1

Space, Time and Motion

1.1 DEFINING SPACE AND TIME

If there is one part of physics that underpins all others, it is the study of motion. The accurate description of the paths of celestial objects, of planets and moons, is historically the most celebrated success of a classical mechanics underpinned by Newton's laws[1]. The range of applicability of these laws is vast, encompassing a scale that extends from the astronomical to the microscopic. We have come to understand that many phenomena not previously associated with motion are in fact linked to the movement of microscopic objects. The absorption and emission spectra of atoms and molecules arise as a result of transitions made by their constituent electrons, and the random motion of ensembles of atoms and molecules forms the basis for the modern statistical description of thermodynamics. Although atomic and subatomic objects are properly described using quantum mechanics, an understanding of the principles of classical mechanics is essential in making the conceptual leap from continuous classical systems with which we are most familiar, to the discretised quantum mechanical systems, which often behave in a manner at odds with our intuition. Indeed, the calculational techniques that are routinely used in quantum mechanics have their roots in the classical mechanics of particles and waves; a close familiarity with their use in classical systems is an asset when facing problems of an inherently quantum mechanical nature.

As we shall see in the second part of this book, when objects move at speeds approaching the speed of light classical notions about the nature of space and time fail us. As a result, the classical mechanics of Newton should be viewed as a low-velocity approximation to the more accurate relativistic theory of Einstein[2]. To look carefully at the differences between relativistic and non-relativistic theories

[1] After Isaac Newton (1643–1727).

[2] Albert Einstein (1879–1955).

Dynamics and Relativity Jeffrey R. Forshaw and A. Gavin Smith

forces us to recognise that our intuitive ideas about how things move are often incorrect. At the most fundamental level, mechanics of either the classical or the quantum kind, in either the relativistic or non-relativistic limit, is a study of motion and to study motion is to ask some fundamental questions about the nature of space and time. In this book we will draw out explicitly the different underlying structures of space and time used in the approaches of Newton and Einstein.

1.1.1 Space and the classical particle

We all have strong intuitive ideas about space, time and motion and it is precisely because of this familiarity that we must take special care in our attempts to define these fundamental concepts, so as not to carry too many unrecognised assumptions along with us as we develop the physics. So let us start by picking apart what we mean by position. We can usually agree what it means for London to be further away than Inverness and we all know that in order to go to London from Inverness we must also know the direction in which to travel. It may also seem to be fairly uncontentious that an object, such as London, has a position that can be specified, i.e. it is assumed that given enough information there will be no ambiguity about where it is. Although this seems reasonable, there is immediately a problem: day-to-day objects such as tennis balls and cities have finite size; there are a number of 'positions' for a given object that describe different parts of the object. Having directions to London may not be enough to find Kings Cross station, and having directions to Kings Cross station may not be enough to find platform number nine. To unambiguously give the position of an object is therefore only possible if the object is very small – vanishingly small, in fact. This hypothetical, vanishingly small object is called a particle. It might be suggested that with the discovery of the substructure of the atom, true particles, with mass but no spatial extent, have been identified. However, at this level, the situation becomes complicated by quantum uncertainty which makes the simultaneous specification of position and momentum impossible. The classical particle is therefore an idealisation, a limit in which the size of an object tends to zero but in which we ignore quantum phenomena. Later we shall see that it is possible to define a point called the centre of mass of an extended object and that this point behaves much like a classical particle. The collection of all possible positions for a particle forms what we call space.

The mathematical object possessing the properties we require for the description of position is called the vector. A vector has both magnitude and direction and we must be careful to distinguish it from a pure number which has a magnitude, but no directional properties. The paradigm for the vector comes from the displacement of a particle from point A to point B as shown in Figure 1.1. The displacement from A to B is represented by the directed-line-segment AB. We can imagine specifying the displacement as, for example, "start at A and move 3 km to the northeast" or "start at A and go 1 parsec in the direction of Alpha Centuri". Once we have specified a displacement between the two points A and B we can imagine sliding each end of the line segment in space until it connects another two points C and D. To do this, we move each end through the same distance and in the same direction,

