

1

Simple Harmonic Motion

In the physical world there are many examples of things that vibrate or oscillate, i.e. perform periodic motion. Everyday examples are a swinging pendulum, a plucked guitar string and a car bouncing up and down on its springs. The most basic form of periodic motion is called simple harmonic motion (SHM). In this chapter we develop quantitative descriptions of SHM. We obtain equations for the ways in which the displacement, velocity and acceleration of a simple harmonic oscillator vary with time and the ways in which the kinetic and potential energies of the oscillator vary. To do this we discuss two particularly important examples of SHM: a mass oscillating at the end of a spring and a swinging pendulum. We then extend our discussion to electrical circuits and show that the equations that describe the movement of charge in an oscillating electrical circuit are identical in form to those that describe, for example, the motion of a mass on the end of a spring. Thus if we understand one type of harmonic oscillator then we can readily understand and analyse many other types. The universal importance of SHM is that to a good approximation many real oscillating systems behave like simple harmonic oscillators when they undergo oscillations of small amplitude. Consequently, the elegant mathematical description of the simple harmonic oscillator that we will develop can be applied to a wide range of physical systems.

1.1 PHYSICAL CHARACTERISTICS OF SIMPLE HARMONIC OSCILLATORS

Observing the motion of a pendulum can tell us a great deal about the general characteristics of SHM. We could make such a pendulum by suspending an apple from the end of a length of string. When we draw the apple away from its *equilibrium position* and release it we see that the apple swings back towards the equilibrium position. It starts off from rest but steadily picks up speed. We notice that it *overshoots* the equilibrium position and does not stop until it reaches the

other extreme of its motion. It then swings back toward the equilibrium position and eventually arrives back at its initial position. This pattern then repeats with the apple swinging backwards and forwards *periodically*. Gravity is the *restoring force* that attracts the apple back to its equilibrium position. It is the *inertia* of the mass that causes it to overshoot. The apple has kinetic energy because of its motion. We notice that its velocity is zero when its displacement from the equilibrium position is a maximum and so its kinetic energy is also zero at that point. The apple also has potential energy. When it moves away from the equilibrium position the apple's vertical height increases and it gains potential energy. When the apple passes through the equilibrium position its vertical displacement is zero and so all of its energy must be kinetic. Thus at the point of zero displacement the velocity has its maximum value. As the apple swings back and forth there is a continuous exchange between its potential and kinetic energies. These characteristics of the pendulum are common to all simple harmonic oscillators: (i) periodic motion; (ii) an equilibrium position; (iii) a restoring force that is directed towards this equilibrium position; (iv) inertia causing overshoot; and (v) a continuous flow of energy between potential and kinetic. Of course the oscillation of the apple steadily dies away due to the effects of dissipative forces such as air resistance, but we will delay the discussion of these effects until Chapter 2.

1.2 A MASS ON A SPRING

1.2.1 A mass on a horizontal spring

Our first example of a simple harmonic oscillator is a mass on a horizontal spring as shown in Figure 1.1. The mass is attached to one end of the spring while the other end is held fixed. The equilibrium position corresponds to the unstretched length of the spring and x is the displacement of the mass from the equilibrium position along the x -axis. We start with an idealised version of a real physical situation. It is idealised because the mass is assumed to move on a frictionless surface and the spring is assumed to be weightless. Furthermore because the motion is in the horizontal direction, no effects due to gravity are involved. In physics it is quite usual to start with a simplified version or model because real physical situations are normally complicated and hard to handle. The simplification makes the problem tractable so that an initial, idealised solution can be obtained. The complications, e.g. the effects of friction on the motion of the oscillator, are then added in turn and at each stage a modified and improved solution is obtained. This process invariably provides a great deal of physical understanding about the real system and about the relative importance of the added complications.

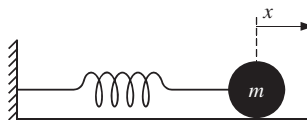


Figure 1.1 A simple harmonic oscillator consisting of a mass m on a horizontal spring.

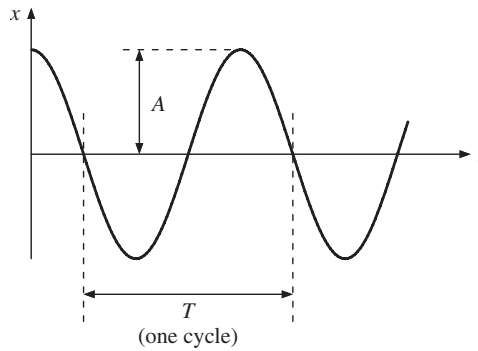


Figure 1.2 Variation of displacement x with time t for a mass undergoing SHM.

Experience tells us that if we pull the mass so as to extend the spring and then release it, the mass will move back and forth in a periodic way. If we plot the displacement x of the mass with respect to time t we obtain a curve like that shown in Figure 1.2. The *amplitude* of the oscillation is A , corresponding to the maximum excursion of the mass, and we note the *initial condition* that $x = A$ at time $t = 0$. The time for one complete cycle of oscillation is the period T . The frequency ν is the number of cycles of oscillation per unit time. The relationship between period and frequency is

$$\nu = \frac{1}{T}. \quad (1.1)$$

The units of frequency are hertz (Hz), where

$$1 \text{ Hz} \equiv 1 \text{ cycle per second} \equiv 1 \text{ s}^{-1}.$$

For small displacements the force produced by the spring is described by Hooke's law which says that the strength of the force is proportional to the extension (or compression) of the spring, i.e. $F \propto x$ where x is the displacement of the mass. The constant of proportionality is the spring constant k which is defined as the force per unit displacement. When the spring is extended, i.e. x is positive, the force acts in the opposite direction to x to pull the mass back to the equilibrium position. Similarly when the spring is compressed, i.e. x is negative, the force again acts in the opposite direction to x to push the mass back to the equilibrium position. This situation is illustrated in Figure 1.3 which shows the direction of the force at various points of the oscillation. We can therefore write

$$F = -kx \quad (1.2)$$

where the minus sign indicates that the force always acts in the opposite direction to the displacement. *All* simple harmonic oscillators have forces that act in this way: (i) the magnitude of the force is directly proportional to the displacement; and (ii) the force is always directed towards the equilibrium position.

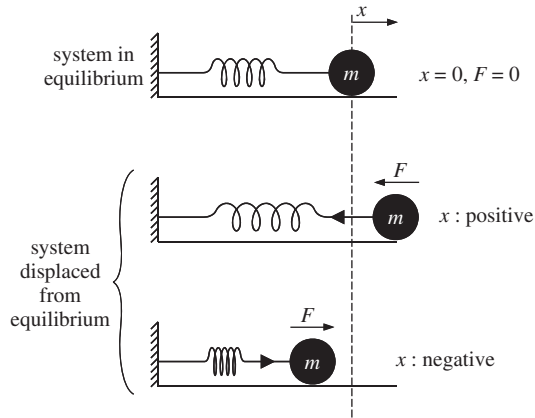


Figure 1.3 The direction of the force acting on the mass m at various values of displacement x .

The system must also obey Newton's second law of motion which states that the force is equal to mass m times acceleration a , i.e. $F = ma$. We thus obtain the equation of motion of the mass

$$F = ma = -kx. \quad (1.3)$$

Recalling that velocity v and acceleration a are, respectively, the first and second derivatives of displacement with respect to time, i.e.

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}, \quad (1.4)$$

we can write Equation (1.3) in the form of the differential equation

$$m \frac{d^2x}{dt^2} = -kx \quad (1.5)$$

or

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x} \quad (1.6)$$

where

$$\omega^2 = \frac{k}{m} \quad (1.7)$$

is a constant. Equation (1.6) is the equation of SHM and *all* simple harmonic oscillators have an equation of this form. It is a linear second-order differential equation; linear because each term is proportional to x or one of its derivatives and second order because the highest derivative occurring in it is second order. The reason for writing the constant as ω^2 will soon become apparent but we note that ω^2 is equal to the restoring force per unit displacement per unit mass.

1.2.2 A mass on a vertical spring

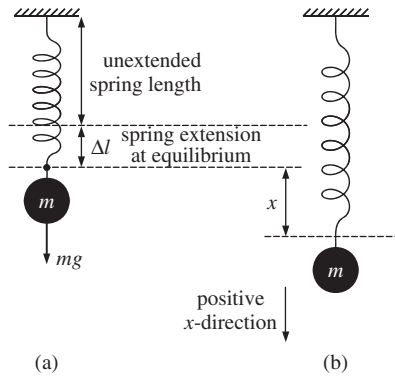


Figure 1.4 An oscillating mass on a vertical spring. (a) The mass at its equilibrium position. (b) The mass displaced by a distance x from its equilibrium position.

