^+, \tag{1.3}$$

where the contractual parameters are the strike K, the expiration time T and the type ϕ, a binary variable which takes the value $+1$ in the case of a call and -1 in the case of a put. The symbol x^+ denotes the positive part of x, i.e., $x^+ \stackrel{\Delta}{=} \max(0, x) \stackrel{\Delta}{=} 0 \vee x$. We generally use the symbol $\stackrel{\Delta}{=}$ to *define* a quantity. Most commonly, vanilla options on foreign exchange are of *European style*, i.e. the holder can only exercise the option at time T. *American style options*,

[1] Generated with Tino Kluge's shape price simulator at www.mathfinance.com/TinoKluge/tools/sharesim/black-scholes.php

where the holder can exercise any time, or *Bermudian style options*, where the holder can exercise at selected times, are not used very often except for time options, see Section 2.1.18.

1.2.1 Value

In the Black-Scholes model the value of the payoff F at time t if the spot is at x is denoted by $v(t, x)$ and can be computed either as the solution of the *Black-Scholes partial differential equation* (see [5])

$$v_t - r_d v + (r_d - r_f) x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0, \tag{1.4}$$

$$v(T, x) = F. \tag{1.5}$$

or equivalently (*Feynman-Kac-Theorem*) as the discounted expected value of the payoff-function,

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = e^{-r_d \tau} \mathbb{E}[F]. \tag{1.6}$$

This is the reason why basic financial engineering is mostly concerned with solving partial differential equations or computing expectations (numerical integration). The result is the *Black-Scholes formula*

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = \phi e^{-r_d \tau}[f \mathcal{N}(\phi d_+) - K \mathcal{N}(\phi d_-)]. \tag{1.7}$$

We abbreviate

- x: current price of the underlying
- $\tau \stackrel{\Delta}{=} T - t$: time to maturity
- $f \stackrel{\Delta}{=} \mathbb{E}[S_T | S_t = x] = x e^{(r_d - r_f)\tau}$: forward price of the underlying
- $\theta_\pm \stackrel{\Delta}{=} \frac{r_d - r_f}{\sigma} \pm \frac{\sigma}{2}$
- $d_\pm \stackrel{\Delta}{=} \frac{\ln \frac{x}{K} + \sigma\theta_\pm \tau}{\sigma\sqrt{\tau}} = \frac{\ln \frac{f}{K} \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}$
- $n(t) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} = n(-t)$
- $\mathcal{N}(x) \stackrel{\Delta}{=} \int_{-\infty}^{x} n(t)\, dt = 1 - \mathcal{N}(-x)$

The Black-Scholes formula can be derived using the integral representation of Equation (1.6)

$$\begin{aligned} v &= e^{-r_d \tau} \mathbb{E}[F] \\ &= e^{-r_d \tau} \mathbb{E}[[\phi(S_T - K)]^+] \\ &= e^{-r_d \tau} \int_{-\infty}^{+\infty} \left[\phi \left(x e^{(r_d - r_f - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau} y} - K \right)\right]^+ n(y)\, dy. \end{aligned} \tag{1.8}$$

Next one has to deal with the positive part and then complete the square to get the Black-Scholes formula. A derivation based on the partial differential equation can be done using results about the well-studied *heat-equation*.

1.2.2 A note on the forward

The *forward price* f is the strike which makes the time zero value of the *forward contract*

$$F = S_T - f \tag{1.9}$$

equal to zero. It follows that $f = \mathbb{E}[S_T] = xe^{(r_d - r_f)T}$, i.e. the forward price is the expected price of the underlying at time T in a risk-neutral setup (drift of the geometric Brownian motion is equal to cost of carry $r_d - r_f$). The situation $r_d > r_f$ is called *contango*, and the situation $r_d < r_f$ is called *backwardation*. Note that in the Black-Scholes model the class of forward price curves is quite restricted. For example, no seasonal effects can be included. Note that the value of the forward contract after time zero is usually different from zero, and since one of the counterparties is always short, there may be risk of default of the short party. A *futures contract* prevents this dangerous affair: it is basically a forward contract, but the counterparties have to maintain a *margin account* to ensure the amount of cash or commodity owed does not exceed a specified limit.

1.2.3 Greeks

Greeks are derivatives of the value function with respect to model and contract parameters. They are an important information for traders and have become standard information provided by front-office systems. More details on Greeks and the relations among Greeks are presented in Hakala and Wystup [3] or Reiss and Wystup [6]. For vanilla options we list some of them now.

(Spot) delta

$$\frac{\partial v}{\partial x} = \phi e^{-r_f \tau} \mathcal{N}(\phi d_+) \tag{1.10}$$

Forward delta

$$\frac{\partial v}{\partial f} = \phi e^{-r_d \tau} \mathcal{N}(\phi d_+) \tag{1.11}$$

Driftless delta

$$\phi \mathcal{N}(\phi d_+) \tag{1.12}$$

Gamma

$$\frac{\partial^2 v}{\partial x^2} = e^{-r_f \tau} \frac{n(d_+)}{x\sigma\sqrt{\tau}} \tag{1.13}$$

Speed

$$\frac{\partial^3 v}{\partial x^3} = -e^{-r_f \tau} \frac{n(d_+)}{x^2\sigma\sqrt{\tau}} \left(\frac{d_+}{\sigma\sqrt{\tau}} + 1 \right) \tag{1.14}$$

Theta

$$\begin{aligned} \frac{\partial v}{\partial t} = & -e^{-r_f \tau} \frac{n(d_+) x \sigma}{2\sqrt{\tau}} \\ & + \phi [r_f x e^{-r_f \tau} \mathcal{N}(\phi d_+) - r_d K e^{-r_d \tau} \mathcal{N}(\phi d_-)] \end{aligned} \tag{1.15}$$

Charm

$$\frac{\partial^2 v}{\partial x \partial \tau} = -\phi r_f e^{-r_f \tau} \mathcal{N}(\phi d_+) + \phi e^{-r_f \tau} n(d_+) \frac{2(r_d - r_f)\tau - d_- \sigma \sqrt{\tau}}{2\tau\sigma\sqrt{\tau}} \tag{1.16}$$

Color

$$\frac{\partial^3 v}{\partial x^2 \partial \tau} = -e^{-r_f \tau} \frac{n(d_+)}{2x\tau\sigma\sqrt{\tau}} \left[2r_f \tau + 1 + \frac{2(r_d - r_f)\tau - d_- \sigma\sqrt{\tau}}{2\tau\sigma\sqrt{\tau}} d_+ \right] \tag{1.17}$$

Vega

$$\frac{\partial v}{\partial \sigma} = x e^{-r_f \tau} \sqrt{\tau} n(d_+) \tag{1.18}$$

Volga

$$\frac{\partial^2 v}{\partial \sigma^2} = x e^{-r_f \tau} \sqrt{\tau} n(d_+) \frac{d_+ d_-}{\sigma} \tag{1.19}$$

Volga is also sometimes called *vomma* or *volgamma*.

Vanna

$$\frac{\partial^2 v}{\partial \sigma \partial x} = -e^{-r_f \tau} n(d_+) \frac{d_-}{\sigma} \tag{1.20}$$

Rho

$$\frac{\partial v}{\partial r_d} = \phi K \tau e^{-r_d \tau} \mathcal{N}(\phi d_-) \tag{1.21}$$

$$\frac{\partial v}{\partial r_f} = -\phi x \tau e^{-r_f \tau} \mathcal{N}(\phi d_+) \tag{1.22}$$

Dual delta

$$\frac{\partial v}{\partial K} = -\phi e^{-r_d \tau} \mathcal{N}(\phi d_-) \tag{1.23}$$

Dual gamma

$$\frac{\partial^2 v}{\partial K^2} = e^{-r_d \tau} \frac{n(d_-)}{K\sigma\sqrt{\tau}} \tag{1.24}$$

Dual theta

$$\frac{\partial v}{\partial T} = -v_t \tag{1.25}$$

1.2.4 Identities

$$\frac{\partial d_\pm}{\partial \sigma} = -\frac{d_\mp}{\sigma} \tag{1.26}$$

$$\frac{\partial d_\pm}{\partial r_d} = \frac{\sqrt{\tau}}{\sigma} \tag{1.27}$$

$$\frac{\partial d_\pm}{\partial r_f} = -\frac{\sqrt{\tau}}{\sigma} \tag{1.28}$$

$$x e^{-r_f \tau} n(d_+) = K e^{-r_d \tau} n(d_-). \tag{1.29}$$

$$\mathcal{N}(\phi d_-) = I\!P[\phi S_T \geq \phi K] \tag{1.30}$$

$$\mathcal{N}(\phi d_+) = I\!P\left[\phi S_T \leq \phi \frac{f^2}{K}\right] \tag{1.31}$$

The *put-call-parity* is the relationship

$$v(x, K, T, t, \sigma, r_d, r_f, +1) - v(x, K, T, t, \sigma, r_d, r_f, -1) = xe^{-r_f\tau} - Ke^{-r_d\tau}, \qquad (1.32)$$

which is just a more complicated way to write the trivial equation $x = x^+ - x^-$. The *put-call delta parity* is

$$\frac{\partial v(x, K, T, t, \sigma, r_d, r_f, +1)}{\partial x} - \frac{\partial v(x, K, T, t, \sigma, r_d, r_f, -1)}{\partial x} = e^{-r_f\tau}. \qquad (1.33)$$

In particular, we learn that the absolute value of a put delta and a call delta are not exactly adding up to one, but only to a positive number $e^{-r_f\tau}$. They add up to one approximately if either the time to expiration τ is short or if the foreign interest rate r_f is close to zero.

Whereas the choice $K = f$ produces identical values for call and put, we seek the *delta-symmetric strike* $\check{K}$ which produces absolutely identical deltas (spot, forward or driftless). This condition implies $d_+ = 0$ and thus

$$\check{K} = fe^{\frac{\sigma^2}{2}T}, \qquad (1.34)$$

in which case the absolute delta is $e^{-r_f\tau}/2$. In particular, we learn, that always $\check{K} > f$, i.e., there can't be a put and a call with identical values *and* deltas. Note that the strike $\check{K}$ is usually chosen as the middle strike when trading a straddle or a butterfly. Similarly the dual-delta-symmetric strike $\hat{K} = fe^{-\frac{\sigma^2}{2}T}$ can be derived from the condition $d_- = 0$.

1.2.5 Homogeneity based relationships

We may wish to measure the value of the underlying in a different unit. This will obviously effect the option pricing formula as follows.

$$av(x, K, T, t, \sigma, r_d, r_f, \phi) = v(ax, aK, T, t, \sigma, r_d, r_f, \phi) \text{ for all } a > 0. \qquad (1.35)$$

Differentiating both sides with respect to a and then setting $a = 1$ yields

$$v = xv_x + Kv_K. \qquad (1.36)$$

Comparing the coefficients of x and K in Equations (1.7) and (1.36) leads to suggestive results for the delta v_x and dual delta v_K. This *space-homogeneity* is the reason behind the simplicity of the delta formulas, whose tedious computation can be saved this way.

We can perform a similar computation for the time-affected parameters and obtain the obvious equation

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = v\left(x, K, \frac{T}{a}, \frac{t}{a}, \sqrt{a}\sigma, ar_d, ar_f, \phi\right) \text{ for all } a > 0. \qquad (1.37)$$

Differentiating both sides with respect to a and then setting $a = 1$ yields

$$0 = \tau v_t + \frac{1}{2}\sigma v_\sigma + r_d v_{r_d} + r_f v_{r_f}. \qquad (1.38)$$

Of course, this can also be verified by direct computation. The overall use of such equations is to generate double checking benchmarks when computing Greeks. These homogeneity methods can easily be extended to other more complex options.

By *put-call symmetry* we understand the relationship (see [7], [8],[9] and [10])

$$v(x, K, T, t, \sigma, r_d, r_f, +1) = \frac{K}{f} v\left(x, \frac{f^2}{K}, T, t, \sigma, r_d, r_f, -1\right). \qquad (1.39)$$

The strike of the put and the strike of the call result in a geometric mean equal to the forward f. The forward can be interpreted as a *geometric mirror* reflecting a call into a certain number of puts. Note that for at-the-money options ($K = f$) the put-call symmetry coincides with the special case of the put-call parity where the call and the put have the same value.

Direct computation shows that the *rates symmetry*

$$\frac{\partial v}{\partial r_d} + \frac{\partial v}{\partial r_f} = -\tau v \tag{1.40}$$

holds for vanilla options. This relationship, in fact, holds for all European options and a wide class of path-dependent options as shown in [6].

One can directly verify the relationship the *foreign-domestic symmetry*

$$\frac{1}{x} v(x, K, T, t, \sigma, r_d, r_f, \phi) = K v\left(\frac{1}{x}, \frac{1}{K}, T, t, \sigma, r_f, r_d, -\phi\right). \tag{1.41}$$

This equality can be viewed as one of the faces of put-call symmetry. The reason is that the value of an option can be computed both in a domestic as well as in a foreign scenario. We consider the example of S_t modeling the exchange rate of EUR/USD. In New York, the call option $(S_T - K)^+$ costs $v(x, K, T, t, \sigma, r_{usd}, r_{eur}, 1)$ USD and hence $v(x, K, T, t, \sigma, r_{usd}, r_{eur}, 1)/x$ EUR. This EUR-call option can also be viewed as a USD-put option with payoff $K(\frac{1}{K} - \frac{1}{S_T})^+$. This option costs $K v(\frac{1}{x}, \frac{1}{K}, T, t, \sigma, r_{eur}, r_{usd}, -1)$ EUR in Frankfurt, because S_t and $\frac{1}{S_t}$ have the same volatility. Of course, the New York value and the Frankfurt value must agree, which leads to (1.41). We will also learn later, that this symmetry is just one possible result based on *change of numeraire*.

1.2.6 Quotation

Quotation of the underlying exchange rate

Equation (1.1) is a model for the exchange rate. The quotation is a constantly confusing issue, so let us clarify this here. The exchange rate means how much of the *domestic* currency is needed to buy one unit of *foreign* currency. For example, if we take EUR/USD as an exchange rate, then the default quotation is EUR-USD, where USD is the domestic currency and EUR is the foreign currency. The term *domestic* is in no way related to the location of the trader or any country. It merely means the *numeraire* currency. The terms *domestic*, *numeraire* or *base currency* are synonyms as are *foreign* and *underlying*. Throughout this book we denote with the slash (/) the currency pair and with a dash (–) the quotation. The slash (/) does *not* mean a division. For instance, EUR/USD can also be quoted in either EUR-USD, which then means how many USD are needed to buy one EUR, or in USD-EUR, which then means how many EUR are needed to buy one USD. There are certain market standard quotations listed in Table 1.1.

Trading floor language

We call one million a *buck*, one billion a *yard*. This is because a billion is called ‘milliarde’ in French, German and other languages. For the British Pound one million is also often called a *quid*.

Certain currency pairs have names. For instance, GBP/USD is called *cable*, because the exchange rate information used to be sent through a cable in the Atlantic ocean between

Table 1.1 Standard market quotation of major currency pairs with sample spot prices

Currency pair	Default quotation	Sample quote
GBP/USD	GPB-USD	1.8000
GBP/CHF	GBP-CHF	2.2500
EUR/USD	EUR-USD	1.2000
EUR/GBP	EUR-GBP	0.6900
EUR/JPY	EUR-JPY	135.00
EUR/CHF	EUR-CHF	1.5500
USD/JPY	USD-JPY	108.00
USD/CHF	USD-CHF	1.2800

America and England. EUR/JPY is called the *cross*, because it is the cross rate of the more liquidly traded USD/JPY and EUR/USD.

Certain currencies also have names, e.g. the New Zealand Dollar NZD is called a *kiwi*, the Australian Dollar AUD is called *Aussie*, the Scandinavian currencies DKR, NOK and SEK are called *Scandies*.

Exchange rates are generally quoted up to five relevant figures, e.g. in EUR-USD we could observe a quote of 1.2375. The last digit ‘5’ is called the *pip*, the middle digit ‘3’ is called the *big figure*, as exchange rates are often displayed in trading floors and the big figure, which is displayed in bigger size, is the most relevant information. The digits left to the big figure are known anyway, the pips right of the big figure are often negligible. To make it clear, a rise of USD-JPY 108.25 by 20 pips will be 108.45 and a rise by 2 big figures will be 110.25.

Quotation of option prices

Values and prices of vanilla options may be quoted in the six ways explained in Table 1.2.

The Black-Scholes formula quotes **d pips**. The others can be computed using the following instruction.

$$\textbf{d pips} \xrightarrow{\times \frac{1}{S_0}} \%\textbf{f} \xrightarrow{\times \frac{S_0}{K}} \%\textbf{d} \xrightarrow{\times \frac{1}{S_0}} \textbf{f pips} \xrightarrow{\times S_0 K} \textbf{d pips} \tag{1.42}$$

Table 1.2 Standard market quotation types for option values

Name	Symbol	Value in units of	Example
domestic cash	**d**	DOM	29,148 USD
foreign cash	**f**	FOR	24,290 EUR
% domestic	**% d**	DOM per unit of DOM	2.3318 % USD
% foreign	**% f**	FOR per unit of FOR	2.4290 % EUR
domestic pips	**d pips**	DOM per unit of FOR	291.48 USD pips per EUR
foreign pips	**f pips**	FOR per unit of DOM	194.32 EUR pips per USD

In this example we take FOR = EUR, DOM = USD, $S_0 = 1.2000$, $r_d = 3.0\,\%$, $r_f = 2.5\,\%$, $\sigma = 10\,\%$, $K = 1.2500$, $T = 1$ year, $\phi = +1$ (call), notional = 1,000,000 EUR = 1,250,000 USD. For the pips, the quotation 291.48 USD pips per EUR is also sometimes stated as 2.9148 % USD per 1 EUR. Similarly, the 194.32 EUR pips per USD can also be quoted as 1.9432 % EUR per 1 USD.

Delta and premium convention

The spot delta of a European option without premium is well known. It will be called *raw spot delta* δ_{raw} now. It can be quoted in either of the two currencies involved. The relationship is

$$\delta_{raw}^{reverse} = -\delta_{raw} \frac{S}{K}. \tag{1.43}$$

The delta is used to buy or sell spot in the corresponding amount in order to hedge the option up to first order.

For consistency the premium needs to be incorporated into the delta hedge, since a premium in foreign currency will already hedge part of the option's delta risk. To make this clear, let us consider EUR-USD. In the standard arbitrage theory, $v(x)$ denotes the value or premium in USD of an option with 1 EUR notional, if the spot is at x, and the raw delta v_x denotes the number of EUR to buy for the delta hedge. Therefore, xv_x is the number of USD to sell. If now the premium is paid in EUR rather than in USD, then we already have $\frac{v}{x}$ EUR, and the number of EUR to buy has to be reduced by this amount, i.e. if EUR is the premium currency, we need to buy $v_x - \frac{v}{x}$ EUR for the delta hedge or equivalently sell $xv_x - v$ USD.

The entire FX quotation story becomes generally a mess, because we need to first sort out which currency is domestic, which is foreign, what is the notional currency of the option, and what is the premium currency. Unfortunately this is not symmetrial, since the counterpart might have another notion of domestic currency for a given currency pair. Hence in the professional inter bank market there is one notion of delta per currency pair. Normally it is the left hand side delta of the *Fenics* screen if the option is traded in left hand side premium, which is normally the standard and right hand side delta if it is traded with right hand side premium, e.g. EUR/USD lhs, USD/JPY lhs, EUR/JPY lhs, AUD/USD rhs, etc . . . Since OTM options are traded most of time the difference is not huge and hence does not create a huge spot risk.

Additionally the standard delta per currency pair [left hand side delta in *Fenics* for most cases] is used to quote options in volatility. This has to be specified by currency.

This standard inter bank notion must be adapted to the real delta-risk of the bank for an automated trading system. For currencies where the risk–free currency of the bank is the base currency of the currency it is clear that the delta is the raw delta of the option and for risky premium this premium must be included. In the opposite case the risky premium and the market value must be taken into account for the base currency premium, so that these offset each other. And for premium in underlying currency of the contract the market-value needs to be taken into account. In that way the delta hedge is invariant with respect to the risky currency notion of the bank, e.g. the delta is the same for a USD-based bank and a EUR-based bank.

Examples

We consider two examples in Tables 1.3 and 1.4 to compare the various versions of deltas that are used in practice.

1.2.7 Strike in terms of delta

Since $v_x = \Delta = \phi e^{-r_f \tau} \mathcal{N}(\phi d_+)$ we can retrieve the strike as

$$K = x \exp\left\{-\phi \mathcal{N}^{-1}(\phi \Delta e^{r_f \tau}) \sigma \sqrt{\tau} + \sigma \theta_+ \tau\right\}. \tag{1.44}$$

Table 1.3 1y EUR call USD put strike $K = 0.9090$ for a EUR-based bank

Delta ccy	Prem ccy	Fenics	Formula	Delta
% EUR	EUR	lhs	$\delta_{raw} - P$	44.72
% EUR	USD	rhs	δ_{raw}	49.15
% USD	EUR	rhs [flip F4]	$-(\delta_{raw} - P)S/K$	−44.72
% USD	USD	lhs [flip F4]	$-(\delta_{raw})S/K$	−49.15

Market data: spot $S = 0.9090$, volatility $\sigma = 12\,\%$, EUR rate $r_f = 3.96\,\%$, USD rate $r_d = 3.57\,\%$. The raw delta is 49.15 % EUR and the value is 4.427 % EUR.

Table 1.4 1y call EUR call USD put strike $K = 0.7000$ for a EUR-based bank

Delta ccy	Prem ccy	Fenics	Formula	Delta
% EUR	EUR	lhs	$\delta_{raw} - P$	72.94
% EUR	USD	rhs	δ_{raw}	94.82
% USD	EUR	rhs [flip F4]	$-(\delta_{raw} - P)S/K$	−94.72
% USD	USD	lhs [flip F4]	$-\delta_{raw}S/K$	−123.13

Market data: spot $S = 0.9090$, volatility $\sigma = 12\,\%$, EUR rate $r_f = 3.96\,\%$, USD rate $r_d = 3.57\,\%$. The raw delta is 94.82 % EUR and the value is 21.88 % EUR.

1.2.8 Volatility in terms of delta

The mapping $\sigma \mapsto \Delta = \phi e^{-r_f\tau}\mathcal{N}(\phi d_+)$ is not one-to-one. The two solutions are given by

$$\sigma_\pm = \frac{1}{\sqrt{\tau}}\left\{\phi\mathcal{N}^{-1}(\phi\Delta e^{r_f\tau}) \pm \sqrt{(\mathcal{N}^{-1}(\phi\Delta e^{r_f\tau}))^2 - \sigma\sqrt{\tau}(d_+ + d_-)}\right\}. \tag{1.45}$$

Thus using just the delta to retrieve the volatility of an option is not advisable.

1.2.9 Volatility and delta for a given strike

The determination of the volatility and the delta for a given strike is an iterative process involving the determination of the delta for the option using at-the-money volatilities in a first step and then using the determined volatility to re–determine the delta and to continuously iterate the delta and volatility until the volatility does not change more than $\epsilon = 0.001\,\%$ between iterations. More precisely, one can perform the following algorithm. Let the given strike be K.

1. Choose σ_0 = at-the-money volatility from the volatility matrix.
2. Calculate $\Delta_{n+1} = \Delta(\text{Call}(K, \sigma_n))$.
3. Take $\sigma_{n+1} = \sigma(\Delta_{n+1})$ from the volatility matrix, possibly via a suitable interpolation.
4. If $|\sigma_{n+1} - \sigma_n| < \epsilon$, then quit, otherwise continue with step 2.

In order to prove the convergence of this algorithm we need to establish convergence of the recursion

$$\begin{aligned}\Delta_{n+1} &= e^{-r_f\tau}\mathcal{N}(d_+(\Delta_n)) \\ &= e^{-r_f\tau}\mathcal{N}\left(\frac{\ln(S/K)+(r_d-r_f+\frac{1}{2}\sigma^2(\Delta_n))\tau}{\sigma(\Delta_n)\sqrt{\tau}}\right)\end{aligned} \tag{1.46}$$

for sufficiently large $\sigma(\Delta_n)$ and a sufficiently smooth volatility smile surface. We must show that the sequence of these Δ_n converges to a fixed point $\Delta^* \in [0, 1]$ with a fixed volatility $\sigma^* = \sigma(\Delta^*)$.

This proof has been carried out in [11] and works like this. We consider the derivative

$$\frac{\partial \Delta_{n+1}}{\partial \Delta_n} = -e^{-r_f\tau}n(d_+(\Delta_n))\frac{d_-(\Delta_n)}{\sigma(\Delta_n)}\cdot\frac{\partial}{\partial \Delta_n}\sigma(\Delta_n). \tag{1.47}$$

The term

$$-e^{-r_f\tau}n(d_+(\Delta_n))\frac{d_-(\Delta_n)}{\sigma(\Delta_n)}$$

converges rapidly to zero for very small and very large spots, being an argument of the standard normal density n. For sufficiently large $\sigma(\Delta_n)$ and a sufficiently smooth volatility surface in the sense that $\frac{\partial}{\partial \Delta_n}\sigma(\Delta_n)$ is sufficiently small, we obtain

$$\left|\frac{\partial}{\partial \Delta_n}\sigma(\Delta_n)\right| \stackrel{\Delta}{=} q < 1. \tag{1.48}$$

Thus for any two values $\Delta_{n+1}^{(1)}$, $\Delta_{n+1}^{(2)}$, a continuously differentiable smile surface we obtain

$$|\Delta_{n+1}^{(1)} - \Delta_{n+1}^{(2)}| < q|\Delta_n^{(1)} - \Delta_n^{(2)}|, \tag{1.49}$$

due to the mean value theorem. Hence the sequence Δ_n is a contraction in the sense of the fixed point theorem of Banach. This implies that the sequence converges to a unique fixed point in [0, 1], which is given by $\sigma^* = \sigma(\Delta^*)$.

1.2.10 Greeks in terms of deltas

In Foreign Exchange markets the moneyness of vanilla options is always expressed in terms of deltas and prices are quoted in terms of volatility. This makes a ten-delta call a financial object as such independent of spot and strike. This method and the quotation in volatility makes objects and prices transparent in a very intelligent and user-friendly way. At this point we list the Greeks in terms of deltas instead of spot and strike. Let us introduce the quantities

$$\Delta_+ \stackrel{\Delta}{=} \phi e^{-r_f\tau}\mathcal{N}(\phi d_+) \text{ spot delta}, \tag{1.50}$$

$$\Delta_- \stackrel{\Delta}{=} -\phi e^{-r_d\tau}\mathcal{N}(\phi d_-) \text{ dual delta}, \tag{1.51}$$

which we assume to be given. From these we can retrieve

$$d_+ = \phi\mathcal{N}^{-1}(\phi e^{r_f\tau}\Delta_+), \tag{1.52}$$

$$d_- = \phi\mathcal{N}^{-1}(-\phi e^{r_d\tau}\Delta_-). \tag{1.53}$$

Interpretation of dual delta

The dual delta introduced in (1.23) as the sensitivity with respect to strike has another – more practical – interpretation in a foreign exchange setup. We have seen in Section 1.2.5 that the domestic value

$$v(x, K, \tau, \sigma, r_d, r_f, \phi) \tag{1.54}$$

corresponds to a foreign value

$$v\left(\frac{1}{x}, \frac{1}{K}, \tau, \sigma, r_f, r_d, -\phi\right) \tag{1.55}$$

up to an adjustment of the nominal amount by the factor xK. From a foreign viewpoint the delta is thus given by

$$\begin{aligned} & -\phi e^{-r_d\tau}\mathcal{N}\left(-\phi\frac{\ln(\frac{K}{x}) + (r_f - r_d + \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}}\right) \\ &= -\phi e^{-r_d\tau}\mathcal{N}\left(\phi\frac{\ln(\frac{x}{K}) + (r_d - r_f - \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}}\right) \\ &= \Delta_-, \end{aligned} \tag{1.56}$$

which means the dual delta is the delta from the foreign viewpoint. We will see below that foreign rho, vega and gamma do not require to know the dual delta. We will now state the Greeks in terms of $x, \Delta_+, \Delta_-, r_d, r_f, \tau, \phi$.

Value

$$v(x, \Delta_+, \Delta_-, r_d, r_f, \tau, \phi) = x\Delta_+ + x\Delta_-\frac{e^{-r_f\tau}n(d_+)}{e^{-r_d\tau}n(d_-)} \tag{1.57}$$

(Spot) delta

$$\frac{\partial v}{\partial x} = \Delta_+ \tag{1.58}$$

Forward delta

$$\frac{\partial v}{\partial f} = e^{(r_f - r_d)\tau}\Delta_+ \tag{1.59}$$

Gamma

$$\frac{\partial^2 v}{\partial x^2} = e^{-r_f\tau}\frac{n(d_+)}{x(d_+ - d_-)} \tag{1.60}$$

Taking a trader's gamma (change of delta if spot moves by 1 %) additionally removes the spot dependence, because

$$\Gamma_{trader} = \frac{x}{100}\frac{\partial^2 v}{\partial x^2} = e^{-r_f\tau}\frac{n(d_+)}{100(d_+ - d_-)} \tag{1.61}$$

Speed

$$\frac{\partial^3 v}{\partial x^3} = -e^{-r_f\tau}\frac{n(d_+)}{x^2(d_+ - d_-)^2}(2d_+ - d_-) \tag{1.62}$$

Theta

$$\frac{1}{x}\frac{\partial v}{\partial t} = -e^{-r_f\tau}\frac{n(d_+)(d_+ - d_-)}{2\tau} + \left[r_f\Delta_+ + r_d\Delta_-\frac{e^{-r_f\tau}n(d_+)}{e^{-r_d\tau}n(d_-)}\right] \tag{1.63}$$

Charm

$$\frac{\partial^2 v}{\partial x\partial\tau} = -\phi r_f e^{-r_f\tau}\mathcal{N}(\phi d_+) + \phi e^{-r_f\tau}n(d_+)\frac{2(r_d - r_f)\tau - d_-(d_+ - d_-)}{2\tau(d_+ - d_-)} \tag{1.64}$$

Color

$$\frac{\partial^3 v}{\partial x^2\partial\tau} = -\frac{e^{-r_f\tau}n(d_+)}{2x\tau(d_+ - d_-)}\left[2r_f\tau + 1 + \frac{2(r_d - r_f)\tau - d_-(d_+ - d_-)}{2\tau(d_+ - d_-)}d_+\right] \tag{1.65}$$

Vega

$$\frac{\partial v}{\partial\sigma} = xe^{-r_f\tau}\sqrt{\tau}n(d_+) \tag{1.66}$$

Volga

$$\frac{\partial^2 v}{\partial\sigma^2} = xe^{-r_f\tau}\tau n(d_+)\frac{d_+d_-}{d_+ - d_-} \tag{1.67}$$

Vanna

$$\frac{\partial^2 v}{\partial\sigma\partial x} = -e^{-r_f\tau}n(d_+)\frac{\sqrt{\tau}d_-}{d_+ - d_-} \tag{1.68}$$

Rho

$$\frac{\partial v}{\partial r_d} = -x\tau\Delta_-\frac{e^{-r_f\tau}n(d_+)}{e^{-r_d\tau}n(d_-)} \tag{1.69}$$

$$\frac{\partial v}{\partial r_f} = -x\tau\Delta_+ \tag{1.70}$$

Dual delta

$$\frac{\partial v}{\partial K} = \Delta_- \tag{1.71}$$

Dual gamma

$$K^2\frac{\partial^2 v}{\partial K^2} = x^2\frac{\partial^2 v}{\partial x^2} \tag{1.72}$$

Dual theta

$$\frac{\partial v}{\partial T} = -v_t \tag{1.73}$$

As an important example we consider vega.

Table 1.5 Vega in terms of Delta for the standard maturity labels and various deltas

Mat/Δ	50 %	45 %	40 %	35 %	30 %	25 %	20 %	15 %	10 %	5 %
1D	2	2	2	2	2	2	1	1	1	1
1W	6	5	5	5	5	4	4	3	2	1
1W	8	8	8	7	7	6	5	5	3	2
1M	11	11	11	11	10	9	8	7	5	3
2M	16	16	16	15	14	13	11	9	7	4
3M	20	20	19	18	17	16	14	12	9	5
6M	28	28	27	26	24	22	20	16	12	7
9M	34	34	33	32	30	27	24	20	15	9
1Y	39	39	38	36	34	31	28	23	17	10
2Y	53	53	52	50	48	44	39	32	24	14
3Y	63	63	62	60	57	53	47	39	30	18

It shows that one can vega hedge a long 9M 35 delta call with 4 short 1M 20 delta puts.

Vega in terms of delta

The mapping $\Delta \mapsto v_\sigma = xe^{-r_f\tau}\sqrt{\tau}n(\mathcal{N}^{-1}(e^{r_f\tau}\Delta))$ is important for trading vanilla options. Observe that this function does not depend on r_d or σ, just on r_f. Quoting vega in % foreign will additionally remove the spot dependence. This means that for a moderately stable foreign term structure curve, traders will be able to use a moderately stable vega matrix. For $r_f = 3\,\%$ the vega matrix is presented in Table 1.5.

1.3 VOLATILITY

Volatility is the *annualized standard deviation of the log-returns*. It is *the* crucial input parameter to determine the value of an option. Hence, the crucial question is where to derive the volatility from. If no active option market is present, the only source of information is estimating the historic volatility. This would give some clue about the *past*. In liquid currency pairs volatility is often a traded quantity on its own, which is quoted by traders, brokers and real-time data pages. These quotes reflect views of market participants about the *future*.

Since volatility normally does not stay constant, option traders are highly concerned with hedging their volatility exposure. Hedging vanilla options' vega is comparatively easy, because vanilla options have convex payoffs, whence the vega is always positive, i.e. the higher the volatility, the higher the price. Let us take for example a EUR-USD market with spot 1.2000, USD- and EUR rate at 2.5 %. A 3-month at-the-money call with 1 million EUR notional would cost 29,000 USD at a volatility of 12 %. If the volatility now drops to a value of 8 %, then the value of the call would be only 19,000 USD. This monotone dependence is not guaranteed for non-convex payoffs as we illustrate in Figure 1.2.

1.3.1 Historic volatility

We briefly describe how to compute the historic volatility of a time series

$$S_0, S_1, \ldots, S_N \tag{1.74}$$

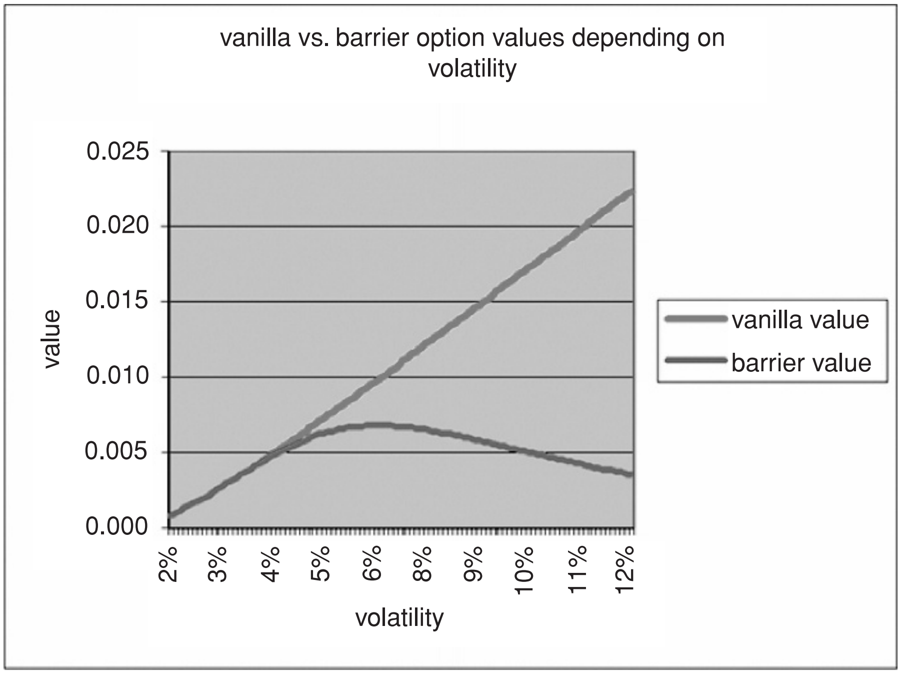


Figure 1.2 Dependence of a vanilla call and a reverse knock-out call on volatility
The vanilla value is monotone in the volatility, whereas the barrier value is not. The reason is that as the spot gets closer to the upper knock-out barrier, an increasing volatility would increase the chance of knock-out and hence decrease the value.

of daily data. First, we create the sequence of log-returns

$$r_i = \ln \frac{S_i}{S_{i-1}}, \quad i = 1, \dots, N. \tag{1.75}$$

Then, we compute the average log-return

$$\bar{r} = \frac{1}{N} \sum_{i=1}^{N} r_i, \tag{1.76}$$

their variance

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (r_i - \bar{r})^2, \tag{1.77}$$

and their standard deviation

$$\hat{\sigma} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (r_i - \bar{r})^2}. \tag{1.78}$$

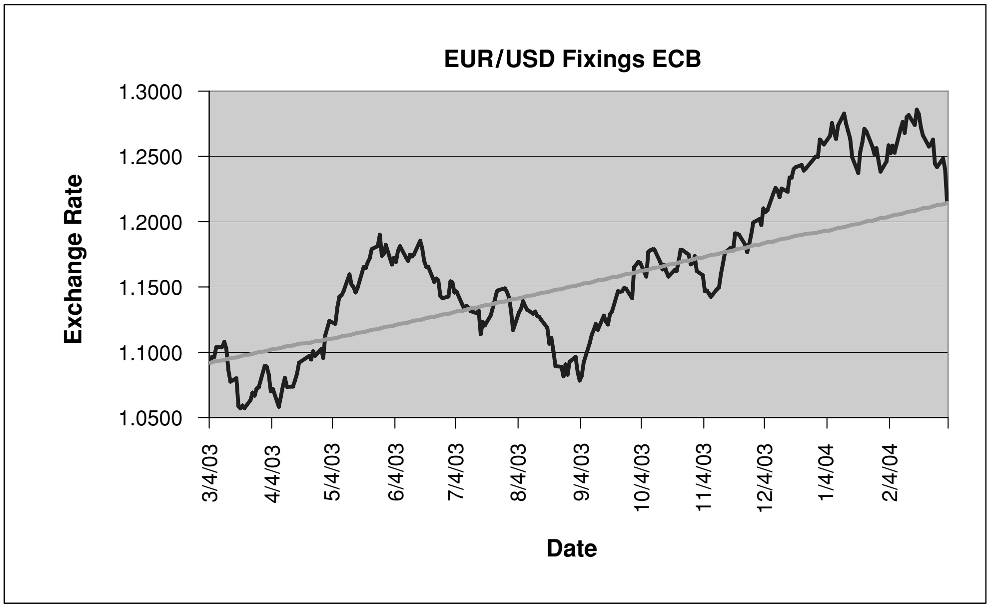


Figure 1.3 ECB-fixings of EUR-USD from 4 March 2003 to 3 March 2004 and the line of average growth

The annualized standard deviation, which is the volatility, is then given by

$$\hat{\sigma}_a = \sqrt{\frac{B}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2}, \tag{1.79}$$

where the *annualization factor* B is given by

$$B = \frac{N}{k}d, \tag{1.80}$$

and k denotes the number of calendar days within the time series and d denotes the number of calendar days per year.

Assuming normally distributed log-returns, we know that $\hat{\sigma}^2$ is χ^2-distributed. Therefore, given a confidence level of p and a corresponding error probability $\alpha = 1 - p$, the p-confidence interval is given by

$$\left[\hat{\sigma}_a\sqrt{\frac{N-1}{\chi^2_{N-1;1-\frac{\alpha}{2}}}}, \hat{\sigma}_a\sqrt{\frac{N-1}{\chi^2_{N-1;\frac{\alpha}{2}}}}\right], \tag{1.81}$$

where $\chi^2_{n;p}$ denotes the p-quantile of a χ^2-distribution[2] with n degrees of freedom.

As an example let us take the 256 ECB-fixings of EUR-USD from 4 March 2003 to 3 March 2004 displayed in Figure 1.3. We get $N = 255$ log-returns. Taking $k = d = 365$,

[2] values and quantiles of the χ^2-distribution and other distributions can be computed on the internet, e.g. at http://eswf.uni-koeln.de/allg/surfstat/tables.htm.

we obtain

$$\bar{r} = \frac{1}{N}\sum_{i=1}^{N} r_i = 0.0004166,$$

$$\hat{\sigma}_a = \sqrt{\frac{B}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2} = 10.85\,\%,$$

and a 95 % confidence interval of [9.99 %, 11.89 %].

1.3.2 Historic correlation

As in the preceding section we briefly describe how to compute the historic correlation of two time series

$$x_0, x_1, \ldots, x_N,$$
$$y_0, y_1, \ldots, y_N,$$

of daily data. First, we create the sequences of log-returns

$$X_i = \ln\frac{x_i}{x_{i-1}}, \quad i = 1, \ldots, N,$$
$$Y_i = \ln\frac{y_i}{y_{i-1}}, \quad i = 1, \ldots, N. \tag{1.82}$$

Then, we compute the average log-returns

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i,$$
$$\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i, \tag{1.83}$$

their variances and covariance

$$\hat{\sigma}_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2, \tag{1.84}$$

$$\hat{\sigma}_Y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^2, \tag{1.85}$$

$$\hat{\sigma}_{XY} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y}), \tag{1.86}$$

and their standard deviations

$$\hat{\sigma}_X = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2}, \tag{1.87}$$

$$\hat{\sigma}_Y = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^2}. \tag{1.88}$$

The estimate for the correlation of the log-returns is given by

$$\hat{\rho} = \frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X \hat{\sigma}_Y}. \tag{1.89}$$

This correlation estimate is often not very stable, but on the other hand, often the only available information. More recent work by Jaekel [12] treats robust estimation of correlation. We will revisit FX correlation risk in Section 1.6.7.

1.3.3 Volatility smile

The Black-Scholes model assumes a constant volatility throughout. However, market prices of traded options imply different volatilities for different maturities and different deltas. We start with some technical issues how to imply the volatility from vanilla options.

Retrieving the volatility from vanilla options

Given the value of an option. Recall the Black-Scholes formula in Equation (1.7). We now look at the function $v(\sigma)$, whose derivative (vega) is

$$v'(\sigma) = xe^{-r_f T}\sqrt{T}n(d_+). \tag{1.90}$$

The function $\sigma \mapsto v(\sigma)$ is

1. strictly increasing,
2. concave up for $\sigma \in [0, \sqrt{2|\ln F - \ln K|/T})$,
3. concave down for $\sigma \in (\sqrt{2|\ln F - \ln K|/T}, \infty)$

and also satisfies

$$v(0) = [\phi(xe^{-r_f T} - Ke^{-r_d T})]^+, \tag{1.91}$$
$$v(\infty, \phi = 1) = xe^{-r_f T}, \tag{1.92}$$
$$v(\sigma = \infty, \phi = -1) = Ke^{-r_d T}, \tag{1.93}$$
$$v'(0) = xe^{-r_f T}\sqrt{T}/\sqrt{2\pi}\,\mathbb{I}_{\{F=K\}}, \tag{1.94}$$

In particular the mapping $\sigma \mapsto v(\sigma)$ is invertible. However, the starting guess for employing Newton's method should be chosen with care, because the mapping $\sigma \mapsto v(\sigma)$ has a saddle point at

$$\left(\sqrt{\frac{2}{T}|\ln\frac{F}{K}|}, \phi e^{-r_d T}\left\{F\mathcal{N}\left(\phi\sqrt{2T[\ln\frac{F}{K}]^+}\right) - K\mathcal{N}\left(\phi\sqrt{2T[\ln\frac{K}{F}]^+}\right)\right\}\right), \tag{1.95}$$

as illustrated in Figure 1.4.

To ensure convergence of Newton's method, we are advised to use initial guesses for σ on the same side of the saddle point as the desired implied volatility. The danger is that a large initial guess could lead to a negative successive guess for σ. Therefore one should start with small initial guesses at or below the saddle point. For at-the-money options, the saddle point is degenerate for a zero volatility and small volatilities serve as good initial guesses.

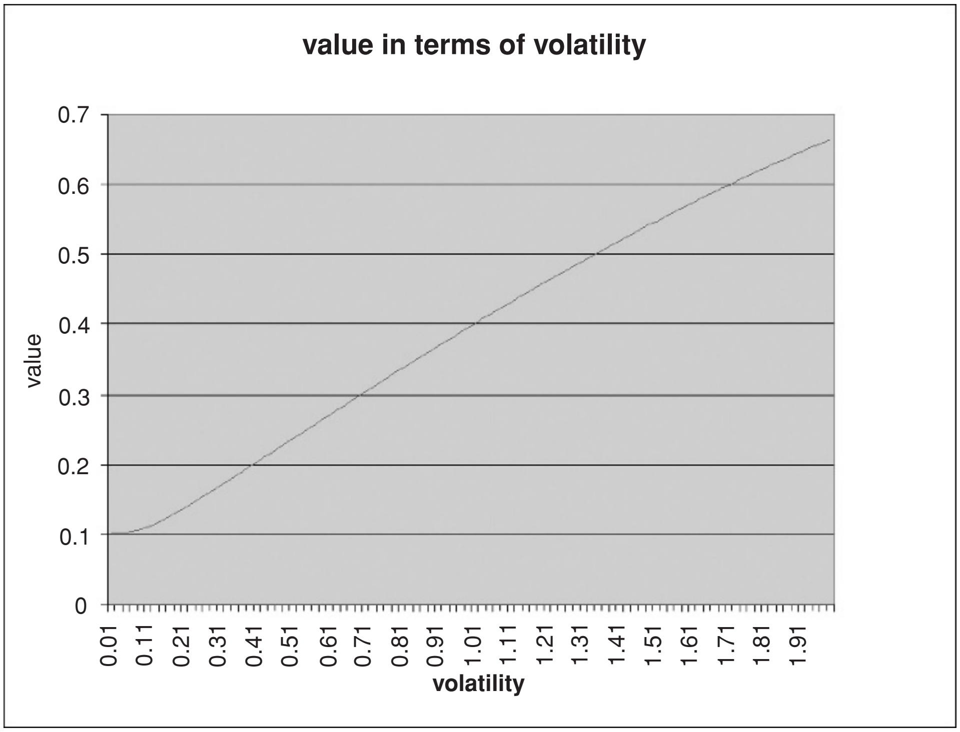


Figure 1.4 Value of a European call in terms of volatility with parameters $x = 1$, $K = 0.9$, $T = 1$, $r_d = 6\,\%$, $r_f = 5\,\%$. The saddle point is at $\sigma = 48\,\%$

Visual basic source code

```
Function VanillaVolRetriever(spot As Double, rd As Double, _
rf As Double, strike As Double, T As Double, _
type As Integer, GivenValue As Double) As Double
Dim func As Double
Dim dfunc As Double
Dim maxit As Integer 'maximum number of iterations
Dim j As Integer
Dim s As Double
'first check if a volatility exists, otherwise set result to zero
If GivenValue < Application.Max _
   (0, type * (spot * Exp(-rf * T) - strike * Exp(-rd * T))) Or _
   (type = 1 And GivenValue > spot * Exp(-rf * T)) Or _
   (type = -1 And GivenValue > strike * Exp(-rd * T)) Then
   VanillaVolRetriever = 0
Else
   ' there exists a volatility yielding the given value,
   ' now use Newton's method:
   ' the mapping vol to value has a saddle point.
   ' First compute this saddle point:
```

```
   saddle = Sqr(2 / T * Abs(Log(spot / strike) + (rd - rf) * T))
   If saddle > 0 Then
      VanillaVolRetriever = saddle * 0.9
   Else
      VanillaVolRetriever = 0.1
   End If
   maxit = 100
   For j = 1 To maxit Step 1
      func = Vanilla(spot, strike, VanillaVolRetriever, _
      rd, rf, T, type, value) - GivenValue
      dfunc = Vanilla(spot, strike, VanillaVolRetriever, _
      rd, rf, T, type, vega)
      VanillaVolRetriever = VanillaVolRetriever - func / dfunc
      If VanillaVolRetriever <= 0 Then VanillaVolRetriever = 0.01
      If Abs(func / dfunc) <= 0.0000001 Then j = maxit
   Next j
End If
End Function
```

Market data

Now that we know how to imply the volatility from a given value, we can take a look at the market. We take EUR/GBP at the beginning of April 2005. The at-the-money volatilities for various maturities are listed in Table 1.6. We observe that implied volatilities are not constant, but depend on the time to maturity of the option as well as on the current time. This shows that the Black-Scholes assumption of a constant volatility is not fully justified looking at market data. We have a *term structure of volatility* as well as a stochastic nature of the term structure curve as time passes.