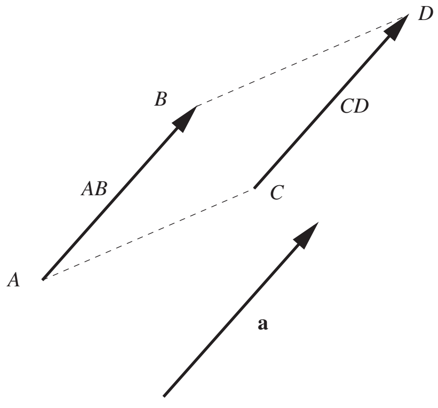

Figure 1.1 Displacement of a particle from point A to point B is illustrated by the directed line segment AB. Parallel transport of this line gives the displacement from point C to point D. The displacement vector **a** is not associated with any particular starting point.

an operation that is known as parallel transport. Now the displacement is denoted CD but its direction and magnitude are the same. It should be clear that there is an infinity of such displacements that may be obtained by parallel transport of the directed line segment. The displacement vector **a** has the magnitude and direction common to this infinite set of displacements but is not associated with a particular position in space. This is an important point which sometimes causes confusion since vectors are illustrated as directed line segments, which appear to have a well defined beginning and an end in space: A vector has magnitude and direction but not location. The position of a particle in space may be given generally by a position vector **r** only in conjunction with a fixed point of origin.

Now, all of this assumes that we understand what it means for lines to be parallel. At this point we assume that we are working in Euclidean space, which means that parallel lines remain equidistant everywhere, i.e. they never intersect. In non-Euclidean spaces, such as the two-dimensional surface of a sphere, parallel lines do intersect[3] and extra mathematics is required to specify how local geometries are transported to different locations in the space. For the moment, since we have no need of non-Euclidean geometry, we will rest our discussion of vectors firmly on the familiar Euclidean notion of parallel lines. Later, when we consider the space-time geometry associated with relativistic motion we will be forced to drop this deep-rooted assumption about the nature of space.

So far, we have been considering only vectors that are associated with displacements from one point to another. Their utility is far more wide ranging than that though: vectors are used to represent other interesting quantities in physics. For example the electric field strength in the vicinity of an electric charge is correctly represented by specifying both its magnitude and direction, i.e. it is a vector. Since it is important to maintain the distinction between vectors and ordinary numbers

[3] For example lines of longitude meet at the poles.

(called scalars) we identify vector quantities in this book by the use of bold font. When writing vectors by hand it is usual to either underline the vector, or to put an arrow over the top. Thus

$$\underline{a} \equiv \overrightarrow{a} \equiv \mathbf{a}.$$

Use the notation that you find most convenient, but always maintain the distinction between vector and scalar quantities. In this book both upper case (**A**) and lower case (**a**) notion will be used for vectors where **A** is in general a different vector from **a**. When a vector has zero magnitude it is impossible to define its direction; we call such a vector the null vector **0**.

1.1.2 Unit vectors

The length of a vector **a** is known as its magnitude, often denoted $|\mathbf{a}|$. To simplify the notation we shall adopt the convention that vectors are printed in bold and their magnitudes are indicated by dropping the bold font, thus $a \equiv |\mathbf{a}|$. Often we will separate the magnitude and direction of a vector, writing

$$\mathbf{a} = a\hat{\mathbf{a}},$$

where $\hat{\mathbf{a}}$ is the vector of unit magnitude with the same direction as **a**. Unit vectors, of which $\hat{\mathbf{a}}$ is an example, are often used to specify directions such as the directions of the axes of a co-ordinate system (see below).

1.1.3 Addition and subtraction of vectors

The geometrical rules for adding and subtracting vectors are illustrated in Figure 1.2. Addition of the vectors **A** and **B** involves sliding the vectors until they are "head-to-tail", so that the resultant vector connects the tail of **A** to the head of **B**. The vector $-\mathbf{A}$ is defined as a vector with the same magnitude, but opposite direction to **A**. The difference $\mathbf{B} - \mathbf{A}$ is constructed by adding **B** and $-\mathbf{A}$ as shown. Subtraction of a vector from itself gives the null vector:

$$\mathbf{A} - \mathbf{A} = \mathbf{0}.$$

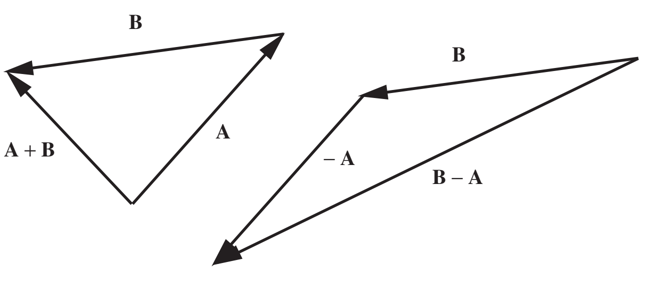

Figure 1.2 Adding and subtracting vectors.