If we suspend a mass from a vertical spring, as shown in Figure 1.4, we have gravity also acting on the mass. When the mass is initially attached to the spring, the length of the spring increases by an amount Δl . Taking displacements in the downward direction as positive, the resultant force on the mass is equal to the gravitational force minus the force exerted upwards by the spring, i.e. the resultant force is given by $mg - k\Delta l$. The resultant force is equal to zero when the mass is at its equilibrium position. Hence

$$k\Delta l = mg.$$

When the mass is displaced downwards by an amount x , the resultant force is given by

$$F = m \frac{d^2x}{dt^2} = mg - k(\Delta l + x) = mg - k\Delta l - kx$$

i.e.

$$m \frac{d^2x}{dt^2} = -kx. \quad (1.8)$$

Perhaps not surprisingly, this result is identical to the equation of motion (1.5) of the horizontal spring: we simply need to measure displacements from the equilibrium position of the mass.

1.2.3 Displacement, velocity and acceleration in simple harmonic motion

To describe the harmonic oscillator, we need expressions for the displacement, velocity and acceleration as functions of time: $x(t)$, $v(t)$ and $a(t)$. These can be obtained by solving Equation (1.6) using standard mathematical methods. However,

we will use our physical intuition to deduce them from the observed behaviour of a mass on a spring.

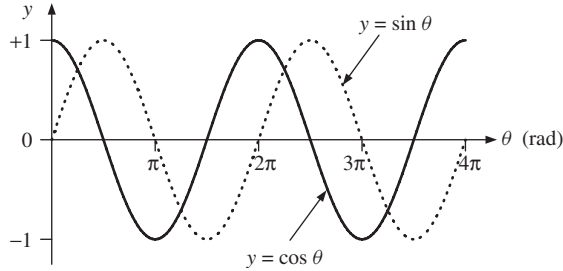


Figure 1.5 The functions $y = \cos \theta$ and $y = \sin \theta$ plotted over two complete cycles.

Observing the periodic motion shown in Figure 1.2, we look for a function $x(t)$ that also repeats periodically. Periodic functions that are familiar to us are $\sin \theta$ and $\cos \theta$. These are reproduced in Figure 1.5 over two complete cycles. Both functions repeat every time the angle θ changes by 2π . We can notice that the two functions are identical except for a shift of $\pi/2$ along the θ axis. We also note the initial condition that the displacement x of the mass equals A at $t = 0$. Comparison of the actual motion with the mathematical functions in Figure 1.5 suggests the choice of a cosine function for $x(t)$. We write it as

$$x = A \cos \left(\frac{2\pi t}{T} \right) \quad (1.9)$$

which has the correct form in that $(2\pi t/T)$ is an angle (in radians) that goes from 0 to 2π as t goes from 0 to T , and so repeats with the correct period. Moreover x equals A at $t = 0$ which matches the initial condition. We also require that $x = A \cos(2\pi t/T)$ is a solution to our differential equation (1.6). We define

$$\omega = \frac{2\pi}{T} \quad (1.10)$$

where ω is the *angular frequency* of the oscillator, with units of rad s^{-1} , to obtain

$$x = A \cos \omega t. \quad (1.11)$$

Then

$$\frac{dx}{dt} = v = -\omega A \sin \omega t, \quad (1.12)$$

and

$$\frac{d^2x}{dt^2} = a = -\omega^2 A \cos \omega t = -\omega^2 x. \quad (1.13)$$

So, the function $x = A \cos \omega t$ is a solution of Equation (1.6) and correctly describes the physical situation. The reason for writing the constant as ω^2 in Equation (1.6) is now apparent: the constant is equal to the square of the angular frequency of oscillation. We have also obtained expressions for the velocity v and acceleration a of the mass as functions of time. All three functions are plotted in Figure 1.6. Since they relate to different physical quantities, namely displacement, velocity and acceleration, they are plotted on separate sets of axes, although the time axes are aligned with respect to each other.

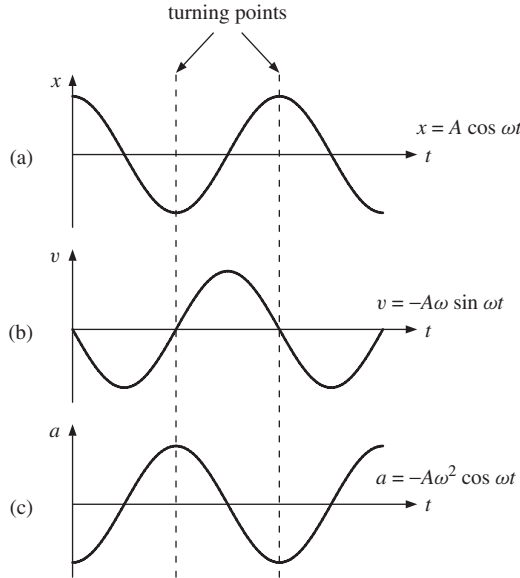


Figure 1.6 (a) The displacement x , (b) the velocity v and (c) the acceleration a of a mass undergoing SHM as a function of time t . The time axes of the three graphs are aligned.

Figure 1.6 shows that the behaviour of the three functions (1.11)–(1.13) agree with our observations. For example, when the displacement of the mass is greatest, which occurs at the *turning points* of the motion ($x = \pm A$), the velocity is zero. However, the velocity is at a maximum when the mass passes through its equilibrium position, i.e. $x = 0$. Looked at in a different way, we can see that the maximum in the velocity curve occurs before the maximum in the displacement curve by one quarter of a period which corresponds to an angle of $\pi/2$. We can understand at which points the maxima and minima of the acceleration occur by recalling that acceleration is directly proportional to the force. The force is maximum at the turning points of the motion but is of opposite sign to the displacement. The acceleration does indeed follow this same pattern, as is readily seen in Figure 1.6.

1.2.4 General solutions for simple harmonic motion and the phase angle ϕ

In the example above, we had the particular situation where the mass was released from rest with an initial displacement A , i.e. x equals A at $t = 0$. For the more

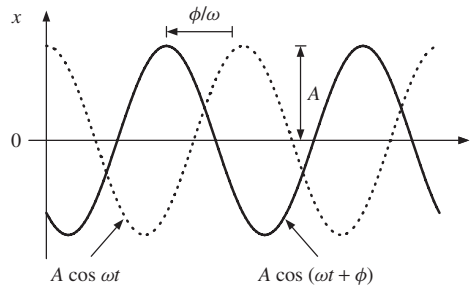


Figure 1.7 General solution for displacement x in SHM showing the phase angle ϕ , where $x = A \cos(\omega t + \phi)$.

general case, the motion of the oscillator will give rise to a displacement curve like that shown by the solid curve in Figure 1.7, where the displacement and velocity of the mass have arbitrary values at $t = 0$. This solid curve looks like the cosine function $x = A \cos \omega t$, that is shown by the dotted curve, but it is displaced horizontally to the left of it by a time interval $\phi/\omega = \phi T/2\pi$. The solid curve is described by

$$\boxed{x = A \cos(\omega t + \phi)} \quad (1.14)$$

where again A is the amplitude of the oscillation and ϕ is called the *phase angle* which has units of radians. [Note that changing ωt to $(\omega t - \phi)$ would shift the curve to the right in Figure 1.7.] Equation (1.14) is also a solution of the equation of motion of the mass, Equation (1.6), as the reader can readily verify. In fact Equation (1.14) is the *general solution* of Equation (1.6). We can state here a property of second-order differential equations that they always contain two arbitrary constants. In this case A and ϕ are the two constants which are determined from the initial conditions, i.e. from the position and velocity of the mass at time $t = 0$.

We can cast the general solution, Equation (1.14), in the alternative form

$$x = a \cos \omega t + b \sin \omega t, \quad (1.15)$$

where a and b are now the two constants. Equations (1.14) and (1.15) are entirely equivalent as we can show in the following way. Since

$$A \cos(\omega t + \phi) = A \cos \omega t \cos \phi - A \sin \omega t \sin \phi \quad (1.16)$$

and $\cos \phi$ and $\sin \phi$ have constant values, we can rewrite the right-hand side of this equation as

$$a \cos \omega t + b \sin \omega t,$$

where

$$a = A \cos \phi \text{ and } b = -A \sin \phi. \quad (1.17)$$

We see that if we add sine and cosine curves of the *same* angular frequency ω , we obtain another cosine (or corresponding sine curve) of angular frequency ω .

This is illustrated in Figure 1.8 where we plot $A \cos \omega t$ and $A \sin \omega t$, and also $(A \cos \omega t + A \sin \omega t)$ which is equal to $\sqrt{2}A \cos(\omega t - \pi/4)$. As the motion of a simple harmonic oscillator is described by sines and cosines it is called harmonic and because there is only a single frequency involved, it is called simple harmonic.