Besides the dependence on the time to maturity (term structure) we also observe different implied volatilities for different degrees of moneyness. This effect is called the *volatility smile*. The term structure and smile together are called a *volatility matrix* or *volatility surface*, if it is graphically displayed. Various possible reasons for this empirical phenomenon are discussed among others by Bates, e.g. in [8].

In Foreign Exchange Options markets implied volatilities are generally quoted and plotted against the deltas of out-of-the-money call and put options. This allows market participants to

Table 1.6 EUR/GBP implied volatilities in % for at-the-money vanilla options

Date	Spot	1 Week	1 Month	3 Month	6 Month	1 Year	2 Years
1-Apr-05	0.6864	4.69	4.83	5.42	5.79	6.02	6.09
4-Apr-05	0.6851	4.51	4.88	5.34	5.72	5.99	6.07
5-Apr-05	0.6840	4.66	4.95	5.34	5.70	5.97	6.03
6-Apr-05	0.6847	4.65	4.91	5.39	5.79	6.05	6.12
7-Apr-05	0.6875	4.78	4.97	5.39	5.79	6.01	6.10
8-Apr-05	0.6858	4.76	5.00	5.41	5.78	6.00	6.09

Source: BBA (British Bankers Association), http://www.bba.org.uk.

ask various partners for quotes on a 25-Delta call, which is spot independent. The actual strike will be set depending on the spot if the trade is close to being finalized. The at-the-money option is taken to be the one that has a strike equal to the forward, which is equivalent to the value of the call and the put being equal. Other types of *at-the-money* are discussed in Section 1.3.6. Their delta is

$$\frac{\partial v}{\partial x} = \phi e^{-r_f \tau} \mathcal{N}\left(\phi \frac{1}{2} \sigma \sqrt{\tau}\right), \tag{1.96}$$

for a small volatility σ and short time to maturity τ, a number near $\phi 50\,\%$. This is no more true for long-term vanilla options. Further market information consists of the implied volatilities for puts and calls with a delta of $\phi 25\,\%$. Other or additional implied volatilities for other deltas such as $\phi 10\,\%$ and $\phi 35\,\%$ are also quoted. Volatility matrices for more delta pillars are usually interpolated.

Symmetric decomposition

Generally in Foreign Exchange, volatilities are decomposed into a *symmetric* part of the smile reflecting the *convexity* and a *skew-symmetric* part of the smile reflecting the *skew*. The way this works is that the market quotes *risk reversals (RR)* and *butterflies (BF)* or strangles, see Sections 1.4.2 and 1.4.5 for the description of the *products* and Figure 1.5 for the payoffs. Here we are talking about the respective *volatilities* to use to price the products. Sample quotes are listed in Tables 1.7 and 1.8. The relationship between risk reversal and strangle/butterfly quotes and the volatility smile are explained graphically in Figure 1.6.

The relationship between risk reversal quoted in terms of volatility (RR) and butterfly/strangle (BF) quoted in terms of volatility and the volatilities of 25-delta calls and puts

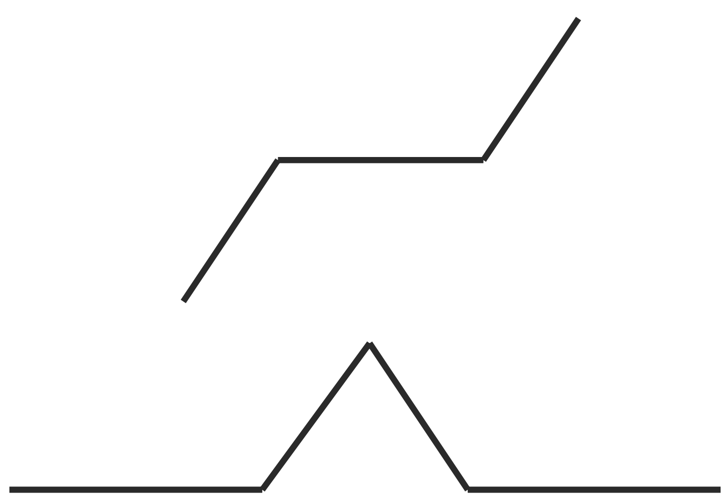


Figure 1.5 The risk reversal (upper payoff) is a skew symmetric product, the butterfly (lower payoff) a symmetric product

Table 1.7 EUR/GBP 25 Delta Risk Reversal in %

Date	Spot	1 Month	3 Month	1 Year
1-Apr-05	0.6864	0.18	0.23	0.30
4-Apr-05	0.6851	0.15	0.20	0.29
5-Apr-05	0.6840	0.11	0.19	0.28
6-Apr-05	0.6847	0.08	0.19	0.28
7-Apr-05	0.6875	0.13	0.19	0.28
8-Apr-05	0.6858	0.13	0.19	0.28

Source: BBA (British Bankers Association). This means that for example on 4 April 2005, the 1-month 25-delta EUR call was priced with a volatility of 0.15 % higher than the EUR put. At that moment the market apparently favored calls indicating a belief in an upward movement.

Table 1.8 EUR/GBP 25 Delta Strangle in %

Date	Spot	1 Month	3 Month	1 Year
1-Apr-05	0.6864	0.15	0.16	0.16
4-Apr-05	0.6851	0.15	0.16	0.16
5-Apr-05	0.6840	0.15	0.16	0.16
6-Apr-05	0.6847	0.15	0.16	0.16
7-Apr-05	0.6875	0.15	0.16	0.16
8-Apr-05	0.6858	0.15	0.16	0.16

Source: BBA (British Bankers Association). This means that for example on 4 April 2005, the 1-month 25-delta EUR call and the 1-month 25-delta EUR put are on average quoted with a volatility of 0.15 % higher than the 1-month at-the-money calls and puts. The result is that the 1-month EUR call is quoted with a volatility of $4.88\,\% + 0.075\,\%$ and the 1-month EUR put is quoted with a volatility of $4.88\,\% - 0.075\,\%$.

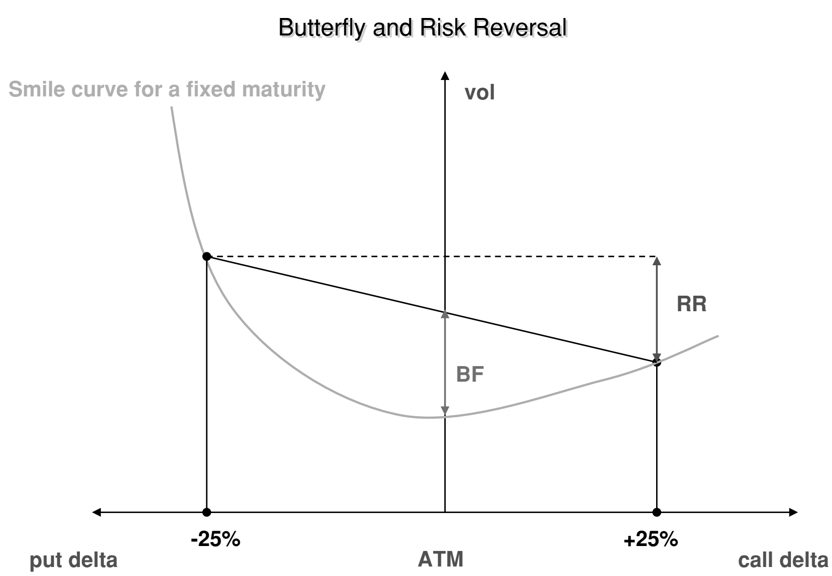


Figure 1.6 Risk reversal and butterfly in terms of volatility for a given FX vanilla option smile

Table 1.9 EUR/GBP implied volatilities in % of 1 April 2005

Maturity	25 delta put	At-the-money	25 delta call
1M	4.890	4.830	5.070
3M	5.465	5.420	5.695
1Y	6.030	6.020	6.330

Source: BBA (British Bankers Association). They are computed based on the market data displayed in Tables 1.6, 1.7 and 1.8 using Equations (1.97) and (1.98).

are given by

$$\sigma_+ = \text{ATM} + BF + \frac{1}{2}RR, \tag{1.97}$$

$$\sigma_- = \text{ATM} + BF - \frac{1}{2}RR, \tag{1.98}$$

$$RR = \sigma_+ - \sigma_-, \tag{1.99}$$

$$BF = \frac{\sigma_+ + \sigma_-}{2} - \sigma_0, \tag{1.100}$$

where σ_0 denotes the at-the-money volatility of both put and call, σ_+ the volatility of an out-of-the-money call (usually 25-Δ) and σ_- the volatility of an out-of-the-money put (usually 25-Δ). Our sample market data is given in terms of RR and BF. Translated into implied volatilities of vanillas we obtain the data listed in Table 1.9 and Figure 1.7.

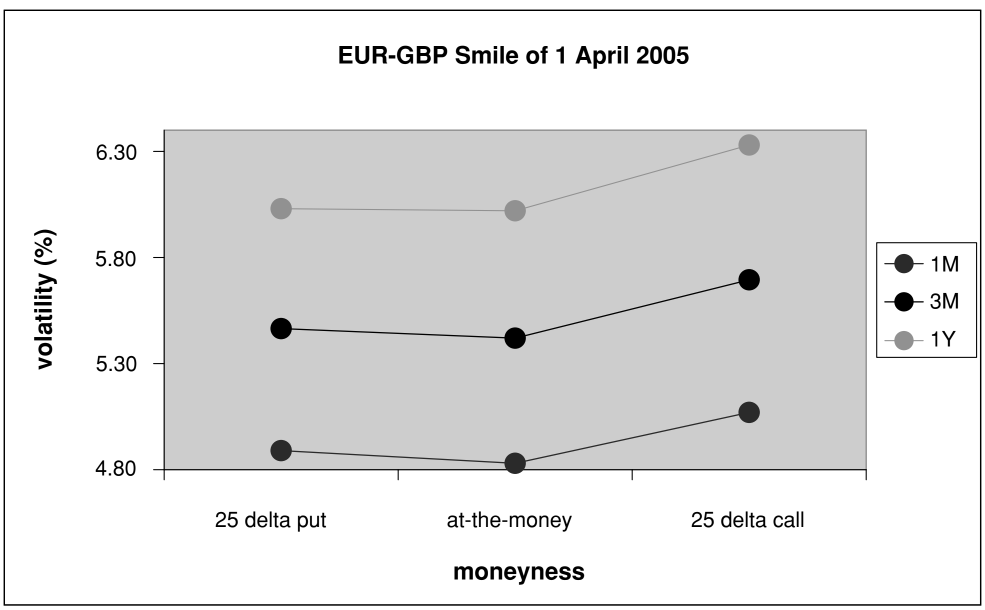


Figure 1.7 Implied volatilities for EUR-GBP vanilla options as of 1 April 2005
Source: BBA (British Bankers Association).

1.3.4 At-the-money volatility interpolation

The interpolation takes into account the effect of reduced volatility on weekends and on days closed in the global main trading centers London or New York and the local market, e.g. Tokyo for JPY-trades. The change is done for the one-day forward volatility. There is a reduction in the one-day forward variance of 25 % for each London and New York closed day. For local market holidays there is a reduction of 25 %, where local holidays for EUR are ignored. Weekends are accounted by a reduction to 15 % variance. The variance on trading days is adjusted to match the volatility on the pillars of the ATM-volatility curve exactly.

The procedure starts from the two pillars t_1, t_2 surrounding the date t_r in question. The ATM forward volatility for the period is calculated based on the consistency condition

$$\sigma^2(t_1)(t_1 - t_0) + \sigma_f^2(t_1, t_2)(t_2 - t_1) = \sigma^2(t_2)(t_2 - t_0), \tag{1.101}$$

whence

$$\sigma_f(t_1, t_2) = \sqrt{\frac{\sigma^2(t_2)(t_2 - t_0) - \sigma^2(t_1)(t_1 - t_0))}{t_2 - t_1}}. \tag{1.102}$$

For each day the factor is determined and from the constraint that the sum of one-day forward variances matches exactly the total variance the factor for the enlarged one day business variances $\alpha(t)$ with t business day is determined.

$$\sigma^2(t_1, t_2)(t_2 - t_1) = \sum_{t=t_1}^{t_r} \alpha(t)\sigma_f^2(t, t+1) \tag{1.103}$$

The variance for the period is the sum of variances to the start and sum of variances to the required date.

$$\sigma^2(t_r) = \sqrt{\frac{\sigma^2(t_1)(t_1 - t_0) + \sum_{t=t_1}^{t_r} \alpha(t)\sigma_f^2(t, t+1)}{t_r - t_0}} \tag{1.104}$$

1.3.5 Volatility smile conventions

The volatility smile is quoted in terms of delta and one at-the-money pillar. We recall that there are several notions of delta

- spot delta $e^{-r_f\tau}N(d_+)$,
- forward delta $e^{-r_d\tau}N(d_+)$,
- driftless delta $N(d_+)$,

and there is the premium which might be included in the delta. It is important to specify the notion that is used to quote the smile. There are three different deltas concerning plain vanilla options.

1.3.6 At-the-money definition

There is one specific at-the-money pillar in the middle. There are at least three notions for the meaning of *at-the-money (ATM)*.

Delta parity: delta call = − delta put

Value parity: call value = put value

Fifty delta: call delta = 50 % and put delta = 50 %

Moreover, these notions use different versions of delta, namely either spot, forward, or driftless and premium included or excluded.

The standard for all currencies one can stick to is spot delta parity with premium included [left hand side *Fenics* delta for call and put is the same] or excluded [right hand side *Fenics* delta] is used.

1.3.7 Interpolation of the volatility on maturity pillars

To determine the spread to at-the-money we can take a kernel interpolation in one dimension to compute the volatility on the delta pillars. Given N points (X_n, y_n), $n = 1, \ldots, N$, where $X = (x^1, x^2) \in \mathbb{R}^2$ and $y \in \mathbb{R}$, a "smooth" interpolation of these points is given by a "smooth" function $g : \mathbb{R}^2 \to \mathbb{R}$ which suffices

$$g(X_n) = y_n \ \ (n = 1, \ \ldots, N)\,. \tag{1.105}$$

The kernel approach is

$$g(X) = g_{[\lambda,\alpha_1,\ldots,\alpha_N]}(X) \stackrel{\Delta}{=} \frac{1}{\Gamma_\lambda(X)} \sum_{n=1}^{N} \alpha_n K_\lambda(\|X - X_n\|), \tag{1.106}$$

where

$$\Gamma_\lambda(x) \stackrel{\Delta}{=} \sum_{n=1}^{N} K_\lambda(\|X - X_n\|) \tag{1.107}$$

and $\|.\|$ denotes the Euclidean norm. The required smoothness may be achieved by using analytic kernels K_λ, for instance $K_\lambda(u) \stackrel{\Delta}{=} e^{-\frac{u^2}{2\lambda^2}}$.

The idea behind this approach is as follows. The parameters which solve the interpolation conditions (1.105) are $\alpha_1, \ldots, \alpha_n$. The parameter λ determines the "smoothness" of the resulting interpolation g and should be fixed according to the nature of the points (X_n, y_n). If these points yield a smooth surface, a "large" λ might yield a good fit, whereas in the opposite case when for neighboring points X_k, X_n the appropriate values y_k, y_n vary significantly, only a small λ, that means $\lambda << \min_{n,k} \|X_k - X_n\|$, can provide the needed flexibility.

For the set of delta pillars of 10 %, 25 %, *ATM*, −25 %, −10 % one can use $\lambda = 25\,\%$ for a smooth interpolation.

1.3.8 Interpolation of the volatility spread between maturity pillars

The interpolation of the volatility spread to ATM uses the interpolation of the spread on the two surrounding maturity pillars for the initial Black–Scholes delta of the option. The spread is interpolated using square root of time where $\tilde{\sigma}$ is the volatility spread,

$$\tilde{\sigma}(t) = \tilde{\sigma}_1 + \frac{\sqrt{t} - \sqrt{t_1}}{\sqrt{t_2} - \sqrt{t_1}}(\tilde{\sigma}_2 - \tilde{\sigma}_1). \tag{1.108}$$

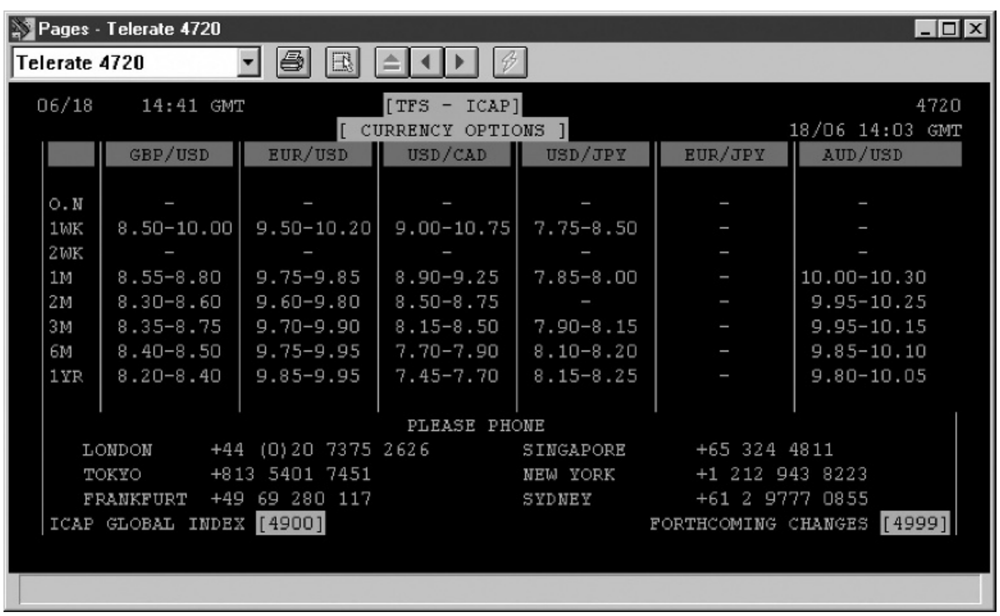
The spread is added to the interpolated ATM volatility as calculated above.

1.3.9 Volatility sources

1. BBA, the *British Bankers Association*, provides historic smile data for all major currency pairs in spread sheet format at http://www.bba.org.uk.
2. Olsen Associates (http://www.olsen.ch) can provide tic data of historic spot rates, from which the historic volatilities can be computed.
3. Bloomberg, not really the traditional FX data source, contains both implied volatilities and historic volatilities.
4. Reuters pages such as FXMOX, SGFXVOL01, and others are commonly used and contain mostly implied volatilities. JYSKEOPT is a common reference for volatilities of Scandinavia (scandie-vols). NMRC has some implied volatilities for precious metals.
5. Telerate pages such as 4720, see Figure 1.8, delivers implied volatilities.
6. Cantorspeed 90 also provides implied volatilities.

1.3.10 Volatility cones

Volatility cones visualize whether current at-the-money volatility levels for various maturities are high or low compared to a recent history of these implied volatilities. This indicates to a



Pages - Telerate 4720

Telerate 4720

06/18 14:41 GMT [TFS - ICAP] 4720
[CURRENCY OPTIONS] 18/06 14:03 GMT

	GBP/USD	EUR/USD	USD/CAD	USD/JPY	EUR/JPY	AUD/USD
O.N	-	-	-	-	-	-
1WK	8.50-10.00	9.50-10.20	9.00-10.75	7.75-8.50	-	-
2WK	-	-	-	-	-	-
1M	8.55-8.80	9.75-9.85	8.90-9.25	7.85-8.00	-	10.00-10.30
2M	8.30-8.60	9.60-9.80	8.50-8.75	-	-	9.95-10.25
3M	8.35-8.75	9.70-9.90	8.15-8.50	7.90-8.15	-	9.95-10.15
6M	8.40-8.50	9.75-9.95	7.70-7.90	8.10-8.20	-	9.85-10.10
1YR	8.20-8.40	9.85-9.95	7.45-7.70	8.15-8.25	-	9.80-10.05

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TOKYO +813 5401 7451 NEW YORK +1 212 943 8223
FRANKFURT +49 69 280 117 SYDNEY +61 2 9777 0855
ICAP GLOBAL INDEX [4900] FORTHCOMING CHANGES [4999]

Figure 1.8 Telerate page 4720 quoting currency option volatilities

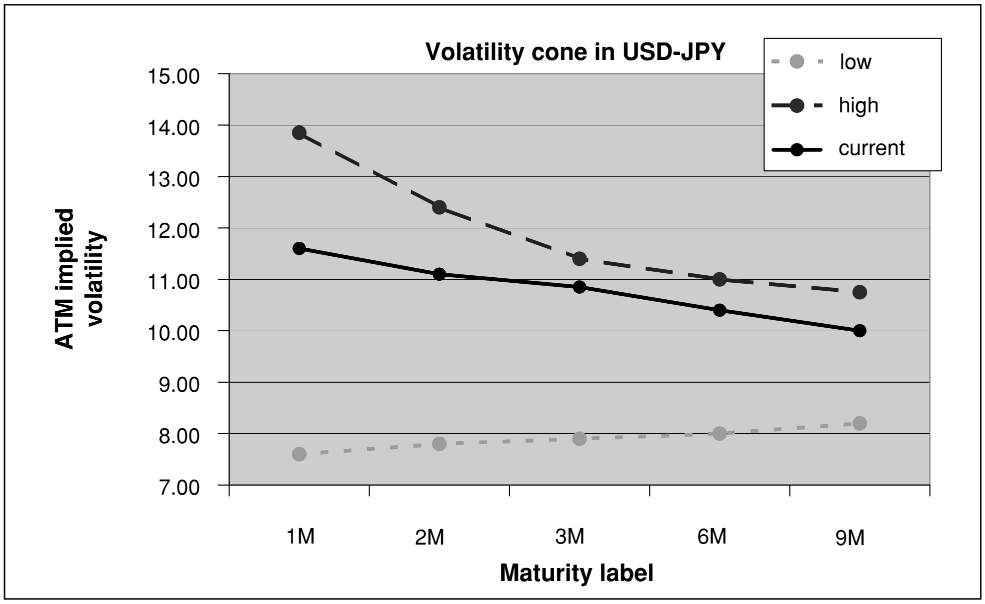

Figure 1.9 Example of a volatility cone in USD-JPY for a 9 months time horizon from 6 Sept 2003 to 24 Feb 2005

trader or risk taker whether it is currently advisable to buy volatility or sell volatility, i.e. to buy vanilla options or to sell vanilla options. We fix a time horizon of historic observations of mid market at-the-money implied volatility and look at the maximum, the minimum that traded over this time horizon and compare this with the current volatility level. Since long term volatilities tend to fluctuate less than short term volatility levels, the chart of the minimum and the maximum typically looks like a part of a cone. We illustrate this in Figure 1.9 based on the data provided in Table 1.10.

1.3.11 Stochastic volatility

Stochastic volatility models are very popular in FX Options, whereas *jump diffusion models* can be considered as the cherry on the cake. The most prominent reason for the popularity is very simple: FX volatility *is* stochastic as is shown for instance in Figure 1.10. Treating stochastic

Table 1.10 Sample data of a volatility cone in USD-JPY for a 9 months time horizon from 6 Sept 2003 to 24 Feb 2005

Maturity	Low	High	Current
1M	7.60	13.85	11.60
2M	7.80	12.40	11.10
3M	7.90	11.40	10.85
6M	8.00	11.00	10.40
12M	8.20	10.75	10.00

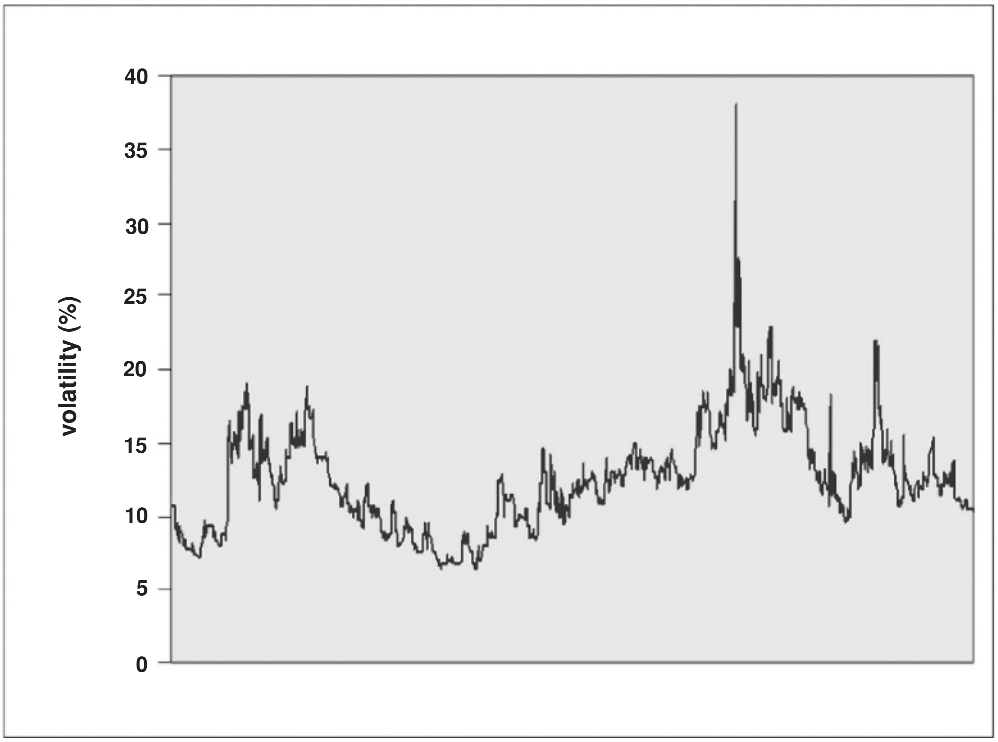

Figure 1.10 Implied volatilities for USD-JPY 1-month vanilla at-the-money options for the period of 1994 to 2000

volatility in detail here is way beyond the scope of this book. A more recent overview can be found in the article *The Heston Model and the Smile* by Weron and Wystup in [13].

1.3.12 Exercises

1. For the market data in Tables 1.6, 1.7 and 1.8 determine a smile matrix for at-the-money and the 25-deltas. Also compute the corresponding strikes for the three pillars or moneyness.
2. Taking the smile of the previous exercise, implement the functions for interpolation to generate a suitable implied volatility for any given time to maturity and any strike or delta.
3. Using the historic data, generate a volatility cone for USD-JPY.
4. It is often believed that an at-the-money (in the sense that the strike is set equal to the forward) vanilla call has a delta near 50 %. What can you say about the delta of a 15 year at-the-money USD-JPY call if USD rates are at 5 %, JPY rates are at 1 % and the volatility is at 11 %?

1.4 BASIC STRATEGIES CONTAINING VANILLA OPTIONS

Linear Combinations of vanillas are quite well known and have been explained in several text books including the one by Spies [14]. Therefore, we will restrict our attention in this section to the most basic strategies.

1.4.1 Call and put spread

A Call spread is a combination of a long and a short Call option. It is also called *capped call*. The motivation to do this is the fact that buying a simple call may be too expensive and the buyer wishes to lower the premium. At the same time he does not expect the underlying exchange rate to appreciate above the strike of the short Call option.

The Call spread entitles the holder to buy an agreed amount of a currency (say EUR) on a specified date (maturity) at a pre-determined rate (long strike) as long as the exchange rate is above the long strike at maturity. However, if the exchange rate is above the short strike at this time, the holder's profit is limited to the spread as defined by the short and long strikes (see example below). Buying a Call spread provides protection against a rising EUR with full participation in a falling EUR. The holder has to pay a premium for this protection. The holder will exercise the option at maturity if the spot is above the long strike.

Advantages

- Protection against stronger EUR/weaker USD
- Low cost product
- Maximum loss is the premium paid

Disadvantages

- Protection is limited when the exchange rate is above the long strike at maturity

The buyer has the chance of full participation in a weaker EUR/stronger USD. However, in case of very high EUR at maturity the protection works only up to the higher strike.

For example, a company wants to buy 1 Million EUR. At maturity:

1. If $S_T < K_1$, it will not exercise the option. The overall loss will be the option's premium. But instead the company can buy EUR at a lower spot in the market.
2. If $K_1 < S_T < K_2$, it will exercise the option and buy EUR at strike K_1.
3. If $S_T > K_2$, it will buy the 1 Million EUR at a rate $K_2 - K_1$ below S_T.

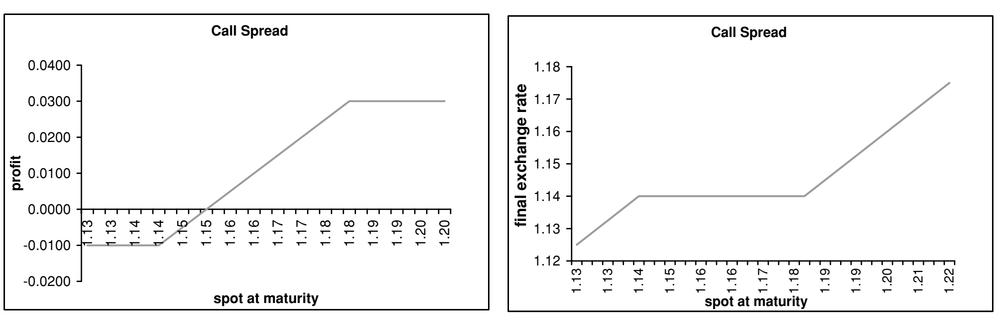

Figure 1.11 Payoff and Final Exchange Rate of a Call spread

Table 1.11 Example of a Call spread

Spot reference	1.1500 EUR-USD
Company buys	EUR call USD put with lower strike
Company sells	EUR call USD put with higher strike
Maturity	1 year
Notional of both the Call option	EUR 1,000,000
Strike of the long Call option	1.1400 EUR-USD
Strike of the short Call option	1.1800 EUR-USD
Premium	EUR 14,500.00
Premium of the long EUR call only	EUR 40,000.00

Example

A company wants to hedge receivables from an export transaction in USD due in 12 months time. It expects a stronger EUR/weaker USD. The company wishes to be able to buy EUR at a lower spot rate if the EUR weakens on the one hand, but on the other be protected against a stronger EUR. The Vanilla Call is too expensive, but the company does not expect a large upward movement of the EUR.

In this case a possible form of protection that the company can use is to buy a Call spread as, for example, listed in Table 1.11.

If the company's market expectation is correct, it can buy EUR at maturity at the strike of 1.1400.

If the EUR-USD exchange rate is below the strike at maturity the option expires worthless. However, the company would benefit from a lower spot when buying EUR.

If the EUR-USD exchange rate is above the short strike of 1.1800 at maturity, the company can buy the EUR amount 400 pips below the spot. Its risk is that the spot at maturity is very high.

The EUR seller can buy a EUR Put spread in a similar fashion.

1.4.2 Risk reversal

Very often corporates seek so-called zero-cost products to hedge their international cash-flows. Since buying a call requires a premium, the buyer can sell another option to finance the purchase of the call. A popular liquid product in FX markets is the Risk Reversal or collar. A Risk Reversal is a combination of a long call and a short put. It entitles the holder to buy an agreed amount of a currency (say EUR) on a specified date (maturity) at a pre-determined rate (long strike) assuming the exchange rate is above the long strike at maturity. However, if the exchange rate is below the strike of the short put at maturity, the holder is obliged to buy the amount of EUR determined by the short strike. Therefore, buying a Risk Reversal provides full protection against rising EUR. The holder will exercise the option only if the spot is above the long strike at maturity. The risk on the upside is financed by a risk on the downside. Since the risk is reversed, the product is named Risk Reversal.

Advantages

- Full protection against stronger EUR/weaker USD
- Zero cost product

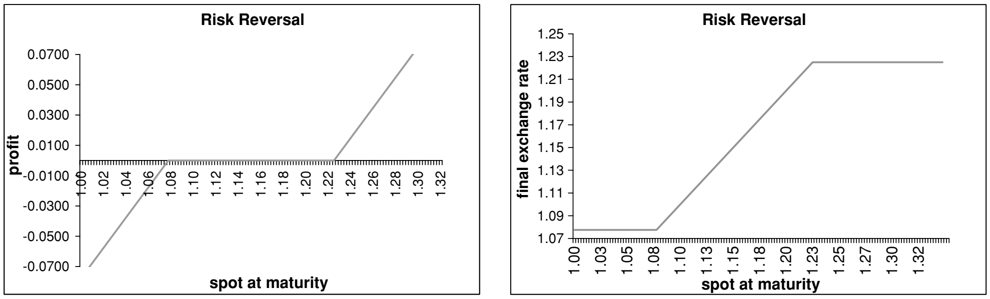

Figure 1.12 Payoff and Final Exchange Rate of a Risk Reversal

Disadvantages

- Participation in weaker EUR/stronger USD is limited to the strike of the sold put

For example, a company wants to sell 1 Million USD. At maturity T:

1. If $S_T < K_1$, it will be obliged to sell USD at K_1. Compared to the market spot the loss can be large. However, compared to the outright forward rate at inception of the trade, K_1 is usually only marginally worse.
2. If $K_1 < S_T < K_2$, it will not exercise the call option. The company can trade at the prevailing spot level.
3. If $S_T > K_2$, it will exercise the option and sell USD at strike K_2.

Example

A company wants to hedge receivables from an export transaction in USD due in 12 months time. It expects a stronger EUR/weaker USD. The company wishes to be fully protected against a stronger EUR. But it finds that the corresponding plain vanilla EUR call is too expensive and would prefer a zero cost strategy by financing the call with the sale of a put. In this case a possible form of protection that the company can use is to buy a Risk Reversal as for example indicated in Table 1.12.

If the company's market expectation is correct, it can buy EUR at maturity at the strike of 1.2250.

The risk is when EUR-USD exchange rate is below the strike of 1.0775 at maturity, the company is obliged to buy 1 Mio EUR at the rate of 1.0775. K_2 is the guaranteed worst case, which can be used as a budget rate.

Table 1.12 Example of a Risk Reversal

Spot reference	1.1500 EUR-USD
Company buys	EUR call USD put with higher strike
Company sells	EUR put USD call with lower strike
Maturity	1 year
Notional of both the Call option	EUR 1,000,000
Strike of the long Call option	1.2250 EUR-USD
Strike of the short Put option	1.0775 EUR-USD
Premium	EUR 0.00

1.4.3 Risk reversal flip

As a variation of the standard risk reversal, we consider the following trade on EUR/USD spot reference 1.2400 with a tenor of two months.

1. Long 1.2500/1.1900 risk reversal (long 1.2500 EUR call, short 1.1900 EUR put).
2. If 1.3000 trades before expiry, it flips into a 1.2900/1.3100 risk reversal (long 1.2900 EUR put, short 1.3100 EUR call).
3. Zero premium.

The corresponding view is that EUR/USD looks bullish and may break on the upside of a recent trading range. However, a runaway higher EUR/USD setting new all-time high within 2 months looks unlikely. However, if EUR/USD overshoots to 1.30, then it will likely retrace afterwards.

The main thrust is to long EUR/USD for zero cost, with a safe cap at 1.30. So the initial risk is EUR/USD below 1.19. If 1.30 is breached, then all accrued profit from the 1.25/1.19 risk reversal is lost, and the maximum risk becomes levels above 1.31. Therefore, this trade is not suitable for EUR bulls who feel there is scope above 1.30 within 2 months. On the other hand, this trade is suitable for those who feel that if spot overshoots to 1.30, then it will retrace down quickly. For early profit taking: with two weeks to go and spot at 1.2800, this trade should be worth approximately 0.84 % EUR. Maximum profit occurs at the trade's maturity.

Composition

This risk reversal flip is rather a proprietary trading strategy than a corporate hedging structure, but may work for corporates as well if the treasurer takes the above view.

The composition is presented in Table 1.13. The options used are standard barrier options, see Section 1.5.1.

1.4.4 Straddle

A straddle is a combination of a put and a call option with the same strike. It entitles the holder to buy an agreed amount of a currency (say EUR) on a specified date (maturity) at a pre-determined rate (strike) if the exchange rate is above the strike at maturity. Alternatively, if the exchange rate is below the strike at maturity, the holder is entitled to sell the amount at this strike. Buying a straddle provides participation in both and upward and a downward movement where the direction of the rate is unclear. The holder has to pay a premium for this product.

Advantages

- Full protection against market movement or increasing volatility
- Maximum loss is the premium paid

Table 1.13 Example of a Risk Reversal Flip

client buys	1.2500 EUR call up-and-out at 1.3000
client sells	1.1900 EUR put up-and-out at 1.3000
client buys	1.2900 EUR put up-and-in at 1.3000
client sells	1.3100 EUR call up-and-in at 1.3000

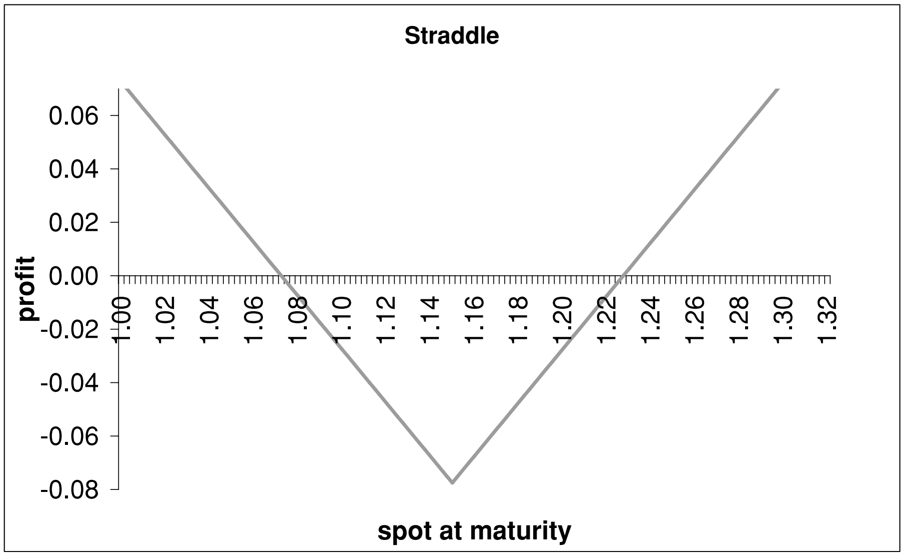

Figure 1.13 Profit of a long straddle

Disadvantages

- Expensive product
- Not suitable for hedge-accounting as it should be clear if the client wants to sell or buy EUR.

Potential profits of a long straddle arise from movements in the spot and also from increases in implied volatility. If the spot moves, the call or the put can be sold before maturity with profit. Conversely, if a quiet market phase persists the option is unlikely to generate much revenue.

Figure 1.13 shows the payoff of a long straddle. The payoff of a short straddle looks like the straddle below a seesaw on a children's playground, which is where the name straddle originated.

For example, a company has bought a straddle with a nominal of 1 Million EUR. At maturity T:

1. If $S_T < K$, it would sell 1 Mio EUR at strike K.
2. If $S_T > K$, it would buy 1 Mio EUR at strike K.

Example

A company wants to benefit from believing that the EUR-USD exchange rate will move far from a specified strike (straddle's strike). In this case a possible product to use is a straddle as, for example, listed in Table 1.14.

If the spot rate is above the strike at maturity, the company can buy 1 Mio EUR at the strike of 1.1500.

If the spot rate is below the strike at maturity, the company can sell 1 Mio EUR at the strike of 1.1500.

The break even points are 1.0726 for the put and 1.2274 for the call. If the spot is between the break even points at maturity, then the company will make an overall loss.

Table 1.14 Example of a straddle

Spot reference	1.1500 EUR-USD
Company buys	EUR call USD put
Company buys	EUR put USD call
Maturity	1 year
Notional of both the options	EUR 1,000,000
Strike of both options	1.1500 EUR-USD
Premium	EUR 77,500.00

1.4.5 Strangle

A strangle is a combination of an out-of-the-money put and call option with two different strikes. It entitles the holder to buy an agreed amount of currency (say EUR) on a specified date (maturity) at a pre-determined rate (call strike), if the exchange rate is above the call strike at maturity. Alternatively, if the exchange rate is below the put strike at maturity, the holder is entitled to sell the amount at this strike. Buying a strangle provides full participation in a strongly moving market, where the direction is not clear. The holder has to pay a premium for this product.

Advantages

- Full protection against a highly volatile exchange rate or increasing volatility
- Maximum loss is the premium paid
- Cheaper than the straddle

Disadvantages

- Expensive product
- Not suitable for hedge-accounting as it should be clear if the client wants to sell or buy EUR.

As in the straddle the chance of the strangle lies in spot movements. If the spot moves significantly, the call or the put can be sold before maturity with profit. Conversely, if a quiet market phase persists the option is unlikely to generate much revenue.

Figure 1.14 shows the profit diagram of a long strangle.

For example, a company has bought a strangle with a nominal of 1 Million EUR. At maturity T:

1. If $S_T < K_1$, it would sell 1 Mio EUR at strike K_1.
2. If $K_1 < S_T < K_2$, it would not exercise either of the two options. The overall loss will be the option's premium.
3. If $S_T > K_2$, it would buy 1 Mio EUR at strike K_2.

Example

A company wants to benefit from believing that the EUR-USD exchange rate will move far from two specified strikes (Strangle's strikes). In this case a possible product to use is a strangle as, for example, listed in Table 1.15.

If the spot rate is above the call strike at maturity, the company can buy 1 Mio EUR at the strike of 1.2000.

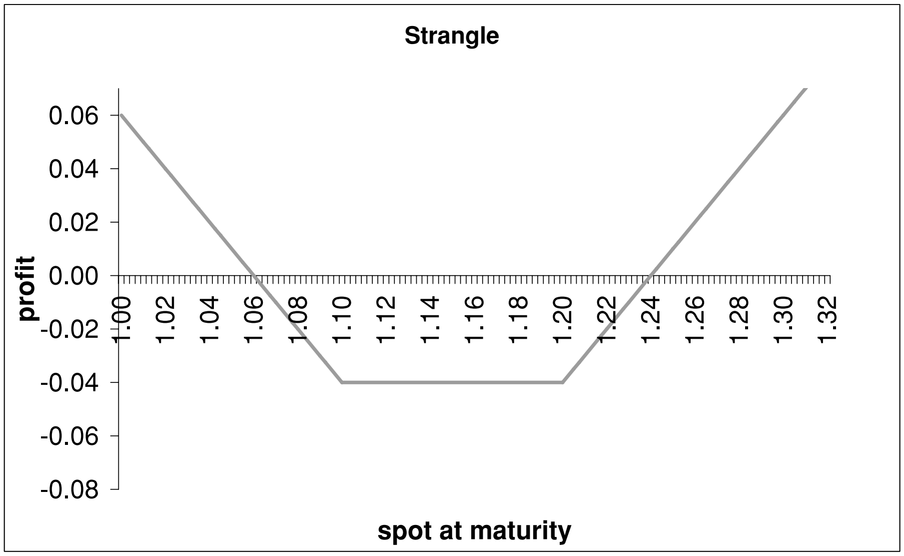

Figure 1.14 Profit of a strangle

If the spot rate is below the put strike at maturity, the company can sell 1 Mio EUR at the strike of 1.1000.

However, the risk is that, if the spot rate is between the put strike and the call strike at maturity, the option expires worthless.

The break even points are 1.0600 for the put and 1.2400 for the call. If the spot is between these points at maturity, then the company makes an overall loss.

1.4.6 Butterfly

A long Butterfly is a combination of a long strangle and a short straddle. Buying a long Butterfly provides participation where a highly volatile exchange rate condition exists. The holder has to pay a premium for this product.

Advantages

- Limited protection against market movement or increasing volatility
- Maximum loss is the premium paid
- Cheaper than the straddle

Table 1.15 Example of a strangle

Spot reference	1.1500 EUR-USD
Company buys	EUR call USD put
Company buys	EUR put USD call
Maturity	1 year
Notional of both the options	EUR 1,000,000
Put Strike	1.1000 EUR-USD
Call Strike	1.2000 EUR-USD
Premium	EUR 40,000.00

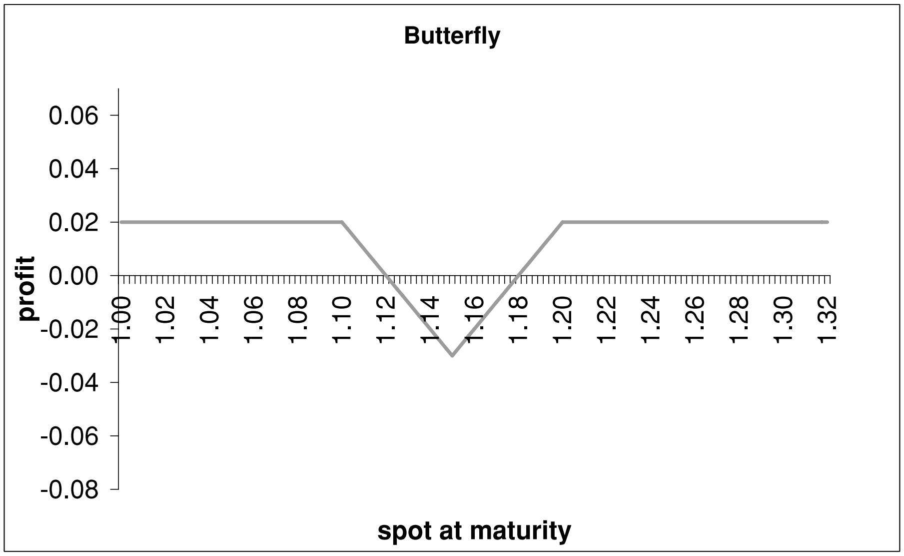

Figure 1.15 Profit of a Butterfly

Disadvantages

- Limited profit
- Not suitable for hedge-accounting as it should be clear if the client wants to sell or buy EUR

If the spot will remain volatile, the call or the put can be sold before maturity with profit. Conversely, if a quiet market phase persists the option is unlikely to be exercised.

Figure 1.15 shows the profit diagram of a long Butterfly.

For example, a company has bought a long Butterfly with a nominal of 1 Million EUR and strikes $K_1 < K_2 < K_3$. At maturity T:

1. If $S_T < K_1$, it would sell 1 Mio EUR at a rate $K_2 - K_1$ higher than the market.
2. If $K_1 < S_T < K_2$, it would sell 1 Mio EUR at strike K_2.
3. If $K_2 < S_T < K_3$, it would buy 1 Mio EUR at strike K_2.
4. If $S_T > K_3$, it would buy 1 Mio EUR at a rate $K_3 - K_2$ less than the market.

Example

A company wants to benefit from believing that the EUR-USD exchange rate will remain volatile from a specified strike (the middle strike K_2).

In this case a possible product to use is a long Butterfly as for example listed in Table 1.16.

If the spot rate is between the lower and the middle strike at maturity, the company can sell 1 Mio EUR at the strike of 1.1500.

If the spot rate is between the middle and the higher strike at maturity, the company can buy 1 Mio EUR at the strike of 1.1500.

If the spot rate is above the higher strike at maturity, the company will buy EUR 100 points below the spot.

If the spot rate is below the lower strike at maturity, the company will sell EUR 100 points above the spot.

Table 1.16 Example of a Butterfly

Spot reference	1.1500 EUR-USD
Maturity	1 year
Notional of both the options	EUR 1,000,000
Lower strike K_1	1.1400 EUR-USD
Middle strike K_2	1.1500 EUR-USD
Upper strike K_3	1.1600 EUR-USD
Premium	EUR 30,000.00

1.4.7 Seagull

A long Seagull Call strategy is a combination of a long call, a short call and a short put. It is similar to a Risk Reversal. So it entitles its holder to purchase an agreed amount of a currency (say EUR) on a specified date (maturity) at a pre-determined long call strike if the exchange rate at maturity is between the long call strike and the short call strike (see below for more information). If the exchange rate is below the short put strike at maturity, the holder must buy this amount in EUR at the short put strike. Buying a Seagull Call strategy provides good protection against a rising EUR.

Advantages

- Good protection against stronger EUR/weaker USD
- Better strikes than in a risk reversal
- Zero cost product

Disadvantages

- Maximum loss depending on spot rate at maturity and can be arbitrarily large

The protection against a rising EUR is limited to the interval from the long call strike and the short call strike. The biggest risk is a large upward movement of EUR.

Figure 1.16 shows the payoff and final exchange rate diagram of a Seagull. Rotating the payoff clockwise by about 45 degrees shows the shape of a flying seagull.