1.1.4 Multiplication of vectors

There are two types of vector multiplication that are useful in classical physics. The scalar (or dot) product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined to be

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta, \tag{1.1}$$

This scalar quantity (a pure number) has a simple geometrical interpretation. It is the projection of $\mathbf{B}$ on $\mathbf{A}$, i.e. $B\cos\theta$, multiplied by the length of $\mathbf{A}$ (see Figure 1.3). Equally, it may be thought of as the projection of $\mathbf{A}$ on $\mathbf{B}$, i.e. $A\cos\theta$, multiplied by the length of $\mathbf{B}$. Clearly the scalar product is insensitive to the order of the vectors and hence $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$. The scalar product takes its maximum value of AB when the two vectors are parallel, and it is zero when the vectors are mutually perpendicular.

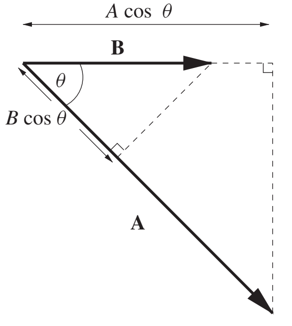

Figure 1.3 Geometry of the scalar product. $\mathbf{A} \cdot \mathbf{B}$ is the product of the length of $\mathbf{A}$, and the projection of $\mathbf{B}$ onto $\mathbf{A}$ or alternatively the product of the length of $\mathbf{B}$, and the projection of $\mathbf{A}$ onto $\mathbf{B}$.

The vector (or cross) product is another method of multiplying vectors that is frequently used in physics. The cross product of vectors $\mathbf{A}$ and $\mathbf{B}$ is defined to be

$$\mathbf{A} \times \mathbf{B} = AB\sin\theta\,\hat{\mathbf{n}}, \tag{1.2}$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$ and $\hat{\mathbf{n}}$ is a unit vector normal to the plane containing both $\mathbf{A}$ and $\mathbf{B}$. Whether $\hat{\mathbf{n}}$ is 'up' or 'down' is determined by convention and in our case we choose to use the right-hand screw rule; turning the fingers of the right hand from $\mathbf{A}$ to $\mathbf{B}$ causes the thumb to point in the sense of $\hat{\mathbf{n}}$ as is shown in Figure 1.4. Interchanging the order of the vectors in the product means that the fingers of the right hand curl in the opposite sense and the direction of the thumb is reversed. So we have

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}. \tag{1.3}$$

The magnitude of the vector product $AB\sin\theta$ also has a simple geometrical interpretation. It is the area of the parallelogram formed by the vectors $\mathbf{A}$ and $\mathbf{B}$. Alternatively it can be viewed as the magnitude of one vector times the projection

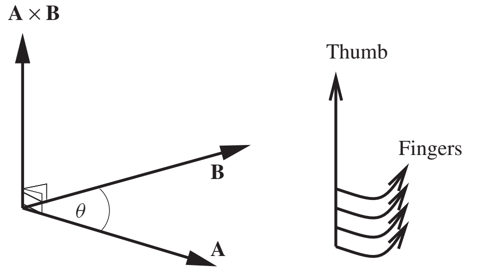

Figure 1.4 Vector product of **A** and **B**.

of the second on an axis which is perpendicular to the first and which lies in the plane of the two vectors. It is this second geometric interpretation that has most relevance in dynamics. As we shall see later, moments of force and momentum involve this type of perpendicular projection. In this book the vector product will find its principal application in the study of rotational dynamics.