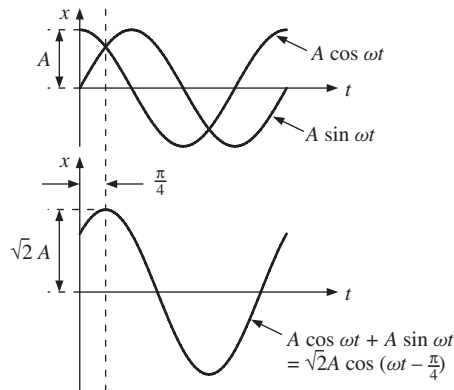


Figure 1.8 The addition of sine and cosine curves with the same angular frequency ω . The resultant curve also has angular frequency ω .

There is an important difference between the constants A and ϕ in the general solution for SHM given in Equation (1.14) and the angular frequency ω . The constants are determined by the initial conditions of the motion. However, the angular frequency of oscillation ω is determined only by the properties of the oscillator: the oscillator has a *natural frequency of oscillation* that is independent of the way in which we start the motion. This is reflected in the fact that the SHM equation, Equation (1.6), already contains ω which therefore has nothing to do with any particular solutions of the equation. This has important practical applications. It means, for example, that the period of a pendulum clock is independent of the amplitude of the pendulum so that it keeps time to a high degree of accuracy.¹ It means that the pitch of a note from a piano does not depend on how hard you strike the keys. For the example of the mass on a spring, $\omega = \sqrt{k/m}$. This expression tells us that the angular frequency becomes lower as the mass increases and becomes higher as the spring constant increases.

Worked example

In the example of a mass on a horizontal spring (cf. Figure 1.1) m has a value of 0.80 kg and the spring constant k is 180 N m⁻¹. At time $t = 0$ the mass is observed to be 0.04 m further from the wall than the equilibrium position and is moving away from the wall with a velocity of 0.50 m s⁻¹. Obtain an

¹ This assumes that the pendulum is operating as an ideal harmonic oscillator which is a good approximation for oscillations of small amplitude.

expression for the displacement of the mass in the form $x = A (\cos \omega t + \phi)$, obtaining numerical values for A , ω and ϕ .

Solution

The angular frequency ω depends only on the oscillator parameters k and m , and not on the initial conditions. Substituting their values gives

$$\omega = \sqrt{k/m} = 15.0 \text{ rad s}^{-1}$$

To find the amplitude A : From $x = A \cos(\omega t + \phi)$ we obtain

$$v = -A\omega \sin(\omega t + \phi).$$

Substituting the initial values (i.e. at time $t = 0$), of x and v into these equations gives

$$0.04 = A \cos \phi, \quad 0.50 = -15A \sin \phi.$$

From $\cos^2 \phi + \sin^2 \phi = 1$, we obtain $A = 0.052 \text{ m}$.

To find the phase angle ϕ : Substituting the value for A leads to two equations for ϕ :

$$\begin{aligned} \cos \phi &= 0.04/0.052, & \text{giving } \phi &= 39.8^\circ \text{ or } 320^\circ, \\ \sin \phi &= -0.50/(15 \times 0.052), & \text{giving } \phi &= -39.8^\circ \text{ or } 320^\circ. \end{aligned}$$

Since ϕ must satisfy both equations, it must have the value $\phi = 320^\circ$.

The angular frequency ω is given in rad s^{-1} . To convert ϕ to radians:

$$\phi = (\pi/180) \times 320 \text{ rad} = 5.59 \text{ rad. Hence, } x = 0.052 \cos(15t + 5.59) \text{ m.}$$

1.2.5 The energy of a simple harmonic oscillator

Consideration of the energy of a system is a powerful tool in solving physical problems. For one thing, scalar rather than vector quantities are involved which usually simplifies the analysis. For the example of a mass on a spring, (Figure 1.1), the mass has kinetic energy K and potential energy U . The kinetic energy is due to the motion and is given by $K = \frac{1}{2}mv^2$. The potential energy U is the energy stored in the spring and is equal to the work done in extending or compressing it, i.e. 'force times distance'. The work done on the spring, extending it from x' to $x' + dx'$, is $kx'dx'$. Hence the work done extending it from its unstretched length by an amount x , i.e. its potential energy when extended by this amount, is

$$U = \int_0^x kx'dx' = \frac{1}{2}kx^2. \quad (1.18)$$

Similarly, when the spring is compressed by an amount x the stored energy is again equal to $\frac{1}{2}kx^2$.

Conservation of energy for the harmonic oscillator follows from Newton's second law, Equation (1.5). In terms of the velocity v , this becomes

$$m \frac{dv}{dt} = -kx.$$

Multiplying this equation by $dx = vdt$ gives

$$mv dv = -kx dx$$

and since $d(x^2) = 2x dx$ and $d(v^2) = 2v dv$, we obtain

$$d\left(\frac{1}{2}mv^2\right) = -d\left(\frac{1}{2}kx^2\right).$$

Integrating this equation gives

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant},$$

where the right-hand term is a constant of integration. The two terms on the left-hand side of this equation are just the kinetic energy K and the potential energy U of the oscillator. It follows that the constant on the right-hand side is the total energy E of the oscillator, i.e. we have derived conservation of energy for this case:

$$\boxed{E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2} \quad (1.19)$$

Equation (1.19) enables us to calculate the energy E of the harmonic oscillator for any solution of the oscillator. If we take the general solution $x = A \cos(\omega t + \phi)$, we obtain the velocity

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \quad (1.20)$$

and the potential and kinetic energies

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi) \quad (1.21)$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2}kA^2 \sin^2(\omega t + \phi) \quad (1.22)$$

where we substituted $\omega^2 = k/m$. Hence the total energy E is given by

$$\begin{aligned} E = K + U &= \frac{1}{2}kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] \\ &= \frac{1}{2}kA^2. \end{aligned} \quad (1.23)$$

Equation (1.23) shows that the energy of a harmonic oscillator is proportional to the square of the amplitude of the oscillation: the more we initially extend the spring the more potential energy we store in it. The first line of Equation (1.23) also shows that the energy of the system flows between kinetic and potential energies although the total energy remains constant. This is illustrated in Figure 1.9, which shows the variation of the potential and kinetic energies with time. We have taken $\phi = 0$ in this figure. We can also plot the kinetic and potential energies as functions of the displacement x . The potential energy $U = \frac{1}{2}kx^2$ is a parabola in x as shown in Figure 1.10. We do not need to work out the equivalent expression for the variation in kinetic energy since this must be equal to $(E - \frac{1}{2}kx^2)$ and is also shown in the figure.

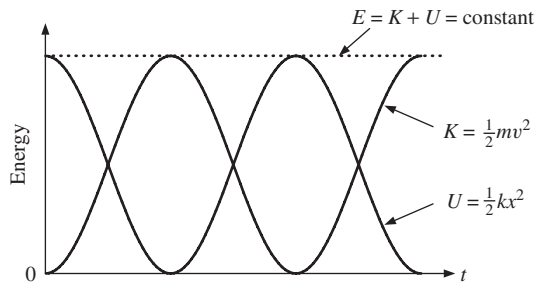


Figure 1.9 The variations of kinetic energy K and potential energy U with time t for a simple harmonic oscillator. The total energy of the oscillator E is the sum of the kinetic and potential energies and remains constant with time.

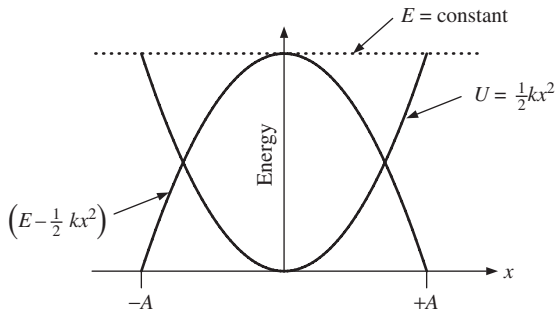


Figure 1.10 The variation of kinetic energy K and potential energy U with displacement x for a simple harmonic oscillator.

1.2.6 The physics of small vibrations

A mass on a spring is an example of a system in stable equilibrium. When the mass moves away from its equilibrium position the restoring force pulls or pushes it back. We found that the potential energy of a mass on a spring is proportional to x^2 so that the potential energy curve has the shape of a parabola given by

$U(x) = \frac{1}{2}kx^2$ (cf. Figure 1.10). This curve has a minimum when $x = 0$, which corresponds to the unstretched length of the spring. The movement of the mass is constrained by the spring and the mass is said to be confined in a potential well. The parabolic shape of this potential well gives rise to SHM. Any system that is in stable equilibrium will oscillate if it is displaced from its equilibrium state. We may think of a marble in a round-bottomed bowl. When the marble is pushed to one side it rolls back and forth in the bowl. The universal importance of the harmonic oscillator is that nearly all the potential wells we encounter in physical situations have a shape that is parabolic when we are sufficiently close to the equilibrium position. Thus, *most oscillating systems will oscillate with SHM when the amplitude of oscillation is small* as we shall prove in a moment. This situation is illustrated in Figure 1.11, which shows as a solid line the potential energy of a simple pendulum as a function of the angular displacement θ . (We will discuss the example of the simple pendulum in detail in Section 1.3.) Superimposed on it as a dotted line is a parabolic-shaped potential well, i.e. proportional to θ^2 . Close to the equilibrium position ($\theta = 0$), the two curves lie on top of each other. So long as the amplitude of oscillation falls within the range where the two curves coincide the pendulum will execute SHM.