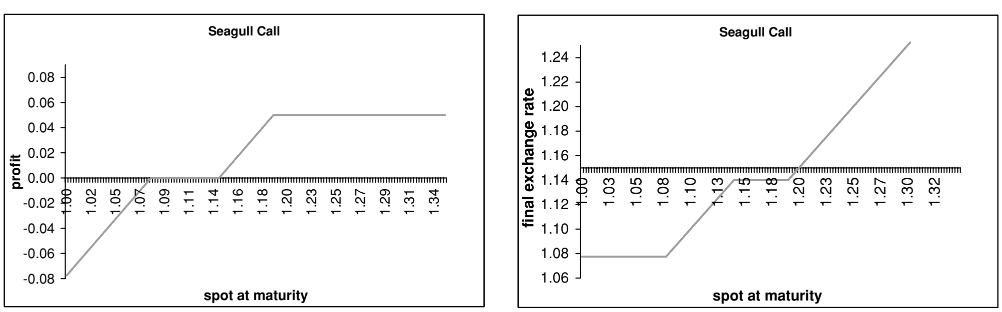

Figure 1.16 Payoff and Final Exchange Rate of a Seagull Call

Table 1.17 Example of a Seagull Call

Spot reference	1.1500 EUR-USD
Maturity	1 year
Notional	USD 1,000,000
Company buys	EUR call USD put strike 1.1400
Company sells	EUR call USD put strike 1.1900
Company sells	EUR put USD call strike 1.0775
Premium	USD 0.00

For example, a company wants to sell 1 Million USD and buy EUR. At maturity T:

1. If $S_T < K_1$, the company must sell 1 Mio USD at rate K_1.
2. If $K_1 < S_T < K_2$, all involved options expire worthless and the company can sell USD in the spot market.
3. If $K_2 < S_T < K_3$, the company would buy EUR at strike K_2.
4. If $S_T > K_3$, the company would sell 1 Mio USD at a rate $K_2 - K_1$ less than the market.

Example

A company wants to hedge receivables from an export transaction in USD due in 12 months time. It expects a stronger EUR/weaker USD but not a large upward movement of the EUR. The company wishes to be protected against a stronger EUR and finds that the corresponding plain vanilla is too expensive and would prefer a zero cost strategy and is willing to limit protection on the upside.

In this case a possible form of protection that the company can use is to buy a Seagull Call as for example presented in Table 1.17.

If the company's market expectation is correct, it can buy EUR at maturity at the strike of 1.1400.

If the EUR-USD exchange rate will be above the short call strike of 1.1900 at maturity, the company will sell USD at 500 points less than the spot.

However the risk is that, if the EUR-USD exchange rate is below the strike of 1.0775 at maturity, it will have to sell 1 Mio USD at the strike of 1.0775.

1.4.8 Exercises

1. For EUR/GBP spot ref 0.7000, volatility 8 %, EUR rate 2.5 %, GBP rate 4 % and flat smile find the strike of the short EUR put for a 6 months zero cost seagull put, where the strike of the long EUR put is 0.7150, the strike of the short call is 0.7300 and the desired sales margin is 0.1 % of the GBP notional. What is the value of the seagull put after three months if the spot is at 0.6900 and the volatility is at 7.8 %?

1.5 FIRST GENERATION EXOTICS

We consider EUR/USD – the most liquidly traded currency pair in the foreign exchange market. Internationally active market participants are always subject to changing foreign exchange rates. To hedge this exposure an immense variety of options are traded worldwide. Besides vanilla (European style put and call) options, the so-called first generation exotics have become

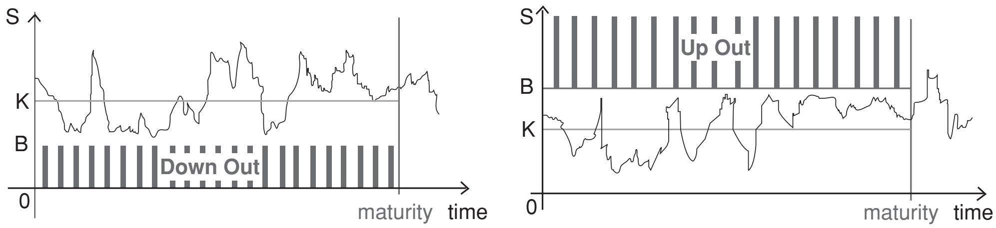

Figure 1.17 *Down-and-out* American barrier
If the exchange rate is never at or below B between the trade date and maturity, the option can be exercised. *Up-and-out* American barrier: If the exchange rate is never at or above B between the trade date and maturity, the option can be exercised.

standard derivative instruments. These are (a) vanilla options that knock in or out if the underlying hits a barrier (or one of two barriers) and (b) all kind of touch options: a one-touch [no-touch] pays a fixed amount of either USD or EUR if the spot ever [never] trades at or beyond the touch-level and zero otherwise. Double one-touch and no-touch options work the same way but have two barriers.

1.5.1 Barrier options

Knock Out Call option (American style barrier)

A Knock-Out Call option entitles the holder to purchase an agreed amount of a currency (say EUR) on a specified expiration date at a pre-determined rate called the strike K provided the exchange rate never hits or crosses a pre-determined barrier level B. However, there is no obligation to do so. Buying a EUR Knock-Out Call provides protection against a rising EUR if no Knock-Out event occurs between the trade date and expiration date whilst enabling full participation in a falling EUR. The holder has to pay a premium for this protection. The holder will exercise the option only if at expiration time the spot is above the strike and if the spot has failed to touch the barrier between the trade date and expiration date (American style barrier) of if the spot at expiration does not touch or cross the barrier (European style barrier), see Figure 1.17. We display the profit and the final exchange rate of an up-and-out call in Figure 1.18.

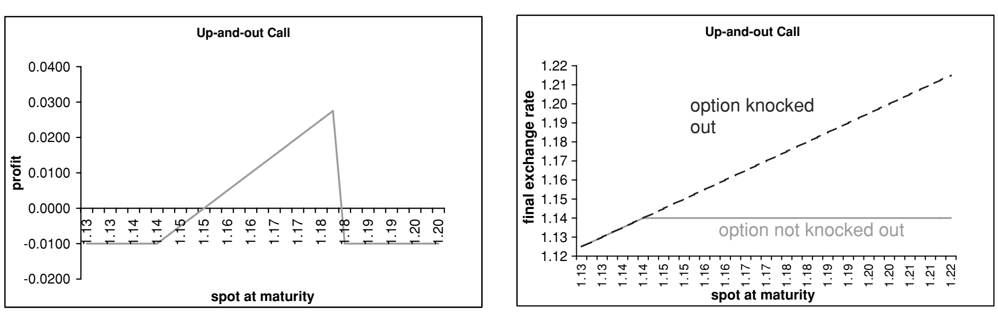

Figure 1.18 *Down-and-out* American barrier
If the exchange rate is never at or below B between the trade date and maturity, the option can be exercised. *Up-and-out* American barrier: If the exchange rate is never at or above B between the trade date and maturity, the option can be exercised.

Advantages

- Cheaper than a plain vanilla
- Conditional protection against stronger EUR/weaker USD
- Full participation in a weaker EUR/stronger USD

Disadvantages

- Option may knock out
- Premium has to be paid

For example the company wants to sell 1 MIO USD. If as usual S_t denotes the exchange rate at time t then at maturity

1. if $S_T < K$, the company would not exercise the option,
2. if $S_T > K$ and if S has respected the conditions pre-determined by the barrier, the company would exercise the option and sell 1 MIO USD at strike K.

Example

A company wants to hedge receivables from an export transaction in USD due in 12 months time. It expects a stronger EUR/weaker USD. The company wishes to be able to buy EUR at a lower spot rate if the EUR becomes weaker on the one hand, but on the other hand be protected against a stronger EUR, and finds that the corresponding vanilla call is too expensive and is prepared to take more risk.

In this case a possible form of protection that the company can use is to buy a EUR Knock-Out Call option as for example listed in Table 1.18.

If the company's market expectation is correct, then it can buy EUR at maturity at the strike of 1.1500.

If the EUR–USD exchange rate touches the barrier at least once between the trade date and maturity the option will expire worthless.

Types of barrier options

Generally the payoff of a standard knock-out option can be stated as

$$[\phi(S_T - K)]^+ \mathbb{I}_{\{\eta S_t > \eta B,\, 0 \leq t \leq T\}}, \tag{1.109}$$

where $\phi \in \{+1, -1\}$ is the usual put/call indicator and $\eta \in \{+1, -1\}$ takes the value $+1$ for a lower barrier (down-and-out) or -1 for an upper barrier (up-and-out). The corresponding

Table 1.18 Example of an up-and-out call

Spot reference	1.1500 EUR-USD
Maturity	1 year
Notional	EUR 1,000,000
Company buys	EUR call USD put
Strike	1.1500 EUR-USD
Up-and-out American barrier	1.3000 EUR-USD
Premium	EUR 12,553.00

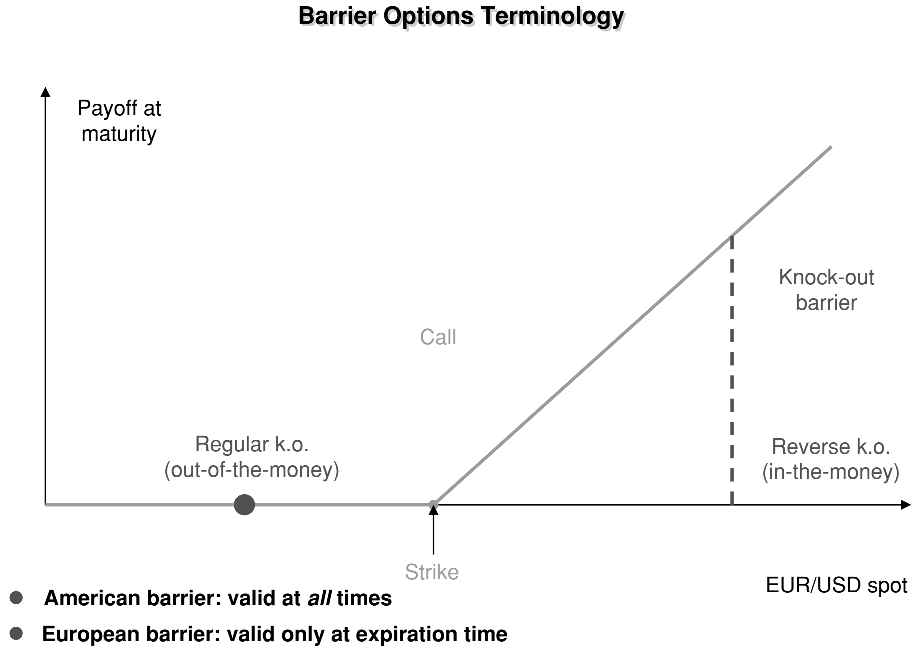

Figure 1.19 Barrier option terminology: regular barriers are out of the money, reverse barriers are in the money

knock-in options only become alive, if the spot ever trades at or beyond the barrier between trade date and expiration date. Naturally,

$$\text{knock-out} + \text{knock-in} = \text{vanilla}. \tag{1.110}$$

Furthermore, we distinguish (see Figure 1.19)

Regular knock out: the barrier is out of the money.

Reverse knock out: the barrier is in the money.

Strike out: the barrier is at the strike.

Losing a reverse barrier option due to the spot hitting the barrier is more painful since the owner already has accumulated a positive intrinsic value.

This means that there are in total 16 different types of barrier options, call or put, in or out, up or down, regular or reverse.

Theoretical value of barrier options

For the standard type of FX barriers options a detailed derivation of values and Greeks can be found in [3].

Barrier option terminology

This paragraph is based on Hakala and Wystup [15]

American vs. European – Traditionally barrier options are of American style, which means that the barrier level is active during the entire duration of the option: at any time between today and maturity the spot hits the barrier, the option becomes worthless. If the barrier level is only active at maturity the barrier option is of European style and can in fact be replicated by a vertical spread and a digital option.

Single, double and outside barriers – Instead of taking just a lower or an upper barrier one could have both if one feels sure about the spot to remain in a range for a while. In this case besides vanillas, constant payoffs at maturity are popular, they are called range binaries. If the barrier and strike are in different exchange rates, the contract is called an outside barrier option or double asset barrier option. Such options traded a few years ago with the strike in USD/DEM and the barrier in USD/FRF taking advantage of the misbalance between implied and historic correlation between the two currency pairs.

Rebates – For knock-in options an amount R is paid at expiration by the seller of the option to the holder of the option if the option failed to kick in during its lifetime. For knock-out options an amount R is paid by the seller of the option to the holder of the option, if the option knocks out. The payment of the rebate is either at maturity or at the first time the barrier is hit. Including such rebate features makes hedging easier for reverse barrier options and serves as a consolation for the holder's disappointment. The rebate part of a barrier option can be completely separated from the barrier contract and can in fact be traded separately, in which case it is called a one-touch (digital) option or hit option (in the knock-out case) and no-touch option (in the knock-in case). We treat the touch options in detail in Section 1.5.2.

Determination of knock-out event – We discuss how breaching the barrier is determined in the beginning of Section 1.5.2.

How the barrier is monitored (Continuous vs. Discrete) and how this influences the value

How often and when exactly do you check whether an option has knocked out or kicked in? This question is not trivial and should be clearly stated in the deal. The intensity of monitoring can create any price between a standard barrier and a vanilla contract. The standard for barrier options is continuous monitoring. Any time the exchange rate hits the barrier the option is knocked out. An alternative is to consider just daily/weekly/monthly fixings which makes the knock-out option more expensive because chances of knocking out are smaller (see Figure 1.20). A detailed discussion of the valuation of discrete barriers can be found in [16].

The popularity of barrier options

- They are less expensive than vanilla contracts: in fact, the closer the spot is to the barrier, the cheaper the knock-out option. Any price between zero and the vanilla premium can be obtained by taking an appropriate barrier level, as we see in Figure 1.21. One must be aware however, that too cheap barrier options are very likely to knock out.
- They allow one to design foreign exchange risk exposure to the special needs of customers. Instead of lowering the premium one can increase the nominal coverage of the vanilla contract by admitting a barrier. Some customers feel sure about exchange rate levels not being hit

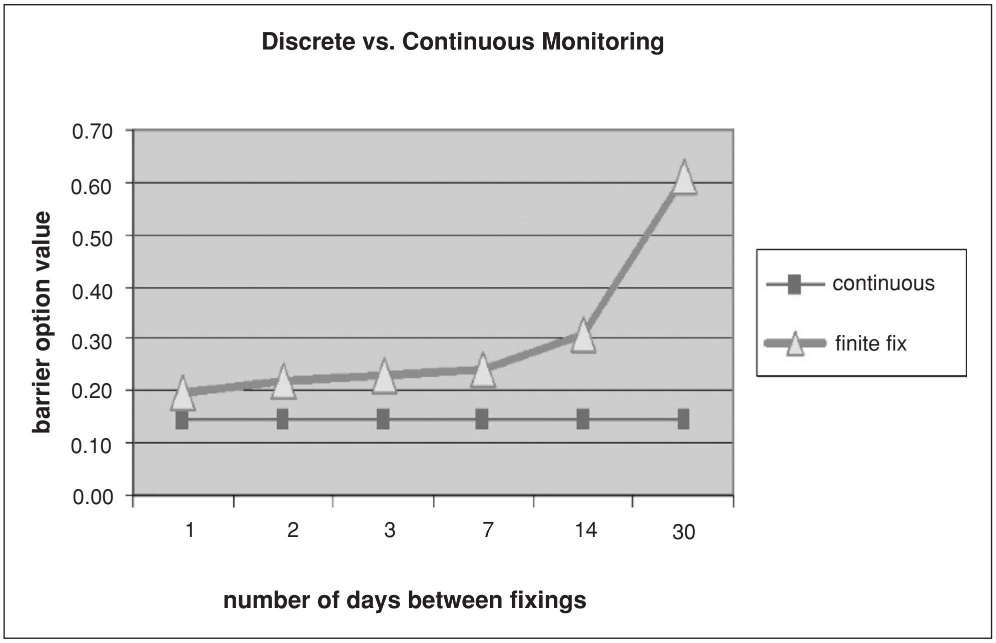

Figure 1.20 Comparison of a discretely and a continuously monitored knock-out barrier option

during the next month which could be exploited to lower the premium. Others really only want to cover their exchange rate exposure if the market moves drastically which would require a knock-in option.
- The savings can be used for another hedge of foreign exchange risk exposure if the first barrier option happened to knock out.
- The contract is easy to understand if one knows about vanillas.
- Many pricing and trading systems include barrier option calculations in their standard.
- Pricing and hedging barriers in the Black-Scholes model is well-understood and most premium calculations use closed-form solutions which allow fast and stable implementation.
- Barrier options are standard ingredients in structured FX forwards, see, for example, the *shark forward* in Section 2.1.5.

Barrier option crisis in 1994–96

In the currency market barrier options became popular in 1994. The exchange rate between USD and DEM was then between 1.50 and 1.70. Since the all time low before 1995 was 1.3870 at September 2 1992 there were a lot of down and out barrier contracts written with a lower knock-out barrier of 1.3800. The sudden fall of the US Dollar in the beginning of 1995 was unexpected and the 1.3800 barrier was hit at 10:30 am on March 29 1995 and fell even more to its all time low of 1.3500 at 9:30 am on April 19 1995. Numerous barrier option holders were shocked to learn that losing the entire option was something that could really happen (see

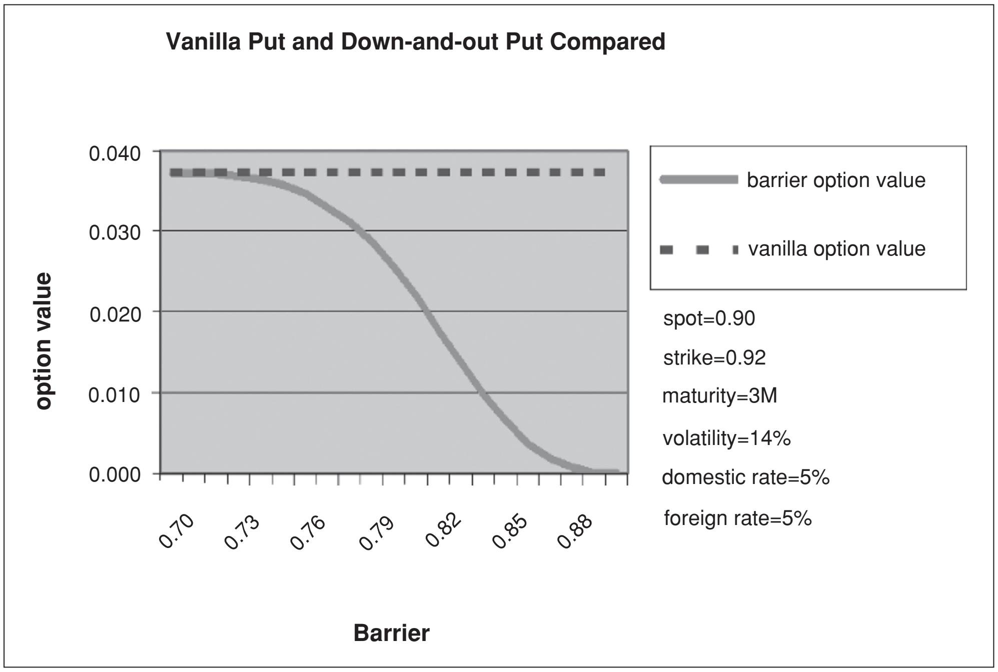

Figure 1.21 Comparison of a vanilla put and a down-and-out put
As the barrier moves far away from the current spot, the barrier option behaves like a vanilla options. As the barrier moves close to the current spot, the barrier option becomes worthless.

Figure 1.22). The shock lasted for more than a year and barrier options were unpopular for a while until many market participants had forgotten that it had happened. Events like this often raise the question about using exotics. Complicated products can in fact lead to unpleasant surprises. However, in order to cover foreign exchange risk in an individual design at minimal cost requires exotic options. Often they appear as an integral part of an investment portfolio. The number of market participants understanding the advantages and pitfalls is growing steadily.

Hedging methods

Several authors claim that barrier options can be hedged statically with a portfolio of vanilla options. These approaches are problematic if the hedging portfolio has to be unwound at hitting time, since volatilities for the vanillas may have changed between the time the hedge is composed and the time the barrier is hit. Moreover, the occasionally high nominals and low deltas can cause a high price for the hedge. The approach by Maruhn and Sachs in [17] appears most promising. For regular barriers a delta and vega hedge is more advisable. A vega hedge can be done almost statically using two vanilla options. In the example we consider a 3-month up-and-out put with strike 1.0100 and barrier 0.9800. The vega minimizing hedge consists of 0.9 short 3-month 50 delta calls and 0.8 long 2-month 25 delta calls. Spot reference for EUR/USD is 0.9400 with rates 3.05 % and 6.50 % and volatility 11.9 %, see Figure 1.23.

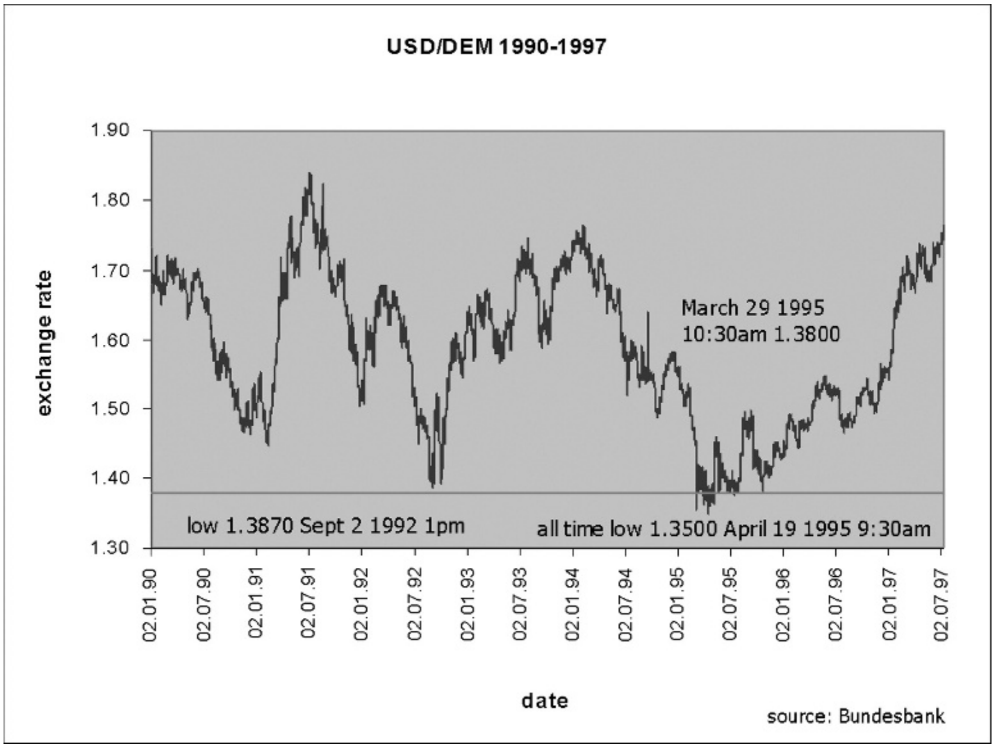

Figure 1.22 Barrier had lost popularity in 1994–96, when USD-DEM had dropped below its historic low

One can also think of statically hedging regular barriers with a risk reversal as indicated in Figure 1.24. The problem is of course, that the value of the risk reversal at knock-out time need not be zero, in fact, considering hedging a down-and-out call with a spot approaching the barrier, the calls will tend to be cheaper than the puts, so to unwind the hedge one would get little for the call one could sell and pay more than expected for the put to be bought back. Therefore, many traders and researchers like to think of stochastic skew models taking exactly this effect into account.

Reverse barrier options have extremely high values for delta, gamma and theta when the spot is near the barrier and the time is close to expiry, see for example the delta in Figure 1.25. This is because the intrinsic value of the option jumps from a positive value to zero when the barrier is hit. In such a situation a simple delta hedge is impractical. However, there are ways to tackle this undesirable state of affairs by moving the barrier or more systematically apply valuation subject to portfolio constraints such as limited leverage, see Schmock, Shreve and Wystup in [18].

How large barrier contracts affect the market

Let's take the example of a reverse up-and-out call in EUR/USD with strike 1.2000 and barrier 1.3000. An investment bank delta-hedging a short position with nominal 10 Million has to

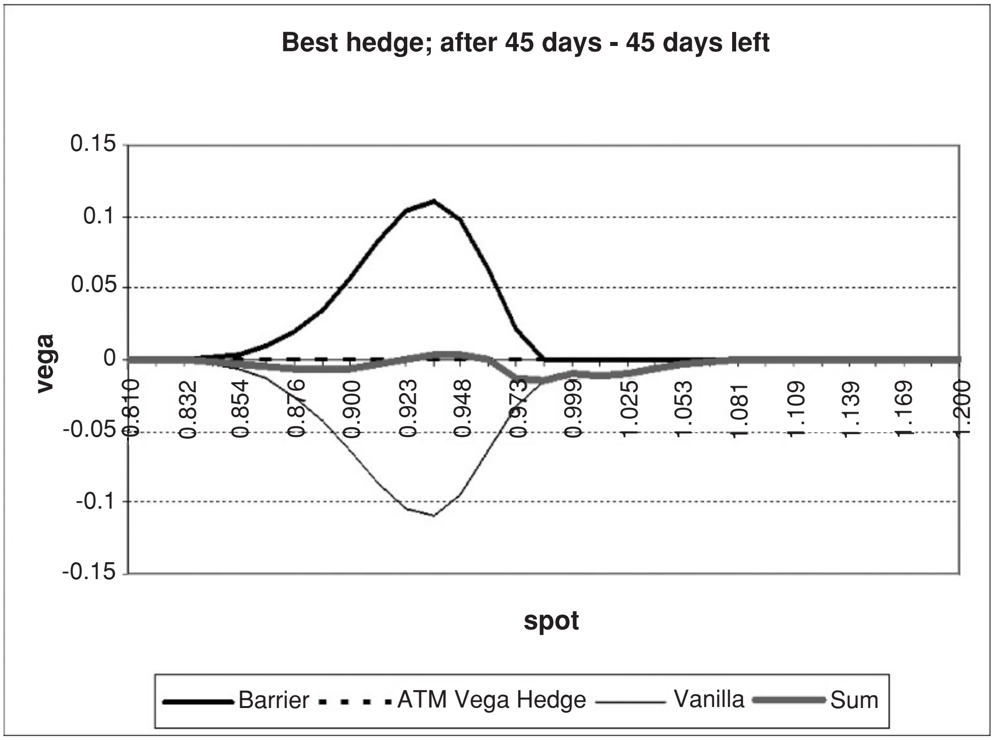

Figure 1.23 Vega depending on spot of an up-and-out put and a vega hedge consisting of two vanilla options

buy 10 Million times delta EUR, which is negative in this example. As the spot goes up to the barrier, delta becomes smaller and smaller requiring the hedging institution to sell more and more EUR. This can influence the market since steadily offering EUR slows down the spot movement towards the barrier and can in extreme cases prevent the spot from crossing the barrier. This is illustrated in Figure 1.26.

On the other hand, if the hedger runs out of breath or the upward market movement can't be stopped by the delta-hedging institution, then the option knocks out and the hedge is unwound.

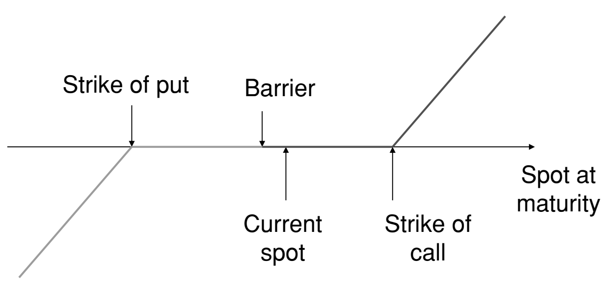

Figure 1.24 Hedging the regular knock-out with a risk reversal
A short down-and-out call is hedged by a long call with the same strike and a short put with a strike chosen in such a way that the value of the call and put portfolio is zero if the spot is at the barrier.

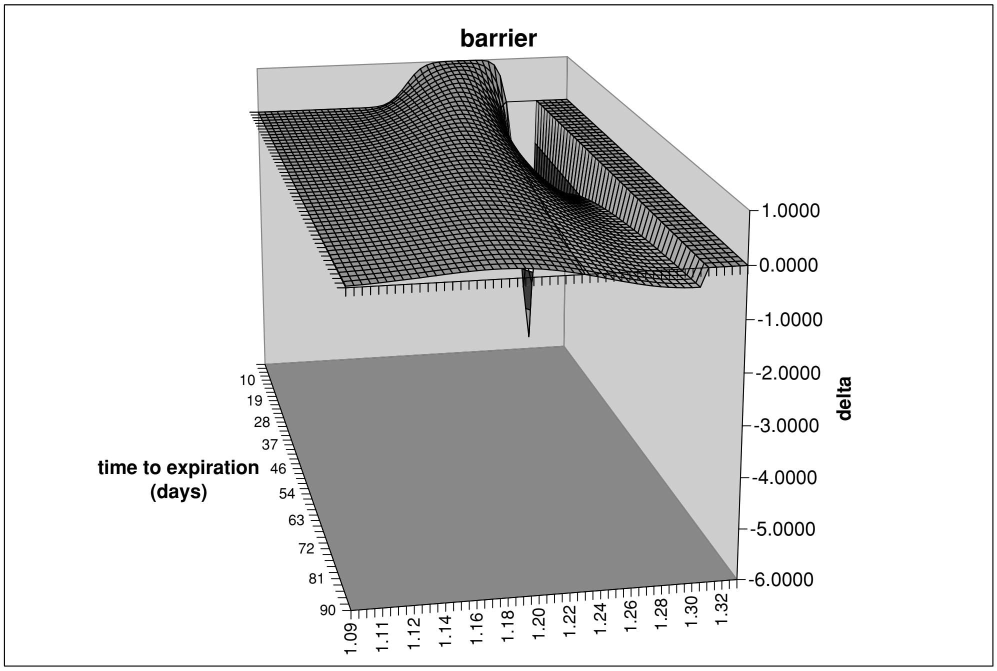

Figure 1.25 Delta of a reverse knock-out call in EUR-USD with strike 1.2000, barrier 1.3000

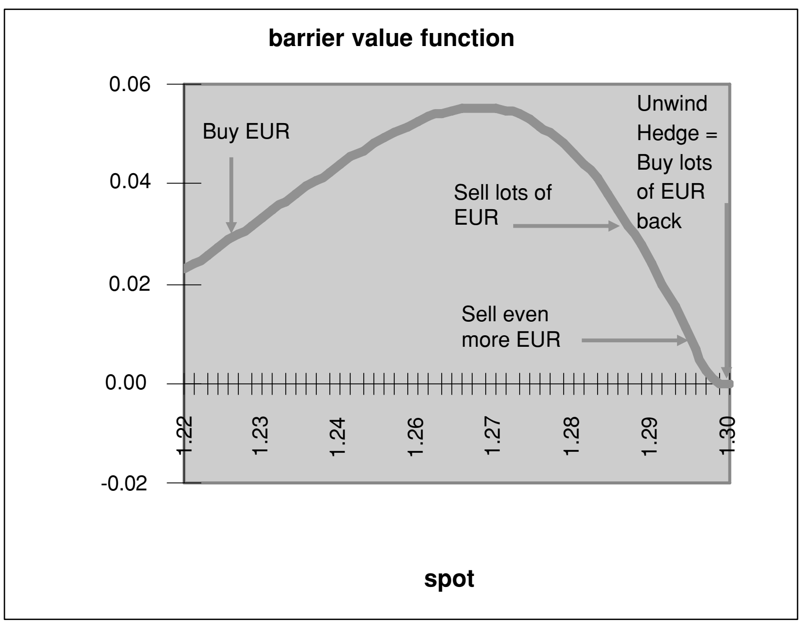

Figure 1.26 Delta hedging a short reverse knock-out call

Then suddenly more EUR are asked whence the upward movement of an exchange rate can be accelerated once a large barrier contract in the market has knocked out. Situations like this have happened to the USD-DEM spot in the early 90s (see again Figure 1.22), where many reverse knock-out puts have been written by banks, as traders are telling.

The reverse situation occurs when the bank hedges a long position, in which case EUR has to be bought when the spot approaches the barrier. This can cause an accelerated movement of the exchange rate towards the barrier and a sudden halt once the barrier is breached.

1.5.2 Digital options, touch options and rebates

Now we take a more detailed look at the pricing of one-touch options – often called (American style) binary or digital options, hit options or rebate options. They trade as listed and over-the-counter products.

The touch-time is the first time the underlying trades at or beyond the touch-level. The barrier determination agent, who is specified in the contract, determines the touch-event. The *Foreign Exchange Committee* recommends to the foreign exchange community a set of best practices for the barrier options market. In the next stage of this project, the Committee is planning on publishing a revision of the *International Currency Options Master Agreement (ICOM) User Guide* to reflect the new recommendations.[3] Some key features are:

- Determination whether the spot has breached the barrier must be due to actual transactions in the foreign exchange markets.
- Transactions must occur between 5:00 a.m. Sydney time on Monday and 5:00 p.m. New York time on Friday.
- Transactions must be of commercial size. In liquid markets, dealers generally accept that commercial size transactions are a minimum of 3 million USD.
- The barrier options determination agent may use cross-currency rates to determine whether a barrier has been breached in respect of a currency pair that is not commonly quoted.

The barrier or touch-level is usually monitored continuously over time. A further contractual issue to specify is the time of the payment of the one-touch. Typically, the notional is paid at the delivery date, which is two business days after the maturity. Another common practice is 2 business days after the touch event. In FX markets the former is the default.

Applications of one-touch options

Market participants of a rather speculative nature like to use one-touch options as bets on a rising or falling exchange rate.[4] Hedging oriented clients often buy one-touch options as a rebate, so they receive a payment as a consolation if the strategy they believe in does not work. One-touch options also often serve as parts of structured products designed to enhance a forward rate or an interest rate.

[3] For details see http://www.ny.frb.org/fxc/fxann000217.html.
[4] Individuals can trade them over the internet, for example at http://www.boxoption.com/.

Theoretical value of a one-touch option

In the standard Black-Scholes model for the underlying exchange rate of EUR/USD,

$$dS_t = S_t[(r_d - r_f)dt + \sigma dW_t], \tag{1.111}$$

where t denotes the running time in years, r_d the USD interest rate, r_f the EUR interest rate, σ the volatility, W_t a standard Brownian motion under the risk-neutral measure, the payoff is given by

$$R I\!I_{\{\tau_B \leq T\}}, \tag{1.112}$$

$$\tau_B \stackrel{\Delta}{=} \inf\{t \geq 0 : \eta S_t \leq \eta B\}. \tag{1.113}$$

This type of option pays a domestic cash amount R USD if a barrier B is hit any time before the expiration time. We use the binary variable η to describe whether B is a lower barrier ($\eta = 1$) or an upper barrier ($\eta = -1$). The stopping time τ_B is called the first hitting time. The option can be either viewed as the rebate portion of a knock-out barrier option or as an American cash-or-nothing digital option. In FX markets it is usually called a *one-touch (option)*, *one-touch-digital* or *hit option*. The modified payoff of a *no-touch (option)*, $R I\!I_{\{\tau_B \geq T\}}$ describes a rebate which is being paid if a knock-in-option has not knocked in by the time it expires and can be valued similarly simply by exploiting the identity

$$R I\!I_{\{\tau_B \leq T\}} + R I\!I_{\{\tau_B > T\}} = R. \tag{1.114}$$

We will further distinguish the cases

$\omega = 0$, rebate paid at hit,

$\omega = 1$, rebate paid at end.

It is important to mention that the payoff is one unit of the base currency. For a payment in the underlying currency EUR, one needs to exchange r_d and r_f, replace S and B by their reciprocal values and change the sign of η.

For the one-touch we will use the abbreviations:

- T: expiration time (in years)
- t: running time (in years)
- $\tau \stackrel{\Delta}{=} T - t$: time to expiration (in years)
- $\theta_{\pm} \stackrel{\Delta}{=} \frac{r_d - r_f}{\sigma} \pm \frac{\sigma}{2}$
- $S_t = S_0 e^{\sigma W_t + \sigma \theta_- t}$: price of the underlying at time t
- $n(t) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$
- $\mathcal{N}(x) \stackrel{\Delta}{=} \int_{-\infty}^{x} n(t)\, dt$
- $\vartheta_- \stackrel{\Delta}{=} \sqrt{\theta_-^2 + 2(1-\omega) r_d}$
- $e_{\pm} \stackrel{\Delta}{=} \frac{\pm \ln \frac{x}{B} - \sigma \vartheta_- \tau}{\sigma \sqrt{\tau}}$

We can describe the value function of the one-touch as a solution to a partial differential equation setup. Let $v(t, x)$ denote the value of the option at time t when the underlying is at x.

Then $v(t,x)$ is the solution of

$$v_t + (r_d - r_f)xv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} - r_d v = 0,\ \ t \in [0,T],\ \eta x \geq \eta B, \tag{1.115}$$

$$v(T,x) = 0,\ \ \eta x \geq \eta B, \tag{1.116}$$

$$v(t,B) = Re^{-\omega r_d \tau},\ \ t \in [0,T]. \tag{1.117}$$

The theoretical value of the one-touch turns out to be

$$v(t,x) = Re^{-\omega r_d \tau}\left[\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} \mathcal{N}(-\eta e_+) + \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} \mathcal{N}(\eta e_-)\right]. \tag{1.118}$$

Note that $\vartheta_- = |\theta_-|$ for rebates paid at end ($\omega = 1$).

Greeks

We list some of the sensitivity parameters of the one-touch here, as they seem hard to find in the existing literature, but many people have asked me for them, so here we go.

Delta

$$\begin{aligned} v_x(t,x) = -\frac{Re^{-\omega r_d \tau}}{\sigma x}\Bigg\{&\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}}\left[(\theta_- + \vartheta_-)\mathcal{N}(-\eta e_+) + \frac{\eta}{\sqrt{\tau}}n(e_+)\right] \\ &+ \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}}\left[(\theta_- - \vartheta_-)\mathcal{N}(\eta e_-) + \frac{\eta}{\sqrt{\tau}}n(e_-)\right]\Bigg\} \end{aligned} \tag{1.119}$$

Gamma can be obtained using $v_{xx} = \frac{2}{\sigma^2 x^2}[r_d v - v_t - (r_d - r_f)xv_x]$ and turns out to be:

$$\begin{aligned} v_{xx}(t,x) = \frac{2Re^{-\omega r_d \tau}}{\sigma^2 x^2}\cdot& \\ \Bigg\{&\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}}\mathcal{N}(-\eta e_+)\left[r_d(1-\omega) + (r_d - r_f)\frac{\theta_- + \vartheta_-}{\sigma}\right] \\ &+ \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}}\mathcal{N}(\eta e_-)\left[r_d(1-\omega) + (r_d - r_f)\frac{\theta_- - \vartheta_-}{\sigma}\right] \\ &+ \eta\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} n(e_+)\left[-\frac{e_-}{\tau} + \frac{r_d - r_f}{\sigma\sqrt{\tau}}\right] \\ &+ \eta\left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} n(e_-)\left[\frac{e_+}{\tau} + \frac{r_d - r_f}{\sigma\sqrt{\tau}}\right]\Bigg\} \end{aligned} \tag{1.120}$$

Theta

$$\begin{aligned} v_t(t,x) &= \omega r_d v(t,x) + \frac{\eta Re^{-\omega r_d \tau}}{2\tau}\left[\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} n(e_+)e_- - \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} n(e_-)e_+\right] \\ &= \omega r_d v(t,x) + \frac{\eta Re^{-\omega r_d \tau}}{\sigma \tau^{(3/2)}}\left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} n(e_+)\ln\left(\frac{B}{x}\right) \end{aligned} \tag{1.121}$$

The computation exploits the identities (1.143), (1.144) and (1.145) derived below.

Vega requires the identities

$$\frac{\partial\theta_-}{\partial\sigma} = -\frac{\theta_+}{\sigma} \tag{1.122}$$

$$\frac{\partial\vartheta_-}{\partial\sigma} = -\frac{\theta_-\theta_+}{\sigma\vartheta_-} \tag{1.123}$$

$$\frac{\partial e_\pm}{\partial\sigma} = \pm\frac{\ln\frac{B}{x}}{\sigma^2\sqrt{\tau}} + \frac{\theta_-\theta_+}{\sigma\vartheta_-}\sqrt{\tau} \tag{1.124}$$

$$A_\pm \stackrel{\Delta}{=} \frac{\partial}{\partial\sigma}\frac{\theta_- \pm \vartheta_-}{\sigma} = -\frac{1}{\sigma^2}\left[\theta_+ + \theta_- \pm \left(\frac{\theta_-\theta_+}{\vartheta_-} + \vartheta_-\right)\right] \tag{1.125}$$

and turns out to be

$$\begin{aligned} v_\sigma(t,x) = {} & Re^{-\omega r_d\tau}\cdot \\ & \left\{\left(\frac{B}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}\left[\mathcal{N}(-\eta e_+)A_+\ln\left(\frac{B}{x}\right) - \eta n(e_+)\frac{\partial e_+}{\partial\sigma}\right]\right. \\ & \left. + \left(\frac{B}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}\left[\mathcal{N}(\eta e_-)A_-\ln\left(\frac{B}{x}\right) + \eta n(e_-)\frac{\partial e_-}{\partial\sigma}\right]\right\}. \end{aligned} \tag{1.126}$$

Vanna uses the identity

$$d_- = \frac{\ln\frac{B}{x} - \sigma\theta_-\tau}{\sigma\sqrt{\tau}} \tag{1.127}$$

and turns out to be

$$\begin{aligned} v_{\sigma x}(t,x) = {} & \frac{Re^{-\omega r_d\tau}}{\sigma x}\cdot \\ & \left\{\left(\frac{B}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}\left[\mathcal{N}(-\eta e_+)A_+\left(-\sigma - (\theta_- + \vartheta_-)\ln\left(\frac{B}{x}\right)\right)\right.\right. \\ & \left.\quad -\frac{\eta n(e_+)}{\sqrt{\tau}}\left(d_-\frac{\partial e_+}{\partial\sigma} + A_+\ln\left(\frac{B}{x}\right) - \frac{1}{\sigma}\right)\right] \\ & + \left(\frac{B}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}\left[\mathcal{N}(\eta e_-)A_-\left(-\sigma - (\theta_- - \vartheta_-)\ln\left(\frac{B}{x}\right)\right)\right. \\ & \left.\left.\quad + \frac{\eta n(e_-)}{\sqrt{\tau}}\left(d_-\frac{\partial e_-}{\partial\sigma} - A_-\ln\left(\frac{B}{x}\right) + \frac{1}{\sigma}\right)\right]\right\} \end{aligned} \tag{1.128}$$

Volga uses the identities

$$g = \frac{1}{\sigma^2 \vartheta_-}\left[-\theta_+^2 - \theta_-^2 - \theta_+\theta_- + \frac{\theta_-^2\theta_+^2}{\vartheta_-^2}\right] \tag{1.129}$$

$$\frac{\partial^2 e_\pm}{\partial \sigma^2} = \mp \frac{2\ln\left(\frac{B}{x}\right)}{\sigma^3\sqrt{\tau}} + g\sqrt{\tau} \tag{1.130}$$

$$\frac{\partial A_\pm}{\partial \sigma} = \frac{\theta_+ + \theta_-}{\sigma^3} - \frac{2A_\pm \pm g}{\sigma} \tag{1.131}$$

and turns out to be

$$\begin{aligned} v_{\sigma\sigma}(t,x) = Re^{-\omega r_d \tau} \cdot \\ \Bigg\{ \left(\frac{B}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} & \left[\mathcal{N}(-\eta e_+)\ln\left(\frac{B}{x}\right)\left(A_+^2 \ln\left(\frac{B}{x}\right) + \frac{\partial A_+}{\partial \sigma}\right)\right. \\ & \left. -\eta n(e_+)\left(2\ln\left(\frac{B}{x}\right)A_+ \frac{\partial e_+}{\partial \sigma} - e_+\left(\frac{\partial e_+}{\partial \sigma}\right)^2 + \frac{\partial^2 e_+}{\partial \sigma^2}\right)\right] \\ + \left(\frac{B}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} & \left[\mathcal{N}(\eta e_-)\ln\left(\frac{B}{x}\right)\left(A_-^2 \ln\left(\frac{B}{x}\right) + \frac{\partial A_-}{\partial \sigma}\right)\right. \\ & \left. +\ \eta n(e_-)\left(2\ln\left(\frac{B}{x}\right)A_- \frac{\partial e_-}{\partial \sigma} - e_-\left(\frac{\partial e_-}{\partial \sigma}\right)^2 + \frac{\partial^2 e_-}{\partial \sigma^2}\right)\right] \Bigg\} \end{aligned} \tag{1.132}$$

The risk-neutral probability of knocking out is given by

$$\begin{aligned} & I\!P[\tau_B \le T] \\ &= I\!E\left[I\!I_{\{\tau_B \le T\}}\right] \\ &= \frac{1}{R} e^{r_d T} v(0, S_0) \end{aligned} \tag{1.133}$$

Properties of the first hitting time τ_B

As derived, e.g., in [19], the first hitting time

$$\tilde{\tau} \stackrel{\Delta}{=} \inf\{t \ge 0 : \theta t + W(t) = x\} \tag{1.134}$$

of a Brownian motion with drift θ and hit level $x > 0$ has the density

$$I\!P[\tilde{\tau} \in dt] = \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\frac{(x - \theta t)^2}{2t}\right\} dt, \quad t > 0, \tag{1.135}$$

the cumulative distribution function

$$I\!P[\tilde{\tau} \le t] = \mathcal{N}\left(\frac{\theta t - x}{\sqrt{t}}\right) + e^{2\theta x}\mathcal{N}\left(\frac{-\theta t - x}{\sqrt{t}}\right), \quad t > 0, \tag{1.136}$$

the Laplace-transform

$$\mathbb{E}e^{-\alpha\tilde{\tau}} = \exp\left\{x\theta - x\sqrt{2\alpha + \theta^2}\right\}, \quad \alpha > 0,\ x > 0, \tag{1.137}$$

and the property

$$\mathbb{P}[\tilde{\tau} < \infty] = \begin{cases} 1 & \text{if } \theta \geq 0 \\ e^{2\theta x} & \text{if } \theta < 0 \end{cases} \tag{1.138}$$

For upper barriers $B > S_0$ we can now rewrite the first passage time τ_B as

$$\begin{aligned} \tau_B &= \inf\{t \geq 0 : S_t = B\} \\ &= \inf\left\{t \geq 0 : W_t + \theta_- t = \frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right)\right\}. \end{aligned} \tag{1.139}$$

The density of τ_B is hence

$$\mathbb{P}[\tilde{\tau_B} \in dt] = \frac{\frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right)}{t\sqrt{2\pi t}} \exp\left\{-\frac{\left(\frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right) - \theta_- t\right)^2}{2t}\right\}, \quad t > 0. \tag{1.140}$$

Derivation of the value function

Using the density (1.140) the value of the paid-at-end ($\omega = 1$) upper rebate ($\eta = -1$) option can be written as the the following integral.

$$\begin{aligned} v(T, S_0) &= Re^{-r_d T}\mathbb{E}\left[\mathbb{I}_{\{\tau_B \leq T\}}\right] \\ &= Re^{-r_d T}\int_0^T \frac{\frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right)}{t\sqrt{2\pi t}} \exp\left\{-\frac{\left(\frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right) - \theta_- t\right)^2}{2t}\right\} dt \end{aligned} \tag{1.141}$$

To evaluate this integral, we introduce the notation

$$e_\pm(t) \stackrel{\Delta}{=} \frac{\pm\ln\frac{S_0}{B} - \sigma\theta_- t}{\sigma\sqrt{t}} \tag{1.142}$$

and list the properties

$$e_-(t) - e_+(t) = \frac{2}{\sqrt{t}}\frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right), \tag{1.143}$$

$$n(e_+(t)) = \left(\frac{B}{S_0}\right)^{-\frac{2\theta_-}{\sigma}} n(e_-(t)), \tag{1.144}$$

$$\frac{\partial e_\pm(t)}{\partial t} = \frac{e_\mp(t)}{2t}. \tag{1.145}$$

We evaluate the integral in (1.141) by rewriting the integrand in such a way that the coefficients of the exponentials are the inner derivatives of the exponentials using properties (1.143), (1.144) and (1.145).

$$
\begin{aligned}
&\int_0^T \frac{\frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right)}{t\sqrt{2\pi t}} \exp\left\{-\frac{\left(\frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right)-\theta_- t\right)^2}{2t}\right\} dt \\
&= \frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right)\int_0^T \frac{1}{t^{(3/2)}} n(e_-(t))\,dt \\
&= \int_0^T \frac{1}{2t} n(e_-(t))[e_-(t)-e_+(t)]\,dt \\
&= -\int_0^T n(e_-(t))\frac{e_+(t)}{2t} + \left(\frac{B}{S_0}\right)^{\frac{2\theta_-}{\sigma}} n(e_+(t))\frac{e_-(t)}{2t}\,dt \\
&= \left(\frac{B}{S_0}\right)^{\frac{2\theta_-}{\sigma}} \mathcal{N}(e_+(T)) + \mathcal{N}(-e_-(T)).
\end{aligned}
\tag{1.146}
$$

The computation for lower barriers ($\eta = 1$) is similar.