The scalar and vector products are interesting to us precisely because they have a geometrical interpretation. That means they represent real things is space. In a sense, we can think of the scalar product as a machine that takes two vectors as input and returns a scalar as output. Similarly the vector product also takes two vectors as its input but instead returns a vector as its output. There are in fact no other significantly different[4] machines that are able to convert two vectors into scalar or vector quantities and as a result you will rarely see anything other than the scalar and vector products in undergraduate/college level physics. There is in fact a machine that is able to take two vectors as its input and return a new type of geometrical object that is neither scalar nor vector. We will even meet such a thing later in this book when we encounter tensors in our studies of advanced dynamics and advanced relativity.

1.1.5 Time

We are constantly exposed to natural phenomena that recur: the beat of a pulse; the setting of the Sun; the chirp of a cricket; the drip of a tap; the longest day of the year. Periodic phenomena such as these give us a profound sense of time and we measure time by counting periodic events. On the other hand, many aspects of the natural world do not appear to be periodic: living things die and decay without rising phoenix-like from their ashes to repeat their life-cycle; an egg dropped on the floor breaks and never spontaneously re-forms into its original state; a candle burns down but never up. There is a sense that disorder follows easily from order, that unstructured things are easily made from structured things but that the reverse is much more difficult to achieve. That is not to say that it is impossible to create order from disorder – you can do that by tidying your room – just that on average

[4] i.e. other than trivial changes such as would occur if we choose instead to define the scalar product to be $\mathbf{A} \cdot \mathbf{B} = \lambda AB \cos\theta$ where λ is a constant. We choose $\lambda = 1$ because it is most convenient but any other choice is allowed provided we take care to revise the geometrical interpretation accordingly.

things go the other way with the passing of time. This idea is central to the study of thermodynamics where the disorder in a system is a measurable quantity called entropy. The total entropy of the Universe appears always to increase with time. It is possible to decrease the entropy (increase the order) of a part of the Universe, but only at the expense of increasing the entropy of the rest of the Universe by a larger amount. This net disordering of the Universe is in accord with our perception that time has a direction. We cannot use natural processes to "wind the clock back" and put the Universe into the state it was in yesterday – yesterday is truly gone forever. That is not to say that the laws of physics forbid the possibility that a cup smashed on the floor will spontaneously re-assemble itself out of the pieces and leap onto the table from which it fell. They do not; it is simply that the likelihood of order forming spontaneously out of disorder like this is incredibly small. In fact, the laws of physics are, to a very good approximation, said to be "time-reversal invariant". The exception occurs in the field of particle physics where "CP-violation" experiments indicate that time-reversal symmetry is not respected in all fundamental interactions. This is evidence for a genuine direction to time that is independent of entropy. Entropy increase is a purely statistical effect, which occurs even when fundamental interactions obey time-reversal symmetry.

Thermodynamics gives us a direction to time and periodic events allow us to measure time intervals. A clock is a device that is constructed to count the number of times some recurring event occurs. A priori there is no guarantee that two clocks will measure the same time, but it is an experimental fact that two clocks that are engineered to be the same and which are placed next to each other, will measure, at least approximately, the same time intervals. This approximate equivalence of clocks leads us to conjecture the existence of absolute time, which is the same everywhere. A real clock is thus an imperfect means of measuring absolute time and a good clock is one that measures absolute time accurately. One problem with this idea is that absolute time is an abstraction, a theoretical idea that comes from an extrapolation of the experimental observation of the similar nature of different clocks. We can only measure absolute time with real clocks and without some notion of which clocks are better than others we have no handle on absolute time. One way to identify a reliable clock is to build lots of copies of it and treat all the copies exactly the same, i.e. put them in the same place, keep them at the same temperature and atmospheric conditions etc. If it is a reliable clock the copies will deviate little from each other over long time intervals. However, a reliable clock is not necessarily a good clock; similarly constructed clocks may run down in similar ways so that, for example, the time intervals between ticks might get longer the longer a clock runs, but in such a way that the similar clocks still read the same time. We can get around this by comparing equally reliable clocks based on different mechanisms. If enough equally-reliable clocks, based on enough different physical processes, all record the same time then we can start to feel confident that there is such a thing as absolute time. It is worth pointing out that in the 17th century reliable clocks were hard to come by and Newton certainly did not come to the idea of absolute time as a result of the observation of the constancy of clocks. Newton had an innate faith in the idea of absolute time and constructed his system of mechanics on that basis.