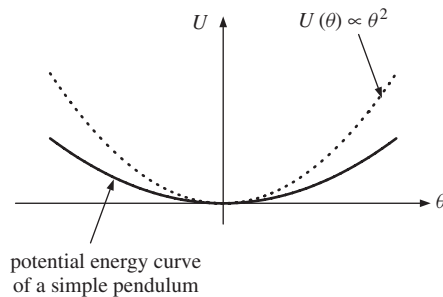


Figure 1.11 The solid curve represents the potential energy U of a simple pendulum as a function of its angular displacement θ . The dotted line represents the potential energy $U(\theta)$ of a simple harmonic oscillator for which the potential energy is proportional to θ^2 . For small angular amplitudes, where the two curves overlap, a simple pendulum behaves as a simple harmonic oscillator.

We can see the above result mathematically using Taylor's theorem which says that any function $f(x)$ which is continuous and possesses derivatives of all orders at $x = a$ can be expanded in a power series in $(x - a)$ in the neighbourhood of the point $x = a$, i.e.

$$f(x) = f(a) + \frac{(x-a)}{1!} \left(\frac{df}{dx} \right)_{x=a} + \frac{(x-a)^2}{2!} \left(\frac{d^2f}{dx^2} \right)_{x=a} + \dots \quad (1.24)$$

where the derivatives df/dx , etc., are evaluated at $x = a$. (In practice all the potential wells that we encounter in physical situations can be described by functions that can be expanded in this way.) We see that Taylor's theorem gives the value of a function $f(x)$ in terms of the value of the function at $x = a$ and the values of

the first and higher derivatives of x evaluated at $x = a$. If we expand $f(x)$ about $x = 0$, we have

$$f(x) = f(0) + x \left(\frac{df}{dx} \right)_{x=0} + \frac{x^2}{2} \left(\frac{d^2f}{dx^2} \right)_{x=0} + \dots$$

In the case of a general potential well $U(x)$, we expand about the equilibrium position $x = 0$ to obtain

$$U(x) = U(0) + x \left(\frac{dU}{dx} \right)_{x=0} + \frac{x^2}{2} \left(\frac{d^2U}{dx^2} \right)_{x=0} + \dots \quad (1.25)$$

The first term $U(0)$ is a constant and has no physical significance in the sense that we can measure potential energy with respect to any position and indeed we can choose it to be equal to zero. The first derivative of U with respect to x is zero because the curve is a minimum at $x = 0$. The second derivative of U with respect to x , evaluated at $x = 0$, will be a constant. Thus if we retain only the first non-zero term in the expansion, which is a good approximation so long as x is small, we have

$$U(x) = \frac{x^2}{2} \left(\frac{d^2U}{dx^2} \right)_{x=0} \quad (1.26)$$

This is indeed the form of the potential energy for the mass on a spring with d^2U/dx^2 playing the role of the spring constant. Then the force close to the equilibrium position takes the general form

$$F = -\frac{dU}{dx} = -x \left(\frac{d^2U}{dx^2} \right)_{x=0} \quad (1.27)$$

The force is directly proportional to x and acts in the opposite direction which is our familiar result for the simple harmonic oscillator.

The fact that a vibrating system will behave like a simple harmonic oscillator when its amplitude of vibration is small means that our physical world is filled with examples of SHM. To illustrate this diversity Table 1.1 gives examples of a variety of physical systems that can oscillate and their associated periods of oscillation. These examples occur in both classical and quantum mechanics. Clearly the more massive the system, the greater is the period of oscillation. For the case of a vibrating tuning fork, we can tell that the ends of the fork are oscillating at a single frequency because we hear a pure note that we can use to tune musical instruments. A plucked guitar string will also oscillate and indeed musical instruments provide a wealth of examples of SHM. These oscillations, however, will in general be more complicated than that of the tuning fork but even here these complex oscillations are a superposition of SHMs as we shall see in Chapter 6. The balance wheel of a mechanical clock, the sloshing of water in a lake and the swaying of a sky scraper in the wind provide further examples of classical oscillators.

TABLE 1.1 Examples of systems that can oscillate and the associated periods of oscillation.

System	Period (s)
Sloshing of water in a lake	$\sim 10^2 - 10^4$
Large bridges and buildings	$\sim 1 - 10$
A clock pendulum or balance wheel	~ 1
String instruments	$\sim 10^{-3} - 10^{-2}$
Piezoelectric crystals	$\sim 10^{-6}$
Molecular vibrations	$\sim 10^{-15}$

A good example of SHM in the microscopic world is provided by the vibrations of the atoms in a crystal. The forces between the atoms result in the regular lattice structure of the crystal. Furthermore, when an atom is slightly displaced from its equilibrium position it is subject to a net restoring force. The shape of the resultant potential well approximates to a parabola for small amplitudes of vibration. Thus when the atoms vibrate they do so with SHM. Einstein used a simple harmonic oscillator model of a crystal to explain the observed variation of heat capacity with temperature (see also Mandl,² Section 6.2). He assumed that the atoms were harmonic oscillators that vibrate independently of each other but with the same angular frequency and he used a quantum mechanical description of these oscillators. As we have seen, in classical mechanics the energy of an oscillator is proportional to the square of the amplitude and can take any value, i.e. the energy is continuous. A fundamental result of quantum mechanics is that the energy of a harmonic oscillator is quantised, i.e. only a discrete set of energies is possible. Einstein's quantum model predicted that the specific heat of a crystal, such as diamond, goes to zero as the temperature of the crystal decreases, unlike the classical result that the specific heat is independent of temperature. Experiment shows that the specific heat of diamond does indeed go to zero at low temperatures.

Another example of SHM in quantum physics is provided by the vibrations of the two nuclei of a hydrogen molecule. The solid curve in Figure 1.12 represents the potential energy U of the hydrogen molecule as a function of the separation r between the nuclei, where we have taken the potential energy to be zero at infinite separation. This potential energy is due to the Coulomb interaction of the electrons and nuclei and the quantum behaviour of the electrons. The curve exhibits a minimum at $r_0 = 0.74 \times 10^{-10}$ m. At small separation ($r \rightarrow 0$) the potential energy tends to infinity, representing the strong repulsion between the nuclei. The nuclei perform oscillations about the equilibrium separation. The dotted line in Figure 1.12 shows the parabolic form of the potential energy of a harmonic oscillator, centred at the equilibrium separation r_0 . For small amplitudes of oscillation (i.e. when the nuclei are not too highly excited) the vibrations occur within the range where the two curves coincide. Again, according to quantum mechanics, only a discrete set of vibrational energies is possible. For a simple harmonic oscillator with angular frequency ω the only allowed values of the energy are $\frac{1}{2}\hbar\omega$, $\frac{3}{2}\hbar\omega$, $\frac{5}{2}\hbar\omega$, \dots , where

² Statistical Physics, F. Mandl, Second Edition, 1988, John Wiley & Sons, Ltd.

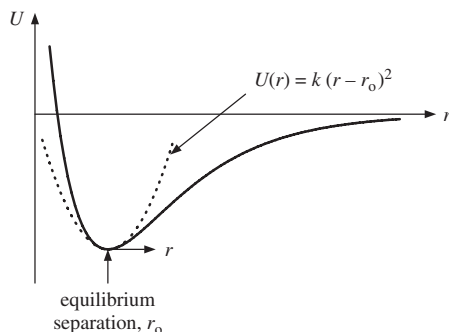


Figure 1.12 The solid curve represents the variation of potential energy of a hydrogen molecule as a function of the separation of the two hydrogen nuclei. The dotted curve represents the potential energy of a simple harmonic oscillator centred on the equilibrium separation r_0 of the two nuclei.

\hbar is Planck's constant divided by 2π . The observed vibrational line spectra of molecules correspond to transitions between these energy levels with the emission of electromagnetic radiation that typically lies in the infrared part of the electromagnetic spectrum. These spectra provide valuable information about the properties of the molecule such as the strength of the molecular bond.

Worked example

The H_2 molecule has a vibrational frequency ν of 1.32×10^{14} Hz. Calculate the strength of the molecular bond, i.e. the 'spring constant', assuming that the molecule can be modelled as a simple harmonic oscillator.