Quotation conventions and bid-ask spreads

If the payoff is at maturity, the undiscounted value of the one-touch is the touch probability under the risk-neutral measure. The market standard is to quote the price of a one-touch in percent of the payoff, a number between 0 and 100 %. The price of a one-touch depends on the theoretical value (TV) of the above formula, the overhedge (explained in Section 3.1) and the bid-ask spread. The spread in turn depends on the currency pair and the client. For interbank trading spreads are usually between 2 % and 4 % for liquid currency pairs, see Section 3.2 for details.

Two-touch

A two-touch pays one unit of currency (either foreign or domestic) if the underlying exchange rate hits both an upper and a lower barrier during its lifetime. This can be structured using basic touch options in the following way. The long two-touch with barriers L and H is equivalent to

1. a long single one-touch with lower barrier L,
2. a long single one-touch with upper barrier H,
3. a short double one-touch with barriers L and H.

This is easily verified by looking at the possible cases.

If the order of touching L and H matters, then the above hedge no longer works, but we have a new product, which can be valued, e.g., using a finite-difference grid or Monte Carlo Simulation.

Double-no-touch

The payoff

$$
I\!I_{\{L \le \min_{[0,T]} S_t < \max_{[0,T]} S_t \le H\}} \tag{1.147}
$$

of a double-no-touch is in units of domestic currency and is paid at maturity T. The lower barrier is denoted by L, the higher barrier by H.

Derivation of the value function

To compute the expectation, let us introduce the stopping time

$$\tau \stackrel{\Delta}{=} \min\{\inf\{t \in [0, T] | S_t = L \text{ or } S_t = H\}, T\} \tag{1.148}$$

and the notation

$$\tilde{\theta}_\pm \stackrel{\Delta}{=} \frac{r_d - r_f \pm \frac{1}{2}\sigma^2}{\sigma} \tag{1.149}$$

$$\tilde{h} \stackrel{\Delta}{=} \frac{1}{\sigma} \ln \frac{H}{S_t} \tag{1.150}$$

$$\tilde{l} \stackrel{\Delta}{=} \frac{1}{\sigma} \ln \frac{L}{S_t} \tag{1.151}$$

$$\theta_\pm \stackrel{\Delta}{=} \tilde{\theta}_\pm \sqrt{T - t} \tag{1.152}$$

$$h \stackrel{\Delta}{=} \tilde{h} / \sqrt{T - t} \tag{1.153}$$

$$l \stackrel{\Delta}{=} \tilde{l} / \sqrt{T - t} \tag{1.154}$$

$$y_\pm \stackrel{\Delta}{=} y_\pm(j) = 2j(h - l) - \theta_\pm \tag{1.155}$$

$$n_T(x) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{x^2}{2T}\right) \tag{1.156}$$

$$n(x) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \tag{1.157}$$

$$\mathcal{N}(x) \stackrel{\Delta}{=} \int_{-\infty}^{x} n(t)\, dt. \tag{1.158}$$

On $[t, \tau]$, the value of the double-no-touch is

$$v(t) = I\!E^t \left[e^{-r_d(T-t)} I\!I_{\{L \le \min_{[0,T]} S_t < \max_{[0,T]} S_t \le H\}} \right], \tag{1.159}$$

on $[\tau, T]$,

$$v(t) = e^{-r_d(T-t)} I\!I_{\{L \le \min_{[0,T]} S_t < \max_{[0,T]} S_t \le H\}}. \tag{1.160}$$

The joint distribution of the maximum and the minimum of a Brownian motion can be taken from [20] and is given by

$$I\!P \left[\tilde{l} \le \min_{[0,T]} W_t < \max_{[0,T]} W_t \le \tilde{h} \right] = \int_{\tilde{l}}^{\tilde{h}} k_T(x)\, dx \tag{1.161}$$

with

$$k_T(x) = \sum_{j=-\infty}^{\infty} \left[n_T(x + 2j(\tilde{h} - \tilde{l})) - n_T(x - 2\tilde{h} + 2j(\tilde{h} - \tilde{l})) \right]. \tag{1.162}$$

Hence the joint density of the maximum and the minimum of a Brownian motion with drift $\tilde{\theta}$, $W_t^{\tilde{\theta}} \stackrel{\Delta}{=} W_t + \tilde{\theta}t$, is given by

$$k_T^{\tilde{\theta}}(x) = k_T(x) \exp\left\{\tilde{\theta}x - \frac{1}{2}\tilde{\theta}^2 T\right\}. \tag{1.163}$$

We obtain for the value of the double-no-touch on $[t, \tau]$

$$\begin{aligned} v(t) &= e^{-r_d(T-t)} I\!E\, I\!I_{\{L \le \min_{[0,T]} S_t < \max_{[0,T]} S_t \le H\}} \\ &= e^{-r_d(T-t)} I\!E\, I\!I_{\{\tilde{l} \le \min_{[0,T]} W_t^{\tilde{\theta}_-} < \max_{[0,T]} W_t^{\tilde{\theta}_-} \le \tilde{h}\}} \\ &= e^{-r_d(T-t)} \int_{\tilde{l}}^{\tilde{h}} k_{(T-t)}^{\tilde{\theta}_-}(x)dx \qquad (1.164) \\ &= e^{-r_d(T-t)} \qquad (1.165) \\ &\quad \cdot \sum_{j=-\infty}^{\infty} \left[e^{-2j\theta_-(h-l)} \{\mathcal{N}(h + y_-) - \mathcal{N}(l + y_-)\}\right. \\ &\quad \left. - e^{-2j\theta_-(h-l)+2\theta_- h} \{\mathcal{N}(h - 2h + y_-) - \mathcal{N}(l - 2h + y_-)\}\right] \end{aligned}$$

and on $[\tau, T]$

$$v(t) = e^{-r_d(T-t)} I\!I_{\{L \le \min_{[0,T]} S_t < \max_{[0,T]} S_t \le H\}}. \tag{1.166}$$

Of course, the value of the double-one-touch on $[t, \tau]$ is given by

$$e^{-r_d(T-t)} - v(t). \tag{1.167}$$

Greeks

We take the space to list some of the sensitivity parameters I have been frequently asked about.

Vega

$$\begin{aligned} v_\sigma(t) = \frac{e^{-r_d(T-t))}}{\sigma} \cdot \sum_{j=-\infty}^{\infty} \\ \Big\{e^{-2j\theta_-(h-l)} & [2j(h-l)(\theta_+ + \theta_-)\{\mathcal{N}(h + y_-) - \mathcal{N}(l + y_-)\} \\ & + n(h + y_-)(-h - y_+) - n(l + y_-)(-l - y_+)] \\ - e^{-2j\theta_-(h-l)+2\theta_- h} & [2(\theta_+ + \theta_-)(j(h-l) - h)\{\mathcal{N}(-h + y_-) - \mathcal{N}(l - 2h + y_-)\} \\ & + n(-h + y_-)(h - y_+) - n(l - 2h + y_-)(-l + 2h - y_+)]\Big\} \end{aligned} \tag{1.168}$$

Vanna

$$v_{\sigma S_t}(t) = \frac{e^{-r_d(T-t))}}{S_t\sigma^2\sqrt{T-t}} \cdot \sum_{j=-\infty}^{\infty} \left\{e^{-2j\theta_-(h-l)}(T_1 - T_2) - e^{-2j\theta_-(h-l)+2\theta_- h}(T_3 + T_4 - T_5)\right\} \quad (1.169)$$

$$T_1 = n(h + y_-)\{1 - 2j(h-l)(\theta_+ + \theta_-) - (h + y_-)(h + y_+)\} \quad (1.170)$$

$$T_2 = n(l + y_-)\{1 - 2j(h-l)(\theta_+ + \theta_-) - (l + y_-)(l + y_+)\} \quad (1.171)$$

$$T_3 = 2(\theta_+ + \theta_-)[-2\theta_- j(h-l) + 2\theta_- h + 1]\{\mathcal{N}(-h + y_-) - \mathcal{N}(l - 2h + y-)\} \quad (1.172)$$

$$T_4 = n(-h + y_-)\{-2\theta_-(h - y_+) + 2(\theta_+ + \theta_-)(j(h-l) - h) + (h - y_-)(h - y_+) - 1\} \quad (1.173)$$

$$T_5 = n(l - 2h + y_-) \cdot \{-2\theta_-(-l + 2h - y_+) + 2(\theta_+ + \theta_-)(j(h-l) - h) + (-l + 2h - y_-)(-l + 2h - y_+) - 1\} \quad (1.174)$$

Volga

$$v_{\sigma\sigma}(t) = \frac{e^{-r_d(T-t))}}{\sigma^2} \cdot \sum_{j=-\infty}^{\infty} \left\{e^{-2j\theta_-(h-l)}(T_1 + T_2) - e^{-2j\theta_-(h-l)+2\theta_- h}(T_3 + T_4)\right\} \quad (1.175)$$

$$T_1 = (2j(\theta_+ + \theta_-)(h-l) - 3)\{2j(h-l)(\theta_+ + \theta_-)[\mathcal{N}(h + y_-) - \mathcal{N}(l + y_-)]\} + (4j(\theta_+ + \theta_-)(h-l) - 1)[n(h + y_-)(-h - y_+) - n(l + y_-)(-l - y_+)] \quad (1.176)$$

$$T_2 = n(h + y_-)(h + y_-)\left[1 - (h + y_+)^2\right] - n(l + y_-)(l + y_-)\left[1 - (l + y_+)^2\right] \quad (1.177)$$

$$T_3 = (2(\theta_+ + \theta_-)(j(h-l) - h) - 3)\{2(\theta_+ + \theta_-)(j(h-l) - h)[\mathcal{N}(-h + y_-)]\} - \mathcal{N}(l - 2h + y_-)]\} + (4(\theta_+ + \theta_-)(j(h-l) - h) - 1)[n(-h + y_-)(h - y_+) - n(l - 2h + y_-)(-l + 2h - y_+)] \quad (1.178)$$

$$T_4 = n(-h + y_-)(h - y_-)\left[(h - y_+)^2 - 1\right] - n(l - 2h + y_-)(-l + 2h - y_-)\left[(-l + 2h - y_+)^2 - 1\right] \quad (1.179)$$

1.5.3 Compound and instalment

Compound options

A Compound call(put) option is the right to buy(sell) a vanilla option. It works similar to a vanilla call, but allows the holder to pay the premium of the call option spread over time. A first payment is made on inception of the trade. On the following payment day the holder of the compound call can decide to turn it into a plain vanilla call, in which case he has to pay the second part of the premium, or to terminate the contract by simply not paying any more.

Advantages

- Full protection against stronger EUR/weaker USD
- Maximum loss is the premium paid
- Initial premium required is less than in the vanilla call
- Easy termination process, specially useful if future cash flows are uncertain

Disadvantages

- Premium required as compared to a zero cost outright forward
- More expensive than the vanilla call

Example

A company wants to hedge receivables from an export transaction in USD due in 12 months time. It expects a stronger EUR/weaker USD. The company wishes to be able to buy EUR at a lower spot rate if the EUR becomes weaker on the one hand, but on the other hand be fully protected against a stronger EUR. The future income in USD is yet uncertain but will be under review at the end of the next half year.

In this case a possible form of protection that the company can use is to buy a EUR Compound call option with 2 equal semiannual premium payments as for example illustrated in Table 1.19.

The company pays 23,000 USD on the trade date. After a half year, the company has the right to buy a plain vanilla call. To do this the company must pay another 23,000 USD.

Of course, besides not paying the premium, another way to terminate the contract is always to sell it in the market or to the seller. So if the option is not needed, but deep in the money, the company can take profit from paying the premium to turn the compound into a plain vanilla call and then selling it.

If the EUR-USD exchange rate is above the strike at maturity, then the company can buy EUR at maturity at a rate of 1.1500.

If the EUR-USD exchange rate is below the strike at maturity the option expires worthless. However, the company would benefit from being able to buy EUR at a lower rate in the market.

Variations

Settlement As vanilla options compound options can be settled in the following two ways:

- Delivery settlement: Both parties deliver the cash flows.
- Cash settlement: the option seller pays cash to the buyer.

Table 1.19 Example of a Compound Call

Spot reference	1.1500 EUR-USD
Maturity	1 year
Notional	USD 1,000,000
Company buys	EUR call USD put strike 1.1500
Premium per half year of the Compound	USD 23,000.00
Premium of the vanilla call	USD 40,000.00

Distribution of payments The payments don't have to be equal. However, the rule is that the more premium is paid later, the higher the total premium. The cheapest distribution of payments is to pay the entire premium in the beginning, which corresponds to a plain vanilla call.

Exercise style Both the mother and the daughter of the compound option can be European and American style. The market default is European style.

Compound strategies One can think of a compound option on any structure, as for instance a compound put on a knock-out call or a compound call on a double shark forward.

Forward volatility

The daughter option of the compound requires knowing the volatility for its lifetime, which starts on the exercise date T_1 of the mother option and ends on the maturity date T_2 of the daughter option. This volatility is not known at inception of the trade, so the only proxy traders can take is the forward volatility $\sigma(T_1, T_2)$ for this time interval. In the Black-Scholes model the consistency equation for the forward volatility is given by Equation (1.102).

The more realistic way to look at this unknown forward volatility is that the fairly liquid market of vanilla compound options could be taken to back out the forward volatilities since this is the only unknown. These should in turn be consistent with other forward volatility sensitive products like forward start options, window barrier options or faders.

In a market with smile the payoff of the compound option can be approximated by a linear combination of vanillas, whose market prices are known. For the payoff of the compound option itself we can take the forward volatility as in Equation (1.102) for the at-the-money value and the smile of today as a proxy. More details on this can be found, e.g. in Schilling [21]. The actual forward volatility however, is a trader's view and can only be taken from market prices.

Instalment options

This section is based on Griebsch, Kühn and Wystup, see [22].

An instalment call option works similar to a compound call, but allows the holder to pay the premium of the call option in instalments spread over time. A first payment is made at inception of the trade. On the following payment days the holder of the instalment call can decide to prolong the contract, in which case he has to pay the second instalment of the premium, or to terminate the contract by simply not paying any more. After the last instalment payment the contract turns into a plain vanilla call. We illustrate two scenarios in Figure 1.27.

Example
A company wants to hedge receivables from an export transaction in USD due in 12 months time. It expects a stronger EUR/weaker USD. The company wishes to be able to buy EUR at a lower spot rate if the EUR becomes weaker on the one hand, but on the other hand be fully protected against a stronger EUR. The future income in USD is yet uncertain but will be under review at the end of each quarter.

In this case a possible form of protection that the company can use is to buy a EUR instalment call option with 4 equal quarterly premium payments as for example illustrated in Table 1.20.

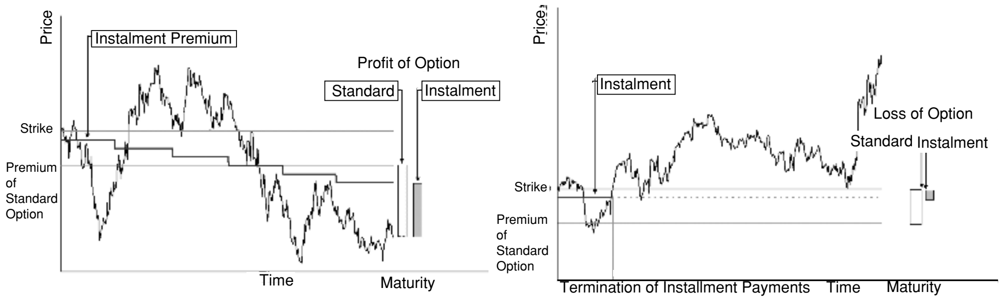

Figure 1.27 Comparison of two scenarios of an instalment option
The left hand side shows a continuation of all instalment payments until expiration. The right hand side shows a scenario where the instalment option is terminated after the first decision date.

The company pays 12,500 USD on the trade date. After one quarter, the company has the right to prolong the instalment contract. To do this the company must pay another 12,500 USD. After 6 months, the company has the right to prolong the contract and must pay 12,500 USD in order to do so. After 9 months the same decision has to be taken. If at one of these three decision days the company does not pay, then the contract terminates. If all premium payments are made, then the contract turns into a plain vanilla EUR call.

Of course, besides not paying the premium, another way to terminate the contract is always to resell it in the market. So if the option is not needed, but deep in the money, the company can take profit from paying the premium to prolong the contract and then selling it.

If the EUR-USD exchange rate is above the strike at maturity, then the company can buy EUR at maturity with a rate of 1.1500.

If the EUR-USD exchange rate is below the strike at maturity the option expires worthless. However, the company would benefit from being able to buy EUR at a lower rate in the market.

Compound options can be viewed as a special case of Instalment options, and the possible variations of compound options apply analogously to instalment options.

Reasons for trading compound and instalment options

We observe that compound and instalment options are always more expensive than buying a vanilla, sometimes substantially more expensive. So why are people buying them? The number one reason is an *uncertainty* about a future cash-flow in a foreign currency. If the cash-flow is certain, then buying a vanilla is, in principle, the better deal. An exception may be the situation in which a treasurer has a budget constraint, i.e. limited funds to spend for foreign exchange risk. With an instalment he can then split the premium over time. The main issue is however, if

Table 1.20 Example of an Instalment Call

Spot reference	1.1500 EUR-USD
Maturity	1year
Notional	USD 1,000,000
Company buys	EUR call USD put strike 1.1500
Premium per quarter of the Instalment	USD 12,500.00
Premium of the vanilla call	USD 40,000.00

a treasurer has to deal with an uncertain cash-flow, and buys a vanilla instead of an instalment, and then is faced with a far out of the money vanilla at time T_1, then selling the vanilla does not give him as much as the savings between the vanilla and the sum of the instalment payments.

The theory of instalment options

This book is not primarily on valuation of options. However, we do want to give some insight into selected topics that come up very often and are of particular relevance to foreign exchange options and have not been published in books so far. We will now take a look at the valuation, the implementation of instalment options and the limiting case of a continuous flow of premium payments.

Valuation in the Black-Scholes model

The intention of this section is to obtain a closed-form formula for the n-variate instalment option in the Black-Scholes model. For the cases $n = 1$ and $n = 2$ the Black-Scholes formula and Geske's compound option formula (see [23]) are already well known.

We consider an exchange rate process S_t modeled by a geometric Brownian motion,

$$S_{t_2} = S_{t_1} \exp((r_d - r_f - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} Z) \quad \text{for } 0 \le t_1 \le t_2 \le T, \tag{1.180}$$

where $\Delta t = t_2 - t_1$ and Z is a standard normal random variable independent of the past of S_t up to time t_1.

Let $t_0 = 0$ be the instalment option inception date and $t_1, t_2, \ldots, t_n = T$ a schedule of decision dates in the contract on which the option holder has to pay the premiums $k_1, k_2, \ldots, k_{n-1}$ to keep the option alive. To compute the price of the instalment option, which is the upfront payment V_0 at t_0 to enter the contract, we begin with the option payoff at maturity T

$$V_n(s) \stackrel{\Delta}{=} [\phi_n(s - k_n)]^+ \stackrel{\Delta}{=} \max[\phi_n(s - k_n), 0],$$

where $s = S_T$ is the price of the underlying asset at T and as usual $\phi_n = +1$ for a call option, $\phi_n = -1$ for a put option.

At time t_i the option holder can either terminate the contract or pay k_i to continue. Therefore by the risk-neutral pricing theory, the holding value is

$$e^{-r_d(t_{i+1}-t_i)} I\!E[V_{i+1}(S_{t_{i+1}}) \mid S_{t_i} = s], \text{ for } i = 0, \ldots, n-1, \tag{1.181}$$

where

$$V_i(s) = \begin{cases} \left[e^{-r_d(t_{i+1}-t_i)} I\!E[V_{i+1}(S_{t_{i+1}}) \mid S_{t_i} = s] - k_i\right]^+ & \text{for } i = 1, \ldots, n-1 \\ V_n(s) & \text{for } i = n \end{cases}. \tag{1.182}$$

Then the unique arbitrage-free value of the initial premium is

$$P \stackrel{\Delta}{=} V_0(s) = e^{-r_d(t_1-t_0)} I\!E[V_1(S_{t_1}) \mid S_{t_0} = s]. \tag{1.183}$$

Figure 1.28 illustrates this context.

One way of pricing this instalment option is to evaluate the nested expectations through multiple numerical integration of the payoff functions via backward iteration. Alternatively, one can derive a solution in closed form in terms of the n-variate cumulative normal.

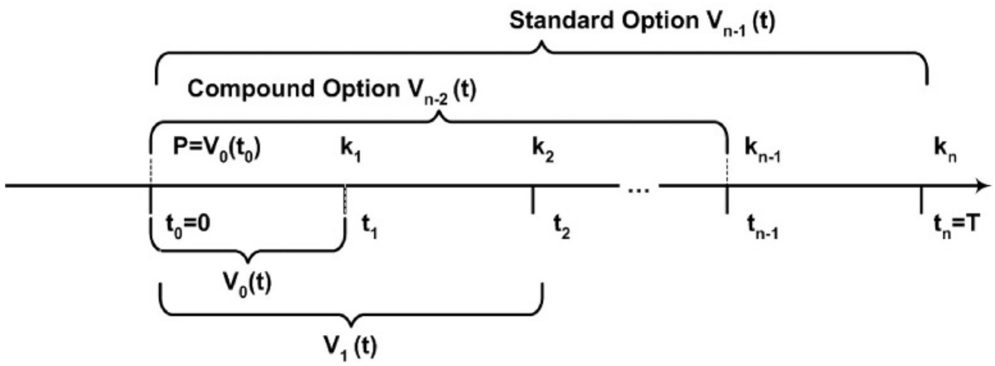

Figure 1.28 Lifetime of the Options V_i

The Curnow and Dunnett integral reduction technique

Denote the n dimensional multivariate normal integral with upper limits $h_1, \ldots, h_n$ and correlation matrix $R_n \triangleq (\rho_{ij})_{i,j=1,\ldots,n}$ by $N_n(h_1, \ldots, h_n; R_n)$, and the univariate normal density function by $n(\cdot)$. Let the correlation matrix be non-singular and $\rho_{11} = 1$.

Under these conditions Curnow and Dunnett [24] derived the following reduction formula for multivariate normal integrals

$$N_n(h_1, \cdots, h_n; R_n) = \int_{-\infty}^{h_1} N_{n-1}\left(\frac{h_2 - \rho_{21}y}{(1-\rho_{21}^2)^{1/2}}, \cdots, \frac{h_n - \rho_{n1}y}{(1-\rho_{n1}^2)^{1/2}}; R_{n-1}^*\right) n(y)dy,$$

$$R_{n-1}^* \triangleq (\rho_{ij}^*)_{i,j=2,\ldots,n},$$

$$\rho_{ij}^* \triangleq \frac{\rho_{ij} - \rho_{i1}\rho_{j1}}{(1-\rho_{i1}^2)^{1/2}(1-\rho_{j1}^2)^{1/2}}. \tag{1.184}$$

A closed form solution for the value of an instalment option

Heuristically, the formula which is given in the theorem below has the structure of the Black-Scholes formula in higher dimensions, namely $S_0 N_n(\cdot) - k_n N_n(\cdot)$ minus the later premium payments $k_i N_i(\cdot)$ $(i = 1, \ldots, n-1)$. This structure is a result of the integration of the vanilla option payoff, which is again integrated minus the next instalment, which in turn is integrated with the following instalment and so forth. By this iteration the vanilla payoff is integrated with respect to the normal density function n times and the i-payment is integrated i times for $i = 1, \ldots, n-1$.

The correlation coefficients ρ_{ij} of these normal distribution functions contained in the formula arise from the overlapping increments of the Brownian motion, which models the price process of the underlying S_t at the particular exercise dates t_i and t_j.

Theorem 1.5.1 *Let $\vec{k} = (k_1, \ldots, k_n)$ be the strike price vector, $\vec{t} = (t_1, \ldots, t_n)$ the vector of the exercise dates of an n-variate instalment option and $\vec{\phi} = (\phi_1, \ldots, \phi_n)$ the vector of the put/call- indicators of these n options.*

The value function of an n-variate instalment option is given by

$$
\begin{aligned}
V_n(S_0, M, \vec{k}, \vec{t}, \vec{\phi}) = {} & e^{-r_f t_n} S_0 \phi_1 \cdot \ldots \cdot \phi_n \\
& \times N_n \left[\frac{\ln \frac{S_0}{S_1^*} + \mu^{(+)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{S_2^*} + \mu^{(+)} t_2}{\sigma \sqrt{t_2}}, \ldots, \frac{\ln \frac{S_0}{S_n^*} + \mu^{(+)} t_n}{\sigma \sqrt{t_n}}; R_n \right] \\
& - e^{-r_d t_n} k_n \phi_1 \cdot \ldots \cdot \phi_n \\
& \times N_n \left[\frac{\ln \frac{S_0}{S_1^*} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{S_2^*} + \mu^{(-)} t_2}{\sigma \sqrt{t_2}}, \ldots, \frac{\ln \frac{S_0}{S_n^*} + \mu^{(-)} t_n}{\sigma \sqrt{t_n}}; R_n \right] \\
& - e^{-r_d t_{n-1}} k_{n-1} \phi_1 \cdot \ldots \cdot \phi_{n-1} \\
& \times N_{n-1} \left[\frac{\ln \frac{S_0}{S_1^*} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{S_2^*} + \mu^{(-)} t_2}{\sigma \sqrt{t_2}}, \ldots, \frac{\ln \frac{S_0}{S_{n-1}^*} + \mu^{(-)} t_{n-1}}{\sigma \sqrt{t_{n-1}}}; R_{n-1} \right] \\
& \vdots \\
& - e^{-r_d t_2} k_2 \phi_1 \phi_2 N_2 \left[\frac{\ln \frac{S_0}{S_1^*} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{S_2^*} + \mu^{(-)} t_2}{\sigma \sqrt{t_2}}; \rho_{12} \right] \\
& - e^{-r_d t_1} k_1 \phi_1 N \left[\frac{\ln \frac{S_0}{S_1^*} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}} \right],
\end{aligned}
\tag{1.185}
$$

where S_i^* $(i = 1, \ldots, n)$ is to be determined as the spot price S_t for which the payoff of the corresponding i-instalment option $(i = 1, \ldots, n)$ is equal to zero and $\mu^{(\pm)}$ is defined as $r_d - r_f \pm \frac{1}{2}\sigma^2$.

The correlation coefficients in R_i of the i-variate normal distribution function can be expressed through the exercise dates t_i,

$$\rho_{ij} = \sqrt{t_i / t_j} \text{ for } i, j = 1, \ldots, n \text{ and } i < j. \tag{1.186}$$

The proof is established with Equation (1.184). Formula (1.185) has been independently derived by Thomassen and Wouve in [25].

Valuation of instalment options with the algorithm of Ben-Ameur, Breton and François

The value of an instalment option at time t is given by the snell envelope of the discounted payoff processes, which is calculated with the dynamic programming method used by the algorithm of Ben-Ameur, Breton and François below. Their original work in [26] deals with instalment options with an additional exercise right at each instalment date. This means that at each decision date the holder can either exercise, terminate or continue.

We examine this algorithm now for the special case of zero value in case it is exercised at $t_1, \ldots, t_{n-1}$. The difference between the above mentioned types of instalment options consists in the (non-)existence of an exercise right at the instalment dates, but this does not change the algorithm in principle.

Model description

The algorithm developed by Ben-Ameur, Breton and François approximates the value of the instalment option in the Black-Scholes Model, which is the premium P paid at time t_0 to enter the contract.

The exercise value of an instalment option at maturity t_n is given by $V_n(s) \stackrel{\Delta}{=} \max[0, \phi_n(s - k_n)]$ and zero at earlier times. The value of a vanilla option at time t_{n-1} is denoted by $V_{n-1}(s) = e^{-r_d \Delta t} I\!E[V_n(s) \mid S_{t_{n-1}} = s]$. At an arbitrary time t_i the holding value is determined as

$$V_i^h(s) = e^{-r_d \Delta t} I\!E[V_{i+1}(S_{t_{i+1}}) \mid S_{t_i} = s] \quad \text{for } i = 0, \ldots, n-1, \tag{1.187}$$

where

$$V_i(s) = \begin{cases} V_0^h(s) & \text{for } i = 0, \\ \max[0, V_i^h(s) - k_i] & \text{for } i = 1, \ldots, n-1, \\ V_n^e(s) & \text{for } i = n. \end{cases} \qquad \text{(DP)} \tag{1.188}$$

The function $V_i^h(s) - k_i$ is called net holding value at t_i, for $i = 1, \ldots, n-1$, which is shown in Figure 1.29.

The option value is the holding value or the exercise value, whichever is greater. The value function V_i, for $i = 0, \ldots, n-1$, is unknown and has to be approximated. Ben-Ameur, Breton and François propose an approximation method, which solves the above dynamic programming (DP)-equation (1.188) in a closed form for all s and i.

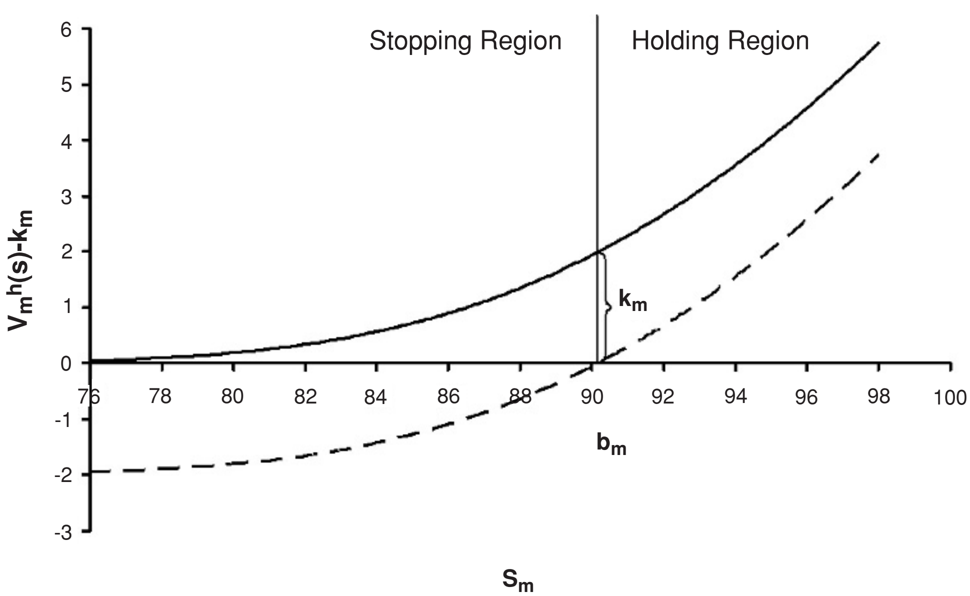

Figure 1.29 The holding value shortly before t_3 for an instalment option with 4 rates is shown by the solid line. The positive slope of this function is less than 1 and the function is continuous and convex. The net holding value of an instalment call option $V_m^h(s) - k_m$ for $(s > 0)$ and a decision time m is presented by the dashed line. This curve intersects the x-axis in the point, where it divides the stopping region and the holding region. The value function is zero in the stopping region $(0, b_i)$ and equal to the net holding value in the holding region $[b_i, \infty)$, where b_i is a threshold for every time t_i, which depends on the parameters of the instalment option

Valuation of instalment options with stochastic dynamic programming

The idea of the above mentioned authors is to partition the positive real axis into intervals and approximate the option value through piecewise linear interpolation. Let $a_0 = 0 < a_1 < \ldots < a_p < a_{p+1} = +\infty$ be points in $\mathbb{R}_0^+ \cup \{\infty\}$ and $(a_j, a_{j+1}]$ for $j = 0, \ldots, p$ a partition of $\mathbb{R}_0^+$ in (p + 1) intervals.

Given approximations $\tilde{V}_i$ of option values V_i at supporting points a_j at the i-th step (at the beginning of the algorithm, at T, this is provided through the input values), this function is piecewise linearly interpolated by

$$\hat{V}_i(s) = \sum_{i=0}^{p} (\alpha_j^i + \beta_j^i s)\mathbf{II}_{\{a_j < s \leq a_{j+1}\}}, \tag{1.189}$$

where $\mathbf{II}$ is the indicator function. The local coefficients of this interpolation in step i, the y-axis intercepts α_j^i and the slopes β_j^i, are obtained by solving the following linear equations

$$\tilde{V}_i(a_j) = \hat{V}_i(a_j) \text{ for } j = 0, \ldots, p-1. \tag{1.190}$$

For $j = p$, one chooses

$$\alpha_p^i = \alpha_{p-1}^i \text{ and } \beta_p^i = \beta_{p-1}^i. \tag{1.191}$$

Now it is assumed, that $\hat{V}_{i+1}$ is known. This is a valid assumption in this context, because the values $\hat{V}_{i+1}$ are known from the previous step. The mean value (1.187) is calculated in step i through

$$\begin{aligned}
\tilde{V}_i^h(a_k) &= e^{-r_d \Delta t} \mathbb{E}[\hat{V}_{i+1}(S_{t_{i+1}}) | S_{t_i} = a_k] \\
&\overset{(1.189)}{=} e^{-r_d \Delta t} \sum_{j=0}^{p} \alpha_j^{i+1} \mathbb{E}\left[\mathbf{II}_{\left\{\frac{a_j}{a_k} < e^{\mu\Delta t + \sigma\sqrt{\Delta t} z} \leq \frac{a_{j+1}}{a_k}\right\}}\right] \\
&+ \beta_j^{i+1} a_k \mathbb{E}\left[e^{\mu\Delta t + \sigma\sqrt{\Delta t} z} \mathbf{II}_{\left\{\frac{a_j}{a_k} < e^{\mu\Delta t + \sigma\sqrt{\Delta t} z} \leq \frac{a_{j+1}}{a_k}\right\}}\right],
\end{aligned} \tag{1.192}$$

where $\mu \stackrel{\Delta}{=} r_d - r_f - \sigma^2/2$ and $\tilde{V}_i^h$ denotes the approximated holding value of the instalment option. Define

$$x_{k,j} \stackrel{\Delta}{=} \frac{\ln\left(\frac{a_j}{a_k}\right) - \mu\Delta t}{\sigma\sqrt{\Delta t}}, \tag{1.193}$$

so for $k = 1, \ldots, p$ and $j = 0, \ldots, p$ the first mean values in Equation (1.192), namely

$$A_{k,j} \stackrel{\Delta}{=} \mathbb{E}\left[\mathbf{II}_{\left\{\frac{a_j}{a_k} < e^{\mu\Delta t + \sigma\sqrt{\Delta t} z} \leq \frac{a_{j+1}}{a_k}\right\}}\right] \tag{1.194}$$

can be expressed as

$$(1.194) = \begin{cases} N(x_{k,1}) & \text{for } j = 0, \\ N(x_{k,j+1}) - N(x_{k,j}) & \text{for } 1 \leq j \leq p-1, \\ 1 - N(x_{k,p}) & \text{for } j = p, \end{cases} \tag{1.195}$$

and similarly

$$B_{k,j} \stackrel{\Delta}{=} I\!\!E\left[a_k e^{\mu\Delta t+\sigma\sqrt{\Delta t}z} \mathrm{II}_{\left\{\frac{a_j}{a_k} < e^{\mu\Delta t+\sigma\sqrt{\Delta t}z} \leq \frac{a_{j+1}}{a_k}\right\}}\right] \tag{1.196}$$

can be expressed in the following way

$$(1.196) = \begin{cases} a_k \mathcal{N}(x_{k,1} - \sigma\sqrt{\Delta t})e^{(r_d - r_f)\Delta t} & \text{for } j = 0, \\ a_k[\mathcal{N}(x_{k,j+1} - \sigma\sqrt{\Delta t}) - \mathcal{N}(x_{k,j} - \sigma\sqrt{\Delta t})]e^{(r_d - r_f)\Delta t} & \text{for } 1 \leq j \leq p-1, \\ a_k[1 - \mathcal{N}(x_{k,p} - \sigma\sqrt{\Delta t})]e^{(r_d - r_f)\Delta t} & \text{for } j = p, \end{cases} \tag{1.197}$$

where $n(z) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ and $\mathcal{N}$ denotes the cumulative normal distribution function.

In the simplifying notation (1.195) and (1.197) the points a_i $(i = 1, \ldots, p)$ can be understood as the quantiles of the log-normal distribution. These are not chosen directly, but are calculated as the quantiles of (e.g. equidistant) probabilities of the log-normal distribution. Thereby the supporting points lie closer together, in areas, where the modification rate of the distribution function is great. The number a_k in Equation (1.193) is the given exchange rate at time t_i and therefore constant. In the implementation it requires an efficient method to calculate the inverse normal distribution function. One possibility is to use the Cody-Algorithm taken from [27].

An algorithm in pseudo code

For a better understanding the described procedure (1.5.3) is sketched in form of an algorithm in this section. The algorithm works according to the dynamic programming principle backwards in time, based on the values of the exercise function of the instalment option at maturity T at predetermined supporting points a_j. Through linear connection of these points an approximation of the exercise function can be obtained. The exercise function at maturity is the payoff function of the vanilla option, which is constant up to the strike price K and in the region behind (i.e. $\geq K$) it is linear. The linear approximation at maturity T is therefore exact, except on the interval $K \in (a_l, a_{l+1})$, in case K does not correspond to one of these supporting points. For this reason the holding value of this linear approximation is calculated by the means of $A_{k,j}$ and $B_{k,j}$ from above. The transition parameters $A_{k,i}$ and $B_{k,i}$ can be calculated before the first iteration, because only values, which are known in the beginning, are required. The advantage of this approach is that the holding value needs only to be calculated at the supporting points a_j and because of linearity, the function values for all s are obtained. The values of the holding value at a_j are used again as approximations of the exercise values at time t_{n-1} and it is proceeded like in the beginning. The output of the algorithm is the value of the instalment option at time t_0.

A description in pseudo code

First the a_k are generated as quantiles of the distribution of the price at maturity of the exchange rate S_T and can be for example approximated by the Cody-Algorithm.

1. Calculate $q_1, \ldots, q_p$-quantiles of the standard normal distribution via the inverse distribution function.

2. Calculate $a_1, \ldots, a_p$-quantiles of the log-normal distribution with mean $\log S_0 - \mu T$ and variance $\sigma\sqrt{T}$ by

$$\exp(q_i \sigma\sqrt{T} + \log S_0 + \mu T) = a_i.$$

In pseudo code the implementation of the theoretical consideration of Section 1.5.3 can be worked out in the following way. The principle of the backward induction is realized as a for-loop that counts backwards from $n - 1$ to 0.

1. Calculate $\hat{V}_n(s)$ for all s, using (1.189), i.e. calculate all α_i^n, β_i^n for $i = 0, \ldots, p$
2. For $j = n$ to 1
   a. Calculate $\tilde{V}_{j-1}^h(a_k)$ for a_k $(k = 1, \ldots, p)$ in closed form using (1.192).
   b. Calculate $\tilde{V}_{j-1}(a_k)$ for $k = 1, \ldots, p$ using (DP) with $\tilde{V}_{j-1}^h(a_k)$ for $V_{j-1}^h(a_k)$.
   c. Calculate $\hat{V}_{j-1}(s)$ for all $s > 0$ using (1.189), i.e. calculate all $\alpha_i^{j-1}, \beta_i^{j-1}$ for $i = 1, \ldots, k$. Unless $j - 1$ is already equal to zero, calculate $\hat{V}_{j-1}(s)$ for $s = S_0$ and break the algorithm.
   d. Substitute $j \leftarrow j - 1$.

Repeat these steps until $\hat{V}_0(S_0)$ is calculated, which is the value of the instalment option at time 0.

This algorithm works with equidistant instalment dates, constant volatility and constant interest rates. Constant volatility and interest rates are assumptions of the applied Black-Scholes Model, but the algorithm would be extendable for piecewise constant volatility- and interest rate as functions of time, with jumps at the instalment dates. The interval length Δt in the calculation can be replaced in every period by arbitrary $t_{i+1} - t_i$. Furthermore the computational time could be decreased by omitting smaller supporting points in the calculation as soon as one of them generates a zero value in the maximum function.

Instalment options with a continuous payment plan

Let $g = (g_t)_{t\in[0,T]}$ be the stochastic process describing the discounted net payoff of an instalment option expressed as multiples of the domestic currency. If the holder stops paying the premium at time t, the difference between the option payoff and premium payments (all discounted to time 0) amounts to

$$g(t) = \begin{cases} e^{-r_d T}(S_T - K)^+ \mathbf{1}_{(t=T)} - \frac{p}{r_d}(1 - e^{-r_d t}) & \text{if } r_d \neq 0 \\ (S_T - K)^+ \mathbf{1}_{(t=T)} - pt & \text{if } r_d = 0 \end{cases}, \tag{1.198}$$

where K is the strike. Given the premium rate p, there is a unique no-arbitrage premium P_0 to be paid at time 0 (supplementary to the rate p) given by

$$P_0 = \sup_{\tau \in \mathcal{T}_{0,T}} I\!E_Q(g_\tau). \tag{1.199}$$

Ideally, p is chosen as the *minimal* rate such that

$$P_0 = 0. \tag{1.200}$$

Note that P_0 from (1.199) can never become negative as it is always possible to stop payments immediately. Thus, besides (1.200), we need a minimality assumption to obtain a unique rate.

We want to compare the instalment option with the American contingent claim $f = (f_t)_{t\in[0,T]}$ given by

$$f_t = e^{-r_d t}(K_t - C_E(T-t, S_t))^+, \quad t \in [0, T], \tag{1.201}$$

where $K_t = \frac{p}{r_d}\left(1 - e^{-r_d(T-t)}\right)$ for $r_d \neq 0$ and $K_t = p(T-t)$ when $r_d = 0$. C_E is the value of a standard European call. Equation (1.201) represents the payoff of an American put on a European call where the variable strike K_t of the put equals the part of the instalments *not* to be paid if the holder decides to terminate the contract at time t. Define by $\tilde{f} = (\tilde{f}_t)_{t\in[0,T]}$ a similar American contingent claim with

$$\tilde{f}(t) = e^{-r_d t}\left[(K_t - C_E(T-t, S_t))^+ + C_E(T-t, S_t)\right], \quad t \in [0, T]. \tag{1.202}$$

As the process $t \mapsto e^{-r_d t}C_E(T-t, S_t)$ is a Q-martingale we obtain that

$$\sup_{\tau\in\mathcal{T}_{0,T}} I\!E_Q(\tilde{f}_\tau) = C_E(T, s_0) + \sup_{\tau\in\mathcal{T}_{0,T}} I\!E_Q(f_\tau). \tag{1.203}$$

The following theorem has been proved in [22] using earlier results of El Karoui, Lepeltier and Millet in [28].

Theorem 1.5.2 *An instalment option is the sum of a European call plus an American put on this European call, i.e.*

$$\underbrace{P_0 + p\int_0^T e^{-r_d s}\, ds}_{\textit{total premium payments}} = C_E(T, s_0) + \sup_{\tau\in\mathcal{T}_{0,T}} I\!E_Q(f_\tau)$$

1.5.4 Asian options

This section is produced in conjunction with Silvia Baumann, Marion Linck, Michael Mohr and Michael Seeberg.

Asian Options are options on the average usually of spot fixings and are very popular and common hedging instruments for corporates. Average options belong to the class of path dependent options. The Term *Asian Options* comes from their origin in the Tokyo office of Banker's Trust in 1987.[5] The payoff of an Asian Option is determined by the path taken by the underlying exchange rate over a fixed period of time. We distinguish the four cases listed in Table 1.21 and compare values of average price options with vanilla options in Figure 1.30. Variations of Asian Options refer particularly to the way the average is calculated.

Kind of average We find geometric, arithmetic or harmonic average of prices. Harmonic averaging originates from a payoff in *domestic* currency and will be treated in Section 1.6.9.

Time interval We need to specify the period over which the prices are taken. The end of the averaging interval can be shorter than or equal to the option's expiration date, the starting value can be any time before. In particular, after an average option is traded, the beginning of the averaging period typically lies in the past, so that parts of the values contributing to the average are already known.

[5] see `http://www.global-derivatives.com` / Options/ Asian Options.

Table 1.21 Types of Asian Options for $T_0 \leq t \leq T$, where $[T_0, T]$ denotes the time interval over which the average is taken, K denotes the strike, S_T the spot price at expiration time and A_T the average

Product name	Payoff
average price call	$(A_T - K)^+$
average price put	$(K - A_T)^+$
average strike call	$(S_T - A_T)^+$
average strike put	$(A_T - S_T)^+$

Sampling style The market generally uses discrete sampling, like daily fixings. In the literature we often find continuous sampling.

Weighting Different weights may be assigned to the prices to account for a non-linear, i.e. skewed, price distribution, see [29], pp. 1116–1117, and the example below under 3.

Variations The wide range of variations covers also the possible right for early execution, Asian options with barriers.

Asian Options are applied in risk management, especially for currencies, for the following reasons.

1. Protection against rapid price movements or manipulation in thinly traded underlyings at maturity, i.e. reduction of significance of the closing price through averaging.
2. Reduction of hedging cost through
   - the lower fair price compared to regular options since an average is less volatile than single prices, and
   - to achieve a similar hedging effect with vanilla options, a chain of such options would have to be bought – an obviously more expensive strategy.

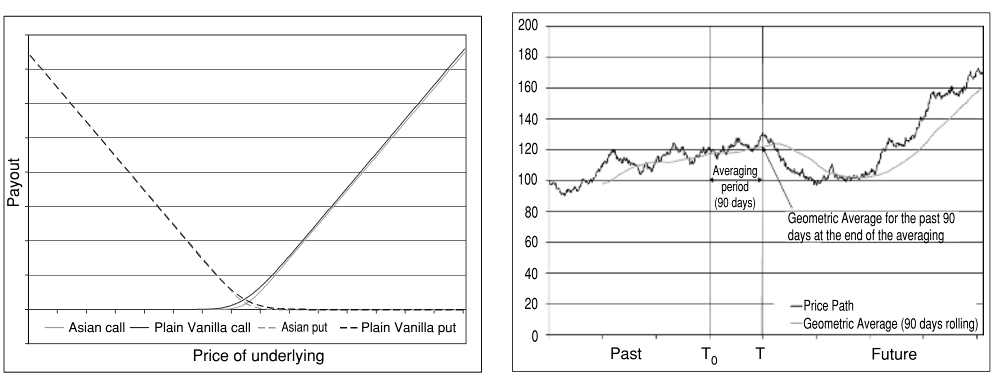

Figure 1.30 lhs: Comparing the value of average price options with vanillas, we see that average price options are cheaper. The reason is that averages are less volatile and hence less risky. rhs: Ingredients for average options: a price path, 90-days rolling price average (here: geometric), and an averaging period for an option with 90-days maturity

3. Adjustment of option payoff to payment structure of the firm
   - Average Price Options can be used to hedge a stream of (received) payments (e.g. a USD average call can be bought to hedge the ongoing EUR revenues of a US-based company). Different amounts of the payments can be reflected in flexible weights, i.e. the prices related to higher payments are assigned a higher weight than those related to smaller cash flows when calculating the average.
   - With Average Strike Options the strike price can be set at the average of the underlying price – a helpful structure in volatile or hardly predictable markets.

Valuation

The pricing approaches developed differ depending on the specific characteristics considered, e.g. averaging method, option style etc. In the following, we present the value formula for a *European Geometric Average Price Call*. Afterwards, two common approaches to evaluate Arithmetic Average Price Options are introduced. Henderson and Wojakowski prove the symmetry between Average Price Options and Average Strike Options in [30] allowing the use of the more established fixed-strike valuation methods to price *Floating Strike Asian Options*.

Geometric average options

Kemna and Vorst [31] derive a closed form solution for Geometric Average Price Options in a geometric Brownian motion model

$$dS_t = S_t[(r_d - r_f)dt + \sigma dW_t]. \tag{1.204}$$

An extension for foreign exchange options can be found in Wystup [32]. A Geometric Average Price Call pays $(A_T - K)^+$, where A_T denotes the geometric average of the price of the underlying. In the discrete case, A_T is calculated as

$$A_T \stackrel{\Delta}{=} \sqrt[n+1]{\prod_{i=0}^{n} S_{t_i}}, \tag{1.205}$$

in the continuous case as

$$A_T \stackrel{\Delta}{=} \exp\left\{\frac{1}{T - T_0}\int_{T_0}^{T} \log S_t \, dt\right\}. \tag{1.206}$$

The random variable $\int_t^T W(u)\,du$ is normally distributed with mean zero variance

$$\Sigma^2 \stackrel{\Delta}{=} \frac{T^3}{3} + \frac{2t^3}{3} - t^2 T \tag{1.207}$$

for any $t \in [T_0, T]$. This can be calculated following the instructions in Shreve's lecture notes [19]. Therefore, the geometric average of a log-normally distributed random variable is log-normally distributed. In the continuous case, the distribution parameters can be derived as

$$\log A_T \sim \mathcal{N}\left[\frac{1}{2}\left(r_d - r_f - \frac{1}{2}\sigma^2\right)(T - T_0) + \log S_0; \frac{1}{3}\sigma^2(T - T_0)\right]. \tag{1.208}$$

The interesting feature of these terms is the replacement of the geometric average by the underlying price S_0 smoothing the way to the option price determination. In the Black-Scholes model the value of the option can be computed as the expected payoff under the risk-neutral probability measure. Using the money market account $e^{-r_d(T-T_0)}$ as numeraire leads to the value of the continuously sampled geometric Asian fixed strike call,

$$C_{G-Asian} = \mathbb{E}\left[e^{-r_d(T-T_0)}(A_T - K)\mathbb{I}_{\{A_T>K\}}\right], \tag{1.209}$$

where we observe that the remaining computation works just like a vanilla. In order to derive a useful general result we need to generalize the payoff of the continuously sampled geometric Asian fixed strike option to

$$[\phi(A(-s,T) - K)]^+, \tag{1.210}$$

$$A(-s,T) \stackrel{\Delta}{=} \exp\left\{\frac{1}{T+s}\int_{-s}^{T} \log S(u)\,du\right\},\ s \geq 0. \tag{1.211}$$

This definition includes the case where parts of the average is already known, which is important to value the option after it has been written.