There is no doubt that absolute time is a useful concept; in this book we shall at first examine the motion of things under the influence of forces, treating time as though it is the same for every observer, and we will get answers accurate to a high degree. However, absolute time is a flawed concept, but flawed in such a way that the cracks only begin to appear under extreme conditions. We shall see later how clocks that are moving at very high relative velocities do *not* record the same time and that time depends on the state of motion of the observer. Einstein's Special Theory of Relativity tells us how to relate the time measured by different observers although the deviations from absolute time are only important when things start to move around at speeds approaching the speed of light. In describing the motion of things that do not approach the speed of light we can ignore relativistic effects with impunity, avoiding the conceptual and computational complications that arise from a full relativistic treatment. This will allow us to focus on concepts such as force, linear and angular momentum and energy. Once the basic concepts of classical mechanics have been established we will move on to study Special Relativity in Part II. Even then we will not completely throw out the concepts that are so successful in classical mechanics. Rather, these shall be adapted into the more general ideas of energy, momentum, space and time that are valid for all speeds.

1.1.6 Absolute space and space-time

At a fundamental level, the natural philosophy of Aristotle and the physics of Newton differ from the physics of Galileo[5] and Einstein in the way that space and time are thought to be connected. One very basic question involves whether space can be thought of as absolute. Consider the corner of the room you might be sitting in. The intersection of the two walls and the ceiling of a room certainly defines a point, but will this point be at the same place a microsecond later? We might be tempted to think so, that is, until the motion of the Earth is considered; the room is hurtling through space and so is our chosen point. Clearly the corner of the room defines a 'different' point at each instant. So would it be better to define a 'fixed' point with reference to some features of the Milky Way? This might satisfy us, at least until we discover that the Milky Way is moving relative to the other galaxies, so such a point cannot really be regarded as fixed. We find it difficult to escape completely from the idea that there is some sort of fixed background framework with respect to which we can measure all motion, but there is, crucially, no experimental evidence for this structure. Such a fixed framework is known as absolute space.

The concept of absolute space, which originates with Aristotle and his contemporaries, can be represented geometrically as shown in Figure 1.5(a). Here we have time as another Cartesian axis, tacked onto the spatial axes to produce a composite space that we call space-time. Consider two things that happen at times and positions that are measured using clocks and co-ordinate axes. We call these happenings 'events' and mark them on our space-time diagram as A and B. In the picture of absolute space, if the spatial co-ordinates of events A and B are identical we say that they represent the same point in space at different times. We can construct a path shown by the dotted line that connects the same point in space for all times. Galilean

[5] Galileo Galilei (1564–1642).

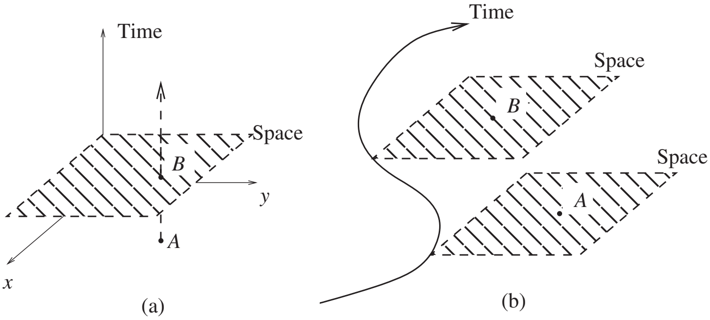

Figure 1.5 Different structures of space and time: (a) absolute space where points A and B are the same point in space; (b) a fibre-bundle structure where each moment in time has its own space.

relativity challenges this picture by rejecting the notion of absolute space and replacing it with the idea that space is defined relative to some chosen set of axes at a given instant in time. This is more like the picture in Figure 1.5(b), a structure that mathematicians call a fibre bundle. The same events A and B now lie in different spaces and the connection between them is no longer obvious. The fibre bundle is a more abstract structure to deal with than the space $\times$ time structure of (a). Imagine, for example, trying to calculate the displacement from A to B. To do this we have to assume some additional structure of space-time that allows us to compare points A and B. It is as if space is erased and redefined at each successive instant and we have no automatic rule for saying how the 'new' space relates to the 'old' one. Notice that this view still treats time as absolute; observers at different points in the $x - y$ plane agree on the common time t. In later chapters we will reconsider the geometry of space and time when we come to study the theory of Special Relativity, where universal time will be rejected in favour of a new space-time geometry in which observers at different positions each have their own local time.