Solution

In previous cases, we considered a mass vibrating at one end of a spring while the other end of the spring was connected to a rigid wall. Now we have two nuclei vibrating against each other, which we model as two equal masses connected by a spring. We can solve this new situation by realising that there is no translation of the molecule during the vibration, i.e. the centre of mass of the molecule does not move. Thus as one hydrogen nucleus moves in one direction by a distance x the other must move in the opposite direction by the same amount and of course both vibrate at the same frequency. The total extension is $2x$ and the tension in the 'spring' is equal to $2kx$ where k represents the 'spring constant' or bond strength. The equation of motion of each nucleus of mass m is then given by

$$m \frac{d^2x}{dt^2} = -2kx$$

or

$$\frac{m}{2} \frac{d^2x}{dt^2} = -kx. \quad (1.28)$$

This equation is analogous to Equation (1.5) where m has been replaced by $m/2$ which is called the *reduced mass* of the system. The classical angular frequency of vibration ω of the molecule is then equal to $\sqrt{2k/m}$. The frequency of vibration $\nu = 1/T = \omega/2\pi$ and $m = 1.67 \times 10^{-27}$ kg. Therefore

$$k = 4\pi^2\nu^2\frac{m}{2} = \frac{4\pi^2(1.32 \times 10^{14})^2 1.67 \times 10^{-27}}{2} = 574 \text{ N m}^{-1}.$$

1.3 THE PENDULUM

1.3.1 The simple pendulum

Timing the oscillations of a pendulum has been used for centuries to measure time accurately. The simple pendulum is the idealised form that consists of a point mass m suspended from a massless rigid rod of length l , as illustrated in Figure 1.13. For an angular displacement θ , the displacement of the mass *along* the arc of the circle of length l is $l\theta$. Hence the angular velocity along the arc is $ld\theta/dt$ and the angular acceleration is $ld^2\theta/dt^2$. At a displacement θ there is a tangential force on the mass acting along the arc that is equal to $-mg \sin \theta$, where as usual the minus sign indicates that it is a restoring force. Hence by Newton's second law we obtain

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin \theta. \quad (1.29)$$

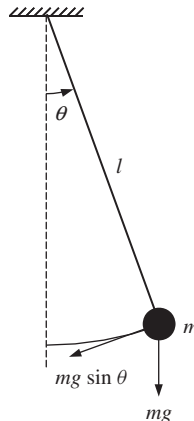


Figure 1.13 The simple pendulum of mass m and length l .

This equation does not have the same form as the equation of SHM, Equation (1.6), as we have $\sin \theta$ on the right-hand side instead of θ . However we can expand $\sin \theta$

in a power series in θ :

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \quad (1.30)$$

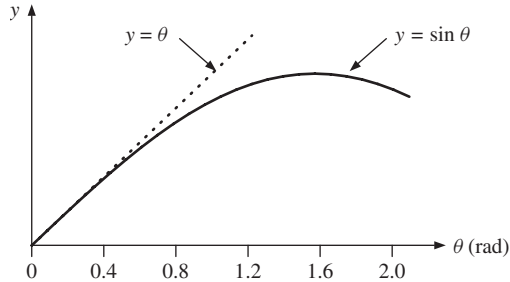


Figure 1.14 A comparison of the functions $y = \theta$ and $y = \sin \theta$ plotted against θ .

For small angular deflections the second and higher terms are much smaller than the first term. For example, if θ is equal to 0.1 rad (5.7°), which is typical for a pendulum clock, then the second term is only 0.17% of the first term and the higher terms are much smaller still. We can see this directly by plotting the functions $y = \sin \theta$ and $y = \theta$ on the same set of axes, as shown in Figure 1.14. The two curves are indistinguishable for values of θ below about $\frac{1}{4}$ rad ($\sim 15^\circ$). Thus for small values of θ , we need retain only the first term in the expansion (1.30) and replace $\sin \theta$ with θ (in radians) to give

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta. \quad (1.31)$$

This is the equation of SHM with $\omega = \sqrt{g/l}$ and $T = 2\pi\sqrt{l/g}$, and we can immediately write down an expression for the angular displacement θ of the pendulum:

$$\theta = \theta_0 \cos(\omega t + \phi) \quad (1.32)$$

where θ_0 is the angular amplitude of oscillation. The period is independent of amplitude for oscillations of small amplitude and this is why the pendulum is so useful as an accurate time keeper. The period does, however, depend on the acceleration due to gravity and so measuring the period of a pendulum provides a way of determining the value of g . (In practice real pendulums do not have their mass concentrated at a point as in the simple pendulum as will be described in Section 1.3.3. So for an accurate determination of g a more sophisticated pendulum has been developed called the *compound pendulum*.) We finally note that for $l = 1.00$ m and for a value of $g = 9.87$ m s $^{-2}$, the period of a simple pendulum is equal to $2\pi\sqrt{1.00/9.87} = 2.00$ s and indeed the second was originally defined as equal to one half the period of a 1 m simple pendulum.

1.3.2 The energy of a simple pendulum

We can also analyse the motion of the simple pendulum by considering its kinetic and potential energies. The geometry of the simple pendulum is shown in Figure 1.15. (The horizontal distance $x = l \sin \theta$ is not exactly the same as the distance along the arc, which is equal to $l\theta$. However, since $\sin \theta \simeq \theta$ for small θ , the difference is negligible.) From the geometry we have

$$l^2 = (l - y)^2 + x^2 \quad (1.33)$$

which gives

$$2ly = y^2 + x^2. \quad (1.34)$$

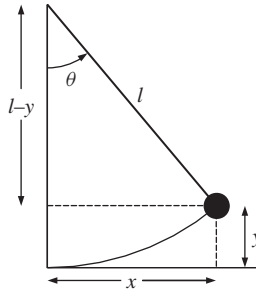


Figure 1.15 The geometry of the simple pendulum.

For small displacements of the pendulum, i.e. $x \ll l$, it follows that $y \ll x$, so that the term y^2 can be neglected and we can write,

$$y = \frac{x^2}{2l}. \quad (1.35)$$

As the mass is displaced from its equilibrium position its vertical height increases and it gains potential energy. This gain in potential energy is equal to $mgy = mgx^2/2l$. The total energy of the system E is given by the sum of the kinetic and potential energies:

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mgx^2}{l}. \quad (1.36)$$

At the turning point of the motion, when x equals A , the velocity v is zero giving

$$E = \frac{1}{2}\frac{mgA^2}{l}. \quad (1.37)$$

From conservation of energy, it follows that

$$\frac{mgA^2}{l} = mv^2 + \frac{mgx^2}{l} \quad (1.38)$$

is true for all times. We can use Equation (1.38) to obtain expressions for velocity v and displacement x :

$$v = \frac{dx}{dt} = \sqrt{\frac{g(A^2 - x^2)}{l}}. \quad (1.39)$$

This expression describes how the velocity changes with the displacement x in SHM in contrast to Equation (1.12) which describes how the velocity changes with time t . Since $v = dx/dt$ we have

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \sqrt{\frac{g}{l}} \int dt. \quad (1.40)$$

The integral on the left-hand side can be evaluated using the substitution $x = A \sin \theta$, giving

$$\sin^{-1} \left(\frac{x}{A} \right) = \sqrt{\frac{g}{l}} t + \phi, \quad (1.41)$$

where ϕ is the constant of integration, and

$$x = A \sin \left(\sqrt{\frac{g}{l}} t + \phi \right). \quad (1.42)$$

Equation (1.42) describes SHM with $\omega = \sqrt{g/l}$ and $T = 2\pi\sqrt{l/g}$ as before.

At this point we note the similarity in the expressions for the total energy of the two examples of SHM that we have considered.

$$\text{For the mass on a spring:} \quad E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \quad (1.43a)$$

$$\text{For the simple pendulum:} \quad E = \frac{1}{2}mv^2 + \frac{1}{2} \frac{mg}{l} x^2. \quad (1.43b)$$

$$\text{Both expressions have the form:} \quad E = \frac{1}{2}\alpha v^2 + \frac{1}{2}\beta x^2, \quad (1.43c)$$

where α and β are constants. It is a universal characteristic of simple harmonic oscillators that their total energy can be written as the sum of two parts, one involving the (velocity)² and the other the (displacement)². Just as $md^2x/dt^2 = -kx$, Equation (1.5), is the signature of SHM in terms of forces, Equation (1.43) is the signature of SHM in terms of energies. If we obtain either of these equations in the analysis of a system then we know we have SHM. We stress that the equations are the same for all simple harmonic oscillators: only the labels for the physical quantities change. We do not need to repeat the analysis again: we can simply take over the results already obtained. The constant α corresponds to the inertia of the system through which it can store kinetic energy. The constant β corresponds to the restoring force per unit displacement through which the system can store

potential energy. When we differentiate the conservation of energy equation for SHM, Equation (1.43c), with respect to time we obtain

$$\frac{dE}{dt} = \alpha v \frac{dv}{dt} + \beta x \frac{dx}{dt} = 0$$

giving

$$\frac{d^2x}{dt^2} = -\frac{\beta}{\alpha}x.$$

Comparing this with Equation (1.6), it follows that the angular frequency of oscillation ω is equal to $\sqrt{\beta/\alpha}$.