With the abbreviations

- T for the expiration time (in years),
- s for the time before valuation date (in years), for which the values and average of the underlying is known,
- K for the strike of the option,
- ϕ taking the values $+1$ or -1 if the option is a call or a put respectively,
- $\alpha \stackrel{\Delta}{=} \frac{T}{T+s} \in [0, 1]$,
- $\theta_\pm \stackrel{\Delta}{=} \frac{r_d - r_f}{\sigma} \pm \frac{\sigma}{2}$,
- $S_t = S_0 e^{\sigma W_t + \sigma\theta_- t}$ for the price of the underlying at time t,
- $d_\pm \stackrel{\Delta}{=} \frac{\ln\frac{S_0}{K} + \sigma\theta_\pm T}{\sigma\sqrt{T}}$,
- $n(t) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$,
- $\mathcal{N}(x) \stackrel{\Delta}{=} \int_{-\infty}^{x} n(t)\,dt$,
- $\text{Vanilla}(S_0, K, \sigma, r_d, r_f, T, \phi) = \phi\left(S_0 e^{-r_f T}\mathcal{N}(\phi d_+) - K e^{-r_d T}\mathcal{N}(\phi d_-)\right)$,
- $H \stackrel{\Delta}{=} \exp\left\{-\frac{\alpha T}{2}\left(r_d - r_f + \frac{\sigma^2}{2}[1 - \frac{2\alpha}{3}]\right)\right\}$,

the value of the continuously sampled geometric Asian fixed strike vanilla is then given by

$$\begin{aligned}\text{Asiangeo}(S_0, K, T, s, \sigma, r_d, r_f, \phi) &= e^{(\alpha-1)r_d T} H \left(\frac{S_0}{A(-s,0)}\right)^{\alpha-1} \\ &\text{Vanilla}\left(S_0, \frac{K}{H}\left(\frac{S_0}{A(-s,0)}\right)^{1-\alpha}, \frac{\alpha\sigma}{\sqrt{3}}, \alpha r_d, \alpha r_f, T, \phi\right).\end{aligned} \tag{1.212}$$

This way a geometric Asian option with fixed strike can be viewed as a multiple of a vanilla option with the same spot and time to maturity but different parameters such as strike, volatility and interest rates. We observe in particular, that as time to maturity becomes smaller, the known part of the average becomes more prominent, α tends to zero and hence the volatility of the auxiliary vanilla option tends to zero. Moreover, the properties known for the function Vanilla carry over to the function Asiangeo. Greeks can also be derived from this relation.

Let us now consider the case, where averaging starts after T_0, i.e., the payoff is changed to

$$[\phi(A(t,T)-K)]^+, \tag{1.213}$$

$$A(t,T) \stackrel{\Delta}{=} \exp\left\{\tfrac{1}{T-t}\int_t^T \log S(u)\,du\right\},\ t\in[0,T]. \tag{1.214}$$

Then the value becomes

$$\begin{aligned}&\text{Asiangeowindow}(S_0,K,T,t,\sigma,r_d,r_f,\phi)=\\ &H\ \text{Vanilla}\left(S_0,\frac{K}{H},\frac{\Sigma\sigma}{(T-t)\sqrt{T}},r_d,r_f,T,\phi\right),\end{aligned} \tag{1.215}$$

$$H \stackrel{\Delta}{=} \exp\left\{-\frac{\sigma\theta_-}{2}(T-t)-\frac{\sigma^2}{2}(t-\frac{\Sigma}{T-t})\right\}. \tag{1.216}$$

Derivation of the value function

First we consider the call without history ($s=0$). We rewrite the geometric average as

$$\begin{aligned}A(0,T) &= \exp\left\{\frac{1}{T}\int_0^T \log S(u)\,du\right\}\\ &= S_0\exp\left\{\frac{\sigma}{2}\theta_- T+\frac{\sigma}{T}\int_0^T W(u)\,du\right\}\end{aligned} \tag{1.217}$$

and compute the value function as

$$\begin{aligned}&\text{Asiangeo}(S_0,K,T,0,\sigma,r_d,r_f,\phi)\\ &= e^{-r_dT}\,\mathbb{E}[(A(0,T)-K)^+]\\ &= e^{-r_dT}\int_{-\infty}^{+\infty}\left(S_0\exp\left\{\frac{\sigma}{2}\theta_-T+\sigma\sqrt{\frac{T}{3}}x\right\}-K\right)^+ n(x)\,dx\\ &= S_0e^{-r_fT}e^{-\frac{T}{2}(r_d-r_f+\frac{\sigma^2}{6})}\mathcal{N}\left(\frac{\ln\frac{S_0}{K}+\frac{\sigma}{2}\theta_-T}{\sigma\sqrt{\frac{T}{3}}}+\sigma\sqrt{\frac{T}{3}}\right)\\ &\quad -Ke^{-r_dT}\mathcal{N}\left(\frac{\ln\frac{S_0}{K}+\frac{\sigma}{2}\theta_-T}{\sigma\sqrt{\frac{T}{3}}}\right),\end{aligned} \tag{1.218}$$

which leads to the desired result. The analysis for the put option is similar. For $s>0$ (real history) note that

$$A(-s,T)=A(-s,0)^{1-\alpha}A(0,T)^{\alpha}. \tag{1.219}$$

The first factor of this product is non-random at time 0, hence the value of a call with history is given by

$$\begin{aligned}
&\text{Asiangeo}(S_0, K, T, s, \sigma, r_d, r_f, \phi) \qquad (1.220)\\
&= e^{-r_d T}\, I\!E[(A(-s, T) - K)^+]\\
&= e^{-r_d T} A(-s, 0)^{1-\alpha}\, I\!E\left[A(0, T)^{\alpha} - \frac{K}{A(-s, 0)^{1-\alpha}})^+\right]\\
&= e^{-r_d T} \int_{-\infty}^{+\infty} \left(S_0^{\alpha} \exp\left\{ \frac{\alpha\sigma}{2}\theta_T + \alpha\sigma\sqrt{\frac{T}{3}}x \right\} - \frac{K}{A(-s, 0)^{1-\alpha}} \right)^+ n(x)\, dx.
\end{aligned}$$

It is now an easy exercise to complete this calculation.

Arithmetic average options

Since the distribution of the arithmetic average of log-normally distributed random variables is not normal, a closed form solution for the frequently used Arithmetic Average Price Options is not immediately available. Some of the approaches to solve this valuation task are

1. Numerical approaches, e.g. Monte Carlo simulations work well, as one can take the geometric Asian option as a highly correlated control variate. Taking a PDE approach is equally fast as Večeř has shown how to reduce the valuation problem to a PDE in one dimension in [33].
2. Modifications of the geometric average approach;
3. Approximations of the density function for the arithmetic average, see [34] on p. 475.

For instance, Turnbull and Wakeman (see [35]) develop an approximation of the density function by defining an alternative distribution for the arithmetic average with moments that match the moments of the true distribution similarly as in Section 1.6.7. One can also match the cumulants up to fourth order: mean, variance, skew and kurtosis. The adjusted mean and variance are finally plugged into the general Black-Scholes formula. Lévy states in [34] that considering only the first two moments delivers acceptable results for typical ranges of volatility and simultaneously reduces the complexity of the Turnbull and Wakeman approach. Hakala and Perissé show in [3] how to include higher moments. We apply a Monte Carlo simulation of price paths to value arithmetic average price options. To improve the quality of the results, we take geometric average options with similar specifications as control variate, see [31], p. 124. For variance reduction techniques see [36], pp. 414–418. For further suggestions on the implementation of pricing models see e.g. [37], pp. 118–123. We show in Table 1.22 that the results are close to the analytical approximations provided by Turnbull and Wakeman as well as Lévy.

Sensitivity analysis

We analyse now the sensitivities of the values with respect to various input parameters and compare them with vanilla options. Throughout we will use the parameters $K = 1.2000$, $S_0 = 1.2000$, $r_d = 3\,\%$, $r_f = 2.5\,\%$, $\sigma = 10\,\%$, $T - T_0 = 3$ months (91 days). The similarity of vanilla and average options, and the effects from averaging prices, which already dominated the derivation of the value formula, are reflected in the *Greeks* as well. Both option types react

Table 1.22 Values of average options

Method	Ar. call	Ar. put	Geo. price call	Geo. price put	Geo. strike call	Geo. strike put
analytical	–	–	271.19	273.63	295.21	248.19
Monte Carlo	295.92	251.95	290.53	256.44	295.38	244.62
with control variate	276.57	269.14	–	–	–	–
Turnbull/Wakeman	276.36	269.02	–	–	–	–
Lévy	276.36	269.02	–	–	–	–

Input parameters are $K = 1.2000$, $S_0 = 1.2000$, $\sigma = 20\,\%$, $r_d = 3\,\%$, $r_f = 2.5\,\%$, $T - T_0 = 90$ days $= 90/365$ years, 90 observations (implying a time step of 0.002739726 years), 10,000 price paths in the Monte Carlo simulation. The arithmetic average options are average price options. All values are in domestic pips.

in the same direction to parameter changes and differ only in the quantity of the option value change. This holds especially for delta, gamma, and vega. These sensitivities, which are related to the underlying, represent best the properties of average options, i.e. initially, the option is very sensitive to price changes in the underlying. Delta, gamma, and vega have accordingly high values. With decreasing time to maturity the impact of single prices on the final payoff diminishes, delta stabilizes, and gamma approaches zero, see [38], pp. 63–64. Figure 1.31 illustrates the similarity between vanilla and average price options with respect to delta and gamma.

For the same level of volatility in the underlying average options have a lower vega compared to vanilla options, because fluctuations of the underlying price are smoothed by the average. Note that the lower the volatility the smaller the value difference between average and vanilla options, see Figure 1.32.

Since single prices – especially at maturity – influence the payoff of average options less significantly than for vanilla options, time, i.e. the chance of a finally favorable performance, plays a less important role in determining the value of average options, leading to a lower theta. The interest rate sensitivity rho of average options is smaller than for vanilla options.

Hedging

With the sensitivity analysis in mind, the question arises, how the writer of an average option should deal with the risks of a short position.

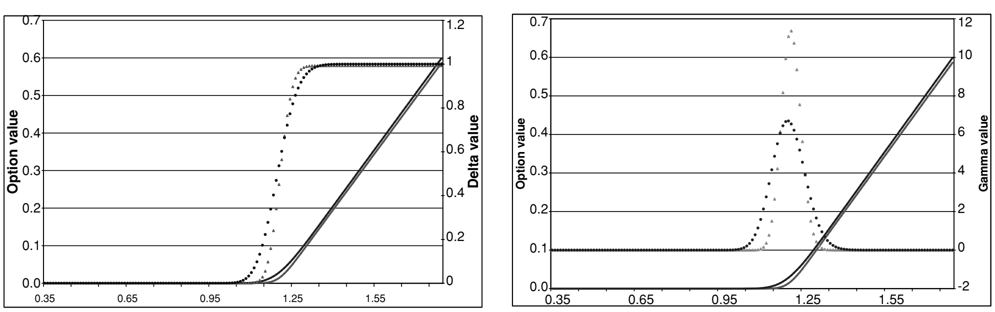

Figure 1.31 lhs: Option values and delta depending on the underlying price; rhs: Option values and gamma depending on the underlying price

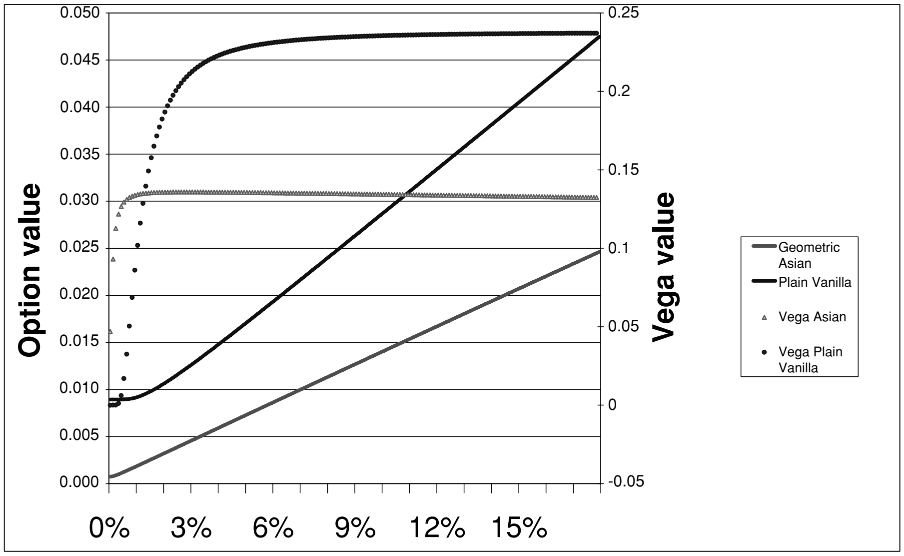

Figure 1.32 Option values and vega depending on volatility for at-the-money options

Dynamic hedging

For a call position, for instance, one way is the replication with an investment in the underlying that is funded by borrowing. The delta of the option suggests how many units of the underlying have to be bought. Since delta changes over time, the amount invested in the underlying has to be adjusted frequently. From the risk analysis it can be inferred that average options are easier to hedge than vanilla options, in particular the delta of average options stabilizes over time. Accordingly, the scope of required rebalancing of the hedge and the related transaction cost decrease over time. The costs of the hedge include interest payments as well as commissions and bid-ask-spreads due at every rebalancing transaction. See [39] and [38] for empirical analyses on the cost of dynamic and static hedging. Dynamic hedging neutralizes the delta exposure inherent in the option position. The volatility exposure has to be hedged with vanilla contracts.

Static hedging

Alternatively, a static hedge involving vanilla options can be set up. The position remains generally unchanged until maturity of the Average Option. Vanilla options are traded in liquid markets at relatively small bid-ask-spreads. Furthermore, not only the delta risk but also the gamma and volatility exposure can be reduced with options as hedge instrument. Static hedges with vanilla options have therefore become common market practice, see [40]. For instance, Lévy suggests in [40] as a rule of thumb to choose a vanilla call with a similar strike as a short average price call and an expiration that is one-third of the averaging period of the exotic, based on the appearance of the factor $\frac{T}{3}$ in Equation (1.218). As the right hand side of Figure 1.33 shows, the sensitivities of the short average price call are at their highest levels in the first third of the averaging period. Hedging with options only during this most critical time period already significantly reduces the sensitivity of the position to underlying price changes. Simultaneously, choosing vanilla calls with shorter maturity saves hedging costs. Nevertheless,

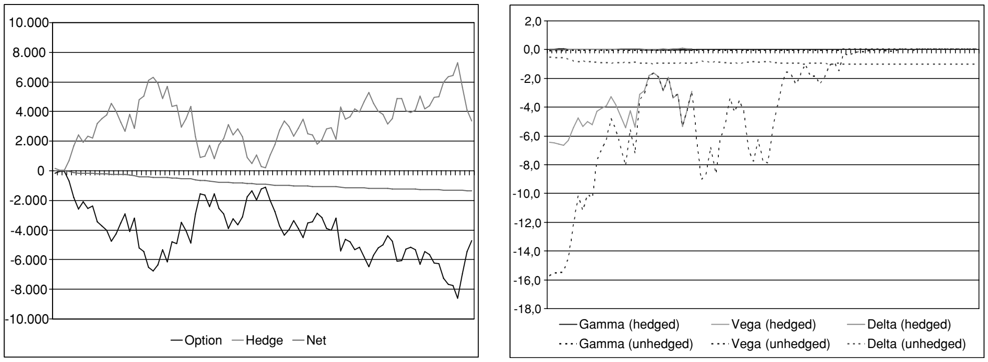

Figure 1.33 lhs: Dynamic hedging: Performance of option position and hedge portfolio; rhs: Static hedging: Comparison of hedged and unhedged "Greek" exposure. For both, sample prices were generated randomly

this approach leaves the option writer with an open position for the remaining time to maturity unless he or she decides to build up a new hedge portfolio (semi-static hedging strategy). Since the stabilized delta in the later life time of the average option reduces the rebalancing effort, a dynamic hedge could be an alternative to a renewed hedge with vanilla options.

1.5.5 Lookback options

This section is produced in conjunction with Silvia Baumann, Marion Linck, Michael Mohr and Michael Seeberg.

Lookback options are, as Asian options, path dependent. At expiration the holder of the option can "look back" over the life time of the option and exercise based upon the optimal underlying value (extremum) achieved during that period. Thus, Lookback options (like Asians) avoid the problem of European options that the underlying performed favorably throughout most of the option's lifetime but moves into a non-favorable direction towards maturity. Moreover, (unlike American Options) Lookback options optimize the market timing, because the investor gets – by definition – the most favorable underlying price. As summarized in Table 1.23 Lookback options can be structured in two different types with the extremum representing either the strike price or the underlying value. Figure 1.34 shows the development of the payoff of Lookback options depending on a sample price path. In detail we define

$$M_{t,T} \triangleq \max_{t \le u \le T} S(u), \tag{1.221}$$

$$M_T \triangleq M_{0,T}, \tag{1.222}$$

$$m_{t,T} \triangleq \min_{t \le u \le T} S(u), \tag{1.223}$$

$$m_T \triangleq m_{0,T}. \tag{1.224}$$

Variations of Lookback options include *Partial Lookback Options*, where the monitoring period for the underlying is shorter than the lifetime of the option. Conze and Viswanathan [41] present further variations like *Limited Risk* and *American Lookback Options*. Since the currency

Table 1.23 Types of lookback options

Payoff	Lookback type	Parameter
$M_T - S_T$	floating strike put	$\phi = -1, \bar{\eta} = -1$
$S_T - m_T$	floating strike call	$\phi = +1, \bar{\eta} = +1$
$(M_T - X)^+$	fixed strike call	$\phi = +1, \bar{\eta} = -1$
$(X - m_T)^+$	fixed strike put	$\phi = -1, \bar{\eta} = +1$

The contract parameters T and X are the time to maturity and the strike price respectively, and S_T denotes the spot price at expiration time. Fixed strike lookback options are also called *hindsight options*.

markets traded lookback options do not fit typical business needs, they are mainly used by speculators, see [42]. An often cited strategy is building *Lookback Straddles* paying

$$M_{t,T} - m_{t,T}, \tag{1.225}$$

(also called *range* or *hi-lo option*), a combination of Lookback put(s) and call(s) that guarantees a payoff equal to the observed range of the underlying asset. In theory, Garman pointed out in [43], that Lookback options can also add value for risk managers, because floating (fixed) strike lookback options are good means to solve the timing problem of market entries (exits), see [44]. For instance, a minimum strike call is suitable to avoid missing the best exchange rate in currency linked security issues. However, this right is very expensive. Since one buys a guarantee for the best possible exchange rate ever, lookback options are generally way too expensive and hardly ever trade. Exceptions are performance notes, where lookback and average features are mixed, e.g. performance notes paying say 50 % of the best of 36 monthly average gold price returns.

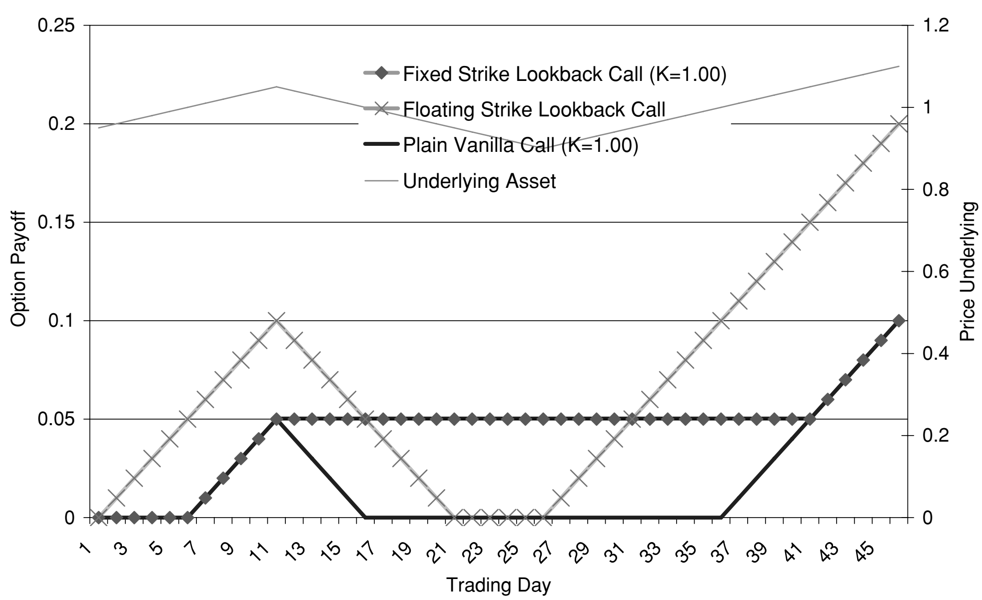

Figure 1.34 Payoff profile of Lookback calls (sample underlying price path, 20 trading days)

Valuation

As in the case of Asian options, closed form solutions only exist for specific products – in this case basically for any lookback option with continuously monitored underlying value. We consider the example of the floating strike lookback call. Again, the value of the option is given by

$$\begin{aligned} v(0, S_0) &= I\!E\left[e^{-r_d T}(S_T - m_T)\right] \\ &= S_0 e^{-r_f T} - e^{-r_d T} I\!E\left[m_T\right]. \end{aligned} \tag{1.226}$$

In the standard Black-Scholes model (1.1), the value can be derived using the reflection principle and results in

$$\begin{aligned} v(t,x) = \phi \Bigg\{ & x e^{-r_f \tau}\mathcal{N}(\phi b_1) - K e^{-r_d \tau}\mathcal{N}(\phi b_2) + \frac{1-\eta}{2}\phi e^{-r_d\tau}[\phi(R-X)]^+ \\ & + \eta x e^{-r_d \tau}\frac{1}{h}\left[\left(\frac{x}{K}\right)^{-h}\mathcal{N}(-\eta\phi(b_1 - h\sigma\sqrt{\tau})) - e^{(r_d - r_f)\tau}\mathcal{N}(-\eta\phi b_1)\right]\Bigg\}. \end{aligned} \tag{1.227}$$

This value function has a removable discontinuity at $h = 0$ where it turns out to be

$$\begin{aligned} v(t,x) = \phi \Bigg\{ & x e^{-r_f \tau}\mathcal{N}(\phi b_1) - K e^{-r_d \tau}\mathcal{N}(\phi b_2) + \frac{1-\eta}{2}\phi e^{-r_d\tau}[\phi(R-X)]^+ \\ & + \eta x e^{-r_d\tau}\sigma\sqrt{\tau}\left[-b_1\mathcal{N}(-\eta\phi b_1) + \eta\phi n(b_1)\right]\Bigg\}. \end{aligned} \tag{1.228}$$

The abbreviations we use are

$$t : \text{running time (in years)}, \tag{1.229}$$

$$x \stackrel{\Delta}{=} S_t : \text{known spot at time of evaluation}, \tag{1.230}$$

$$\tau \stackrel{\Delta}{=} T - t : \text{time to expiration (in years)}, \tag{1.231}$$

$$n(t) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}, \tag{1.232}$$

$$\mathcal{N}(x) \stackrel{\Delta}{=} \int_{-\infty}^{x} n(t)\,dt, \tag{1.233}$$

$$h \stackrel{\Delta}{=} \frac{2(r_d - r_f)}{\sigma^2}, \tag{1.234}$$

$$K \stackrel{\Delta}{=} \begin{cases} R & \text{floating strike lookback} \quad (X \le 0) \\ \bar{\eta}\min(\bar{\eta}X, \bar{\eta}R) & \text{fixed strike lookback} \quad (X > 0) \end{cases}, \tag{1.235}$$

$$\eta \stackrel{\Delta}{=} \begin{cases} +1 & \text{floating strike lookback} \quad (X \le 0) \\ -1 & \text{fixed strike lookback} \quad (X > 0) \end{cases}, \tag{1.236}$$

$$b_1 \stackrel{\Delta}{=} \frac{\ln\frac{x}{K} + (r_d - r_f + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{1.237}$$

$$b_2 \stackrel{\Delta}{=} b_1 - \sigma\sqrt{\tau}. \tag{1.238}$$

Note that this formula basically consists of that for a vanilla call (1st two terms) plus another term. Conze and Viswanathan also show closed form solutions for fixed strike lookback options and the variations mentioned above in [41]. Heynen and Kat develop equations for *Partial*

Table 1.24 Sample output data for lookback options

Payoff	Analytic model	Continuous
$M_T - S_T$	0.0231	0.0255
$S_T - m_T$	0.0310	0.0320
$(M_T - 0.99)^+$	0.0107	0.0131
$(0.97 - m_T)^+$	0.0235	0.0246

For the input data we used spot $S_0 = 0.8900$, $r_d = 3\,\%$, $r_f = 6\,\%$, $\sigma = 10\,\%$, $\tau = \frac{1}{12}$, running min = 0.9500, running max = 0.9900, $m = 22$. We find the analytic results in the continuous case in agreement with the ones published in [47]. We can also reproduce the numerical results for the discretely sampled floating strike lookback put contained in [48].

Fixed and Floating Strike Lookback Options in [45]. For those preferring the PDE-approach of deriving formulae, we refer to [46]. For most practical matters, where we have to deal with fixings and lookback features in combination with averaging, the only reasonable valuation technique is Monte Carlo simulation.

Example
We list some sample results in Table 1.24.

Sensitivity analysis

Delta

$$v_x(t,x) = \phi \Bigg\{ e^{-r_f\tau}\mathcal{N}(\phi b_1) + \eta e^{-r_d\tau}\frac{1}{h}\left[\left(\frac{x}{K}\right)^{-h}\mathcal{N}(-\eta\phi(b_1 - h\sigma\sqrt{\tau}))(1-h) - e^{(r_d-r_f)\tau}\mathcal{N}(-\eta\phi b_1)\right]\Bigg\} \quad (1.239)$$

At $h = 0$ this simplifies to

$$v_x(t,x) = \phi \left\{ e^{-r_f\tau}\mathcal{N}(\phi b_1) + \eta e^{-r_d\tau}\left[\sigma\sqrt{\tau}\,[-b_1\mathcal{N}(-\eta\phi b_1) + \eta\phi n(b_1)] - \mathcal{N}(-\eta\phi b_1)\right]\right\} \quad (1.240)$$

Gamma

$$v_{xx}(t,x) = \frac{2e^{-r_f\tau}}{x\sigma\sqrt{\tau}}n(b_1) - \phi\eta e^{-r_d\tau}\frac{1-h}{x}\mathcal{N}(-\phi\eta(b_1 - h\sigma\sqrt{\tau})) \quad (1.241)$$

Theta

We can use the Black-Scholes partial differential equation to obtain theta from value, delta and gamma.

vega

$$v_\sigma(t,x) = \phi\eta x e^{-r_d\tau}\frac{2}{\sigma}\left[\left(\frac{x}{K}\right)^{-h}\mathcal{N}(-\eta\phi(b_1 - h\sigma\sqrt{\tau}))\left(\frac{1}{h} + \ln\frac{x}{K}\right) - e^{(r_d-r_f)\tau}\frac{1}{h}\mathcal{N}(-\eta\phi b_1)\right] \quad (1.242)$$

At $h = 0$ this simplifies to

$$v(t, x) = \phi\eta x e^{-r_d\tau}\sqrt{\tau}\left[-\sigma\sqrt{\tau}b_1\mathcal{N}(-\eta\phi b_1) + 2\eta\phi n(b_1)\right] \tag{1.243}$$

Discrete sampling

In practice, one cannot take the average over a continuum of exchange rates. The standard is to specify a *fixing calendar* and take only a finite number of fixings into account. Suppose there are m equidistant sample points left until expiration at which we evaluate the extremum. In this case the value can be determined by an approximation described in [49]. We set

$$\beta_1 = 0.5826 = -\zeta(1/2)/\sqrt{2\pi}, \tag{1.244}$$

$$a = e^{\phi\beta_1\sigma\sqrt{\tau/m}}, \tag{1.245}$$

and obtain for fixed strike lookback options

$$v(t, x, r_d, r_f, \sigma, R, X, \phi, \bar{\eta}, m) = v(t, x, r_d, r_f, \sigma, aR, aX, \phi, \bar{\eta})/a, \tag{1.246}$$

and for floating strike lookback options

$$v(t, x, r_d, r_f, \sigma, R, X, \phi, \bar{\eta}, m) = av(t, x, r_d, r_f, \sigma, R/a, X, \phi, \bar{\eta}) - \phi(a-1)xe^{-r_f\tau}. \tag{1.247}$$

One interesting observation is that when the options move deep in the money and have the same strike price, lookback options and vanilla options have the same value, except for extreme risk parameter inputs. This can be explained recalling that a floating strike lookback option has an exercise probability of 1 and buys (sells) at the minimum (maximum). When the strike price of a vanilla option equals the extremum of the exotic and is deep in the money, the holder of the option will also buy (sell) at the extremum with a probability very close to 1. Moreover, recall that the floating strike lookback option consists of a vanilla option and an additional term. Garman names this term a *strike-bonus option*, see [43]. It can be considered as an option that has an increased payoff whenever a new extremum is reached. When the underlying price moves very far away from the current extremum, the strike-bonus option has almost zero value.

The structure of the Greeks delta, rho, theta and vega is comparable for lookback and vanilla calls. Nonetheless, the intensity of these sensitivities against changes differs, see Figure 1.35. Close to or at the money, lookback calls have a significantly lower *delta* than their vanilla counterparts. The reason is that the strike-bonus option in the lookback call has a negative delta when the underlying value is close to the current extremum and a delta next to zero when it is far in the money. Intuitively, the lower Lookback delta is explained by the fact that the closer the current underlying value is to the extremum, the more likely is that the payoff of the lookback option remains unchanged, which is different for vanilla options where the payoff changes with every underlying movement. Note that whenever a new extremum is achieved, the payoff for a lookback option equals zero and remains unchanged until the underlying value moves into the adverse direction.

Floating strike lookback options have a lower *rho* than vanilla options (with equal strikes at the time of observation), which can be explained by the fact that the option holder needs to pay more up-front and thus has a lower principal profiting from favorable interest rate movements. As a rule of thumb, a floating strike lookback option is worth twice as much as a vanilla option.

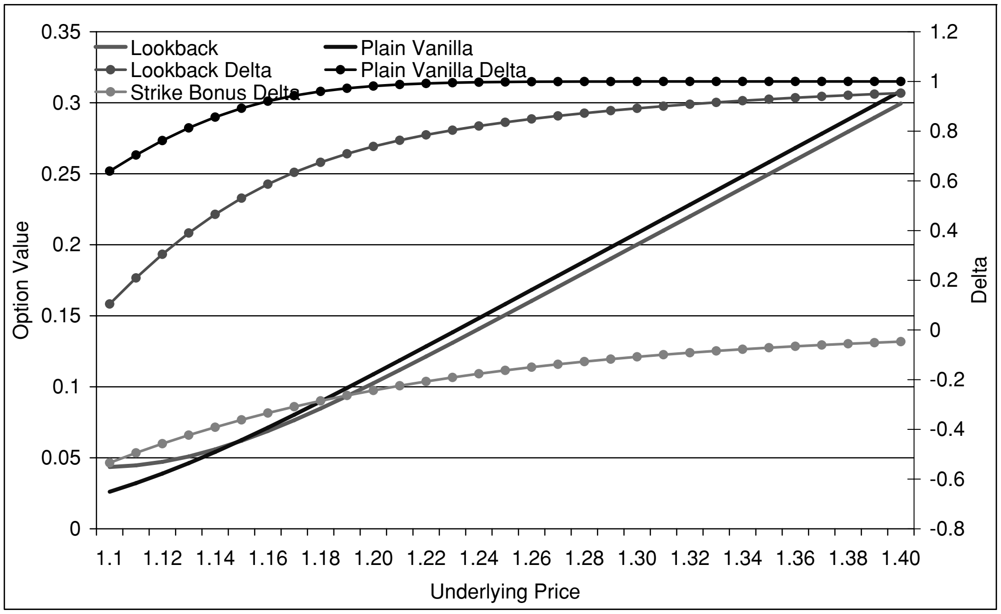

Figure 1.35 Vanilla and lookback call (left hand scale) with deltas (right hand scale) using $\min_t S_t = 1.00$. The Lookback delta equals the sum of the delta of a vanilla option plus the delta of a strike bonus option.

The longer the time to maturity, the more intensively floating strike lookback options react compared to vanilla options.

The higher *theta* for lookback options reflects the fact that the optimal value achieved to date is "locked in" and the longer the time to maturity, the higher the chance to lock in an even better extremum.

Regarding the *vega*, lookback options show a stronger reaction than regular options. The higher the volatility of the underlying, the higher the probability to reach a new extremum. Moreover, having "locked in" this new extremum the option value can benefit even more from the higher chance of adverse price movements.

As pointed out by Taleb in [50], one particularly interesting risk parameter is the *gamma* since it is *one-sided*, while the vanilla gamma changes symmetrically for up- and down movements of the underlying, see Figure 1.36. A lookback option always has its maximum gamma at the extremum which can move over time. Vanilla options, however, have their maximum gamma at the strike price. The *lookback gamma asymmetry* indicates that gamma risk cannot be consistently (statically) hedged with vanilla options. The fact that gamma is considerably higher for lookback options implies that a frequent rebalancing of the hedging portfolio and hence high transaction costs are likely, see [51].

Hedging

Due to the maximum (minimum) function that allows the strike price to change there exists no pure static hedging strategy for floating strike lookback options. Instead, a *semi-static rollover*

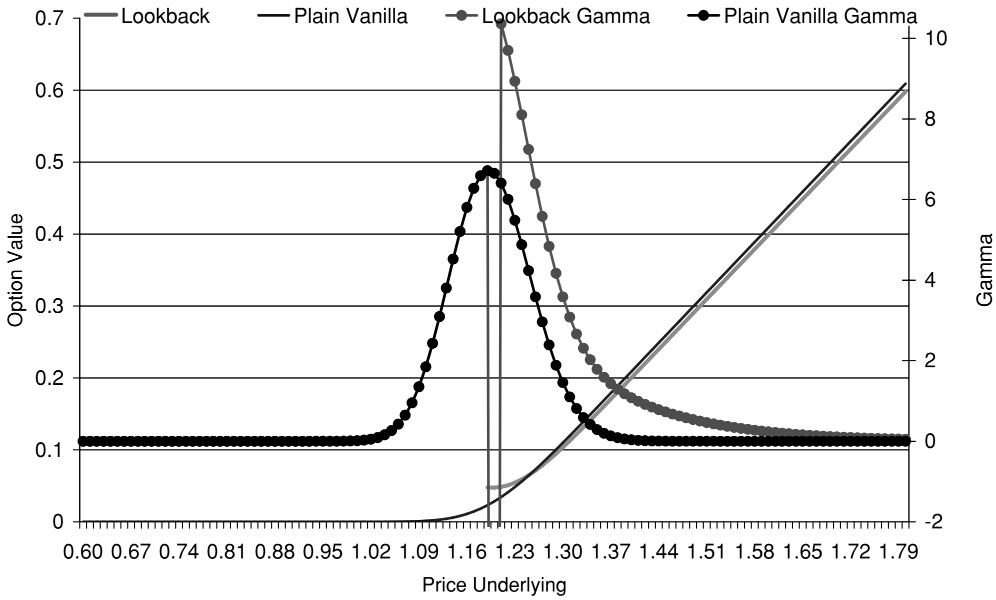

Figure 1.36 Value (left hand scale) and gamma (right hand scale) of an at-the-money floating strike lookback and a vanilla call

strategy can be applied, see [43]. As can be read from the value formula (1.227), we can hedge parts with a vanilla option. Whenever the maximum (minimum) changes, the writer of the option buys a new put (call) struck at the current market price and sells the old put (call). However, this does not work without costs. While the new put (call) is at-the-money, the old put (call) is out-of-the-money at the time of the sale and hence returns less money than the amount necessary to purchase the new option. We encounter vanilla option bid-ask spreads, and smile risk. The strike-bonus option returns exactly the money that is needed for the rollover. This approach, however, is rather theoretic, since strike-bonus options are hardly available in the market.

In practice, floating strike lookback options are usually hedged with a straddle, see Section 1.4.4. Cunningham and Karumanchi also explain a hedging strategy for fixed strike lookback options in [51]. The straddle to use is a combination of a vanilla put and a vanilla call, which have a term to maturity equal to that of the lookback option to be hedged, and a strike equal to the maximum (minimum) achieved by the underlying. At maturity T, the call (put) of this straddle becomes worthless since the strike is below (above) the terminal stock price S_T. The remaining put (call) exactly satisfies the obligation of the lookback option (see Figure 1.37). Over the lifetime of the option, the strike price of the straddle needs to be adapted if the current exchange rate S_t rises above (falls below) the current maximum (minimum). Regarding the intrinsic value, the holder of the hedging portfolio will not lose money since for instance the intrinsic value lost by the put will be exactly gained by the call. However, the deltas of the two options differ, not only in their sign. In addition, attempting to create a hedging portfolio with zero delta, the hedger has to buy a certain number of puts per one call. Figure 1.37 shows that for the latter two reasons this is not a self-financing hedging strategy.

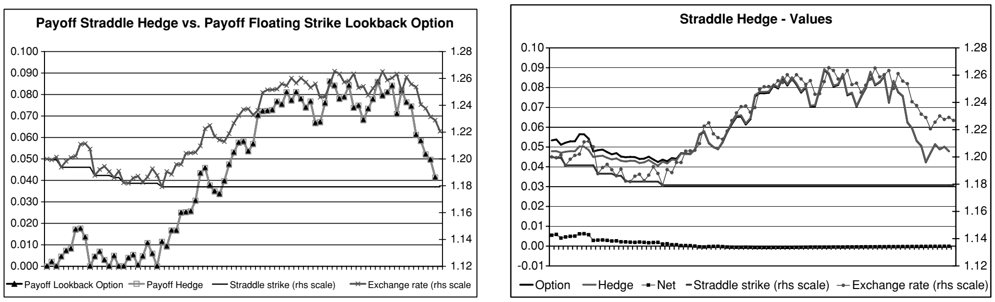

Figure 1.37 Comparison of the payoffs of a floating strike lookback option and a vanilla straddle (lhs) and the values of the positions (rhs) for a random time path (Exchange rate and straddle strike on the rhs scale)

Note that the strategy would not be self-financing even if the straddle was not adapted for a zero delta of the position. The definition of a *re-hedge threshold* and a maximum number of trades per period can help to balance the risk taken with transaction costs and administrative efforts.

Apart from this *semi-static*[6] hedge, a *dynamic hedge* using spot and money market is also possible. Due to the risk parameters, especially gamma and vega, which are difficult to hedge, the hedge appears to deviate considerably in value relative to the option over time.

1.5.6 Forward start, ratchet and cliquet options

A forward start vanilla option is just like a vanilla option, except that the strike is set on a future date t. It pays off

$$[\phi(S_T - K)]^+, \tag{1.248}$$

where K denotes the strike and ϕ takes the values $+1$ for a call and -1 for a put. The strike K is set as αS_t at time $t \in [0, T]$. Very commonly α is set to one.

Advantages

- Protection against spot market movement and against increasing volatility
- Buyer can lock in current volatility level
- Spot risk easy to hedge

Disadvantages

- Protection level not known in advance

[6] We refer to this technique as semi-static since the basic idea of the hedge is that of a static one: Initialize the hedge and wait until maturity. However, due to the changes of the extrema, the static hedge has to be adapted – a characteristic which is usually associated with dynamic hedging.

The value of forward start options

Using the abbreviations

- x for the current spot price of the underlying,
- $\tau \stackrel{\Delta}{=} T - t$,
- $F_s \stackrel{\Delta}{=} I\!E[S_s | S_0] = S_0 e^{(r_d - r_f)s}$ for the outright forward of the underlying,
- $\theta_\pm \stackrel{\Delta}{=} \frac{r_d - r_f}{\sigma} \pm \frac{\sigma}{2}$,
- $d_\pm \stackrel{\Delta}{=} \frac{\ln \frac{x}{K} + \sigma\theta_\pm \tau}{\sigma\sqrt{\tau}} = \frac{\ln \frac{f}{K} \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}$,
- $d_\pm^\alpha \stackrel{\Delta}{=} \frac{-\ln \alpha + \sigma\theta_\pm \tau}{\sigma\sqrt{\tau}}$,
- $n(t) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} = n(-t)$,
- $\mathcal{N}(x) \stackrel{\Delta}{=} \int_{-\infty}^{x} n(t)\,dt = 1 - \mathcal{N}(-x)$,

we recall the value of a vanilla option in Equation (1.7),

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = \phi e^{-r_d \tau}[f\mathcal{N}(\phi d_+) - K\mathcal{N}(\phi d_-)]. \tag{1.249}$$

For the value of a forward start vanilla option in a constant-coefficient geometric Brownian motion model we obtain

$$\begin{aligned} v &= e^{-r_d t} I\!E\, v(S_t, K = \alpha S_t, T, t, \sigma, r_d, r_f, \phi) \\ &= \phi e^{-r_d T}[F_T \mathcal{N}(\phi d_+^\alpha) - \alpha F_t \mathcal{N}(\phi d_-^\alpha)]. \end{aligned} \tag{1.250}$$

Noticeably, the value computation is easy here, because the strike K is set as a *multiple* of the future spot. If we were to choose to set the strike as a constant *difference* of the future spot, the integration would not work in closed form, and we would have to use numerical integration.

The crucial pricing issue here is that one needs to know the volatility, which is the *forward volatility*, i.e. the volatility that will materialize at the future time t for a maturity $T - t$. It is not at all clear in the market which proxy to take for this forward volatility. The standard is to use Equation (1.102).

Greeks

The Greeks are the same as for vanilla options after time t, when the strike has been set. Before time t they are given by

(Spot) delta

$$\frac{\partial v}{\partial S_0} = \frac{v}{S_0} \tag{1.251}$$

Gamma

$$\frac{\partial^2 v}{\partial x^2} = 0 \tag{1.252}$$

Theta

$$\frac{\partial v}{\partial t} = r_f v \tag{1.253}$$

Table 1.25 Value and Greeks of a forward start vanilla in USD on EUR/USD – spot of 0.9000, $\alpha = 99\,\%$, $\sigma = 12\,\%$, $r_d = 2\,\%$, $r_f = 3\,\%$, maturity $T = 186$ days, strike set at $t = 90$ days

	Call	Put
value	0.0251	0.0185
delta	0.0279	0.0206
gamma	0.0000	0.0000
theta	0.0007	0.0005
vega	0.1793	0.1793
rhod	0.1217	−0.1052
rhof	−0.1329	0.0950

Vega

$$\frac{\partial v}{\partial \sigma} = -\frac{e^{-r_d T}}{\sigma}[F_T n(d_+^\alpha)d_-^\alpha - \alpha F_t n(d_-^\alpha)d_+^\alpha] \quad (1.254)$$

Rho

$$\frac{\partial v}{\partial r_d} = \phi e^{-r_d T} \alpha F_t (T - t)\mathcal{N}(\phi d_-^\alpha) \quad (1.255)$$

$$\frac{\partial v}{\partial r_f} = -T v - \phi e^{-r_d T} \alpha F_t (T - t)\mathcal{N}(\phi d_-^\alpha) \quad (1.256)$$

Example
We consider an example in Table 1.25.

Reasons for trading forward start options

The key reason for trading a forward start is trading the forward volatility without any spot exposure. In quiet market phases with low volatility, buying a forward start is cheap. Keeping a long position will allow participation in rising volatility, independent of the spot level.

Variations

Forward start options can be altered in all kind of ways: they can be of American style, they can come with a deferred delivery or deferred premium, they can have barriers or appear as a strip.

A strip of forward start options is generally called a *Cliquet*.

A *Ratchet Option* consists of a series of forward start options, where the strike for the next forward start option is set equal to the spot at maturity of the previous.

1.5.7 Power options

This section is produced in conjunction with Silvia Baumann, Marion Linck, Michael Mohr and Michael Seeberg.

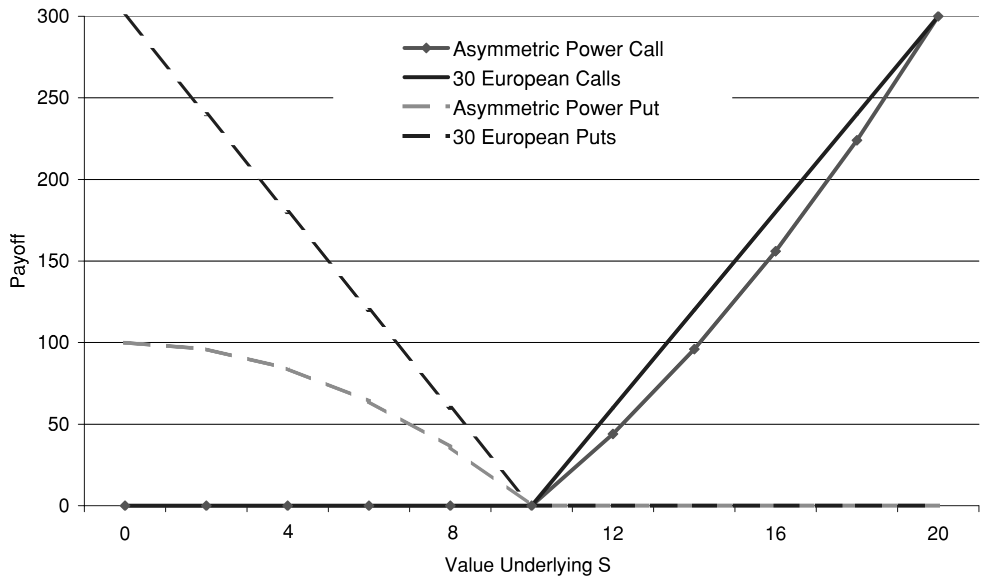

Figure 1.38 Payoff of asymmetric power options vs. vanilla options, using $K = 10$, $n = 2$

For power options, the vanilla option payoff function $[\phi(S_T - K)]^+$ is adjusted by raising the entire function or parts of the function to the n-th power, see, e.g. Zhang ([55]). The result is a non-linear profile with the potential of a higher payoff at maturity with a greater leverage than standard options. If the exponent n is exactly 1, the option is equal to a vanilla option. We distinguish between *asymmetric* and *symmetric* power options. Their payoffs in comparison with vanilla options are illustrated in Figures 1.38 and 1.39.

Asymmetric power options

With an asymmetric power option, the underlying S_T and strike K of a standard option payoff function are individually raised to the n-th power,

$$[\phi(S_T^n - K^n)]^+. \tag{1.257}$$

Figure 1.38 illustrates why this option type is called *asymmetric*. With increasing S_T, the convex call payoff grows exponentially. Given the limited and fixed profit potential of $K^2 = 10^2$, the concave put payoff decreases exponentially. It requires 30 vanilla options to replicate the call payoff if the underlying S_T moves to 20.

Asymmetric power options

In the symmetric type, the entire vanilla option payoff is raised to the n-th power,

$$[[\phi(S_T - K)]^+]^n, \tag{1.258}$$

see [52]. Figure 1.39 insinuates naming this option type *symmetric*, since put and call display the same payoff shape. Here, 10 vanilla options suffice to replicate the symmetric power option if the underlying S_T moves to 20.