1.2 VECTORS AND CO-ORDINATE SYSTEMS

As far as we can tell, space is three-dimensional, which means that three numbers are required to define a unique position. How we specify the three position-giving numbers defines what is known as the co-ordinate system. The co-ordinate system therefore introduces a sort of invisible grid or mesh that maps every point in space onto a unique ordered set of three real numbers. Figure 1.6 shows two commonly-used 3-dimensional co-ordinate systems. The Cartesian system is named after the French philosopher and mathematician René Descartes (1596–1650), who is reputed to have invented it from his bed while considering how he might specify the position of a fly that was buzzing around his room. This co-ordinate system consists of three mutually perpendicular axes, labelled x, y and z, that intersect at the point O, called the origin. The position of a particle at P may be specified by giving the set of three distances (x, y, z). Another frequently used co-ordinate system, the spherical-polar system, is obtained when the position of the particle is given instead by the distance from the origin r and two angles: the polar angle

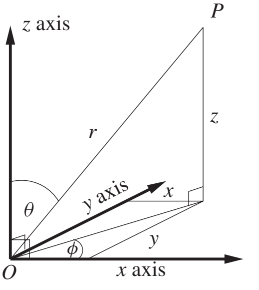

Figure 1.6 Two 3-dimensional co-ordinate systems covering the same space. The Cartesian co-ordinates consist of the set (x, y, z). The spherical polar co-ordinates consist of the set (r, θ, ϕ).

θ and the azimuthal angle ϕ. The Cartesian and the spherical polar systems are just two possible ways of mapping the same space, and it should be clear that for any given physical problem there will be an infinite number of equally-valid co-ordinate systems. The decision as to which one to use is based on the nature of the problem, and the ease or difficulty of the calculation that results from the choice.

Choosing a co-ordinate system immediately gives us a way to represent vectors. Associated with any co-ordinate system are a set of unit vectors known as basis vectors. Each co-ordinate has an associated basis vector that points in the direction in which that co-ordinate is increasing. For example, in the 3D Cartesian system **i** points in the direction of increasing x, i.e. along the x-axis, while **j** and **k** point along the $y-$ and $z-$axes, respectively. Suppose that the position of a particle relative to the origin is given by the vector **r**, known as the 'position vector' of the particle. Then **r** can be written in terms of the Cartesian basis vectors as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \tag{1.4}$$

where the numbers (x, y, z) are the Cartesian co-ordinates of the particle. The magnitude of the position vector, which is the distance between the particle and the origin, can be calculated by Pythagoras' Theorem and is

$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{x^2 + y^2 + z^2}. \tag{1.5}$$

We have focussed upon a position vector in the Cartesian basis but we could have talked about a force, or an acceleration or a magnetic field etc. Any vector **A** can be expressed in terms of its components (A_x, A_y, A_z) according to

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}. \tag{1.6}$$

It is not our aim here to present a full discussion of the algebraic properties of vectors. Some key results, which will prove useful later are listed in Table 1.1.

Very often, the motion of an object may be constrained to a known plane, such as in the case of a ball on a pool table, or a planet in orbit around the Sun. In such situations the full 3D co-ordinate system is not required and a two-dimensional

TABLE 1.1 Vector operations in the Cartesian basis. **A** and **B** are vectors, λ is a scalar.

Operation	Notation	Resultant
Negation	$-\mathbf{A}$	$(-A_x)\mathbf{i} + (-A_y)\mathbf{j} + (-A_z)\mathbf{k}$
Addition	$\mathbf{A} + \mathbf{B}$	$(A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$
Subtraction	$\mathbf{A} - \mathbf{B}$	$(A_x - B_x)\mathbf{i} + (A_y - B_y)\mathbf{j} + (A_z - B_z)\mathbf{k}$
Scalar (Dot) Product	$\mathbf{A} \cdot \mathbf{B}$	$A_x B_x + A_y B_y + A_z B_z$
Vector (Cross) Product	$\mathbf{A} \times \mathbf{B}$	$(A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$
Scalar Multiplication	$\lambda\mathbf{A}$	$\lambda A_x \mathbf{i} + \lambda A_y \mathbf{j} + \lambda A_z \mathbf{k}$