Worked example

A marble of radius r rolls back and forth without slipping in a spherical dish of radius R . Use energy considerations to show that the motion is simple harmonic for small displacements of the marble from its equilibrium position and deduce an expression for the period of the oscillations. The moment of inertia I of a solid sphere of mass m about an axis through its centre is equal to $\frac{2}{5}mr^2$.

Solution

The equilibrium and displaced positions of the marble are shown in Figure 1.16, where the arrows indicate the rotation of the marble when it rotates through an angle ϕ . If the marble were rotating through an angle ϕ on a *flat* surface it would roll a distance $r\phi$. However on a spherical surface as in Figure 1.16, it rolls a distance l along the arc of radius R given by $l = r(\phi + \theta)$. Since $l = R\theta$,

$$\phi = \frac{(R-r)}{r}\theta \text{ and } \frac{d\phi}{dt} = \frac{(R-r)}{r} \left(\frac{d\theta}{dt} \right).$$

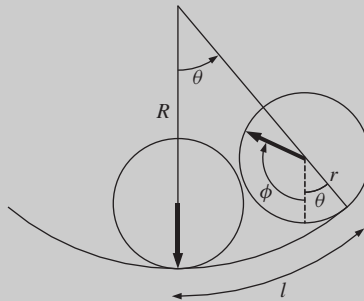


Figure 1.16 A marble of radius r that rolls back and forth without slipping in a spherical dish of radius R .

The total kinetic energy of the marble, as it moves along the surface of the dish, is equal to the kinetic energy of the translational motion of its centre of

mass plus the kinetic energy of its rotational motion about the centre of mass. Hence

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I \left(\frac{d\phi}{dt} \right)^2.$$

The translational kinetic energy is given by

$$\frac{1}{2}mv^2 = \frac{1}{2}m(R-r)^2 \left(\frac{d\theta}{dt} \right)^2.$$

Therefore,

$$K = \frac{1}{2}m \left(\frac{7}{5} \right) (R-r)^2 \left(\frac{d\theta}{dt} \right)^2$$

where we have substituted for I . The potential energy is

$$U = mg(R-r)(1 - \cos\theta) = \frac{1}{2}mg(R-r)\theta^2$$

for small θ . Thus

$$E = \frac{1}{2}m \left(\frac{7}{5} \right) (R-r)^2 \left(\frac{d\theta}{dt} \right)^2 + \frac{1}{2}mg(R-r)\theta^2.$$

This has the general form of the energy equation (1.43c) of a harmonic oscillator

$$E = \frac{1}{2}\alpha \left(\frac{d\theta}{dt} \right)^2 + \frac{1}{2}\beta\theta^2$$

where now θ represents the displacement coordinate. Hence

$$\omega = \sqrt{\frac{\beta}{\alpha}} = \sqrt{\frac{5g}{7(R-r)}} \quad \text{and} \quad T = 2\pi\sqrt{\frac{7(R-r)}{5g}}.$$

This example would be much more difficult to solve from force considerations.

1.3.3 The physical pendulum

In a physical pendulum the mass is not concentrated at a point as in the simple pendulum, but is distributed over the whole body. It is thus more representative of real pendulums. An example of a physical pendulum is shown in Figure 1.17. It consists of a uniform rod of length l that pivots about a horizontal axis at its upper end. This is a rotating system where the pendulum rotates about its point of suspension. For a rotating system, Newton's second law for linear systems,

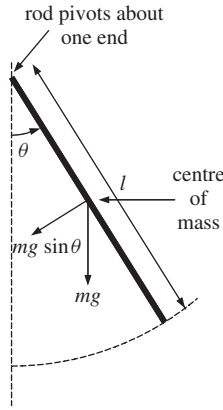


Figure 1.17 A rod that pivots about one of its ends, which is an example of a physical pendulum.

$m d^2x/dt^2 = F$, becomes

$$I \frac{d^2\theta}{dt^2} = \tau \quad (1.44)$$

where I is the moment of inertia of the body about its axis of rotation and τ is the applied torque. The moment of inertia of a uniform rod of length l about an end is equal to $\frac{1}{3}ml^2$ and its centre of mass is located at its mid point. The resultant torque τ on the rod when it is displaced through an angle θ is given by the product of the torque arm $\frac{1}{2}l$ and the component of the force normal to the torque arm ($mg \sin \theta$), i.e.

$$\tau = \left(\frac{1}{2}l\right) \times (-mg \sin \theta).$$

Hence we obtain

$$\frac{1}{3}ml^2 \frac{d^2\theta}{dt^2} = -\frac{1}{2}mgl \sin \theta \quad (1.45)$$

giving

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2l} \sin \theta. \quad (1.46)$$

Again we can use the small-angle approximation to obtain

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2l} \theta. \quad (1.47)$$

This is SHM with $\omega = \sqrt{3g/2l}$ and $T = 2\pi\sqrt{2l/3g}$.

In a simple model we can describe the walking pace of a person in terms of a physical pendulum. We model the human leg as a solid rod that pivots from the hip. Furthermore, when we walk we do so at a comfortable pace that coincides with the natural period of oscillation of the leg. If we assume a value of 0.8 m for l , the length of the leg, then its natural period is ~ 1.5 s. One complete period of the swinging leg corresponds to two strides. Try this yourself. If the length of a stride is, say, 1 m then we would walk at a speed of approximately $2/1.5 \text{ m s}^{-1}$ which corresponds to 4.8 km h^{-1} or about 3 mph which is in good agreement with reality.

1.3.4 Numerical solution of simple harmonic motion³

When solving the equation of motion for an oscillating pendulum we made use of the small-angle approximation, $\sin \theta \simeq \theta$ when θ is small. This made the equation of motion much easier to solve. However an alternative way, without resorting to the small-angle approximation, is to solve the equation numerically. The essential idea is that if we know the position and velocity of the mass at time t and we know the force acting on it then we can use this knowledge to obtain good estimates of these parameters at time $(t + \delta t)$. We then repeat this process, step by step, over the full period of the oscillation to trace out the displacement of the mass with time. We can make these calculations as accurate as we like by making the time interval δt sufficiently small. To demonstrate this approach we apply it to the simple pendulum. Figure 1.18 shows a simple pendulum and the angular position of the mass at three instants of time each separated by δt , i.e. at t , $(t + \delta t)$ and $(t + 2\delta t)$. Using the notation $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ for $d\theta(t)/dt$ and $d^2\theta(t)/dt^2$, respectively, we can write the equation of motion of the mass, Equation (1.29),

$$\ddot{\theta}(t) = -\frac{g}{l} \sin \theta(t). \quad (1.48)$$

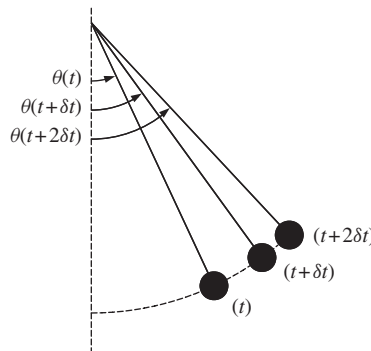


Figure 1.18 A simple pendulum showing the position of the mass at three instants of time separated by time interval δt .

³ This section may be omitted as it is not required later in the book.

If the angular position of the mass is $\theta(t)$ at time t , then its position at time $(t + \delta t)$ will be different by an amount equal to the angular velocity of the mass times the time interval δt (cf. the familiar expression $x = vt$ for linear motion). We might be tempted to use $\dot{\theta}(t)$ for this angular velocity. However, as we know, the angular velocity varies during the time δt . A better estimate for the angular velocity is its *average* value between the times t and $(t + \delta t)$, i.e. $\dot{\theta}(t + \delta t/2)$. Thus to a good approximation we have

$$\theta(t + \delta t) = \theta(t) + \delta t \times \dot{\theta}(t + \delta t/2). \quad (1.49)$$

In a similar way we can relate the angular velocities of the mass at times separated by time δt , i.e. the new velocity will be different from the old value by an amount equal to $\delta t \times \ddot{\theta}(t)$, where $\ddot{\theta}(t)$ is the angular acceleration (cf. the familiar expression $v = u + at$ for linear motion). The acceleration also varies with time and so again we will use its average value during the time interval δt . For the evaluation of $\dot{\theta}(t + \delta t/2)$ this translates to

$$\dot{\theta}(t + \delta t/2) = \dot{\theta}(t - \delta t/2) + \delta t \times \ddot{\theta}(t) \quad (1.50)$$

where $\ddot{\theta}(t)$ is the average value of the angular acceleration between the times $(t - \delta t/2)$ and $(t + \delta t/2)$ which we know from Equation (1.48). For the first step of this calculation we need the value of the angular velocity at time $t = \delta t/2$. For this particular case we use the expression

$$\dot{\theta}(\delta t/2) = (\delta t/2) \times \ddot{\theta}(0). \quad (1.51)$$

Armed with these expressions for angular position, velocity and acceleration we can trace the angular displacement of the mass step by step.