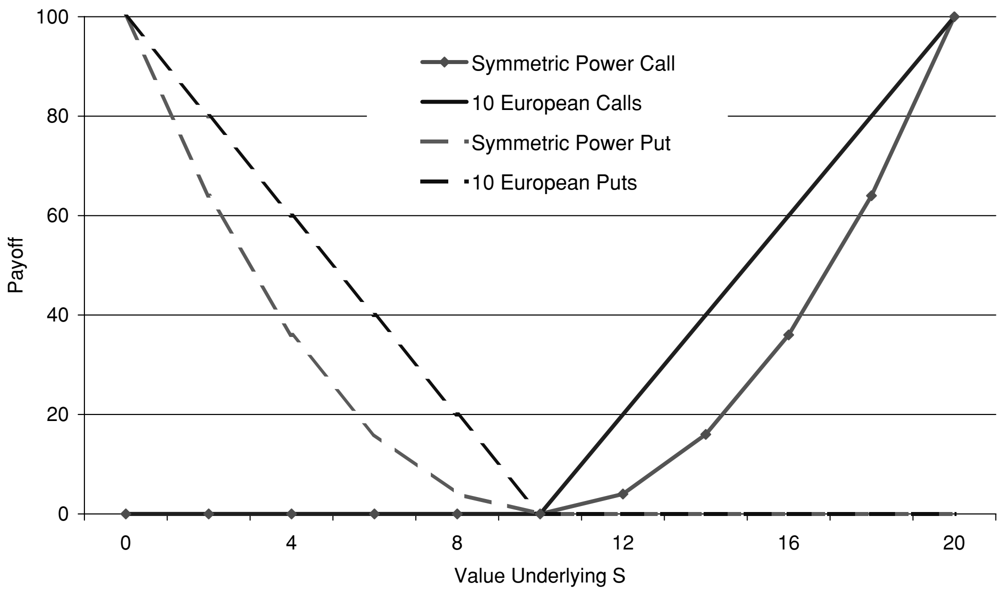

Figure 1.39 Payoff of symmetric power options vs. vanilla options, using $K = 10$, $n = 2$

Combining a symmetric power call and put as in Figure 1.39 leads to a symmetric power straddle, which pays

$$|S_T - K|^n. \tag{1.259}$$

Reasons for trading power options

Power options are often equipped with a payoff cap C to limit the short position risk as well as the option premium for the buyer. For example, the payments of the short position at $n = 3$ for $K = 10$ shoots to 2375(125) for the asymmetric (symmetric) power call if S_T moves to 15. Even with cap, the highly leveraged payoff motivates speculators to invest in the product that demands a considerably higher option premium than a vanilla option. Power options are mostly popular in the listed derivatives and retail market, due to their high leverage and due to their mere name. Besides this obvious reason additionally one can think of the following motives:

1. Hedging future levels of implied volatility. Vega, which is volatility risk, is extremely difficult to hedge as there is no directly observable measure available, see [53]. A power straddle is an effective instrument to do so as it preserves the volatility exposure better than a vanilla straddle when the price of the underlying moves significantly as shown below in the section on sensitivities to risk parameters.
2. Through their exponential, non-linear payoff, power options can hedge non-linear price risks. An example is an importer earning profits merely through a percentage mark-up on imported products. An exchange rate change will lead to a price change, which in turn may affect demand volumes. The importer faces a risk of non-linearly decreasing earnings, see [54].
3. With very large short positions in vanilla options, a rebalancing of a dynamic hedge may require such massive buying (selling) of the underlying that this impacts the price of the

underlying, which in turn requires further hedge adjustments and may "pin" the underlying to the strike price, see p. 37 in [52]. To *smooth this pin risk*, option sellers propose a *soft strike option* with a similarly smooth and continuous payoff curvature as power options. As we will show in the hedging analysis of this section, this payoff curvature can be effectively replicated using vanilla options with different strike prices. The *diversified* range of strikes then softens any effect of a move in the underlying price. For details on *soft strike options* see [52], pp. 37 and [54], pp. 51.

Valuation of the asymmetric power option

The value can be written as the expected payoff value under the risk neutral measure. Using the money market numeraire $e^{-r_d T}$ yields

$$\text{asymmetric power option value } v_{aPC} = e^{-r_d T}\, I\!E\left[\phi(S_T^n - K^n) I\!I_{\{\phi S_T > \phi K\}}\right]. \tag{1.260}$$

As K is a constant S_T is the only random variable which simplifies the equation to

$$v_{aPC} = \phi e^{-r_d T}\, I\!E\left[S_T^n I\!I_{\{\phi S_T > \phi K\}}\right] - \phi e^{-r_d T} K^n\, I\!E\left[I\!I_{\{\phi S_T > \phi K\}}\right]. \tag{1.261}$$

The expectation of an indicator function is just the probability that the event $\{S_T > K\}$ occurs. In the Black-Scholes model, S_T is log-normally distributed and evolves according to a geometric Brownian motion (1.1). Itô's Lemma implies that S_T^n is also a geometric Brownian motion following

$$dS_t^n = \left[n(r_d - r_f) + \frac{1}{2}n(n-1)\sigma^2\right] S_t^n\,dt + n\sigma S_t^n\,dW_t. \tag{1.262}$$

Solving the differential equation and calculating the expected value in Equation (1.261) leads to the desired closed form solution

$$v_{aPC} = \phi e^{-r_d T}\left[f^n e^{\frac{1}{2}n(n-1)\sigma^2 T}\mathcal{N}(\phi d_+^n) - K^n \mathcal{N}(\phi d_-)\right], \tag{1.263}$$

$$f \stackrel{\Delta}{=} S_0 e^{(r_d - r_f)T},$$

$$d_- \stackrel{\Delta}{=} \frac{\ln\frac{f}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}},$$

$$d_+^n \stackrel{\Delta}{=} \frac{\ln\frac{f}{K} + \left(n - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

Valuation of the symmetric power option

Due to the binomial term $(S_T - K)^n$ the general value formula derivation for the symmetric version looks more complicated. That is why a more intuitive approach is taken and the valuation logic is shown based on the asymmetric option discussed above. Taking the example of $n = 2$ the difference between asymmetric and symmetric call is

$$[S_T^2 - K^2] - [S_T^2 - 2S_T K + K^2] = 2K(S_T - K). \tag{1.264}$$

The symmetric version for $n = 2$ is thus exactly equal to the asymmetric power option minus $2K$ vanilla options. This way pricing and hedging the symmetric power option becomes a structuring exercise, see Figure 1.40.

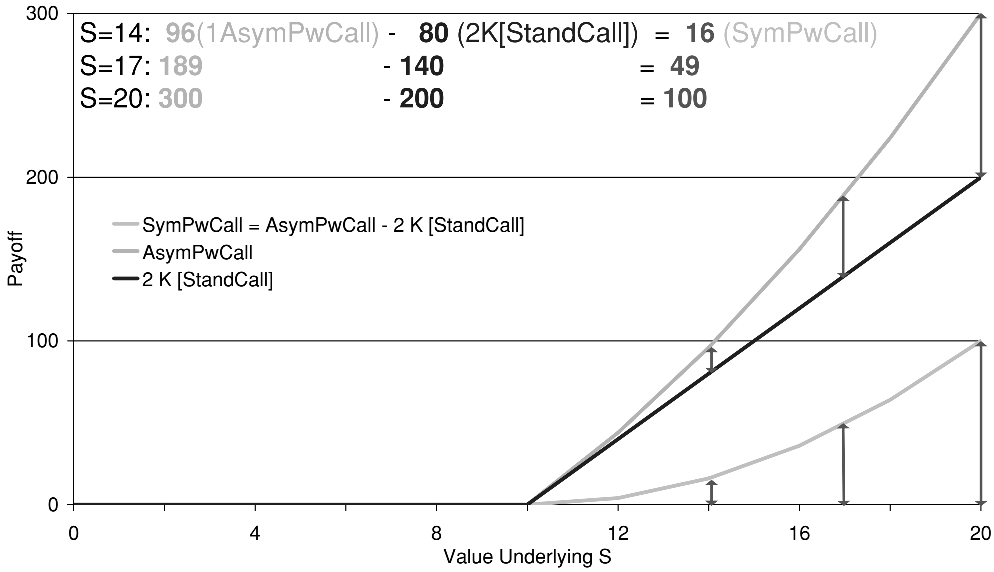

Figure 1.40 Symmetric power call replicated with asymmetric power and vanilla calls, using $K = 10$, $n = 2$

Tompkins and Zhang both discuss the more complicated derivation of the general formula for symmetric power options in [52] and [55]. Tompkins also presents a formula for symmetric power straddles for $n = 2$.

Sensitivity analysis

Looking at the *Greeks* of asymmetric power options compared to vanilla options, the exponential elements of power options are well reflected in the exposures. This is especially true for delta and gamma as can be seen in Figure 1.41, but is also valid for theta and vega. The power option rhos are very similar to the vanilla version.

Contrary to the asymmetric power option, the symmetric power option sensitivities exhibit new features that cannot be found with vanilla options, namely extreme delta values and a

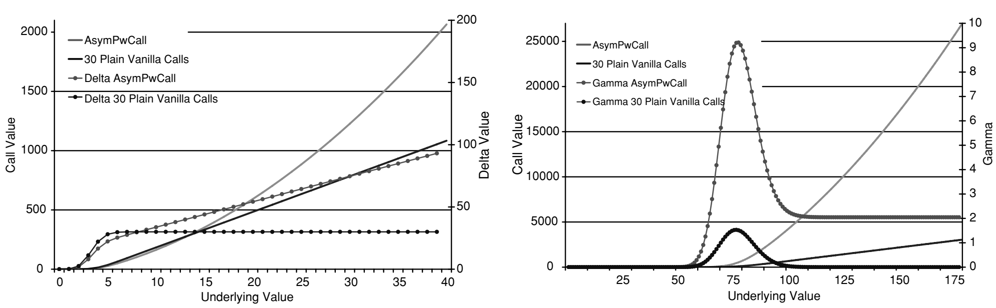

Figure 1.41 Asymmetric power call and vanilla call value and delta (lhs) and gamma (rhs) in relation to the underlying price, using $K = 10$, $n = 2$, $\sigma = 20\,\%$, $r_d = 5\,\%$, $r_f = 0\,\%$, $T = 90$ days

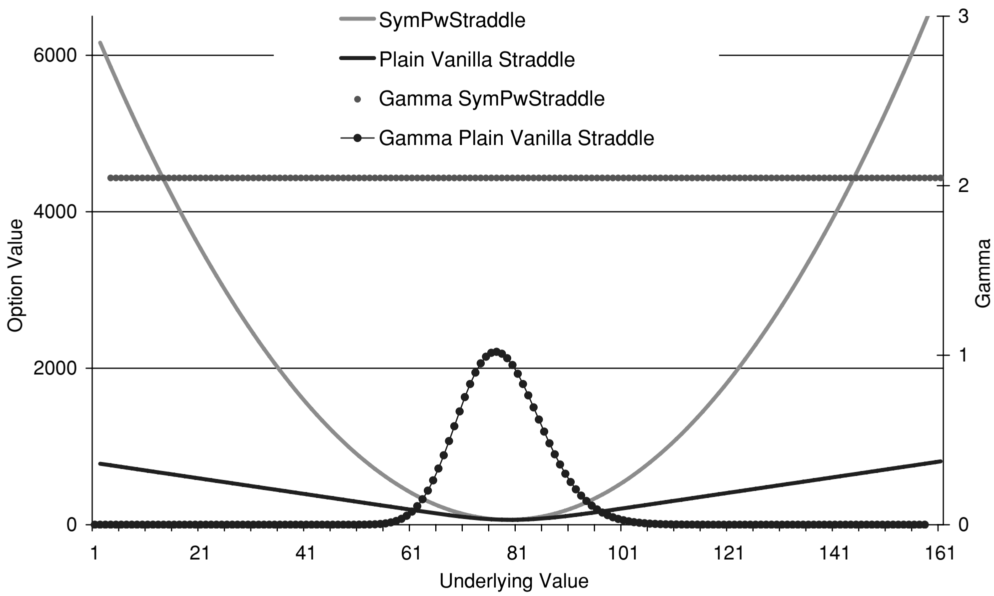

Figure 1.42 Gamma exposure of a symmetric power versus vanilla straddle, using $K = 80$ (at-the-money), $n = 2$, $\sigma = 20\,\%$, $r_d = 5\,\%$, $r_f = 0\,\%$, $T = 90$ days

gamma that resembles the plain vanilla delta. In the form of a straddle this creates a *constant gamma exposure*, see Figure 1.42.

At the same time, if the underlying increases significantly, the symmetric power straddle is able to preserve the exposure to volatility, whereas the vanilla straddle value becomes more and more invariant to the volatility input. Therefore, the power straddle is useful to hedge implied volatility, see Figure 1.43.

Hedging

The insights from the option payoffs, valuation, and the sensitivity analysis provide an effective static hedging strategy for both asymmetric as well as symmetric power options. The respective call values are considered as an example.

Static hedging

The continuous curvature of a power option can be approximated piecewise, adding up linear payoffs of vanilla options with different strike prices, see [52].

The symmetric power call for $n = 2$ is, as explained in the pricing section, just an asymmetric power call less $2K$ vanilla options.

The vanilla option hedge as a piecewise linear approximation is a natural upper boundary for the option price as it overestimates the option value, see Table 1.27. The complexity of a static hedge increases enormously with higher values of n. For the above example, a package of 25499 (3439) vanilla options is required to hedge one asymmetric (symmetric) power call. Overall, the static hedge strategy works very well as can be seen in Figure 1.44.

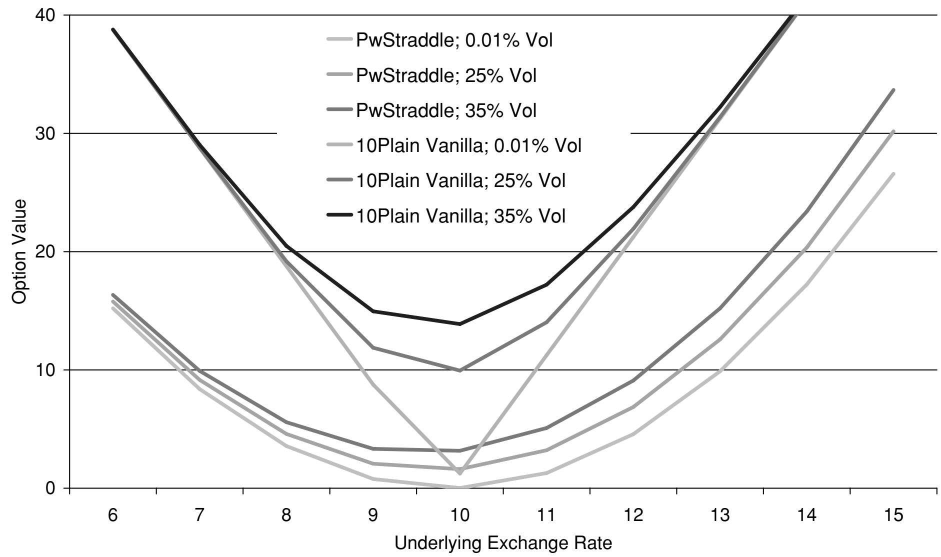

Figure 1.43 Vega exposure of a symmetric power versus vanilla straddle, using $K = 10$ (at-the-money), $n = 2$, $\sigma = 20\,\%$, $r_d = 5\,\%$, $r_f = 0\,\%$, $T = 90$ days

The static hedge of power options with vanillas takes into account the smile correctly, whence the price of the static hedge portfolio can serve as a market price of the power. All the sensitivities are automatically correctly hedged.

Dynamic hedging

A dynamic hedge involves setting up a position in the underlying and cash that offsets any value change in the option position. Usually the difficulty of dynamic hedging lies in second order, that is in gamma, and in vega risk. As the symmetric power straddle has a constant gamma, this simplifies delta hedging activities. In practice however, a static hedge is preferred

Table 1.26 Static hedging for the symmetric power call, using $K = 20$, $n = 2$

Underlying Price		10	11	12	13	14	15	16	17	18	19	20
Asym. Power Call		0	21	44	69	96	125	156	189	224	261	300
Vanilla Call Package	Sum	0	21	44	69	96	125	156	189	224	261	300
Package Components	Strike											
$2K$ Standard Calls	10		20	40	60	80	100	120	140	160	180	200
One Standard Call	10		1	2	3	4	5	6	7	8	9	10
Two Standard Calls	11			2	4	6	8	10	12	14	16	18
Two Standard Calls	12				2	4	6	8	10	12	14	16
Two Standard Calls	13					2	4	6	8	10	12	14
Two Standard Calls	14						2	4	6	8	10	12
Two Standard Calls	15							2	4	6	8	10
Two Standard Calls	16								2	4	6	8
Two Standard Calls	17									2	4	6
Two Standard Calls	18										2	4
Two Standard Calls	19											2

Table 1.27 Asymmetric power call hedge versus formula value, using $K = 10$ (at-the-money), $n = 2$, $\sigma = 15\,\%$, $r_d = 5\,\%$, $r_f = 0\,\%$, $T = 90$ days

Asym. Power Call	Formula Value	11.91
Vanilla Call Package	Sum	12
Package Components	Strike	
$2K$ Standard Calls	10	11.0354
One Standard Call	10	0.55177
Two Standard Calls	11	0.33257
Two Standard Calls	12	0.06928
Two Standard Calls	13	0.01034
Two Standard Calls	14	0.00116
Two Standard Calls	15	0.00010
Two Standard Calls	16	7.55E-06
Two Standard Calls	17	4.74E-07
Two Standard Calls	18	2.64E-08
Two Standard Calls	19	1.33E-09

as it allows banks packaging and thus hedging deeply out-of-the-money vanilla options, which are part of the options portfolio anyway.

1.5.8 Quanto options

A quanto option can be any cash-settled option, whose payoff is converted into a third currency at maturity at a pre-specified rate, called the *quanto factor*. There can be quanto plain vanilla,

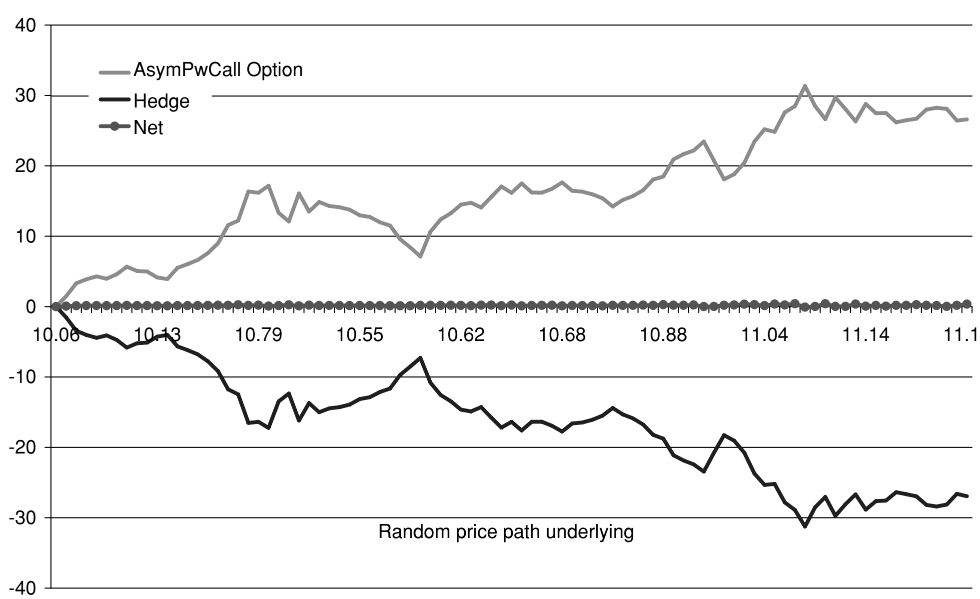

Figure 1.44 Static hedge performance of an asymmetric power call, using $K = 10$, $n = 2$, $\sigma = 150\,\%$, $r_d = 5\,\%$, $r_f = 0\,\%$, $T = 90$ days.

quanto barriers, quanto forward starts, quanto corridors, etc. The valuation theory is covered for example in [2] and [3]. We treat the example of a self-quanto forward in the exercises.

FX quanto drift adjustment

We take the example of a Gold contract with underlying XAU/USD in XAU-USD quotation that is quantoed into EUR. Since the payoff is in EUR, we let EUR be the numeraire or domestic or base currency and consider a Black-Scholes model

$$\text{XAU-EUR: } dS_t^{(3)} = (r_{EUR} - r_{XAU})S_t^{(3)}\,dt + \sigma_3 S_t^{(3)}\,dW_t^{(3)}, \tag{1.265}$$

$$\text{USD-EUR: } dS_t^{(2)} = (r_{EUR} - r_{USD})S_t^{(2)}\,dt + \sigma_2 S_t^{(2)}\,dW_t^{(2)}, \tag{1.266}$$

$$dW_t^{(3)} dW_t^{(2)} = -\rho_{23}\,dt, \tag{1.267}$$

where we use a minus sign in front of the correlation, because both $S^{(3)}$ and $S^{(2)}$ have the same base currency (DOM), which is EUR in this case. The scenario is displayed in Figure 1.45. The actual underlying is then

$$\text{XAU-USD: } S_t^{(1)} = \frac{S_t^{(3)}}{S_t^{(2)}}. \tag{1.268}$$

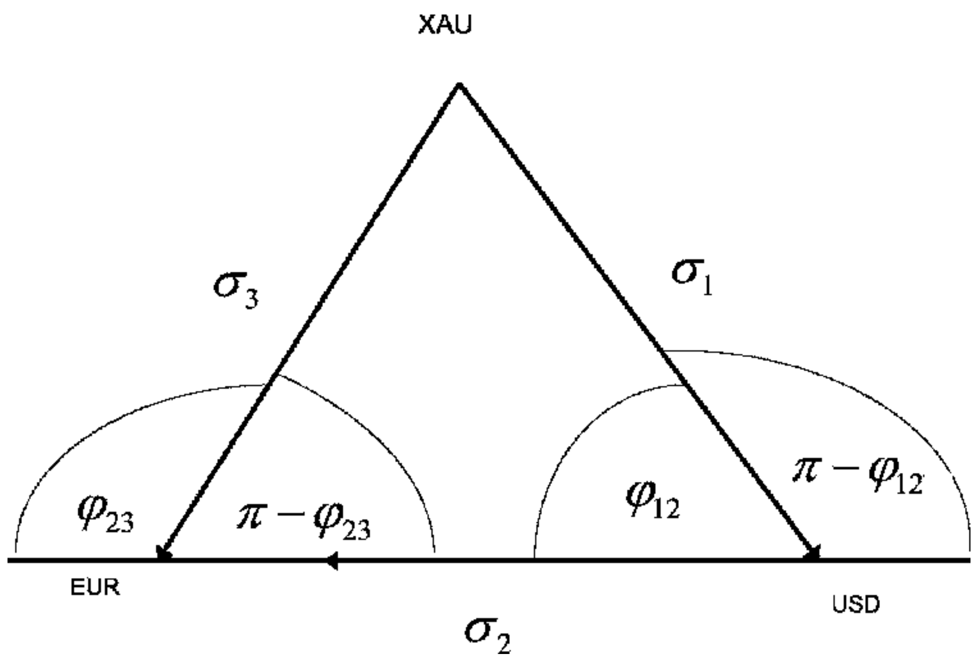

Figure 1.45 XAU-USD-EUR FX Quanto Triangle
The arrows point in the direction of the respective base currencies. The length of the edges represents the volatility. The cosine of the angles $\cos\phi_{ij} = \rho_{ij}$ represents the correlation of the currency pairs $S^{(i)}$ and $S^{(j)}$, *if* the base currency (DOM) of $S^{(i)}$ is the underlying currency (FOR) of $S^{(j)}$. If both $S^{(i)}$ and $S^{(j)}$ have the same base currency (DOM), then the correlation is denoted by $-\rho_{ij} = \cos(\pi - \phi_{ij})$.

Using Itô's formula, we first obtain

$$d\frac{1}{S_t^{(2)}} = -\frac{1}{(S_t^{(2)})^2}\,dS_t^{(2)} + \frac{1}{2}\cdot 2\cdot\frac{1}{(S_t^{(2)})^3}\,(dS_t^{(2)})^2$$
$$= (r_{USD} - r_{EUR} + \sigma_2^2)\frac{1}{S_t^{(2)}}\,dt - \sigma_2\frac{1}{S_t^{(2)}}\,dW_t^{(2)}, \tag{1.269}$$

and hence

$$\begin{aligned} dS_t^{(1)} &= \frac{1}{S_t^{(2)}}\,dS_t^{(3)} + S_t^{(3)}\,d\frac{1}{S_t^{(2)}} + dS_t^{(3)}\,d\frac{1}{S_t^{(2)}} \\ &= \frac{S_t^{(3)}}{S_t^{(2)}}(r_{EUR} - r_{XAU})\,dt + \frac{S_t^{(3)}}{S_t^{(2)}}\sigma_3\,dW_t^{(3)} \\ &\quad + \frac{S_t^{(3)}}{S_t^{(2)}}(r_{USD} - r_{EUR} + \sigma_2^2)\,dt + \frac{S_t^{(3)}}{S_t^{(2)}}\sigma_2\,dW_t^{(2)} + \frac{S_t^{(3)}}{S_t^{(2)}}\rho_{23}\sigma_2\sigma_3\,dt \\ &= (r_{USD} - r_{XAU} + \sigma_2^2 + \rho_{23}\sigma_2\sigma_3)S_t^{(1)}\,dt + S_t^{(1)}(\sigma_3\,dW_t^{(3)} + \sigma_2\,dW_t^{(2)}). \end{aligned}$$

Since $S_t^{(1)}$ is a geometric Brownian motion with volatility σ_1, we introduce a new Brownian motion $W_t^{(1)}$ and find

$$dS_t^{(1)} = (r_{USD} - r_{XAU} + \sigma_2^2 + \rho_{23}\sigma_2\sigma_3)S_t^{(1)}\,dt + \sigma_1 S_t^{(1)}\,dW_t^{(1)}. \tag{1.270}$$

Now Figure 1.45 and the *law of cosine* imply

$$\sigma_3^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2, \tag{1.271}$$
$$\sigma_1^2 = \sigma_2^2 + \sigma_3^2 + 2\rho_{23}\sigma_2\sigma_3, \tag{1.272}$$

which yields

$$\sigma_2^2 + \rho_{23}\sigma_2\sigma_3 = \rho_{12}\sigma_1\sigma_2. \tag{1.273}$$

As explained in Figure 1.45, ρ_{12} is the correlation between XAU-USD and USD-EUR, whence $\rho \stackrel{\Delta}{=} -\rho_{12}$ is the correlation between XAU-USD and EUR-USD. Inserting this into Equation (1.270), we obtain the usual formula for the drift adjustment

$$dS_t^{(1)} = (r_{USD} - r_{XAU} - \rho\sigma_1\sigma_2)S_t^{(1)}\,dt + \sigma_1 S_t^{(1)}\,dW_t^{(1)}. \tag{1.274}$$

This is the risk-neutral process that can be used for the valuation of any derivative depending on $S_t^{(1)}$ which is quantoed into EUR.

Quanto vanilla

With these preparations we can easily determine the value of a vanilla quanto paying

$$Q[\phi(S_T - K)]^+, \tag{1.275}$$

where K denotes the strike, T the expiration time, ϕ the usual put-call indicator, S the underlying in FOR-DOM quotation and Q the quanto factor from the domestic currency into the quanto currency. We let

$$\tilde{\mu} \stackrel{\Delta}{=} r_d - r_f - \rho\sigma\tilde{\sigma}, \tag{1.276}$$

Table 1.28 Example of a quanto digital put

Notional	81,845 EUR
Maturity	3 months
European style Barrier	108.65 USD-JPY
Premium	60,180 EUR
including	1,000 EUR sales margin
Fixing source	ECB

The buyer receives 81,845 EUR if at maturity, the ECB fixing for USD-JPY (computed via EUR-JPY and EUR-USD) is below 108.65. Terms were created on January 12 2004

be the *adjusted drift*, where r_d and r_f denote the risk free rates of the domestic and foreign underlying currency pair respectively, σ the volatility of this currency pair, $\tilde{\sigma}$ the volatility of the currency pair DOM-QUANTO and ρ the correlation between the currency pairs FOR-DOM and DOM-QUANTO in this quotation. Furthermore we let r_Q be the risk free rate of the quanto currency.

Then the formula for the value can be written as

$$v = Qe^{-r_Q T}\phi[S_0 e^{\tilde{\mu}T}\mathcal{N}(\phi d_+) - K\mathcal{N}(\phi d_-)], \tag{1.277}$$

$$d_\pm = \frac{\ln\frac{S_0}{K} + \left(\tilde{\mu} \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}. \tag{1.278}$$

Example

We provide an example of European style digital put in USD/JPY quanto into EUR in Table 1.28.

Applications

The standard applications are performance linked deposit as in Section 2.3.2 or notes as in Section 2.5. Any time the performance of an underlying asset needs to be converted into the notional currency invested, and the exchange rate risk is with the seller, we need a quanto product. Naturally, an underlying like gold, which is quoted in USD, would be a default candidate for a quanto product, when the investment is in currency other than USD.

1.5.9 Exercises

1. Consider a EUR-USD market with spot at 1.2500, EUR rate at 2.5 %, USD rate at 2.0 %, volatility at 10.0 % and the situation of a treasurer expecting 1 Million USD in one year, that he wishes to change into EUR at the current spot rate ot 1.2500. In 6 months he will know if the company gets the definite order. Compute the price of a vanilla EUR call in EUR. Alternatively compute the price of a compound with two thirds of the total premium to be paid at inception and one third to be paid in 6 months. Do the same computations if the sales margin for the vanilla is 1 EUR per 1000 USD notional and for the compound is 2 EUR per 1000 USD notional. After six months the company ends up not getting the order and can waive its hedge. How much would it get for the vanilla if the spot is at 1.1500, at 1.2500 and at 1.3500? Would it be better for the treasurer to own the compound

and not pay the second premium? How would you split up the premia for the compound to persuade the treasurer to buy the compound rather than the vanilla? (After all there is more margin to earn.)

2. Find the fair price and the hedge of a *perpetual one-touch*, which pays 1 unit of the domestic currency if the barrier $H > S_0$ is ever hit, where S_0 denotes the current exchange rate. How about payment in the foreign currency? How about a *perpetual no-touch*? These thoughts are developed further to a *vanilla-one-touch duality* by Peter Carr [56].
3. Find the value of a *perpetual double-one-touch*, which pays a rebate R_H, if the spot reaches the higher level H before the lower level L, and R_L, if the spot reaches the lower level first. Consider as an example the EUR-USD market with a spot of S_0 at time zero between L and H. Let the interest rates of both EUR and USD be zero and the volatility be 10 %. The specified rebates are paid in USD. There is no finite expiration time, but the rebate is paid whenever one of the levels is reached. How do you hedge a short position?
4. A call (put) option is the right to buy (sell) one unit of an underlying asset (stock, commodity, foreign exchange) on a maturity date T at a pre-defined price K, called the strike price. An knock-out call with barrier B is like a call option that becomes worthless, if the underlying ever touches the barrier B at any time between inception of the trade and its expiration time. Let the market parameters be spot $S_0 = 120$, all interest and dividend rates be zero, volatility $\sigma = 10\,\%$. In a liquid and jump-free market, find the value of a one-year *strike-out*, i.e. a down-and-out knock-out call, where $K = B = 100$.

   Suppose now, that the spot price movement can have downward jumps, but the forward price is still constant and equal to the spot (since there are no interest or dividend payments). How do these possible jumps influence the value of the knock-out call?

   The solution to this problem is used for the design of *turbo notes*, see Section 2.5.4.
5. Consider a regular down-and-out call in a Black-Scholes model with constant drift μ and constant volatility σ. Suppose you are allowed to choose time dependent deterministic functions for the drift $\mu(t)$ and the volatility $\sigma(t)$ with the constraint that their average over time coincides with their constant values μ and σ. How can the function $\mu(t)$ be shaped to make the down-and-out call more expensive? How can the function $\sigma(t)$ be shaped to make the down-and-out call more expensive? Justify your answer.
6. Consider a regular up-and-out EUR put USD call with maturity of 6 months. Consider the volatilities for all maturities, monthly up to 6 months. In a scenario with EUR rates lower than USD rates, describe the term structure of vega, i.e. what happens to the value if the k month volatility goes up for $k = 1, 2, \ldots, 6$. What if the rates are equal?
7. Suppose the EUR-USD spot is positively correlated with the EUR rates. How does this change the TV of a strike-out call?
8. What is the vega profile as a function of spot for a strike-out call?
9. Given Equation (1.165), which represents the theoretical value of a double-no-touch in units of domestic currency, where the payoff currency is also domestic. Let us denote this function by

$$v^d(S, r_d, r_f, \sigma, L, H), \tag{1.279}$$

   where the superscript d indicates that the payoff currency is domestic. Using this formula, prove that the corresponding value in domestic currency of a double-no-touch paying one unit of *foreign* currency is given by

$$v^f(S, r_d, r_f, \sigma, L, H) = Sv^d\left(\frac{1}{S}, r_f, r_d, \sigma, \frac{1}{H}, \frac{1}{L}\right). \tag{1.280}$$

Assuming you know the sensitivity parameters of the function v^d, derive the following corresponding sensitivity parameters for the function v^f,

$$\frac{\partial v^f}{\partial S} = v^d\left(\frac{1}{S}, r_f, r_d, \sigma, \frac{1}{H}, \frac{1}{L}\right) - \frac{1}{S}\frac{\partial v^d}{\partial S}\left(\frac{1}{S}, r_f, r_d, \sigma, \frac{1}{H}, \frac{1}{L}\right), \tag{1.281}$$

$$\frac{\partial^2 v^f}{\partial S^2} = \frac{1}{S^3}\frac{\partial^2 v^d}{\partial S^2}\left(\frac{1}{S}, r_f, r_d, \sigma, \frac{1}{H}, \frac{1}{L}\right),$$

$$\frac{\partial v^f}{\partial \sigma} = S\frac{\partial v^d}{\partial \sigma},$$

$$\frac{\partial^2 v^f}{\partial \sigma^2} = S\frac{\partial^2 v^d}{\partial^2 \sigma},$$

$$\frac{\partial^2 v^f}{\partial S \partial \sigma} = S\frac{\partial v^d}{\partial \sigma}\left(\frac{1}{S}, r_f, r_d, \sigma, \frac{1}{H}, \frac{1}{L}\right) - \frac{1}{S}\frac{\partial^2 v^d}{\partial S \partial \sigma}\left(\frac{1}{S}, r_f, r_d, \sigma, \frac{1}{H}, \frac{1}{L}\right),$$

$$\frac{\partial v^f}{\partial r_d} = \frac{\partial v^d}{\partial r_f},$$

$$\frac{\partial v^f}{\partial r_f} = \frac{\partial v^d}{\partial r_d}.$$

10. Suppose your front office application for double-no-touch options is out of order, but you can use double-knock-out options. Replicate a double-no-touch using two double-knock-out options. As shown in Figure 1.46, one can replicate a long double no-touch with

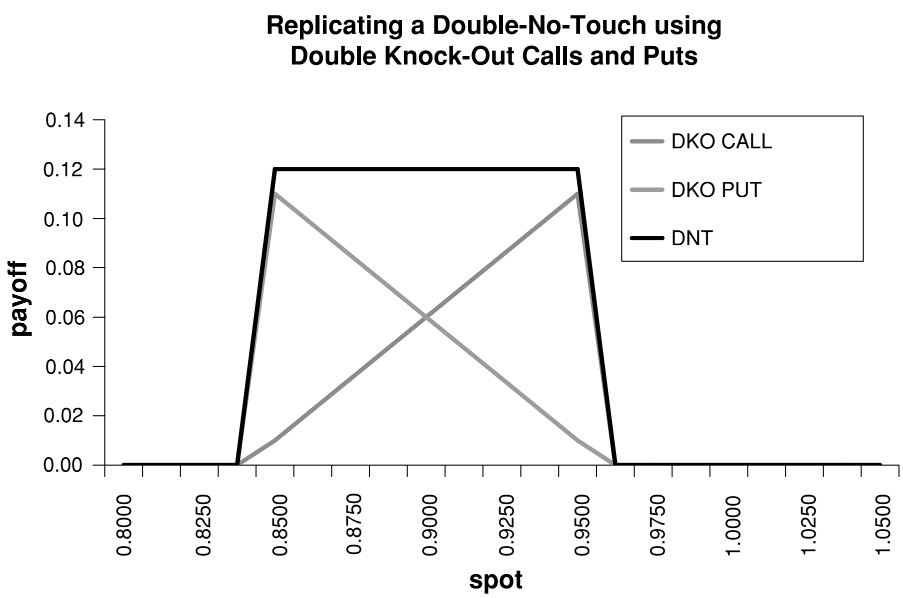

Figure 1.46 Replication of a double-no-touch with two double-knock-out options

barriers L and H using a portfolio of
(a) a long double-knock-out call with barriers L and H and strike L,
(b) a long double-knock-out put with barriers L and H and strike H.
The nominal amounts of the respective double-knock-out options depend on the currency in which the payoff is settled. In case of a EUR-USD double-no-touch paying 1 USD (domestic currency), show that the nominal amounts of the double-knock-out call and put are both $\frac{1}{H-L}$.

Verify the following argument how to replicate a double-no-touch paying one unit of EUR (foreign currency). More precisely, we can price this by taking a USD-EUR double-no-touch with barriers $1/H$ and $1/L$. This double-no-touch can be composed as before using

$$\frac{1}{\frac{1}{L}-\frac{1}{H}} = \frac{LH}{H-L} \tag{1.282}$$

DKO USD Calls with strike $1/H$ and DKO USD Puts with strike $1/L$, both with barriers $1/H$ and $1/L$. Since furthermore

$$\begin{aligned}&1 \text{ DKO USD Call with strike } 1/H \text{ and barriers } 1/H \text{ and } 1/L\\ =\ &(1 \text{ DKO EUR Puts with strike } H \text{ and barriers } L \text{ and } H)/(H\cdot S),\end{aligned}$$

and similarly

$$\begin{aligned}&1 \text{ DKO USD Put with strike } 1/L \text{ and barriers } 1/H \text{ and } 1/L\\ =\ &(1 \text{ DKO EUR Calls with strike } L \text{ and barriers } L \text{ and } H)/(L\cdot S),\end{aligned}$$

we obtain for the EUR-USD double-no-touch paying one unit of EUR (foreign currency)

$$\frac{\text{DKOPut}(H,L,H)\cdot L + \text{DKOCall}(L,L,H)\cdot H}{(H-L)\cdot S}, \tag{1.283}$$

where DKOPut(H, L, H) means a EUR Put with strike H and barriers L and H and DKOCall(L, L, H) means a EUR Call with strike L and barriers L and H. The division by EUR-USD Spot S must be omitted, if the price of the double-no-touch is to be quoted in EUR. If it is quoted in USD, then the formula stays as it is.

11. Derive a closed form solution for the value of a continuously sampled geometric floating strike call and put. How are they related to the fixed strike formulae?
12. Discuss the settlement possibilities of Asian options, i.e. which type of average options can be settled in physical delivery, and which only in cash. In case of cash settlement, specify if both domestic and foreign currency can be paid or just one of them.
13. In theory, a lookback option can be replicated by a continuum of one-touch-options, see, for example, in Poulsen [57]. Find out more details and use this result to determine a market price of lookback options based on the liquid market of one-touch options. This can be approximated by methods of Section 3.1.
14. Implement a valuation of a forward start option (see Section 1.5.6) in a version where the strike K is set as $S_t + d$ at time $t \in [0, T]$. This has to be done using numerical integration. Compare the values you obtain with the standard forward start option values.
15. One can view a market of liquid forward start options as a source of information for the forward volatility. Using similar techniques as in the case of vanilla options, how would you back out the forward volatility smile?

16. Derive a closed form solution for a single barrier option where both the strike and the barrier are set as multiples of the spot S_t at the future time t, following the approach for forward start options.
17. In a standard annuity of n months, total loan amount K, monthly payments, the amount A paid back to the bank every month is constant and is the sum of the interest payment and the amortization. Payments happen at the end of each month, and the first payment is at the end of the first month. Clearly, as time passes the amortization rises. Assuming annual interest rate r compute the remaining debt R after n months. Furthermore, setting $R = 0$, find n. You may assume the time of one month being $1/12$ of a year and the monthly interest rate to be $R/12$. You may check your calculations at www.mathfinance.com/annuity.html.
18. The price of an ounce of Gold is quoted in USD. If the price of Gold drops by 5 %, but the price of Gold in EUR remains constant, determine the change of the EUR-USD exchange rate.
19. Suppose you are long a USD Put JPY Call quanto into AUD. What are the vega profiles of the positions in USD-JPY, USD-AUD and AUD-JPY?
20. Suppose you are short a double-no-touch. Draw the possible vega profiles as a function of the spot and discuss the possible scenarios.
21. Suppose you know the vega of a 2-month at-the-money vanilla. By what factor is the vega of a 4-month at-the-money vanilla bigger? How does this look for a 5-year vanilla in comparison to a 10-year vanilla?
22. Suppose the exchange rate S follows a Brownian motion without drift and constant volatility. How can you hedge a single-one-touch with digitals? Hint: Use the reflection principle.
23. Given vanillas and digitals, how can you structure European style barrier options?
24. Given the following market of bonds

Bond	Tenor in Years	Price	Notional	Coupon
ZeroBond 1	1	94	100	0 %
ZeroBond 2	2	88	100	0 %
CouponBond 1	3	100	100	7 %
CouponBond 2	4	100	100	8 %

write down the cash flows of the four bonds and determine the present value of the cash flow

t_0	1	2	3	4	years
	375	275	575	540	USD

in this market using both a replicating portfolio and a calculation of discount factors.
25. How would you find the Black-Scholes value of a EUR-USD *self-quanto forward* with strike K, which is cash-settled in EUR at maturity? Consider the two cases where either the conversion from USD into EUR of the payoff $S_T - K$ is done using S_T or using S_0.
26. Suppose a client believe very strongly that USD/JPY will reach a level of 120.00 in 3 months time. With a current spot level of 110.00, volatility of 10 %, JPY rate of 0 %,

USD rate of 3 %, find a product with maximum leverage and create a term sheet for the client explaining chances and risks.

27. Prove that a symmetric power straddle has a constant gamma. What does this imply for delta, vega, rhos and theta?
28. In Section 1.5.5 it was stated, that *as a rule of thumb, a floating strike lookback option is worth twice as much as a vanilla option*. Can you prove this rule? Discuss where it applies.
29. Let NT(B) and OT(B) denote the value of a no-touch and a one-touch with barrier B respectively, both paid at the end. Let KOPut(K, B) and KOCall(K, B) denote the value of a regular knock-out put and call with strike K and barrier B respectively. Let SOPut(K) and SOCall(K) denote the value of a strike-out put and call with strike K and barrier K respectively. Finally, let RKOPut(K, B) and RKOCall(K, B) denote the value of a reverse knock-out put and call with strike K and barrier B respectively. How can you replicate reverse knock-outs using touch-options, strike-outs and regular knock-outs? In particular, prove or verify the equation

$$\text{RKOCall}(K, B) = (B - K)\text{NT}(B) - \text{SOPut}(B) + \text{KOPut}(K, B). \tag{1.284}$$

    Support your answer with a suitable figure and state the corresponding equation for the RKOPut. This implies in particular that the market prices for reverse knock-outs can be implied from the market prices of touch and regular barrier options. Moreover, this result also shows how to hedge regular knock-out options.
30. Derive the value function of a quanto forward. Consider next a *self-quanto*, where in a EUR-USD market, a client does a forward where he agrees to receive $S_T - K$ in EUR rather than USD. If the amount he receives is negative, then he pays.

1.6 SECOND GENERATION EXOTICS

1.6.1 Corridors

A European corridor entitles its holder to receive a pre-specified amount of a currency (say EUR) on a specified date (maturity) proportional to the number of fixings inside a lower range and an upper range between the start date and maturity. The buyer has to pay a premium for this product.

Advantages

- High leverage product, high profit potential
- Can take advantage of a quiet market phase
- Easy to price and to understand

Disadvantages

- Not suitable for long-term
- Expensive product
- Price spikes and large market movements can lead to loss

Figure 2.12 shows a sample scenario for a corridor. At delivery, the holder receives $\frac{n}{N} \cdot$ notional, where n is the number of fixings between the lower and the upper range and N denotes to maximum number of fixing possible.

Types of corridors

European style corridor. The corridor is *resurrecting*, i.e. all fixings inside the range count for the accumulation, even if some of the fixings are outside. Given a *fixing schedule* $\{S_{t_1}, S_{t_2}, \ldots, S_{t_N}\}$ the payoff can be specified by

$$\text{notional} \cdot \frac{1}{N} \sum_{i=1}^{N} I\!I_{\{S_{t_i} \in (L,H)\}}, \tag{1.285}$$

where N denotes the total number of fixings, L the lower barrier, H the higher barrier.

American style corridor. The corridor is *non-resurrecting*, i.e. only fixings count for the accumulation that occurs before the first time the exchange rate leaves the range. In this case one needs to specify exactly the time the fixing is set. In particular, if on one day the exchange rate trades at or outside the range, does a fixing inside the range on this day still account for the accumulation? The default is that it does, if the range is left after the fixing time. In any case, the holder of the corridor keeps the accumulated amount.

Introducing the stopping time

$$\tau \stackrel{\Delta}{=} \inf\{t : S_t \notin (L, H)\}, \tag{1.286}$$

the payoff can be specified by

$$\text{notional} \cdot \frac{1}{N} \sum_{i=1}^{N} I\!I_{\{S_{t_i} \in (L,H)\}} I\!I_{\{t_i < \tau\}}. \tag{1.287}$$

As a variation, the fixing range and the knock-out range need not be identical, the ranges can be one-sided or only partially valid over time.

American style corridor with complete knock-out. This is an American style corridor, where all of the accumulated amount is lost once the exchange rate trades at or outside the range. This is equivalent to a double-no-touch. The payoff can be specified by

$$\text{notional} \cdot \frac{1}{N} \sum_{i=1}^{N} I\!I_{\{S_{t_i} \in (L,H)\}} I\!I_{\{L < \min_{0 \le t \le T} S_t \le \max_{0 \le t \le T} S_t < H\}}, \tag{1.288}$$

where T denotes the expiration time. This type of corridor only makes sense if the range for the fixings is strictly smaller than the range for the knock-out.

American style corridor with discrete knock-out. This is like an American style corridor where the knock-out occurs when the fixing is outside the range for the first time, i.e. we replace the stopping time by

$$\tau_d \stackrel{\Delta}{=} \min\{t_i : S_{t_i} \notin (L, H)\}. \tag{1.289}$$

Forward start corridor. In this type, which can be European or American as before, the range will be set relative to a future spot level, see also Section 1.5.6.

Table 1.29 Example of a European corridor

Spot reference	1.1500 EUR-USD
Notional	1,000,000 EUR
Maturity	1 year
European style corridor	1.1000 - 1.18000 EUR-USD
Fixing schedule	monthly
Fixing source	ECB
Premium	500,000 EUR

To compare, the premium for the same corridor in American style would be 100,000 EUR.

Example

An investor wants to benefit from believing that the EUR-USD exchange rate will be often between two ranges during 12 months. In this case an advisable product to use is a European corridor as for example presented in Table 1.29.

If the investor's market expectation is correct, then it will receive 1 Mio EUR at delivery, twice the premium at stake.

Explanations

Fixings are *official* exchange rate sources such as from the European Central Bank, the federal reserve bank or private banks, which takes place on each business day. For details on the impact on pricing see Section 3.4.

Fixing source is the exact source of the fixing, for example Reuters page ECB37, OPTREF, or Bloomberg pages.

Fixing schedule requires a start date, end date and a frequency such as daily, weekly or monthly. It can also be customized. Since there are often disputes about holidays, it is advisable to specify any fixing schedule explicitly in the deal confirmation.

Composition and applications

Obviously, a European style corridor is a sum of digital options. The only issue is that the expiration times are the fixing time and the delivery time is the same for all digital options.

Similarly, an American style corridor is a sum of double-barrier digitals with deferred delivery. We refer the reader to the exercises to work out the details.

Corridors occur very often as part of structured products such as a *range accrual forward* in Section 2.1.9 or a *corridor deposit* in Section 2.3.4.

1.6.2 Faders

Faders are options, whose nominal is directly proportional to the number of fixings the spot stays inside or outside a pre-defined range. A *fade-in option* has a progressive activation of the nominal. In a *fade-out option* the concept of a progressive activation of the nominal is changed to a progressive deactivation. We discuss as an example the fade-in put option, whose characteristics are the pre-defined range and the associated fixing schedule with the maximal number of fixing being M. For each fixing date with the fixing inside the pre-defined range the holder of a fade-in put option receives a vanilla put option with the notional

$$\frac{\text{number of fixings inside the range}}{M}. \tag{1.290}$$

Buying a fade-in put option provides protection against falling EUR and allows full participation in a rising EUR. The holder has to pay a premium for this protection. He will exercise the option only if at maturity the spot is below the strike. The seller of the option receives the premium, but is exposed to market movements and would need to hedge his exposure accordingly.