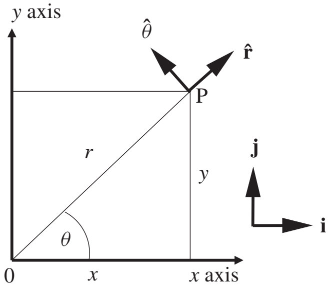

Figure 1.7 2D co-ordinate systems. The Cartesian co-ordinates consist of the set (x, y). The plane polar co-ordinates consist of the set (r, θ).

system may be used. Two of these are shown in Figure 1.7. The Cartesian 2D co-ordinate system has basis vectors **i** and **j** and co-ordinates (x, y). The plane-polar co-ordinates are (r, θ)[6] where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}. \tag{1.7}$$

The plane-polar system has basis vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. These may be expressed in terms of **i** and **j** as

$$\begin{aligned} \hat{\mathbf{r}} &= \mathbf{i}\cos\theta + \mathbf{j}\sin\theta, \\ \hat{\boldsymbol{\theta}} &= -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta. \end{aligned} \tag{1.8}$$

The general position vector in the plane may therefore be written as

$$\mathbf{r} = r\hat{\mathbf{r}} = r(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) = x\mathbf{i} + y\mathbf{j}. \tag{1.9}$$

Some care is required when using polar co-ordinates to describe the motion of a particle since the basis vectors depend on the co-ordinate θ, which may itself depend on time. This means that as the particle moves, the basis vectors change direction.

[6] Note the conventional use of θ for the angle to the x axis rather than ϕ, which is used for the corresponding angle in the spherical (3D) polar system.

This will lead to more complicated expressions for velocity and acceleration in polar co-ordinates than are obtained for Cartesian co-ordinates, as will be seen in the next section.

1.3 VELOCITY AND ACCELERATION

A particle is in motion when its position vector depends on time. The Ancient Greek philosophers had problems accepting the idea of a body being both in motion, and being 'at a point in space' at the same time. Zeno, in presenting his 'runner's paradox', divided up the interval between the start and finish of a race to produce an infinite sum for the total distance covered. He argued that before the runner completes the full distance (l) he must get half-way, and before he gets to the end of the second half he must get to half of that length and so on. The total distance covered can therefore be written as the infinite series

$$l\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right].$$

Zeno argued that it would be impossible for the runner to cover all of the sub-stretches in a finite time, and would therefore never get to the finish line. This contradiction forced him to decide that motion is impossible and that what we perceive as motion must be an illusion. We now know that the resolution of this paradox lies in an understanding of calculus. As the series continues, the steps get shorter and shorter, as do the time intervals taken for the runner to cover each step and we tend to a situation in which a vanishingly short distance is covered in a vanishingly small time.

Assuming that the position is a smooth function of time, we define the velocity as

$$\mathbf{v}(t)=\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}=\operatorname*{limit}_{\Delta t\to 0}\left(\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\right). \tag{1.10}$$

Notice that it involves a difference in the position vector at time $t+\Delta t$ and at time t. This difference, divided by the time interval Δt, only becomes the velocity *in the limit* that Δt goes to zero. Thus the velocity is defined in terms of an infinitesimally small displacement divided by an infinitesimally small time interval. Notice that the vector nature of $\mathbf{v}$ follows directly from the vector nature of $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$, which differs from $\mathbf{v}$ only by division by the scalar Δt. Often it is useful to refer to the magnitude of the velocity; this is known as the speed v, i.e.

$$v=|\mathbf{v}|.$$

With the notion that the ratio of two infinitesimally small quantities can be a finite number, we return to the Runner's Paradox. Zeno's argument does not rely on the particular choice of infinite series stated above. So we can simplify things by instead using a series made of equal-length steps. First we divide l up into

N equal lengths $\Delta x = \frac{l}{N}$. Assuming that the runner has a constant speed v in a straight line, we can write

$$l = \sum_{i=1}^{N} \Delta x = \sum_{i=1}^{N} \frac{\Delta x}{\Delta t} \Delta t, \tag{1.11}$$

where Δt is the time taken for the runner to cover the distance Δx. If we now let $N \to \infty$, then $\Delta t \to 0$ and $\frac{\Delta x}{\Delta t} \to v$, so that

$$l = v \sum_{i=1}^{\infty} \Delta t.$$

The time taken to run the whole race is therefore

$$t = \sum_{i=1}^{\infty} \Delta t = \frac{l}{v}.$$

Thus, provided that we are happy that the limit Eq. (1.10) exists, and that v is a non-zero number, then we can explain why the runner finishes the race in a finite time: there is no paradox. We may have laboured the point rather, the bottom line is of course that the distance travelled involves both integration and differentation:

$$l = \int \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t, \tag{1.12}$$

which works even if the speed is varying from point to point.