We proceed by building up a table of consecutive values of $\theta(t)$, $\dot{\theta}(t)$ and $\ddot{\theta}(t)$. As an example we chose the length of the simple pendulum to give $T = 2.0$ s and $\omega = \pi$. We also chose a time interval δt of 0.02 s (equal to one hundredth of the period) and an angular amplitude θ_0 of 0.10 rad (5.7°). The values obtained in the first 10 steps of the calculation are shown in Table 1.2 and were obtained using a hand calculator. For comparison the final column of Table 1.2 shows the values obtained from the analytic solution $\theta(t) = \theta_0 \cos \omega t$. We see that the numerically calculated values of the displacement are in agreement with the analytic values up to the third significant figure. These two sets of values for a complete period of oscillation are plotted in Figure 1.19 and show the familiar variation of displacement with time. The solid curve corresponds to the values of displacement obtained from the analytic solution $\theta(t) = \theta_0 \cos \omega t$, while the dots (\bullet) correspond to the numerically computed values. The agreement is so good that the dots lie exactly on top of the analytic curve. These results demonstrate that the small-angle approximation is valid in this case and that the numerical method gives accurate results.

This numerical method allows us to explore what happens for large-amplitude oscillations where the small angle approximation is no longer valid. Figure 1.20 shows the results for a very large angular amplitude of 1.0 rad (57°) which were

TABLE 1.2 Computed values of angular displacement, velocity and acceleration of a simple pendulum. The last column on the right shows the values obtained from the analytic solution.

Time (s)	Angular displacement, $\theta(t)$ (rad)	Angular acceleration, $\ddot{\theta}(t)$ (rad s ⁻²)	Angular velocity, $\dot{\theta}(t)$ (rad s ⁻¹)	$\theta(t) = 0.1 \cos \pi t$ (rad)
0.00	0.1000	-0.985	-0.0099	0.1000
0.02	0.0998	-0.983	-0.0295	0.0998
0.04	0.0992	-0.978	-0.0491	0.0992
0.06	0.0982	-0.968	-0.0685	0.0982
0.08	0.0968	-0.954	-0.0876	0.0969
0.10	0.0950	-0.937	-0.106	0.0951
0.12	0.0929	-0.915	-0.124	0.0930
0.14	0.0904	-0.891	-0.142	0.0905
0.16	0.0876	-0.863	-0.159	0.0876

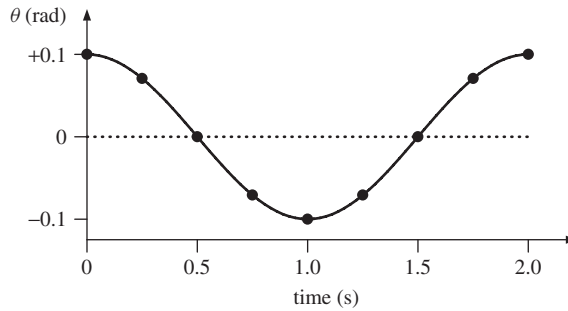


Figure 1.19 The angular displacement θ , plotted against time, for a simple pendulum with a small amplitude of oscillation; $\theta_0 = 0.1$ rad. The solid curve corresponds to the values of displacement obtained from the analytic solution $\theta(t) = \theta_0 \cos \omega t$, while the dots (\bullet) correspond to the numerically computed values. The agreement is so good that the computed values lie on top of the analytical curve.

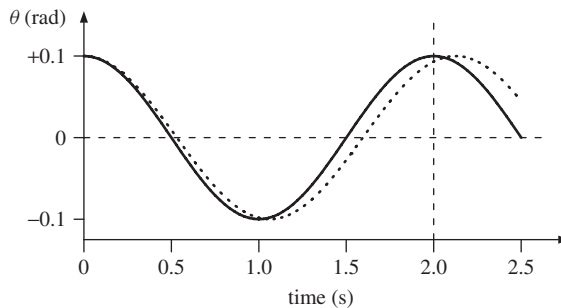


Figure 1.20 The angular displacement θ , plotted against time, of a simple pendulum for a large amplitude of oscillation; $\theta_0 = 1.0$ rad. The solid curve corresponds to the values of displacement obtained from the solution $\theta(t) = \theta_0 \cos \omega t$, while the dotted curve is obtained from the numerically computed results. For large-amplitude oscillations the period of the pendulum is no longer independent of amplitude and increases with amplitude.

obtained using a spreadsheet program. The solid curve corresponds to the values of displacement obtained from the solution $\theta(t) = \theta_0 \cos \omega t$ while the dotted curve is the one obtained from the numerically computed values. There is a significant difference between the two curves: the actual angular displacement of the mass, which is given by the numerical values, no longer closely matches the analytic solution. In particular the time period for the actual oscillations has increased to a value of 2.13 s: an increase of 6.5%. We see that for large-amplitude oscillations the period of the pendulum is no longer independent of amplitude and that it increases with amplitude.

1.4 OSCILLATIONS IN ELECTRICAL CIRCUITS: SIMILARITIES IN PHYSICS

In this section we consider oscillations in an electrical circuit. What we find is that these oscillations are described by a differential equation that is identical in form to Equation (1.6) and so has an identical solution: only the physical quantities associated with the differential equation are different. This illustrates that when we understand one physical situation we can understand many others. It also means that we can simulate one system by another and in this way build analogue computers, i.e. we can build an electrical circuit consisting of resistors, capacitors and inductors that will exactly simulate the operation of a mechanical system.

1.4.1 The LC circuit

The simplest example of an oscillating electrical circuit consists of an inductor L and capacitor C connected together in series with a switch as shown in Figure 1.21.

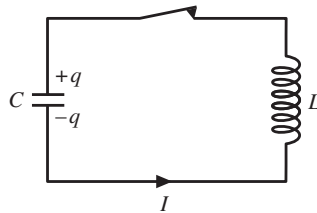


Figure 1.21 An electrical oscillator consisting of an inductor L and a capacitor C connected in series.

As usual we start with an idealised situation where we assume that the resistance in the circuit is negligible. This is analogous to the assumption for mechanical systems that there are no frictional forces present. Initially, the switch is open and the capacitor is charged to voltage V_C . The charge q on the capacitor is given by $q = V_C C$ where C is the capacitance. When the switch is closed the charge begins to flow through the inductor and a current $I = dq/dt$ flows in the circuit. This is a time-varying current and produces a voltage across the inductor given

by $V_L = LdI/dt$. We can analyse the LC circuit using *Kirchhoff's law*, which states that 'the sum of the voltages around the circuit is zero', i.e. $V_C + V_L = 0$. Therefore

$$\frac{q}{C} + L \frac{dI}{dt} = 0 \quad (1.52)$$

giving

$$\frac{q}{C} + L \frac{d^2q}{dt^2} = 0 \quad (1.53)$$

and

$$\frac{d^2q}{dt^2} = -\frac{1}{LC}q. \quad (1.54)$$

This equation describes how the charge on a plate of the capacitor varies with time. It is of the same form as Equation (1.6) and represents SHM. The frequency of the oscillation is given directly by, $\omega = \sqrt{1/LC}$. Since we have the initial condition that the charge on the capacitor has its maximum value at $t = 0$, then the solution to Equation (1.54) is $q = q_0 \cos \omega t$, where q_0 is the initial charge on the capacitor. The variation of charge q with respect to t is shown in Figure 1.22 and is analogous to the way the displacement of a mass on a spring varies with time.

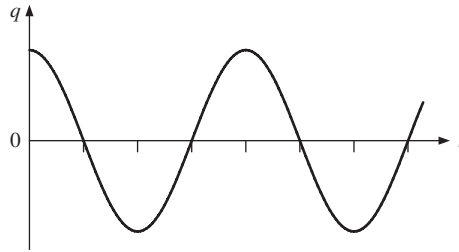


Figure 1.22 The variation of charge q with time on the capacitor in a series LC circuit. The charge oscillates in time in an analogous way to the displacement of a mass oscillating at the end of a spring.

We can also consider the energy of this electrical oscillator. The energy stored in a capacitor charged to voltage V_C is equal to $\frac{1}{2}CV_C^2$. This is electrostatic energy. The energy stored in an inductor is equal to $\frac{1}{2}LI^2$ and this is magnetic energy. Thus the total energy in the circuit is given by

$$E = \frac{1}{2}LI^2 + \frac{1}{2}CV_C^2 \quad (1.55)$$

or

$$E = \frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}. \quad (1.56)$$

For these electrical oscillations the charge flows between the plates of the capacitor and through the inductor, so that there is a continuous exchange between electrostatic and magnetic energy.