Advantages

- Protection against weaker EUR/stronger USD
- Premium not as high as for a Plain Vanilla Put option
- Full participation in a favorable spot movement

Disadvantages

- Selling amount depends on market movements
- No guaranteed worst case exchange rate for the full notional.

Example for the computation of the notional

We explain this product with a EUR Put-USD Call with strike K, which has two ranges and 6 fixings, in Figure 1.47.

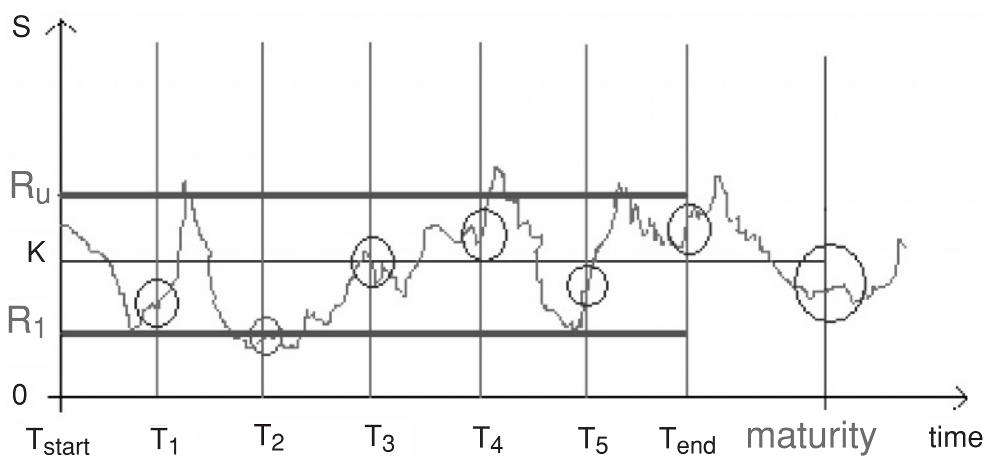

Figure 1.47 Notional of a fade-in put
At T_{end}, the holder would be entitled to sell $\frac{5}{6} \cdot 1$ Mio EUR, where 5 is the number of fixings between the lower and the upper range R_l and R_u on a resurrecting basis (here $n = 5$ because at T_2, the spot fixing is below the lower range). The total number of fixings inside the range will be known only at T_{end}. Hence, the notional of the put will only be known at T_{end}.

Table 1.30 Example of a fade-in put

Spot reference	1.1500 EUR-USD
Company buys	EUR put USD call
Fixing schedule	Monthly
Maturity	1 year
Notional amount	EUR 1,000,000
Strike	1.1600 EUR-USD
Lower Range	1.0000 EUR-USD
Upper Range	1.2000 EUR-USD
Premium	EUR 6,000.00

In comparison the corresponding vanilla put costs 50,000.00 EUR.

At maturity, the fade-in put works like a vanilla put. The holder would exercise the option and sell $\frac{5}{6} \cdot$ 1 Mio EUR at the strike K if the spot is below the strike. If it ends up above, the option would expire worthless. The overall loss of the buyer would be the option's premium.

Example
A company wants to hedge receivables from an export transaction in EUR due in 12 months time. It expects a weaker EUR/stronger USD. The company wishes to be able to sell EUR at a higher spot rate if the EUR becomes stronger on the one hand, but on the other hand be protected against a weaker EUR. The company finds the corresponding vanilla put too expensive and is prepared to take more risk. The treasurer believes that EUR/USD will not trade outside the range 1.1000–1.2000 for a significantly long time.

In this case a possible form of protection that the company can use is to buy a EUR fade-in put option, as for example presented in Table 1.30.
If the EUR-USD exchange rate is below the strike at maturity, then the company can sell EUR at maturity at the strike of 1.1600.

If the EUR-USD exchange rate is above the strike at maturity the option expires worthless. However, the company will benefit from a higher spot when selling EUR.

Variations

Besides puts, there are fade-in call or fade-in forwards, see Table 1.31 or the live trade in Table 2.3 in Section 2.1.3. Also more exotic types of faders can be created by taking exotic options and let them fade in or out.

Faders often have an additional knock-out range just like corridors, see Section 1.6.1. One then classifies faders into *resurrecting*, *non-resurrecting, keeping the accrued amount* and *non-resurrecting loosing parts of all of the accrued amount*.

Faders are most popularly applied in structuring *accumulative forwards*, see Section 2.1.10.

1.6.3 Exotic barrier options

Digital barrier options

Just like barrier options, which are calls or puts with knock-out or knock-in barriers, one can consider digital calls and puts with additional American style knock-out or knock-in barriers.

Table 1.31 Example of a fade-in forward. In comparison the corresponding fade-in call costs 27,000.00 EUR

Spot reference	1.1500 EUR-USD
Company buys	EUR-USD forward
Fixing schedule	Monthly
Maturity	1 year
Notional amount	EUR 1,000,000
Strike	1.0000 EUR-USD
Lower Range	1.0000 EUR-USD
Upper Range	1.1800 EUR-USD
Premium	EUR 9,000.00

Knowing the digitals, we can derive the knock-in digitals form the knock-out digitals. The knock-out digitals can be viewed as the delta of the knock-out vanilla options, and hence the values, prices and hedges from there.

The motivation for such products is to make betting on events cheaper.

Window barriers

Barriers need not be active for the entire lifetime of the option. Window Barrier Options are European Plain Vanilla or Binary Options with Barriers where the Barriers are active during a period of time which is shorter than the whole lifetime of the option. For example only the first 3 months from a 6 months maturity option. One can specify arbitrary time ranges with piecewise constant barrier levels or even non-constant barriers. See Figure 1.48 for the value function of a window barrier option. Linear and exponential barriers are useful if there is a certain drift in the exchange rate caused, e.g., by a high interest rate differential (high swap points).

Step and soft barriers

In case of a knock-out event, a client might argue: "Come on, the spot only crossed the barrier for a very short moment, can't you make an exception and not let my option knock out?" This is a very common concern: how to get protection against price spikes. Such a protection is certainly possible, but surely has its price. One way is to measure the time the spot spends opposite the knock-out barrier and let the option knock out gradually. For instance one could agree that the option's nominal is decreased by 10 % for each day the exchange rate fixing is opposite the barrier. This can be done linearly or exponentially. Such contracts are also referred to as *occupation time derivatives*.

Fluffy barriers

Fluffy Barrier Options are European Options with a Fluffy Barrier which knocks-in or -out in a non-digital way. The knock-in or knock-out is generally linear between the minimum and maximum Fluffy Barrier levels. For instance one can specify a barrier range of 2.20 to 2.30 where the option loses 25 % of its nominal when 2.20 is breached, 50 % when 2.25 is breached, 75 % when 2.275 is breached and 100 % when 2.30 is breached.

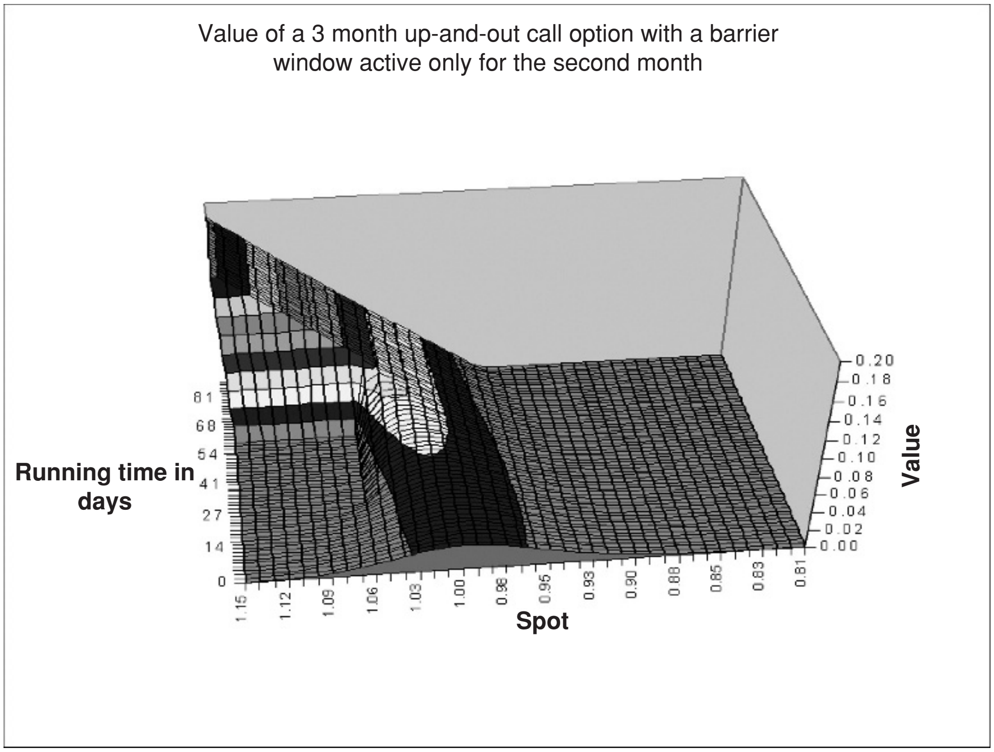

Figure 1.48 Value function $v(t, x)$ of an up-and-out call option with window barrier active only for the second month, with strike $K = 0.9628$, knock-out barrier $B = 1.0590$ and maturity 3 months. We used the interest rates $r_d = 6.68\,\%$, $r_f = 5.14\,\%$, volatility $\sigma = 11.6\,\%$ and $R = 0$

Parisian and Parasian barriers

Another way to get price spike protection is to let the option knock out only if the spot spends a certain pre-specified length of time opposite the barrier – either in total (Parasian) or in a row (Parisian). Clearly the plain barrier option is the least expensive, followed by the Parasian, then the Parisian barrier option and finally the corresponding vanilla contract. See Figure 1.49.

Resetable barriers

This is a way to give the holder of a barrier option a chance to reset the barrier during the life of the option n times at a priori determined N times in the future ($N \geq n$). This kind of extra protection also makes the barrier option more expensive.

Quanto barriers

In foreign exchange options markets option payoffs are often paid in a currency different from the underlying currency pair. For instance a USD/JPY call is designed to be paid in EUR, where the exchange rate for EUR/JPY is determined a priori. Surely such features can be applied to barrier options as well.

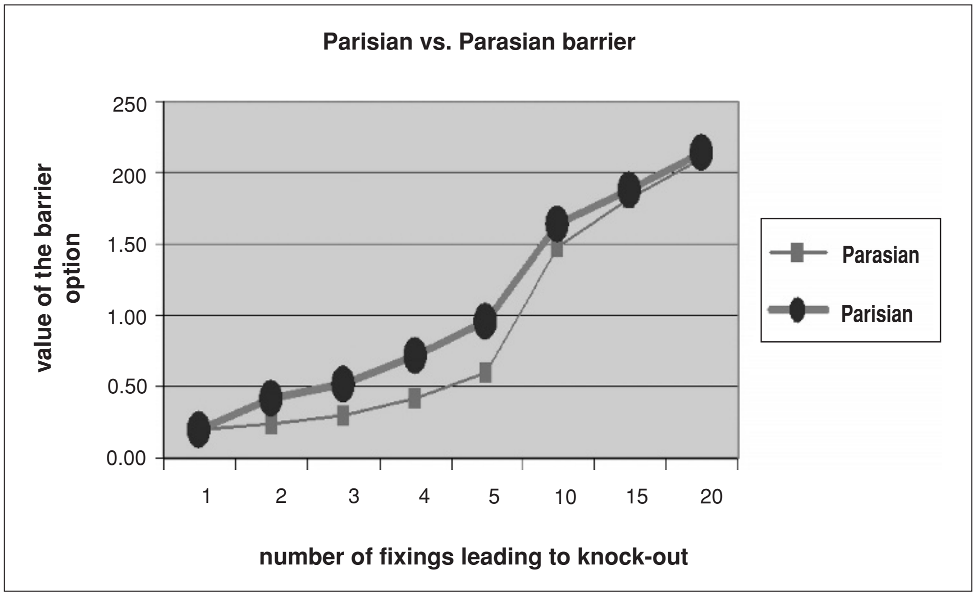

Figure 1.49 Comparison of Parisian and Parasian barrier option values

Transatlantic barrier options

For Transatlantic barrier options one barrier is of American style, the other one of European style. Naturally, the European style barrier is in-the-money, the American style barrier usually out-of-the-money. Therefore, there are essentially two versions,

1. a call with strike K, a European style up-and-out $H > K$ and an American style down-and-out at $L \leq K$,
2. a put with strike K, a European style down-and-out $L < K$ and an American style up-and-out $H \geq K$.

The motivation for such products is of course the savings effect in comparison to vanilla or single barrier options on the one hand and the fear of price spikes and a resulting preference for European style barriers on the other.

The pricing and hedging is comparatively easy provided we have regular and digital barrier options available as basic products. Then we can structure the transatlantic barrier option just like in Equation (3.31), with an additional out-of-the-money knock-out barrier.

Outside barrier options

Outside barrier options are options in one currency pair with one or several barriers or window barriers in another currency pair. In general form the payoff can be written as

$$[\phi(S_T - K)]^+ \, I\!I_{\{\min_{0 \leq t \leq T}(\eta R(t)) > \eta B\}}. \tag{1.291}$$

This is a European put or call with strike K and a knock-out barrier H in a second currency pair, called the *outer* currency pair. As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put and the binary variable η takes the value $+1$ for a lower barrier and -1 for an upper barrier. The positive constants σ_i denote the annual volatilities of the i-th asset or foreign currency, ρ the instantaneous correlation of their log-returns, r the domestic risk free rate and T the expiration time in years. In a risk-neutral setting the drift terms μ_i take the values

$$\mu_i = r - r_i \tag{1.292}$$

where r_i denotes the risk free rate of the i-th foreign currency. Knock-in outside barrier options prices can be obtained by the standard relationship *knock-in plus knock-out = vanilla*.

In the standard two-dimensional Black-Scholes model

$$dS_t = S_t \left[\mu_1 dt + \sigma_1 dW_t^{(1)}\right], \tag{1.293}$$

$$dR_t = R_t \left[\mu_2 dt + \sigma_2 dW_t^{(2)}\right], \tag{1.294}$$

$$\mathbf{Cov}\left[W_t^{(1)}, W_t^{(2)}\right] = \sigma_1 \sigma_2 \rho t, \tag{1.295}$$

Heynen and Kat derive the value in [45].

$$\begin{aligned} V_0 = {} & \phi S_0 e^{-r_1 T} \mathcal{N}_2(\phi d_1, -\eta e_1; \phi\eta\rho) \\ & - \phi S_0 e^{-r_1 T} \exp\left(\frac{2(\mu_2 + \rho\sigma_1\sigma_2)\ln(H/R_0)}{\sigma_2^2}\right) \mathcal{N}_2(\phi d_1', -\eta e_1'; \phi\eta\rho) \\ & - \phi K e^{-rT} \mathcal{N}_2(\phi d_2, -\eta e_2; \phi\eta\rho) \\ & + \phi K e^{-rT} \exp\left(\frac{2\mu_2 \ln(H/R_0)}{\sigma_2^2}\right) \mathcal{N}_2(\phi d_2', -\eta e_2'; \phi\eta\rho), \end{aligned} \tag{1.296}$$

$$d_1 = \frac{\ln(S_0/K) + (\mu_1 + \sigma_1^2)T}{\sigma_1\sqrt{T}}, \tag{1.297}$$

$$d_2 = d_1 - \sigma_1\sqrt{T}, \tag{1.298}$$

$$d_1' = d_1 + \frac{2\rho\ln(H/R_0)}{\sigma_2\sqrt{T}}, \tag{1.299}$$

$$d_2' = d_2 + \frac{2\rho\ln(H/R_0)}{\sigma_2\sqrt{T}}, \tag{1.300}$$

$$e_1 = \frac{\ln(H/R_0) - (\mu_2 + \rho\sigma_1\sigma_2)T}{\sigma_2\sqrt{T}}, \tag{1.301}$$

$$e_2 = e_1 + \rho\sigma_1\sqrt{T}, \tag{1.302}$$

$$e_1' = e_1 - \frac{2\ln(H/R_0)}{\sigma_2\sqrt{T}}, \tag{1.303}$$

$$e_2' = e_2 - \frac{2\ln(H/R_0)}{\sigma_2\sqrt{T}}. \tag{1.304}$$

The bivariate standard normal distribution $\mathcal{N}_2$[7] and density functions n_2 are defined by

$$n_2(x, y; \rho) \stackrel{\Delta}{=} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho x y + y^2}{2(1-\rho^2)}\right), \tag{1.305}$$

$$\mathcal{N}_2(x, y; \rho) \stackrel{\Delta}{=} \int_{-\infty}^{x}\int_{-\infty}^{y} n_2(u, v; \rho)\, du\, dv. \tag{1.306}$$

For the Greeks, most of the calculations of partial derivatives can be simplified substantially by the homogeneity method described in [6], which states for instance, that

$$V_0 = S_0 \frac{\partial V_0}{\partial S_0} + K \frac{\partial V_0}{\partial K}. \tag{1.307}$$

We list some of the sensitivities for reference.

delta(inner spot)

$$\begin{aligned}\frac{\partial V_0}{\partial S_0} &= \phi e^{-r_1 T} \mathcal{N}_2(\phi d_1, -\eta e_1; \phi\eta\rho) \\ &\quad -\phi e^{-r_1 T} \exp\left(\frac{2(\mu_2 + \rho\sigma_1\sigma_2)\ln(H/R_0)}{\sigma_2^2}\right) \mathcal{N}_2(\phi d_1', -\eta e_1'; \phi\eta\rho)\end{aligned} \tag{1.308}$$

dual delta(inner strike)

$$\begin{aligned}\frac{\partial V_0}{\partial K} &= -\phi e^{-rT} \mathcal{N}_2(\phi d_2, -\eta e_2; \phi\eta\rho) \\ &\quad +\phi e^{-rT} \exp\left(\frac{2\mu_2 \ln(H/R_0)}{\sigma_2^2}\right) \mathcal{N}_2(\phi d_2', -\eta e_2'; \phi\eta\rho)\end{aligned} \tag{1.309}$$

gamma(inner spot)

$$\begin{aligned}\frac{\partial^2 V_0}{\partial S_0^2} &= \frac{e^{-r_1 T}}{S_0 \sigma_1 \sqrt{T}} \left[n(d_1) \mathcal{N}\left(\frac{-\phi\rho d_1 - \eta e_1}{\sqrt{1-\rho^2}}\right)\right. \\ &\quad \left. - \exp\left(\frac{2(\mu_2 + \rho\sigma_1\sigma_2)\ln(H/R_0)}{\sigma_2^2}\right) n(d_1') \mathcal{N}\left(\frac{-\phi\rho d_1' - \eta e_1'}{\sqrt{1-\rho^2}}\right)\right]\end{aligned} \tag{1.310}$$

The standard normal density function n and its cumulative distribution function $\mathcal{N}$ are defined in (1.316) and (1.323). Furthermore, we use the relations

$$\frac{\partial}{\partial x} \mathcal{N}_2(x, y; \rho) = n(x) \mathcal{N}\left(\frac{y - \rho x}{\sqrt{1-\rho^2}}\right), \tag{1.311}$$

$$\frac{\partial}{\partial y} \mathcal{N}_2(x, y; \rho) = n(y) \mathcal{N}\left(\frac{x - \rho y}{\sqrt{1-\rho^2}}\right). \tag{1.312}$$

[7] See `http://www.mathfinance.com/frontoffice.html` for a source code to compute $\mathcal{N}_2$.

dual gamma(inner strike) Again, the homogeneity method described in [6] leads to the result

$$S^2 \frac{\partial^2 V_0}{\partial S_0^2} = K^2 \frac{\partial^2 V_0}{\partial K^2}. \tag{1.313}$$

In order to derive the value function we start with a triple integral. We treat the up-and-out call as an example. The value of an outside up-and-out call option is given in Section 24 in [19] by the integral

$$V_0 \stackrel{\Delta}{=} \frac{e^{-rT}}{\sqrt{T}} \int_{\hat{m}=0}^{\hat{m}=m} \int_{\hat{b}=-\infty}^{\hat{b}=\hat{m}} \int_{\tilde{b}=-\infty}^{\tilde{b}=\infty} F(\hat{b}, \tilde{b}) n(\frac{\tilde{b}}{\sqrt{T}}) f(\hat{m}, \hat{b})\, d\tilde{b}\, d\hat{b}\, d\hat{m}, \tag{1.314}$$

where the payoff function F, the normal density function n, the joint density function f and the parameters $m, b, \hat{\theta}, \gamma$ are defined by

$$F(\hat{b}, \tilde{b}) \stackrel{\Delta}{=} \left(S_0 e^{\gamma\sigma_2 T + \rho\sigma_2 \hat{b} + \sqrt{1-\rho^2}\sigma_2 \tilde{b}} - K\right)^+ \tag{1.315}$$

$$n(t) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}, \tag{1.316}$$

$$f(\hat{m}, \hat{b}) \stackrel{\Delta}{=} \frac{2(2\hat{m} - \hat{b})}{T\sqrt{2\pi T}} \exp\left\{-\frac{(2\hat{m} - \hat{b})^2}{2T} + \hat{\theta}\hat{b} - \frac{1}{2}\hat{\theta}^2 T\right\}, \tag{1.317}$$

$$m \stackrel{\Delta}{=} \frac{1}{\sigma_1} \ln \frac{L}{Y_0}, \tag{1.318}$$

$$b \stackrel{\Delta}{=} \frac{1}{\sigma_2} \ln \frac{L}{S_0}, \tag{1.319}$$

$$\hat{\theta} \stackrel{\Delta}{=} \frac{r}{\sigma_1} - \frac{\sigma_1}{2}, \tag{1.320}$$

$$\gamma \stackrel{\Delta}{=} \frac{r}{\sigma_2} - \frac{\sigma_2}{2} - \rho\hat{\theta}. \tag{1.321}$$

The goal is to write the above integral in terms of the bivariate normal distribution function (1.306). For easier comparison we use the identification Table 1.32.

Table 1.32 Relating the notation of Heynen and Kat to the one by Shreve

Heynen/Kat	Shreve
S_0	S_0
R_0	Y_0
σ_1	σ_2
σ_2	σ_1
H	L
K	K
μ_1	$r - \frac{\sigma_2^2}{2}$
μ_2	$r - \frac{\sigma_1^2}{2}$

The solution can be obtained by following the steps

(a) Use a change of variables to prove the identity

$$\int_{-\infty}^{x} \mathcal{N}(az+B)n(z)\,dz = \mathcal{N}_2\left(x, \frac{B}{\sqrt{1+a^2}}; \frac{-a}{\sqrt{1+a^2}}\right), \tag{1.322}$$

where the cumulative normal distribution function $\mathcal{N}$ is defined by

$$\mathcal{N}(x) \stackrel{\Delta}{=} \int_{-\infty}^{x} n(t)\,dt. \tag{1.323}$$

A probabilistic proof is presented in [58].

(b) Extend the identity (1.322) to

$$\int_{-\infty}^{x} e^{Az}\mathcal{N}(az+B)n(z)\,dz = e^{\frac{A^2}{2}}\mathcal{N}_2\left(x-A, \frac{aA+B}{\sqrt{1+a^2}}; \frac{-a}{\sqrt{1+a^2}}\right). \tag{1.324}$$

(c) Change the order of integration in Equation (1.314) and integrate the $\hat{m}$ variable.

(d) Change the order of integration to make $\tilde{b}$ the inner variable and $\hat{b}$ the outer variable. Then use the condition $F(\hat{b}, \tilde{b}) \geq 0$ to find a lower limit for the range of $\tilde{b}$. This will enable you to skip the positive part in F and write Equation (1.314) as a sum of four integrals.

(e) Use (1.322) and (1.324) to write each of these four summands in terms of the bivariate normal distribution function $\mathcal{N}_2$.

(f) Compare your result with the one provided by Heynen and Kat using the identification table given above.

The solution of the integral works like this.

(a) To prove the identity

$$\int_{-\infty}^{x} \mathcal{N}(az+B)n(z)\,dz = \mathcal{N}_2\left(x, \frac{B}{\sqrt{1+a^2}}; \frac{-a}{\sqrt{1+a^2}}\right),$$

we must show that for $\rho = \frac{-a}{\sqrt{1+a^2}}$

$$\int_{v=-\infty}^{x}\int_{u=-\infty}^{av+B} \exp\left(-\frac{1}{2}(u^2+v^2)\right)du\,dv \tag{1.325}$$

$$= \frac{1}{\sqrt{1-\rho^2}}\int_{v=-\infty}^{x}\int_{u=-\infty}^{\frac{B}{\sqrt{1+a^2}}} \exp\left(-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right)du\,dv. \tag{1.326}$$

We start with (1.325), do the change of variable $u = (\tilde{u} - \rho v)/\sqrt{1-\rho^2}$ and obtain

$$\frac{1}{\sqrt{1-\rho^2}} \int_{v=-\infty}^{x} \int_{\tilde{u}=-\infty}^{\sqrt{1-\rho^2}(av+B)+\rho v} \exp\left(-\frac{\tilde{u}^2 - 2\rho\tilde{u}v + v^2}{2(1-\rho^2)}\right) d\tilde{u}\, dv. \tag{1.327}$$

The choice $\rho = \frac{-a}{\sqrt{1+a^2}}$ produces the upper limit of integration $\frac{B}{\sqrt{1+a^2}}$ for $\tilde{u}$ and this leads to (1.326).

(b) Extend the identity (1.322) to

$$\int_{-\infty}^{x} e^{Az}\mathcal{N}(az+B)n(z)\,dz = e^{\frac{A^2}{2}}\mathcal{N}_2\left(x - A, \frac{aA+B}{\sqrt{1+a^2}}; \frac{-a}{\sqrt{1+a^2}}\right).$$

We complete the square, substitute $z - A = u$, use identity (1.322) and obtain

$$\begin{aligned}\int_{-\infty}^{x} e^{Az}\mathcal{N}(az+B)n(z)\,dz &= e^{\frac{A^2}{2}} \int_{-\infty}^{x} \mathcal{N}(az+B)n(z-A)\,dz \\ &= e^{\frac{A^2}{2}} \int_{-\infty}^{x-A} \mathcal{N}(au+aA+B)n(u)\,du \\ &= e^{\frac{A^2}{2}}\mathcal{N}_2\left(x - A, \frac{aA+B}{\sqrt{1+a^2}}; \frac{-a}{\sqrt{1+a^2}}\right).\end{aligned}$$

(c) Change the order of integration in Equation (1.314) and integrate the $\hat{m}$ variable.

$$\begin{aligned}V_0 &= \frac{e^{-rT-\frac{1}{2}\hat{\theta}^2T}}{\sqrt{T}} \int_{\tilde{b}=-\infty}^{\tilde{b}=\infty} n\left(\frac{\tilde{b}}{\sqrt{T}}\right) \int_{\hat{b}=-\infty}^{\hat{b}=m} e^{\hat{\theta}\hat{b}} F(\hat{b},\tilde{b}) \int_{\hat{m}=\hat{b}\vee 0}^{\hat{m}=m} f(\hat{m},\hat{b})\,d\hat{m}\,d\hat{b}\,d\tilde{b} \\ &= \frac{e^{-rT-\frac{1}{2}\hat{\theta}^2T}}{T} \int_{\tilde{b}=-\infty}^{\tilde{b}=\infty} n\left(\frac{\tilde{b}}{\sqrt{T}}\right) \int_{\hat{b}=-\infty}^{\hat{b}=m} e^{\hat{\theta}\hat{b}} F(\hat{b},\tilde{b}) \left[n\left(\frac{\hat{b}}{\sqrt{T}}\right) - n\left(\frac{2m-\hat{b}}{\sqrt{T}}\right)\right] d\hat{b}\,d\tilde{b}.\end{aligned}$$

(d) Change the order of integration to make $\tilde{b}$ the inner variable and $\hat{b}$ the outer variable. Then use the condition $F(\hat{b},\tilde{b}) \geq 0$ to find a lower limit for the range of $\tilde{b}$. This will enable you to skip the positive part in F and write Equation (1.314) as a sum of four integrals.

The condition $F(\hat{b},\tilde{b}) \geq 0$ is satisfied if and only if

$$\tilde{b} \geq \frac{b - \rho\hat{b} - \gamma T}{\sqrt{1-\rho^2}}. \tag{1.328}$$

We may now proceed in our calculation as follows.

$$
\begin{aligned}
V_0 &= \frac{e^{-rT-\frac{1}{2}\hat{\theta}^2 T}}{T} \int_{\hat{b}=-\infty}^{\hat{b}=m} \int_{\tilde{b}=\frac{b-\rho\hat{b}-\gamma T}{\sqrt{1-\rho^2}}}^{\tilde{b}=\infty} n\left(\frac{\tilde{b}}{\sqrt{T}}\right) e^{\hat{\theta}\hat{b}} F\left(\hat{b},\tilde{b}\right) \left[n\left(\frac{\hat{b}}{\sqrt{T}}\right) - n\left(\frac{2m-\hat{b}}{\sqrt{T}}\right)\right] d\tilde{b}\, d\hat{b} \\
&= \frac{S_0 e^{-rT-\frac{1}{2}\hat{\theta}^2 T}}{T} \int_{\hat{b}=-\infty}^{\hat{b}=m} \int_{\tilde{b}=\frac{b-\rho\hat{b}-\gamma T}{\sqrt{1-\rho^2}}}^{\tilde{b}=\infty} n\left(\frac{\tilde{b}}{\sqrt{T}}\right) e^{\hat{\theta}\hat{b}} e^{\gamma\sigma_2 T+\rho\sigma_2\hat{b}+\sqrt{1-\rho^2}\sigma_2\tilde{b}} n\left(\frac{\hat{b}}{\sqrt{T}}\right) d\tilde{b}\, d\hat{b} \\
&- \frac{S_0 e^{-rT-\frac{1}{2}\hat{\theta}^2 T}}{T} \int_{\hat{b}=-\infty}^{\hat{b}=m} \int_{\tilde{b}=\frac{b-\rho\hat{b}-\gamma T}{\sqrt{1-\rho^2}}}^{\tilde{b}=\infty} n\left(\frac{\tilde{b}}{\sqrt{T}}\right) e^{\hat{\theta}\hat{b}} e^{\gamma\sigma_2 T+\rho\sigma_2\hat{b}+\sqrt{1-\rho^2}\sigma_2\tilde{b}} n\left(\frac{2m-\hat{b}}{\sqrt{T}}\right) d\tilde{b}\, d\hat{b} \\
&- \frac{K e^{-rT-\frac{1}{2}\hat{\theta}^2 T}}{T} \int_{\hat{b}=-\infty}^{\hat{b}=m} \int_{\tilde{b}=\frac{b-\rho\hat{b}-\gamma T}{\sqrt{1-\rho^2}}}^{\tilde{b}=\infty} n\left(\frac{\tilde{b}}{\sqrt{T}}\right) e^{\hat{\theta}\hat{b}} n\left(\frac{\hat{b}}{\sqrt{T}}\right) d\tilde{b}\, d\hat{b} \\
&+ \frac{K e^{-rT-\frac{1}{2}\hat{\theta}^2 T}}{T} \int_{\hat{b}=-\infty}^{\hat{b}=m} \int_{\tilde{b}=\frac{b-\rho\hat{b}-\gamma T}{\sqrt{1-\rho^2}}}^{\tilde{b}=\infty} n\left(\frac{\tilde{b}}{\sqrt{T}}\right) e^{\hat{\theta}\hat{b}} n\left(\frac{2m-\hat{b}}{\sqrt{T}}\right) d\tilde{b}\, d\hat{b} \\
&= S_0 e^{(-r-\frac{1}{2}\hat{\theta}^2+\gamma\sigma_2+\frac{1}{2}(1-\rho^2)\sigma_2^2)T} \\
&\quad \int_{y=-\infty}^{y=\frac{m}{\sqrt{T}}} e^{(\hat{\theta}+\rho\sigma_2)\sqrt{T}y} \mathcal{N}\left(\frac{\rho}{\sqrt{1-\rho^2}} y + \frac{-b+\gamma T+(1-\rho^2)\sigma_2 T}{\sqrt{1-\rho^2}\sqrt{T}}\right) n(y)\, dy \\
&- S_0 e^{(-r-\frac{1}{2}\hat{\theta}^2+\gamma\sigma_2+\frac{1}{2}(1-\rho^2)\sigma_2^2)T} e^{(\hat{\theta}+\rho\sigma_2)2m} \\
&\quad \int_{y=-\infty}^{y=-\frac{m}{\sqrt{T}}} e^{(\hat{\theta}+\rho\sigma_2)\sqrt{T}y} \mathcal{N}\left(\frac{\rho}{\sqrt{1-\rho^2}} y + \frac{-b+2\rho m+\gamma T+(1-\rho^2)\sigma_2 T}{\sqrt{1-\rho^2}\sqrt{T}}\right) n(y)\, dy \\
&- K e^{(-r-\frac{1}{2}\hat{\theta}^2)T} \\
&\quad \int_{y=-\infty}^{y=\frac{m}{\sqrt{T}}} e^{\hat{\theta}\sqrt{T}y} \mathcal{N}\left(\frac{\rho}{\sqrt{1-\rho^2}} y + \frac{-b+\gamma T}{\sqrt{1-\rho^2}\sqrt{T}}\right) n(y)\, dy \\
&+ K e^{(-r-\frac{1}{2}\hat{\theta}^2)T} e^{2m\hat{\theta}} \\
&\quad \int_{y=-\infty}^{y=-\frac{m}{\sqrt{T}}} e^{\hat{\theta}\sqrt{T}y} \mathcal{N}\left(\frac{\rho}{\sqrt{1-\rho^2}} y + \frac{-b+2m\rho+\gamma T}{\sqrt{1-\rho^2}\sqrt{T}}\right) n(y)\, dy.
\end{aligned}
$$

(e) Use (1.322) and (1.324) to write each of these four summands in terms of the bivariate normal distribution function $\mathcal{N}_2$.

We take $a = \frac{\rho}{\sqrt{1-\rho^2}}$ in all four summands which implies that $\frac{-a}{\sqrt{1+a^2}} = -\rho$. We choose A and B as suggested by Equation (1.324) and obtain

$$\begin{aligned}
V_0 = {} & S_0 \mathcal{N}_2 \left(\frac{\ln \frac{L}{Y_0} - (r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}} - \rho\sigma_2\sqrt{T}, \frac{\ln \frac{S_0}{K} + (r + \frac{\sigma_2^2}{2})T}{\sigma_2\sqrt{T}}; -\rho \right) \\
& - S_0 e^{2m(\hat{\theta}+\rho\sigma_2)} \mathcal{N}_2 \left(\frac{-\ln \frac{L}{Y_0} - (r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}} - \rho\sigma_2\sqrt{T}, \frac{\ln \frac{S_0}{K} + (r + \frac{\sigma_2^2}{2})T}{\sigma_2\sqrt{T}} + \frac{2m\rho}{\sqrt{T}}; -\rho \right) \\
& - K e^{-rT} \mathcal{N}_2 \left(\frac{\ln \frac{L}{Y_0} - (r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}}, \frac{\ln \frac{S_0}{K} + (r - \frac{\sigma_2^2}{2})T}{\sigma_2\sqrt{T}}; -\rho \right) \\
& + K e^{-rT} e^{2m\hat{\theta}} \mathcal{N}_2 \left(\frac{-\ln \frac{L}{Y_0} - (r - \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}}, \frac{\ln \frac{S_0}{K} + (r - \frac{\sigma_2^2}{2})T}{\sigma_2\sqrt{T}} + \frac{2m\rho}{\sqrt{T}}; -\rho \right).
\end{aligned}$$

(f) Compare your result with the one provided by Heynen and Kat using the identification table given above.

This comparison can be done instantly. We just note that $\mathcal{N}_2(x, y; \rho) = \mathcal{N}_2(y, x; \rho)$.

Inside barrier options can be viewed as a special case. The formula for the (inside) up-and-out call option can be deduced from this result simply by choosing $Y_0 = S_0, \sigma_1 = \sigma_2 \stackrel{\Delta}{=} \sigma$, $\hat{\theta} = \theta_-, \rho = 1$ and using the identity $\mathcal{N}_2(x, y; -1) = \mathcal{N}(x) - \mathcal{N}(-y) = \mathcal{N}(y) - \mathcal{N}(-x)$. Denoting $\theta_\pm \stackrel{\Delta}{=} \frac{r}{\sigma} \pm \frac{\sigma}{2}$, it follows that

$$\begin{aligned}
V_0 = {} & S_0 \left[\mathcal{N}\left(\frac{m - \theta_+ T}{\sqrt{T}} \right) - \mathcal{N}\left(\frac{b - \theta_+ T}{\sqrt{T}} \right) \right] \\
& - S_0 e^{2m\theta_+} \left[\mathcal{N}\left(\frac{m + \theta_+ T}{\sqrt{T}} \right) - \mathcal{N}\left(\frac{2m - b + \theta_+ T}{\sqrt{T}} \right) \right] \\
& - K e^{-rT} \left[\mathcal{N}\left(\frac{m - \theta_- T}{\sqrt{T}} \right) - \mathcal{N}\left(\frac{b - \theta_- T}{\sqrt{T}} \right) \right] \\
& + K e^{-rT} e^{2m\theta_-} \left[\mathcal{N}\left(\frac{m + \theta_- T}{\sqrt{T}} \right) - \mathcal{N}\left(\frac{2m - b + \theta_- T}{\sqrt{T}} \right) \right].
\end{aligned}$$

Knock-in-knock-out options

Knock-In-Knock-Out Options are barriers with both a knock-out and a knock-in barrier. However, it is not so simple, because there are three fundamentally different types, namely,

1. the knock-out can happen *any time*,
2. the knock-out can happen only *after* the knock-in,
3. the knock-out can happen only *before* the knock-in.

The first one is the market standard, but when dealing one should always clarify which type of knock-in-knock-out is agreed upon. For example, let the lower barrier L be a knock-out barrier and the upper barrier U be a knock-out barrier. Standard type 1 KIKO can only be exercised if L is never touched *and* U has been touched at least once. This can be replicated by standard barrier options via

$$\mathrm{KIKO}(L, U) = \mathrm{KO}(L) - \mathrm{DKO}(L, U). \tag{1.329}$$

Therefore, pricing and hedging of this KIKO is straightforward.

The second type is a special case of a *knock-in on strategy* option. Any structure can be equipped with a *global* knock-in barrier, that has to be touched before the structure becomes alive. Knock-out events in the structure are only active *after* the structure knocks in. This is a product of its own and requires an individual valuation, pricing and hedging approach.

In the third type of KIKO a knock-out can only happen before the knock-in. Once the option is knocked in, the knock-out barrier is no longer active. This is also a product of its own and requires an individual valuation, pricing and hedging approach.

James Bond Range

As James Bond can only live twice, the *James Bond Range* is a double-no-touch type of an option. Given an upper barrier H and a lower barrier L, it pays one unit of currency, if the spot remains inside (L, H) at all times until expiry T, or if the spot hits L the spot thereafter remains in a new range to be set around L or similarly if the spot hits H the spot thereafter remains in a new range to be set around H.

1.6.4 Pay-later options

A pay-later option is a vanilla option, whose premium is only paid if the option is exercised, i.e. if the spot is in-the-money at the expiration time. If the spot is not in-the-money, the holder of the option cannot exercise the option, and will end up not having paid anything. However, if the spot is in-the-money, the holder of the option has to pay the option premium, which will then be noticeably higher than the plain vanilla. For this reason pay-later options are not traded very often.

Advantages

- Full protection against spot market movement
- Premium is only paid if the options ends up in-the-money
- Premium is paid only at maturity

Disadvantages

- More expensive than a plain vanilla
- Credit risk for the seller as payoff can be negative

The valuation for the pay-later option

The payoff of pay-later option is defined as

$$[\phi(S_T - K) - P]\, \mathbb{I}_{\{\phi S_T \geq \phi K\}} \tag{1.330}$$

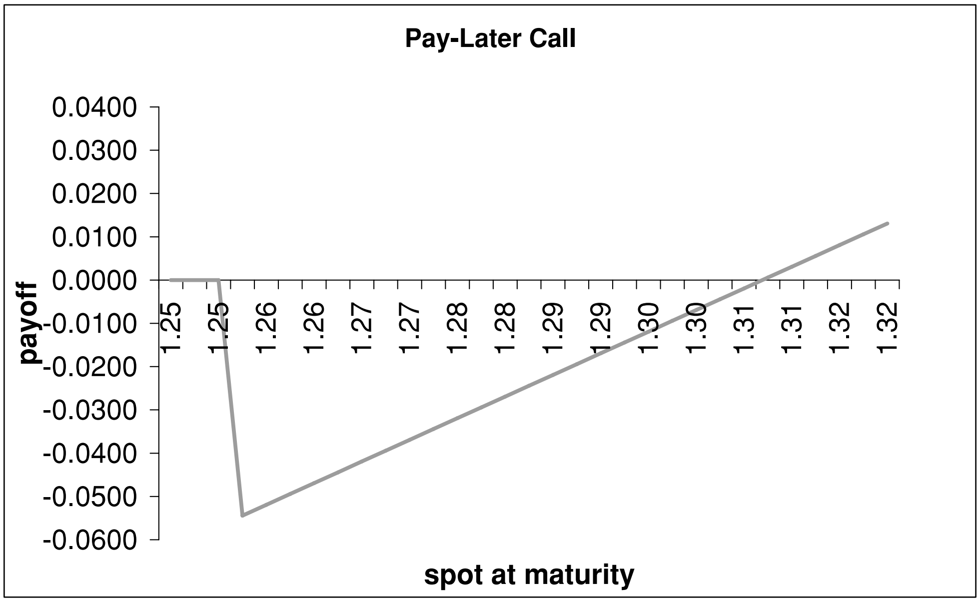

Figure 1.50 Payoff of a pay-later EUR call USD put
We use the market input spot $S_0 = 1.2000$, volatility $\sigma = 10\,\%$, EUR rate $r_f = 2\,\%$, USD rate $r_d = 2.5\,\%$, strike $K = 1.2500$, time to maturity $T = 0.5$ years. The vanilla value is 0.0158 USD, the digital value is 0.2781 USD, the resulting pay-later price is 0.0569 USD, which is substantially higher than the plain vanilla value. Consequently the break-even point is at 1.3075, which is quite far off. For this reason pay-later type structures do not trade very often.

and illustrated in Figure 1.50. As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put, K the strike in units of the domestic currency, and T the expiration time in years. The *price* P of the pay-later option is paid at time T, but it is set at time zero in such a way that the time zero *value* of the above payoff is zero. Take care to notice the difference between price and value. After the option is written, the price P does not change anymore.

We denote the current spot by x and the current time by t and define, furthermore, the abbreviations

$$n(t) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}, \tag{1.331}$$

$$\mathcal{N}(x) \stackrel{\Delta}{=} \int_{-\infty}^{x} n(t)\,dt, \tag{1.332}$$

$$\tau \stackrel{\Delta}{=} T - t, \tag{1.333}$$

$$f = x e^{(r_d - r_f)\tau}, \tag{1.334}$$

$$d_\pm \stackrel{\Delta}{=} \frac{\log \frac{f}{K} \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \tag{1.335}$$

$$\text{vanilla}(x, K, T, t, \sigma, r_d, r_f, \phi) = \phi e^{-r_d\tau}[f\mathcal{N}(\phi d_+) - K\mathcal{N}(\phi d_-)], \tag{1.336}$$

$$\text{digital}(x, K, T, t, \sigma, r_d, r_f, \phi) = e^{-r_d\tau}\mathcal{N}(\phi d_-). \tag{1.337}$$

The formulae of vanilla and digital options have been derived in Sections 1.2 and 1.5.2.

The payoff can be rewritten as

$$[\phi(S_T - K)]^+ - P\,I\!I_{\{\phi S_T \geq \phi K\}}, \tag{1.338}$$

whence the value of the pay-later option in the Black-Scholes model

$$dS_t = S_t\left[(r_d - r_f)dt + \sigma\, dW_t\right] \tag{1.339}$$

is easily read as

$$\begin{aligned}\text{paylater}(x, K, P, T, t, \sigma, r_d, r_f, \phi) = {} & \text{vanilla}(x, K, T, t, \sigma, r_d, r_f, \phi) \\ & - P \cdot \text{digital}(x, K, T, t, \sigma, r_d, r_f, \phi).\end{aligned} \tag{1.340}$$

In particular, this leads to a quick implementation of the value and all the Greeks having the functions vanilla and digital at hand.

For the *pay-later price* setting

$$\text{paylater}(x, K, P, T, 0, \sigma, r_d, r_f, \phi) = 0 \tag{1.341}$$

yields

$$P = \frac{\text{vanilla}(x, K, T, 0, \sigma, r_d, r_f, \phi)}{\text{digital}(x, K, T, 0, \sigma, r_d, r_f, \phi)} \tag{1.342}$$

$$= \text{vanilla}(x, K, T, 0, \sigma, r_d, r_f, \phi)\frac{e^{r_d T}}{\mathcal{N}(\phi d_-)}. \tag{1.343}$$

This can be interpreted as follows. The value P is like the value of a vanilla option, except that

- we must pay interest $e^{r_d T}$, since the premium is due only at time T and
- the premium only needs to be paid if the option is exercised, which is why we divide by the (risk-neutral) probability that the option is exercised $\mathcal{N}(\phi d_-)$.

We observe that the pay-later option can be viewed as a *structured product*. All we need are vanilla and digital options. The structurer will easily replicate a short pay-later with a long vanilla and a short digital. We learn that several types of options can be composed from existing ones, which is the actual job of structuring. This way it is also straightforward to determine a market price, given a vanilla market.

Variations

Pay-later options are an example of the family of contingent or deferred payment options. We can also simply defer the payment of a vanilla without any conditions on the moneyness. Another variation is paying back the vanilla premium if the spot stays inside some range, see the exercises in Section 2.1.19. Naturally, the pay-later effect can be extended beyond vanilla options to all kind of options.

1.6.5 Step up and step down options

The Step Option is an option where the strike of the option is readjusted at predefined fixing dates, but only if the spot is more favorable than that of the previous fixing date. The step option can either be a plain vanilla option or single barrier option. The concept of a progressive *step up* or *step down* could be changed also to a progressive *step up* or *step down* for a forward rate.

1.6.6 Spread and exchange options

A spread option compensates a spread in exchange rates and pays off

$$\left[\phi\left(aS_T^{(1)} - bS_T^{(2)} - K\right)\right]^+ . \tag{1.344}$$

This is a European spread put ($\phi = -1$) or call ($\phi = +1$) with strike $K > 0$ and the expiration time in years T. We assume without loss of generality that the weights a and b are positive. These weights are needed to make the two exchange rates comparable, as USD-CHF and USD-JPY differ by a factor of the size of 100. A standard for the weights are the reciprocals of the initial spot rates, i.e. $a = \frac{1}{S_0^{(1)}}$ and $b = \frac{1}{S_0^{(2)}}$.

Spread options are not traded very often in FX markets. If they are they are usually cash-settled. Exchange options come up more often as they entitle the owner to exchange one currency for another, which is very similar like a vanilla option, which is reflected in the valuation formula.

In the two-dimensional Black-Scholes model

$$dS_t^{(1)} = S_t^{(1)}\left[\mu_1 dt + \sigma_1 dW_t^{(1)}\right], \tag{1.345}$$

$$dS_t^{(2)} = S_t^{(2)}\left[\mu_2 dt + \sigma_2 dW_t^{(2)}\right], \tag{1.346}$$

$$\mathbf{Cov}\left[W_t^{(1)}, W_t^{(2)}\right] = \rho t, \tag{1.347}$$

with positive constants σ_i denoting the annual volatilities of the i-th foreign currency, ρ the instantaneous correlation of their log-returns, r the domestic risk free rate and risk-neutral drift terms

$$\mu_i = r - r_i, \tag{1.348}$$

where r_i denotes the risk free rate of the i-th foreign currency, the value is given by (see [59])

$$\text{spread} = \int_{-\infty}^{+\infty} \text{vanilla}\left(S(x), K(x), \sigma_1\sqrt{1-\rho^2}, r, r_1, T, \phi\right) n(x)\,dx \tag{1.349}$$

$$S(x) \stackrel{\Delta}{=} aS_0^{(1)} e^{\rho\sigma_1\sqrt{T}x - \frac{1}{2}\sigma_1^2\rho^2 T} \tag{1.350}$$

$$K(x) \stackrel{\Delta}{=} bS_0^{(2)} e^{\sigma_2\sqrt{T}x + \mu_2 T - \frac{1}{2}\sigma_2^2 T} + K. \tag{1.351}$$

Notes

1. The integration can be done by the Gauß-Legendre-algorithm using integration limits −5 and 5. A corresponding source code and sample figures can be found in the Front Office section of www.mathfinance.com. The function vanilla (European put and call) can be found in Section 1.2.
2. The integration can be done analytically if $K = 0$. This is the case of *exchange options*, the right to exchange one currency for another.
3. To compute Greeks one may want to use homogeneity relations as discussed in [6].
4. In a foreign exchange setting, the correlation can be computed in terms of known volatilities. This can be found in Section 1.6.7.