1.4.2 Similarities in physics

We note the similarities between the equations for the mechanical and electrical cases

$$m \frac{d^2x}{dt^2} = -kx, \quad L \frac{d^2q}{dt^2} = -\frac{1}{C}q \quad (1.57a)$$

and

$$E = \frac{1}{2}m \left(\frac{dx}{dt} \right)^2 + \frac{1}{2}kx^2, \quad E = \frac{1}{2}L \left(\frac{dq}{dt} \right)^2 + \frac{1}{2} \frac{q^2}{C}, \quad (1.57b)$$

where we have written dx/dt for the velocity v and dq/dt for the current I , in order to bring out more sharply the similarity of the two cases. In both cases we have the identical forms

$$\alpha \frac{d^2Z}{dt^2} = -\beta Z, \quad E = \frac{1}{2}\alpha \left(\frac{dZ}{dt} \right)^2 + \frac{1}{2}\beta Z^2, \quad (1.58)$$

where α and β are constants and $Z = Z(t)$ is the oscillating quantity (see also Equations 1.43). In the mechanical case Z stands for the displacement x , and in the electrical case for the charge q . Thus all we have learned about mechanical oscillators can be carried over to electrical oscillators. Moreover we can see a direct correspondence between the two sets of physical quantities involved:

- q takes the place of x ;
- L takes the place of m ;
- $1/C$ takes the place of k .

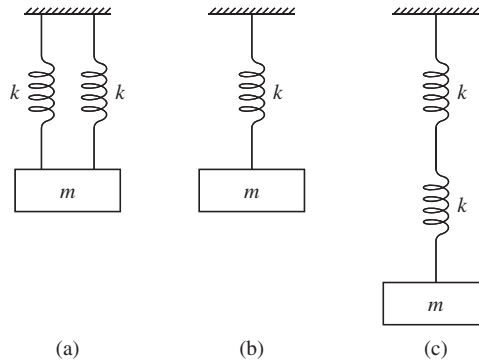
For example, the inductance L is the electrical analogue of mechanical inertia m . These analogies enable us to build an electrical circuit that exactly mimics the operation of a mechanical system. This is useful because in the development of a mechanical system it is much easier to change, for example, the value of a capacitor in the analogue circuit than to manufacture a new mechanical component.

PROBLEMS 1

- 1.1 A mass of 0.50 kg hangs from a light spring and executes SHM so that its position x is given by $x = A \cos \omega t$. It is found that the mass completes 20 cycles of oscillation in 80 s. (a) Determine (i) the period of the oscillations, (ii) the angular frequency of the oscillations and (iii) the spring constant k . (b) Using a value of $A = 2$ mm, make sketches of the variations with time t of the displacement, velocity and acceleration of the mass.

- 1.2 The ends of a tuning fork oscillate at a frequency of 440 Hz with an amplitude of 0.50 mm. Determine (a) the maximum velocity and (b) the maximum acceleration of the ends.
- 1.3 A platform oscillates in the vertical direction with SHM. Its amplitude of oscillation is 0.20 m. What is the maximum frequency (Hz) of oscillation for a mass placed on the platform to remain in contact with the platform? (Assume $g = 9.81 \text{ m s}^{-2}$.)
- 1.4 A mass executes SHM at the end of a light spring. (a) What fraction of the total energy of the system is potential and what fraction is kinetic at the instant when the displacement of the mass is equal to half the amplitude? (b) If the maximum amplitude of the oscillations is doubled, what will be the change in (i) the total energy of the system, (ii) the maximum velocity of the mass and (iii) the maximum acceleration of the mass. Will the period of oscillation change?
- 1.5 A mass of 0.75 kg is attached to one end of a horizontal spring of spring constant 400 N m^{-1} . The other end of the spring is attached to a rigid wall. The mass is pushed so that at time $t = 0$ it is 4.0 cm closer to the wall than the equilibrium position and is travelling towards the wall with a velocity of 0.50 m s^{-1} . (a) Determine the total energy of the oscillating system. (b) Obtain an expression for the displacement of the mass in the form $x = A \cos(\omega t + \phi)$ m, giving numerical values for A , ω and ϕ .

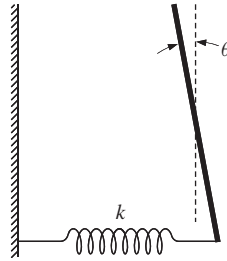
1.6



The figure shows three systems of a mass m suspended by light springs that all have the same spring constant k . Show that the frequencies of oscillation for the three systems are in the ratio $\omega_a : \omega_b : \omega_c = \sqrt{2} : 1 : \sqrt{1/2}$.

- 1.7 A test tube is weighted by some lead shot and floats upright in a liquid of density ρ . When slightly displaced from its equilibrium position and released, the test tube oscillates with SHM. (a) Show that the angular frequency of the oscillations is equal to $\sqrt{A\rho g/m}$ where g is the acceleration due to gravity, A is the cross-sectional area of the test tube and m is its mass. (b) Show that the potential energy of the system is equal to $\frac{1}{2}A\rho g x^2$ where x is the displacement from equilibrium. Hence give an expression for the total energy of the oscillating system in terms of the instantaneous displacement and velocity of the test tube.
- 1.8 We might assume that the period of a simple pendulum depends on the mass m , the length l of the string and g the acceleration due to gravity, i.e. $T \propto m^\alpha l^\beta g^\gamma$, where α , β and γ are constants. Consider the dimensions of the quantities involved to deduce the values of α , β and γ and hence show $T \propto \sqrt{l/g}$.
- 1.9 A simple pendulum has a length of 0.75 m. The pendulum mass is displaced horizontally from its equilibrium position by a distance of 5.0 mm and then released. Calculate (a) the maximum speed of the mass and (b) the time it takes to reach this speed. (Assume $g = 9.81 \text{ m s}^{-2}$.)

1.10



The figure shows a thin uniform rod of mass M and length $2L$ that is pivoted without friction about an axis through its mid point. A horizontal light spring of spring constant k is attached to the lower end of the rod. The spring is at its equilibrium length when the angle θ with respect to the vertical is zero. Show that for oscillations of small amplitude, the rod will undergo SHM with a period of $2\pi\sqrt{M/3k}$. The moment of inertia of the rod about its mid point is $ML^2/3$. (Assume the small angle approximations: $\sin \theta \simeq \theta$ and $\cos \theta \simeq 1$.)

- 1.11 The potential energy $U(x)$ between two atoms in a diatomic molecule can be expressed approximately by

$$U(x) = -\frac{a}{x^6} + \frac{b}{x^{12}}$$

where x is the separation of the atoms and a and b are constants. (a) Obtain an expression for the force between the two atoms and hence show that the equilibrium separation x_0 of the atoms is equal to $(2b/a)^{1/6}$. (b) Show that the system will oscillate with SHM when slightly displaced from equilibrium with a frequency equal to $\sqrt{k/m}$, where m is the reduced mass and $k = 36a(a/2b)^{4/3}$.

- 1.12 A mass M oscillates at the end of a spring that has spring constant k and finite mass m . (a) Show that the total energy E of the system for oscillations of small amplitude is given by

$$E = \frac{1}{2}(M + m/3)v^2 + \frac{1}{2}kx^2$$

where v and x are the velocity and displacement of the mass M , respectively. (Hint: To find the kinetic energy of the spring, consider it to be divided into infinitesimal elements of length dl and find the total kinetic energy of these elements, assuming that the mass of the spring is evenly distributed along its length. The total energy E of the system is the sum of the kinetic energies of the spring and the mass M and the potential energy of the extended spring.) (b) Hence show that the frequency of the oscillations is equal to $\sqrt{k/(M + m/3)}$.

- 1.13 A particle oscillates with amplitude A in a one-dimensional potential $U(x)$ that is symmetric about $x = 0$, i.e. $U(x) = U(-x)$. (a) Show, from energy considerations, that the velocity v of the particle at displacement x from the equilibrium position ($x = 0$), is given by

$$v = \sqrt{2[U(A) - U(x)]/m}.$$

(b) Hence show that the period of oscillation T is given by

$$T = 4\sqrt{\frac{m}{2U(A)}} \int_0^A \frac{dx}{\sqrt{[1 - U(x)/U(A)]}}.$$

(c) If the potential $U(x)$ is given by

$$U(x) = \alpha x^n$$

where α is a constant and $n = 2, 4, 6, \dots$, obtain the dependence of the period T on the amplitude A for different values of $n = 2, 4, \dots$. (Hint: Introduce the new variable of integration $\xi = x/A$ in the above expression for the period T .)