Derivation of the value function

We use Equation (1.7) for the value of vanilla options along with the abbreviations thereafter.

We rewrite the model in terms of independent new Brownian motions $W^{(1)}$ and $W^{(2)}$ and get

$$S_T^{(1)} = S_0^{(1)} \exp\left[\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)T + \sigma_1\rho W_T^{(2)} + \sigma_1\sqrt{1-\rho^2}W_T^{(1)}\right], \tag{1.352}$$

$$S_T^{(2)} = S_0^{(2)} \exp\left[\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)T + \sigma_2 W_T^{(2)}\right]. \tag{1.353}$$

This allows us to write $S_T^{(1)}$ in terms of $S_T^{(2)}$, i.e.,

$$S_T^{(1)} = \exp\left[\hat{\mu}_1 + \frac{\sigma_1\rho}{\sigma_2}\left(\ln S_T^{(2)} - \hat{\mu}_2\right) + \sigma_1\sqrt{1-\rho^2}W_T^{(1)}\right], \tag{1.354}$$

$$\hat{\mu}_i \stackrel{\Delta}{=} \ln S_0^{(i)} + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)T, \tag{1.355}$$

which shows that given $S_T^{(2)}$, $\ln S_T^{(1)}$ is normally distributed with mean and variance

$$\mu = \hat{\mu}_1 + \frac{\sigma_1\rho}{\sigma_2}\left(\ln S_T^{(2)} - \hat{\mu}_2\right), \tag{1.356}$$

$$\sigma^2 = \sigma_1^2(1-\rho^2)T. \tag{1.357}$$

We recall from the derivation of the Black-Scholes formula for vanilla options that (and in fact, for $\rho = 0$ this *is* the Black-Scholes formula)

$$\begin{aligned} &I\!E\left[\left(\phi\left(S_T^{(1)} - K\right)\right)^+\right] \qquad (1.358)\\ &= \phi\left[e^{\mu+\frac{\sigma^2}{2}}\mathcal{N}\left(\phi\frac{-\ln K + \mu + \sigma^2}{\sigma}\right) - K\mathcal{N}\left(\phi\frac{-\ln K + \mu + \sigma^2}{\sigma}\right)\right], \end{aligned}$$

which allows to compute the value of a spread option as

$$\begin{aligned} &e^{-rT} I\!E\left[\left(\phi\left(aS_T^{(1)} - bS_T^{(2)} - K\right)\right)^+\right] \qquad (1.359)\\ &= a I\!E\left[e^{-rT} I\!E\left[\left(\phi\left(S_T^{(1)} - \left(\frac{b}{a}S_T^{(2)} + \frac{K}{a}\right)\right)\right)^+ \middle| S_T^{(2)}\right]\right] \qquad (1.360)\\ &= a \cdot I\!E\left[\text{vanilla}\left(S_0^{(1)} \exp\left\{\frac{\sigma_1\rho}{\sigma_2}\left(\ln S_T^{(2)} - \hat{\mu}_2\right) - \frac{1}{2}\sigma_1^2\rho^2 T\right\}\right.\right., \\ &\qquad\left.\left.\frac{b}{a}S_T^{(2)} + \frac{K}{a}, \sigma_1\sqrt{1-\rho^2}, r, r_1, T, \phi\right)\right] \qquad (1.361)\\ &= \int_{\infty}^{\infty} \text{vanilla}\left(aS_0^{(1)} \exp\left\{\sigma_1\rho\sqrt{T}x - \frac{1}{2}\sigma_1^2\rho^2 T\right\}\right., \\ &\qquad\left. b\exp\{\sigma_2\sqrt{T}x + \hat{\mu}_2\} + K, \sigma_1\sqrt{1-\rho^2}, r, r_1, T, \phi\right) n(x)\,dx \\ &= \int_{-\infty}^{+\infty} \text{vanilla}\left(S(x), K(x), \sigma_1\sqrt{1-\rho^2}, r, r_1, T, \phi\right) n(x)\,dx. \qquad (1.362) \end{aligned}$$

Table 1.33 Example of a spread option

	EUR	GBP
Spot in USD	1.2000	1.8000
Interest rates	2 %	4 %
Volatility	10 %	9 %
Weights	1/1.2000	1/1.8000
USD rate	3 %	
Correlation	20 %	
Maturity	0.5 years	
Strike	0.0020	
Value	0.0375 USD	

Example

We consider the example in Table 1.33. An investor or corporate believes that EUR/USD will out perform GBP/USD in 6 months. To make it concrete we first normalize both exchange rates by dividing by their current spot and then want to reward the investor one pip for each pip the normalized EUR/USD will be more than 20 pips higher than normalized GBP/USD.

1.6.7 Baskets

This section is produced jointly with Jürgen Hakala and appeared first in [60].

In many cases corporate and institutional currency managers are faced with an exposure in more than one currency. Generally these exposures would be hedged using individual strategies for each currency. These strategies are composed of spot transactions, forwards, and in many cases options on a single currency. Nevertheless, there are instruments that include several currencies, and these can be used to build a multi-currency strategy that is almost always cheaper than the portfolio of the individual strategies. As a prominent example we now consider basket options in detail.

Protection with currency baskets

Basket options are derivatives based on a common base currency, say EUR, and several other risky currencies. The option is actually written on the basket of risky currencies. Basket options are European options paying the difference between the basket value and the strike, if positive, for a basket call, or the difference between strike and basket value, if positive, for a basket put respectively at maturity. The risky currencies have different weights in the basket to reflect the details of the exposure.

For example, a basket call on two currencies USD and JPY pays off

$$\max\left(a_1\frac{S_1(T)}{S_1(0)}+a_2\frac{S_2(T)}{S_2(0)}-K,0\right) \tag{1.363}$$

at maturity T, where $S_1(t)$ denotes the exchange rate of EUR-USD and $S_2(t)$ denotes the exchange rate of EUR-JPY at time t, a_i the corresponding weights and K the basket strike. A basket option protects against a drop in both currencies at the same time. Individual options on each currency cover some cases that are not protected by a basket option (shaded triangular areas in Figure 1.51) and that's why they cost more than a basket.

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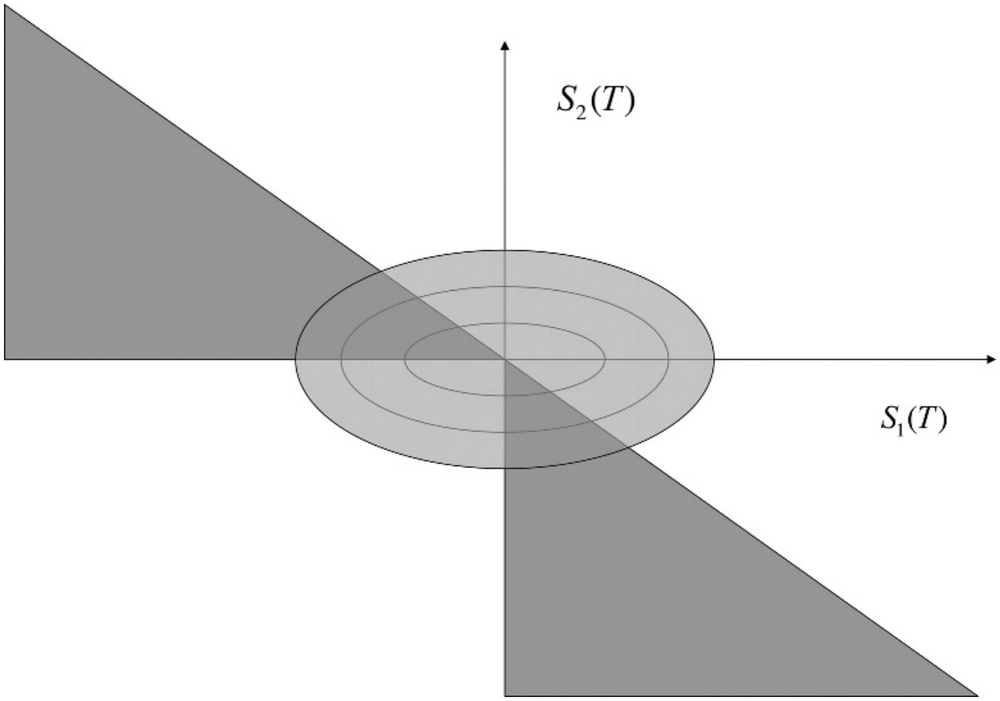

Figure 1.51 Protection with a basket option in two currencies
The ellipsoids connect the points that are reached with the same probability assuming that the forward prices are at the center.

Pricing basket options

Basket options should be priced in a consistent way with plain vanilla options. In the Black-Scholes model we assume a log-normal process for the individual correlated basket constituents. A decomposition into uncorrelated constituents of the exchange rate processes

$$dS_i = \mu_i S_i \, dt + S_i \sum_{j=1}^{N} \Omega_{ij} \, dW_j \tag{1.364}$$

is the basis for pricing. Here μ_i denotes the difference between the foreign and the domestic interest rate of the i-th currency pair, dW_j the j-th component of independent Brownian increments. The covariance matrix is given by

$$C_{ij} = (\Omega\Omega^T)_{ij} = \rho_{ij}\sigma_i\sigma_j. \tag{1.365}$$

Here σ_i denotes the volatility of the i-th currency pair and ρ_{ij} the correlation coefficients.

Exact Method. Starting with the uncorrelated components the pricing problem is reduced to the N-dimensional integration of the payoff. This method is accurate but rather slow for more than two or three basket components.

A Simple Approximation method assumes that the basket spot itself is a log-normal process with drift μ and volatility σ driven by a Wiener Process $W(t)$,

$$dS(t) = S(t)[\mu\,dt + \sigma\,dW(t)] \tag{1.366}$$

with solution

$$S(T) = S(t)e^{\sigma W(T-t)+\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)}, \tag{1.367}$$

given we know the spot $S(t)$ at time t. It is a fact that the sum of log-normal processes is not log-normal, but as a crude approximation it is certainly a quick method that is easy to implement. In order to price the basket call the drift and the volatility of the basket spot need to be determined. This is done by matching the first and second moment of the basket spot with the first and second moment of the log-normal model for the basket spot. The moments of log-normal spot are

$$\mathbb{E}[S(T)] = S(t)e^{\mu(T-t)}, \tag{1.368}$$
$$\mathbb{E}[S(T)^2] = S(t)^2 e^{(2\mu+\sigma^2)(T-t)}. \tag{1.369}$$

We solve these equations for the drift and volatility,

$$\mu = \frac{1}{T-t}\ln\left(\frac{\mathbb{E}[S(T)]}{S(t)}\right), \tag{1.370}$$
$$\sigma = \sqrt{\frac{1}{T-t}\ln\left(\frac{\mathbb{E}[S(T)^2]}{S(t)^2}\right)}. \tag{1.371}$$

In these formulae we now use the moments for the basket spot,

$$\mathbb{E}[S(T)] = \sum_{j=1}^{N}\alpha_j S_j(t)e^{\mu_j(T-t)}, \tag{1.372}$$
$$\mathbb{E}[S(T)^2] = \sum_{i,j=1}^{N}\alpha_i\alpha_j S_i(t)S_j(t)e^{\left(\mu_i+\mu_j+\sum_{k=1}^{N}\Omega_{ki}\Omega_{jk}\right)(T-t)}. \tag{1.373}$$

The value is given by the well-known Black-Scholes-Merton formula for plain vanilla call options,

$$v(0) = e^{-r_d T}\left(f\mathcal{N}(d_+) - K\mathcal{N}(d_-)\right), \tag{1.374}$$
$$f = S(0)e^{\mu T}, \tag{1.375}$$
$$d_\pm = \frac{\ln\frac{f}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \tag{1.376}$$

where $\mathcal{N}$ denotes the cumulative standard normal distribution function and r_d the domestic interest rate.

A more accurate and equally fast approximation. The previous approach can be taken one step further by introducing one more term in the Itô-Taylor expansion of the basket spot, which results in

$$v(0) = e^{-r_d T} \left(F\mathcal{N}(d_1) - K\mathcal{N}(d_2) \right), \tag{1.377}$$

$$F = \frac{S(0)}{\sqrt{1-\lambda T}} e^{\left(\mu - \frac{\lambda}{2} + \frac{\lambda\sigma^2}{2(1-\lambda T)}\right)T}, \tag{1.378}$$

$$d_2 = \frac{\sigma - \sqrt{\sigma^2 + \lambda\left(\left(1 + \frac{\lambda}{1-\lambda T}\right)\sigma^2 T - 2\ln\frac{F\sqrt{1-\lambda T}}{K}\right)}}{\lambda\sqrt{T}}, \tag{1.379}$$

$$d_1 = \sqrt{1-\lambda T}\, d_2 + \frac{\sigma\sqrt{T}}{\sqrt{1-\lambda T}}. \tag{1.380}$$

The new parameter λ is determined by matching the third moment of the basket spot and the model spot. For details see [3]. Most remarkably this major improvement in the accuracy only requires a marginal additional computation effort.

Correlation risk

Correlation coefficients between market instruments are usually not obtained easily. Either historical data-analysis or implied calibrations need to be done. However, in the foreign exchange market the cross instrument is traded as well, for the example above the USD-JPY spot and options are traded, and the correlation can be determined from this contract. In fact, denoting the volatilities as in the tetrahedron in Figure 1.52, we obtain formulae for the correlation coefficients in terms of known market implied volatilities

$$\rho_{12} = \frac{\sigma_3^2 - \sigma_1^2 - \sigma_2^2}{2\sigma_1\sigma_2}, \tag{1.381}$$

$$\rho_{34} = \frac{\sigma_1^2 + \sigma_6^2 - \sigma_2^2 - \sigma_5^2}{2\sigma_3\sigma_4}. \tag{1.382}$$

This method also allows hedging correlation risk by trading FX implied volatility. For details see [3].

Pricing basket options with smile

The previous calculations are all based on the Black-Scholes model with constant market parameters for rates and volatility. This can all be made time-dependent and can then include the term structure of volatility. If we wish to include the smile in the valuation, then we can either switch to a more appropriate model or perform a Monte Carlo simulation where the probabilities of the exchange rate paths are computed in such a way that the individual vanilla prices are correctly determined. This *weighted Monte Carlo approach* has been discussed by Avellaneda et al. in [61].

Practical Example
We want to find out how much one can save using a basket option. We take EUR as a base currency and consider a basket of three currencies USD, GBP and JPY. We list the contract data and the amount of option premium one can save using a basket call rather than three individual call options in Table 1.34 and the market data in Table 1.35.

The amount of premium saved essentially depends on the correlation of the currency pairs. In Figure 1.53 we take the parameters of the previous scenario, but restrict ourselves to the currencies USD and JPY.

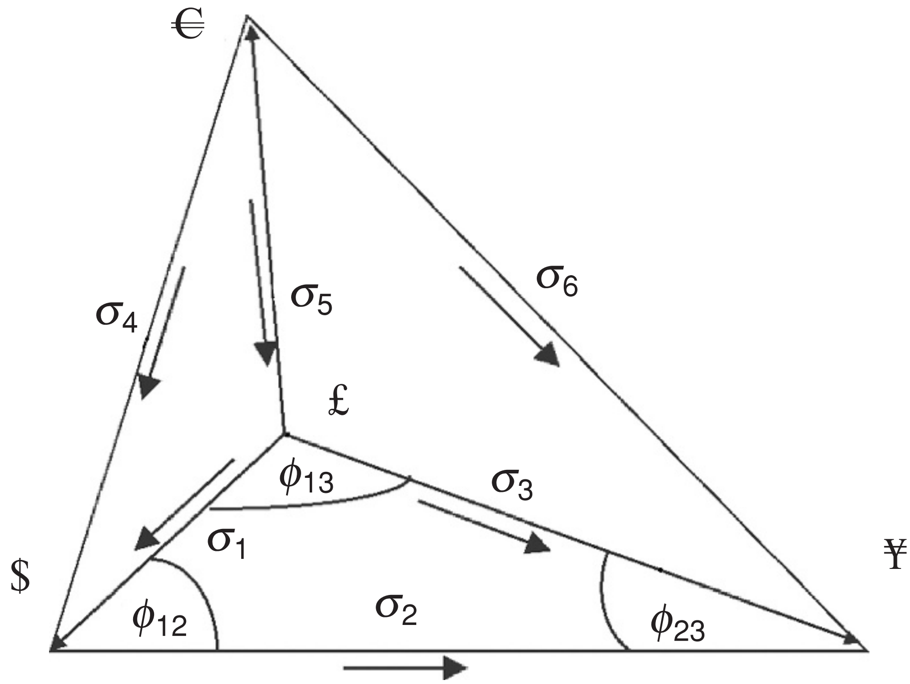

Figure 1.52 Relationship between volatilities σ(edges) and correlations ρ (cosines of angles) in a tetrahedron with 4 currencies and 6 currency pairs. The arrows mark the market standard quotation direction, i.e. in EUR-USD the base currency is USD and the arrow points to USD

Conclusions

Many corporate clients are exposed to multi-currency risk. One way to turn this fact into an advantage is to use multi-currency hedge instruments. We have shown that basket options are convenient instruments protecting against exchange rates of most of the basket components changing in the same direction. A rather unlikely market move of half of the currencies' exchange rates in opposite directions is not protected by basket options, but when taking this residual risk into account the hedging cost is reduced substantially. Another example how to use currency basket options is discussed in Section 2.5.2.

Table 1.34 Sample contact data of a EUR call basket put

Contract data	Strikes	Weights	Single option Prices
EUR/USD	1.1390	33.33 %	4.94 %
EUR/GBP	0.7153	33.33 %	2.50 %
EUR/JPY	125.00	33.33 %	3.87 %
sum		100 %	3.77 %
basket price			2.90 %

The value of the basket is noticeably less than the value of 3 vanilla EUR calls

Table 1.35 Sample market data of 21 October 2003 of four currencies EUR, GBP, USD and JPY

		Correlation						
Vol	Spot	ccy pair	GBP/USD	USD/JPY	GBP/JPY	EUR/USD	EUR/GBP	EUR/JPY
8.80	1.6799	GBP/USD	1.00	−0.49	0.42	0.72	−0.15	0.29
9.90	109.64	USD/JPY	−0.49	1.00	0.59	−0.55	−0.21	0.41
9.50	184.17	GBP/JPY	0.42	0.59	1.00	0.09	−0.35	0.70
10.70	1.1675	EUR/USD	0.72	−0.55	0.09	1.00	0.58	0.54
7.50	0.6950	EUR/GBP	−0.15	−0.21	−0.35	0.58	1.00	0.42
9.80	128.00	EUR/JPY	0.29	0.41	0.70	0.54	0.42	1.00

The correlation coefficients are implied from the volatilities based on Equations (1.381) and (1.382).

1.6.8 Best-of and worst-of options

Options on the maximum or minimum of two or more exchange rates pay in their simple version

$$\left[\phi\left(\eta \min\left(\eta S_T^{(1)}, \eta S_T^{(2)}\right) - K\right)\right]^+ . \tag{1.383}$$

This is a European put or call with expiration time T in years on the minimum ($\eta = +1$) or maximum ($\eta = -1$) of the two underlyings $S_T^{(1)}$ and $S_T^{(2)}$ with strike K. As usual, the binary variable ϕ takes the value $+1$ for a call and -1 for a put.

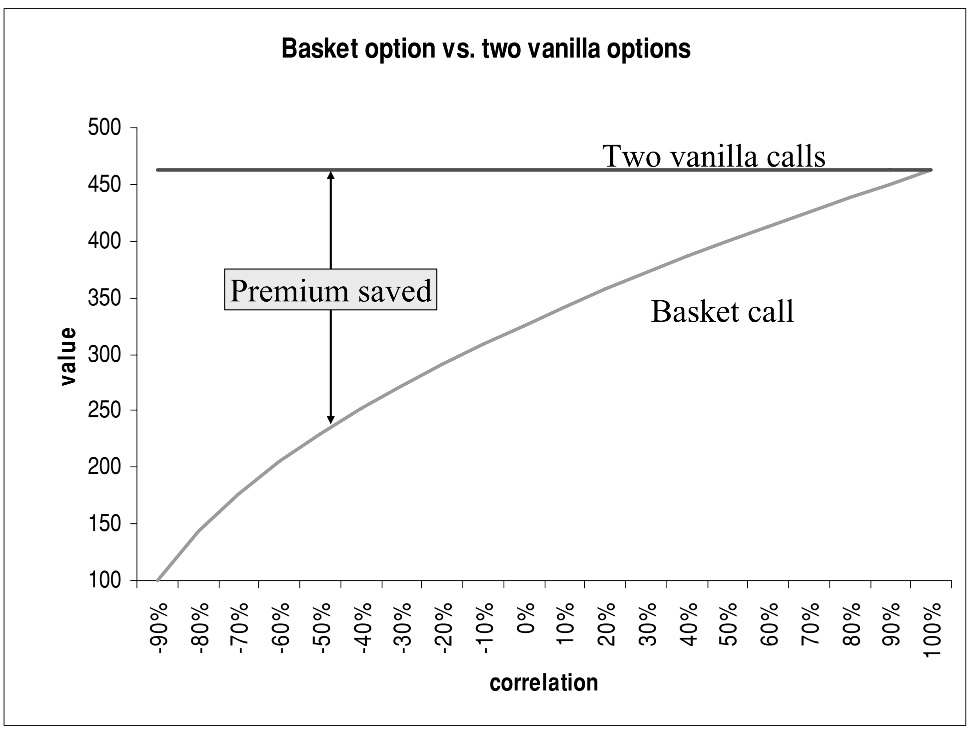

Figure 1.53 Amount of premium saved in a basket of two currencies compared to two single vanillas as a function of correlation: The smaller the correlation, the higher the premium savings effect

Valuation in the Black Scholes model

In the two-dimensional Black-Scholes model

$$dS_t^{(1)} = S_t^{(1)} \left[\mu_1 dt + \sigma_1 dW_t^{(1)} \right], \tag{1.384}$$

$$dS_t^{(2)} = S_t^{(2)} \left[\mu_2 dt + \sigma_2 dW_t^{(2)} \right], \tag{1.385}$$

$$\mathbf{Cov}\left[W_t^{(1)}, W_t^{(2)} \right] = \sigma_1 \sigma_2 \rho t, \tag{1.386}$$

we let the positive constants σ_i denote the volatilities of the i-th foreign currency, ρ the instantaneous correlation of their log-returns, r the domestic risk free rate. In a risk-neutral setting the drift terms μ_i take the values

$$\mu_i = r - r_i, \tag{1.387}$$

where r_i denotes the risk free rate of the i-th foreign currency.

The value has been published originally by Stulz in [62] and happens to be

$$\begin{aligned} & v\big(t, S_t^{(1)}, S_t^{(2)}, K, T, r_1, r_2, r, \sigma_1, \sigma_2, \rho, \phi, \eta\big) \\ &= \phi \Big[S_t^{(1)} e^{-r_1 \tau} \mathcal{N}_2(\phi d_1, \eta d_3; \phi \eta \rho_1) \\ &+ S_t^{(2)} e^{-r_2 \tau} \mathcal{N}_2(\phi d_2, \eta d_4; \phi \eta \rho_2) \\ &- K e^{-r\tau} \left(\frac{1 - \phi\eta}{2} + \phi\eta \mathcal{N}_2(\eta(d_1 - \sigma_1 \sqrt{\tau}), \eta(d_2 - \sigma_2 \sqrt{\tau}); \rho) \right) \Big], \end{aligned} \tag{1.388}$$

$$\sigma^2 \triangleq \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2, \tag{1.389}$$

$$\rho_1 \triangleq \frac{\rho\sigma_2 - \sigma_1}{\sigma}, \tag{1.390}$$

$$\rho_2 \triangleq \frac{\rho\sigma_1 - \sigma_2}{\sigma}, \tag{1.391}$$

$$\tau \triangleq T - t, \tag{1.392}$$

$$d_1 \triangleq \frac{\ln\big(S_t^{(1)}/K\big) + \big(\mu_1 + \frac{1}{2}\sigma_1^2\big)\tau}{\sigma_1 \sqrt{\tau}}, \tag{1.393}$$

$$d_2 \triangleq \frac{\ln\big(S_t^{(2)}/K\big) + \big(\mu_2 + \frac{1}{2}\sigma_2^2\big)\tau}{\sigma_2 \sqrt{\tau}}, \tag{1.394}$$

$$d_3 \triangleq \frac{\ln\big(S_t^{(2)}/S_t^{(1)}\big) + \big(r_1 - r_2 - \frac{1}{2}\sigma^2\big)\tau}{\sigma \sqrt{\tau}}, \tag{1.395}$$

$$d_4 \triangleq \frac{\ln\big(S_t^{(1)}/S_t^{(2)}\big) + \big(r_2 - r_1 - \frac{1}{2}\sigma^2\big)\tau}{\sigma \sqrt{\tau}}. \tag{1.396}$$

The bivariate standard normal distribution and density functions $\mathcal{N}_2$ and n_2 are defined in Equations (1.306) and (1.305).

Greeks

Most of the calculations of partial derivatives can be simplified substantially by the homogeneity method described in [6], which states for instance, that

$$v = S_t^{(1)} \frac{\partial v}{\partial S_t^{(1)}} + S_t^{(2)} \frac{\partial v}{\partial S_t^{(2)}} + K \frac{\partial v}{\partial K}. \tag{1.397}$$

Using this equation we can immediately write down the deltas.

deltas

$$\frac{\partial v}{\partial S_t^{(1)}} = \phi e^{-r_1 \tau} \mathcal{N}_2(\phi d_1, \eta d_3; \phi \eta \rho_1) \tag{1.398}$$

$$\frac{\partial v}{\partial S_t^{(2)}} = \phi e^{-r_2 \tau} \mathcal{N}_2(\phi d_3, \eta d_4; \phi \eta \rho_2) \tag{1.399}$$

dual delta (strike)

$$\frac{\partial v}{\partial K} = -\phi e^{-r\tau} \left(\frac{1 - \phi\eta}{2} + \phi\eta \mathcal{N}_2(\eta(d_1 - \sigma_1 \sqrt{\tau}), \eta(d_2 - \sigma_2 \sqrt{\tau}); \rho) \right) \tag{1.400}$$

gammas We use the identities

$$\frac{\partial}{\partial x} \mathcal{N}_2(x, y; \rho) = n(x) \mathcal{N} \left(\frac{y - \rho x}{\sqrt{1 - \rho^2}} \right), \tag{1.401}$$

$$\frac{\partial}{\partial y} \mathcal{N}_2(x, y; \rho) = n(y) \mathcal{N} \left(\frac{x - \rho y}{\sqrt{1 - \rho^2}} \right), \tag{1.402}$$

and obtain

$$\frac{\partial^2 v}{\partial (S_t^{(1)})^2} = \frac{\phi e^{-r_1 \tau}}{S_t^{(1)} \sqrt{\tau}} \left[\frac{\phi}{\sigma_1} n(d_1) \mathcal{N} \left(\eta\sigma \frac{d_3 - d_1 \rho_1}{\sigma_2 \sqrt{1 - \rho^2}} \right) - \frac{\eta}{\sigma} n(d_3) \mathcal{N} \left(\phi\sigma \frac{d_1 - d_3 \rho_1}{\sigma_2 \sqrt{1 - \rho^2}} \right) \right], \tag{1.403}$$

$$\frac{\partial^2 v}{\partial (S_t^{(2)})^2} = \frac{\phi e^{-r_2 \tau}}{S_t^{(2)} \sqrt{\tau}} \left[\frac{\phi}{\sigma_2} n(d_2) \mathcal{N} \left(\eta\sigma \frac{d_4 - d_2 \rho_2}{\sigma_1 \sqrt{1 - \rho^2}} \right) - \frac{\eta}{\sigma} n(d_4) \mathcal{N} \left(\phi\sigma \frac{d_2 - d_4 \rho_2}{\sigma_1 \sqrt{1 - \rho^2}} \right) \right], \tag{1.404}$$

$$\frac{\partial^2 v}{\partial S_t^{(1)} \partial S_t^{(2)}} = \frac{\phi\eta e^{-r_1 \tau}}{S_t^{(2)} \sigma \sqrt{\tau}} n(d_3) \mathcal{N} \left(\phi\sigma \frac{d_1 - d_3 \rho_1}{\sigma_2 \sqrt{1 - \rho^2}} \right). \tag{1.405}$$

sensitivity with respect to correlation Direct computations require the identity

$$\frac{\partial}{\partial\rho}\mathcal{N}_2(x, y; \rho) = \frac{1}{\sqrt{1-\rho^2}} n(y) n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right) \tag{1.406}$$

$$= \frac{1}{\sqrt{1-\rho^2}} n(x) n\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right) \tag{1.407}$$

$$= n_2(x, y, \rho). \tag{1.408}$$

However, it is easier to use the following identity relating correlation risk and cross gamma outlined in [6].

$$\frac{\partial v}{\partial\rho} = \sigma_1\sigma_2\tau S_t^{(1)} S_t^{(2)} \frac{\partial^2 v}{\partial S_t^{(1)} \partial S_t^{(2)}} \tag{1.409}$$

vegas Again, we refer to [6] to get the following formulas for the vegas.

$$\frac{\partial v}{\partial\sigma_1} = \frac{\rho v_\rho + \sigma_1^2\tau (S_t^{(1)})^2 v_{S_t^{(1)} S_t^{(1)}}}{\sigma_1} \tag{1.410}$$

$$= S_t^{(1)} e^{-r_1\tau}\sqrt{\tau}\left[\rho_1\phi\eta n(d_3)\mathcal{N}\left(\phi\sigma\frac{d_1 - d_3\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right) + n(d_1)\mathcal{N}\left(\eta\sigma\frac{d_3 - d_1\rho_1}{\sigma_2\sqrt{1-\rho^2}}\right)\right] \tag{1.411}$$

$$\frac{\partial v}{\partial\sigma_2} = \frac{\rho v_\rho + \sigma_2^2\tau (S_t^{(2)})^2 v_{S_t^{(2)} S_t^{(2)}}}{\sigma_2} \tag{1.412}$$

$$= S_t^{(2)} e^{-r_2\tau}\sqrt{\tau}\left[\rho_2\phi\eta n(d_4)\mathcal{N}\left(\phi\sigma\frac{d_2 - d_4\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right) + n(d_2)\mathcal{N}\left(\eta\sigma\frac{d_4 - d_2\rho_2}{\sigma_1\sqrt{1-\rho^2}}\right)\right] \tag{1.413}$$

rhos Again, we refer to [6] to get the following formulas for the rhos.

$$\frac{\partial v}{\partial r_1} = -S_t^{(1)}\tau\frac{\partial v}{\partial S_t^{(1)}} \tag{1.414}$$

$$\frac{\partial v}{\partial r_2} = -S_t^{(2)}\tau\frac{\partial v}{\partial S_t^{(2)}} \tag{1.415}$$

$$\frac{\partial v}{\partial r} = -K\tau\frac{\partial v}{\partial K} \tag{1.416}$$

theta Among the various ways to compute theta one may use the one based on [6].

$$\frac{\partial v}{\partial t} = -\frac{1}{\tau}\left[r_1 v_{r_1} + r_2 v_{r_2} + r v_r + \frac{\sigma_1}{2} v_{\sigma_1} + \frac{\sigma_2}{2} v_{\sigma_2}\right] \tag{1.417}$$

More general results about best-of and worst of options can be found in detail in Chapter 7 of [3].

Variations

Options on the maximum and minimum generalize in various ways. For instance, they can be quantoed or have individual strikes for each currency pair. We consider some examples.

Multiple strike option

This variation of best-of/worst-of options deals with individual strikes, i.e. they pay off

$$\max_i \left[0; M_i(\phi(S_T^{(i)} - K_i))\right]. \tag{1.418}$$

Madonna option

This one pays the *Euclidian distance*,

$$\max \left[0; \sqrt{\sum_i (S_T^{(i)} - K_i)^2}\right]. \tag{1.419}$$

Pyramid option

This one pays the *maximum norm*,

$$\max \left[0; \sum_i |S_T^{(i)} - K_i| - K\right]. \tag{1.420}$$

Mountain range and Himalaya option. This type of option comes in various flavors and is rather popular in equity markets, whence we will not discuss them here. A reference is the thesis by Mahomed [63].

Quanto best-of/worst-of options. These options come up naturally if an investor wants to participate in several exchange rate movements with a payoff in other than the base currency.

Barrier best-of/worst-of options. One can also add knock-out and knock-in features to all the previous types discussed.

Application for re-insurance

Suppose you want to protect yourself against a weak USD compared to several currencies for a period of one year. As USD seller and buyer of EUR, GBP and JPY you need simultaneous protection of all three rising against the USD. Of course, you can buy three put options, but if you only need one of the three, then you can save considerably on the premium, as shown in Table 1.36. We can imagine a situation like this if a re-insurance company insures ships in various oceans. If a ship sinks near the coast of Japan, the client will have to be paid an amount in JPY. The re-insurance company is long USD and assumes only one ship to sink at most in one year and ready to take the residual risk of more than one sinking.

Table 1.36 Example of a triple strike best-of call (American style) with 100 M USD notional and one year maturity. Compared to buying vanilla options one saves 800,000 USD or 20 %

Currency pair	Spot	Strikes	Vanilla premium	Best-of premium
EUR/USD	0.9750	1.0500	1.4 M	
USD/JPY	119.00	110.00	1.7 M	
GBP/USD	1.5250	1.6300	0.9 M	
		Total in USD	4.0 M	3.2 M

Since the accidents can occur any time, all options are of American style, i.e. they can be exercised any time. The holder of the option can choose the currency pair to exercise. Hence, he can decide for the one with the highest profit, even if the currency of accident is a different one. It would be difficult to incorporate and hedge this event insurance into the product, whence the protection needs to assume the worst case scenario that is still acceptable to the re-insurance company. For example, if the re-insurance needs GBP and the spots at exercise time are at EUR/USD = 1.1200, USD/JPY = 134.00 and GBP/USD = 1.6400, you will find both the EUR and GBP constituents in-the-money. However, exercising in GBP would pay a net of 613,496.93 USD, in EUR 6,666,666.67 USD. The client would then exercise in EUR, buy the desired GBP in the EUR/GBP spot market and keep the rest of the EUR.

Application for corporate and retail investors

Just like a *dual currency deposit* described in Section 2.3.1, one can use a worst-of put to structure a *multi-currency deposit* with a coupon even higher. We refer the reader to the exercises.

1.6.9 Options and forwards on the harmonic average

Let there be a time schedule of observation times $t_1, \ldots, t_n$ of some underlying. Options and Forwards on the arithmetic average

$$\frac{1}{n}\sum_{i=1}^{n} S(t_i) \tag{1.421}$$

have been analyzed and traded for some time, see Section 1.5.4. The geometric average

$$\sqrt[n]{\prod_{i=1}^{n} S(t_i)} \tag{1.422}$$

has often been used as control variate for the arithmetic average, whose distribution in a multiplicative model like Black-Scholes is cumbersome to deal with. The *harmonic average*

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{R(t_i)}} \tag{1.423}$$

comes up if a client wants to exchange an amount of *domestic* currency into the *foreign* currency at an average rate of the currency pair FOR-DOM, e.g. wants to exchange USD into EUR at a rate, which is an average of observed EUR-USD rates. In this case the USD is the base or numeraire currency and we need to actually look at the exchange rate of $R = 1/S$ in DOM-FOR quotation in order to allow the domestic currency as a notional amount. As in the case of standard Asian contracts, there can be forwards and options on the harmonic average, both with fixed and floating strike. We treat one possible example in the next section.

Harmonic Asian swap

We consider a EUR-USD market with spot reference 1.0070, swap points for time T_1 of −45, swap points for time $T_2 > T_1$ of −90. As a contract specification, the client buys N USD at the daily average of the period of one months before T_1, denoted by A_1. Then the client sells N USD at the daily average of a period of one month before T_2, denoted by A_2. The payoff in EUR of this structure (cash settled two business days after T_2) is

$$\frac{N}{A_2} - \frac{N}{A_1}. \tag{1.424}$$

To replicate this using the fixed strike Asian Forward we can decompose it as follows.

1. We sell to the client the payoff $1 - \frac{1}{A_1}$ (using strike 1 by default) with notional N.
2. We buy from the client the payoff $1 - \frac{1}{A_2}$ (using strike 1 by default) with notional N.

On a notional of $N = 5$ million USD this could have a theoretical value of −23.172 EUR. This is what we should charge the client in addition to overhedge and sales margin. One problem is that the structure is very transparent for the client. If we take the forward for mid February, we have −45 swap points, for mid June −90 swap points. This means that the client would know that in a first order approximation he owes the bank 45 swap points, which is

$$5 \text{ MIO USD} \cdot 0.0045 = 22{,}500\text{EUR}.$$

If the swap ticket requires entering a strike, one can use 1.0000 in both tickets, but this value does not influence the value of the swap.

1.6.10 Variance and volatility swaps

A variance swap is a contract that pays the difference of a pre-determined fixed variance (squared volatility), which is usually determined in such a way that the trading price is zero, and a realized historic annualized variance, which can be computed only at maturity of the trade. Therefore, the variance swap is an ideal instrument to hedge volatility exposure, a need for funds and institutional clients. Of course one can hedge vega with vanilla options, but is then also subject to spot movements and time decay of the hedge instruments. The variance swap also serves as a tool for speculating on volatility.

Advantages

- Insurance against changing volatility levels
- Independence of spot
- Zero cost product
- Fixed volatility (break-even point) easy to approximate as average of smile

Table 1.37 Example of a Variance Swap in EUR-USD

Spot reference	1.0075 EUR-USD
Notional M	USD 10,000,000
Start date	19 November 2002
Expiry date	19 December 2002
Delivery of cash settlement	23 December 2002
Fixing period	Every weekday from 19-Nov-02 to 19-Dec-02
Fixing source	ECB fixings $F_0, F_1, \ldots, F_N$
Number of fixing days N	23 (32 actual days)
Annualization factor B	262.3 = 23/32*365
Fixed strike K	85.00 % % corresponding to a volatility of 9.22 %
Payoff	$M *$ (realized variance $- K$)
Realized variance	$\frac{B}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2; \bar{r} = \frac{1}{N}\sum_{i=1}^{N} r_i; r_i = \ln \frac{F_i}{F_{i-1}}$
Premium	none

The quantity r_i is called the log-return from fixing day $i - 1$ to day i and the average log-return is denoted by $\bar{r}$. The notation %% means a multiplication with 0.0001. It is also sometimes denoted as $\%^2$.

Disadvantages

- Difficult to understand
- Many details in the contract to be set
- Variance harder to capture than volatility
- Volatility swaps are harder to price than variance swaps

Example

Suppose the 1-month implied volatility for EUR/USD at-the-money options are close to its one-year historic low. This can easily be noticed by looking at *volatility cones*, see Section 1.3.10. Suppose further that you are expecting a period of higher volatility during the next month. Your are looking for a zero cost strategy, where you would be rewarded if your expectation turns out to be correct, but you are ready to encounter a loss otherwise. In this case an advisable strategy to trade is a variance or volatility swap. We consider an example of a variance swap in Table 1.37.

To make this clear we consider the following two scenarios with possible fixing results listed in Table 1.38 and Figure 1.54.

- If the realized variance is 0.41 % (corresponding to a volatility of 6.42 %), then the market was quieter than expected and you need to pay 10 MIO USD * (0.85 % – 0.41 %) = 44,000 USD.
- If the realized variance is 1.15 % (corresponding to a volatility of 10.7 %), then your market expectation turned out to be correct and you will receive 10 MIO USD * (1.15 %–0.85 %) = 30,000 USD.

A volatility swap trades

$$\sqrt{\frac{B}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2} \tag{1.425}$$

Table 1.38 Example of two variance scenarios in EUR-USD

Date	Fixing (low vol)	Fixing (high vol)
19/11/02	1.0075	1.0075
20/11/02	1.0055	1.0055
21/11/02	1.0111	1.0111
22/11/02	1.0086	1.0086
25/11/02	1.0027	1.0027
26/11/02	1.0019	1.0067
27/11/02	1.0033	0.9997
28/11/02	1.0096	1.0113
29/11/02	1.0077	1.0062
2/12/02	1.0094	1.0094
3/12/02	1.0029	0.9999
4/12/02	1.0043	1.0043
5/12/02	0.9977	0.9977
6/12/02	0.9953	1.0037
9/12/02	0.9966	0.9962
10/12/02	0.9986	0.9986
11/12/02	1.0003	0.9907
12/12/02	0.9956	1.0018
13/12/02	0.9981	1.0000
16/12/02	0.9963	0.9963
17/12/02	1.0040	1.0040
18/12/02	1.0045	1.0017
19/12/02	1.0085	1.0114
variance	0.41 %	1.15 %
volatility	6.42 %	10.70 %

The left column shows a possible fixing set with a lower realized variance, the right column a scenario with a higher variance.

against a fixed volatility, which is usually determined in such a way that the trading price is zero. Since the square root is not a linear function of the variance, this product is more difficult to price than a standard variance swap. For details on pricing and hedging we refer to [64]. As a rule of thumb, the fixed variance or volatility to make the contract worth zero is the average of the volatilities in the volatility smile matrix for the maturity under consideration as there exists a static hedging portfolio consisting of vanilla options with the same maturity.

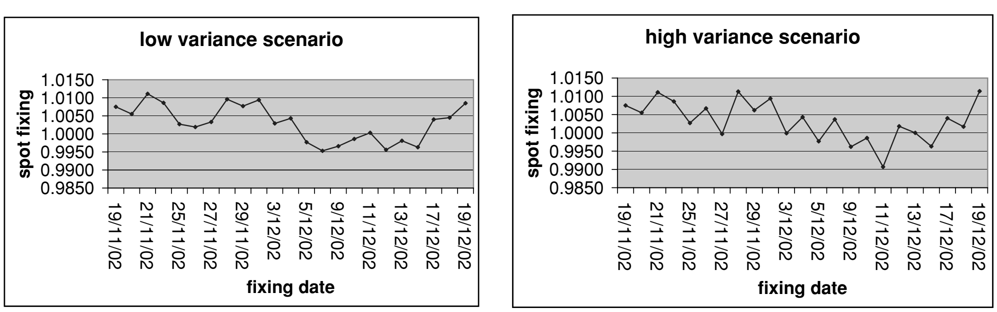

Figure 1.54 Comparison of scenarios for a low variance (left column) and a higher variance (right column)

Forward variance swap

In a standard variance swap, the first spot fixing is at inception of the trade or two business days thereafter. However, there may be situations where a client needs to hedge a forward volatility exposure that originates from a compound, instalment, forward start, cliquet or other exotic option with a significant forward volatility dependence. We will illustrate now how to structure a forward variance swap, where the first fixing is at some time in the future, using standard variance swaps. Let there be J fixings in the initial period and M fixings in the second period. The total number of Fixings is hence $M + J$. We can then split the payoff

$$\frac{B}{M-1}\sum_{i=J+1}^{J+M}(r_i - \bar{r})^2 - K \tag{1.426}$$

into the two parts

$$\begin{aligned} &\frac{B}{M-1}\sum_{i=1}^{J+M}(r_i - \bar{r})^2 - K - \left[\frac{B}{M-1}\sum_{i=1}^{J}(r_i - \bar{r})^2 - 0\right] \\ &= \frac{C}{J+M-1}\sum_{i=1}^{J+M}(r_i - \bar{r})^2 - K - \left[\frac{D}{J-1}\sum_{i=1}^{J}(r_i - \bar{r})^2 - 0\right] \end{aligned} \tag{1.427}$$

and find as the only solution for the numbers C and B

$$\begin{aligned} C &= \frac{(J+M-1)B}{M-1}, \\ D &= \frac{(J-1)B}{M-1}. \end{aligned} \tag{1.428}$$

Modifications

When computing the variance of a random variable X whose mean is small, we can take the second moment $\mathbb{E}X^2$ as an approximation of the variance

$$\mathbf{var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2. \tag{1.429}$$

Following this idea and keeping in mind that the average of log-returns of FX fixings is indeed often close to zero, the variance swap is sometimes understood as a second moment swap rather than an actual variance swap. To clarify traders specify in their dialogue whether the product is *mean subtracted* or not. We have presented here the variance swap with the mean subtracted.

1.6.11 Exercises

1. Sometimes buyers of options prefer to pay for their option only at the delivery date of the contract, rather than the spot value date, which is by default two business days after the trade date. How do the value formulae for say vanilla options change if we include this *deferred payment* style? You need to consider carefully the currency in which the premium is paid. To be precise, let the sequence of dates $t_0 < t_{sv} < t_e < t_d$ and t_{pv} denote the *trade date*, the *spot value date*, *expiration date*, *delivery date* and the *premium value date* respectively,

with all of the dates generally having different interest rates. How does the vanilla formula generalize?

2. Starting with the value for digital options, derive exactly the value of a European style corridor in the Black-Scholes model. Discuss how to find a market price based on the market of vanilla options. How does this extend to American style corridors?
3. How would you structure a *fade-out call* that starts with a nominal amount of M. As the exchange rate evolves, the notional will be decreased by $\frac{M}{N}$ for each of the N fixings that is outside a pre-defined range?
4. Similar to the corridors in Section 1.6.1 write down the exact payoff formulae for the various variations of faders in Section 1.6.2.
5. Describe a possible client view that could lead to trading a fade-in forward in Table 1.31.
6. What is wrong in Equation (1.427) in the decomposition of the forward variance swap? How can we fix it?
7. Implement the static hedge for a variance swap following [64] using the approximation of the logarithm by vanilla options. Then take the current smile of USD-JPY and find the fair fixed variance of a variance swap for 6 months maturity. How does the fixed variance change if the seller wants to earn a sales margin of 0.1 % of the notional amount? Compare the fixed strike with the average of the implied volatilities for 6 months. Discuss the impact of changing interest rates on the price and on the hedge.
8. Compute the integral in Equation (1.349) to get a closed form solution for the exchange option.
9. The value of a spread option presented in Section 1.6.6 works for the case of a joint base currency, like USD/CHF and USD/JPY. How does the formula extend if the quotation differs, like USD-CHF and EUR-USD, so there is a joint currency in both exchange rates, but the base currencies are different? More generally, consider arbitrary exchange rate pairs like EUR-GBP and USD-JPY.
10. How would delivery-settlement of a spread option work in practice?
11. Compute the correlation coefficients implied from the volatilities in Table 1.39 based on Equations (1.381) and (1.382). What are the upper and lower limits for the EUR/USD volatility to guarantee all correlation coefficients being contained in the interval $[-1, +1]$, assuming all the other volatilities are fixed?
12. As a variation of the James Bond range in Section 7, we consider barriers A, B, C, D as illustrated in Figure 1.55.

    A rather *tolerant double no-touch* knocks out after the second barrier is touched or crossed. How would you hedge it statically using standard barrier and touch options?

Table 1.39 Sample market data of four currencies EUR, GBP, USD and CHF

ccy pair	Volatility
GBP/USD	9.20 %
USD/CHF	11.00 %
GBP/CHF	8.80 %
EUR/USD	10.00 %
EUR/GBP	7.80 %
EUR/CHF	5.25 %

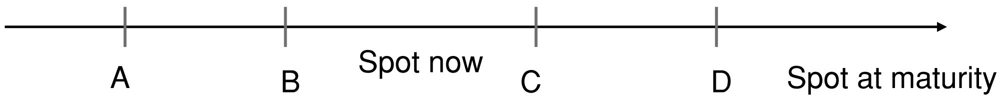

Figure 1.55 Nested double-no-touch ranges

13. Derive closed form solutions to all knock-in-knock-out types of barrier options in the Black-Scholes model.
14. The formula for the theoretical value of outside barrier options in Section 1.6.3 only works for two currency pairs with the same base or domestic currency such as for example EUR-USD and GBP-USD. Why? How does it extend if the second currency pair is USD-JPY? And how does it extend if the second currency pair is AUD-JPY?
15. The pricing of Parisian barrier options can be done with Monte Carlo and PDE methods. Implement the approach by Bernard, le Courtois and Quittard-Pinon using characteristic functions described in [65].
16. The pay-later price in Equation (1.342) is measured in units of domestic currency. Does this change if the premium is specified to be paid in foreign currency? If no, argue why. If yes, specify how.
17. Derive the pay-later price of a digital option.
18. Derive the pay-later price of a call spread.
19. How would you structure an up-and-out call whose premium is only paid if the spot is in-the-money at the expiration time?
20. A *chooser option* lets the buyer decide at expiration time, if he wants to either exercise α calls with strike K_c or β puts with strike K_p. Discuss how to find a market price and how to statically hedge it. (Hint: Straddle.). Moreover, if the decision of which option to take is taken at time t strictly before the expiration time T, how would you price and hedge the chooser? How does it simplify if $\alpha = \beta = 1$ and $K_c = K_p$?