Just as velocity captures the rate at which a displacement changes so we introduce the acceleration, in order to quantify the rate of change of velocity:

$$\mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = \underset{\Delta t \to 0}{\text{limit}} \left(\frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} \right). \tag{1.13}$$

Again, $\mathbf{a}$ is a vector since it is defined as a vector divided by a scalar. In the Cartesian system, the velocity and acceleration take on a particularly simple form, since the basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ do not depend on time. Thus, if

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

use of the definitions Eq. (1.10) and Eq. (1.13) leads to

$$\begin{aligned} \mathbf{v}(t) &= \frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{k}, \\ \mathbf{a}(t) &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{i} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\mathbf{j} + \frac{\mathrm{d}^2z}{\mathrm{d}t}\mathbf{k}. \end{aligned} \tag{1.14}$$

Example 1.3.1 *Calculate the acceleration of a particle with the time-dependent position vector given by*

$$\mathbf{r}(t) = A\sin(\omega t)\mathbf{i} + \frac{1}{2}at^2\mathbf{j}.$$

Solution 1.3.1 *Differentiation once gives*

$$\mathbf{v}(t) = A\omega\cos(\omega t)\mathbf{i} + at\mathbf{j}$$

and again to obtain

$$\mathbf{a}(t) = -A\omega^2\sin(\omega t)\mathbf{i} + a\mathbf{j}.$$

1.3.1 Frames of reference

To describe the motion of a particle we need a position vector and a point of origin. We have seen that a position vector may be represented by components in a given co-ordinate system. The question immediately arises as to how one chooses a co-ordinate system to best suit a given physical situation.

For example, consider a cabin attendant who pushes a trolley along the aisle of an aircraft in flight. For a passenger on the aircraft a natural co-ordinate system to use would be one fixed to the aircraft, perhaps a Cartesian system with one axis pointed along the aisle. On the other hand, an observer on the ground might prefer a co-ordinate system fixed to the Earth. The reason why the observers tend to choose different co-ordinate systems is that each observer is surrounded by a different collection of objects that appear to be stationary. The passenger on the aircraft regards the structure of the aircraft as fixed whereas the observer on the ground regards objects on the Earth as stationary. We say that the passenger and the observer on the ground have different frames of reference.

A frame of reference is an abstraction of a rigid structure. We might think of a collection of particles whose relative positions do not change with time. However, it is not necessary for the particles to actually exist in order to define a frame of reference, we simply understand that the particles *could* exist in some sort of static arrangement that defines the frame of reference. Within a particular frame of reference there is always an infinite choice of co-ordinate systems. For example, if the observer on the ground chooses Cartesian co-ordinates, there are an infinite number of ways in which the axes may be oriented. Alternatively, latitude, longitude and distance from the centre of the Earth may be chosen as the three co-ordinates, with an arbitrary choice of where the meridian lines lie. The choice of co-ordinate system implies a particular frame of reference, but we can discuss frames of reference without commitment to a particular co-ordinate system.

1.3.2 Relative motion

In describing the motion of two particles it is often advantageous to use relative position and velocity vectors. The relative position vector $\mathbf{R}_{ab}(t) = \mathbf{r}_b(t) - \mathbf{r}_a(t)$ is the displacement from the position of particle a to that of particle b and it is, in general, a function of time. We differentiate with respect to time to obtain the relative velocity, $\mathbf{V}_{ab}(t)$,

$$\mathbf{V}_{ab}(t) = \frac{\mathrm{d}\mathbf{R}_{ab}(t)}{\mathrm{d}t} = \mathbf{v}_b(t) - \mathbf{v}_a(t), \tag{1.15}$$

where $\mathbf{v}_a(t)$ and $\mathbf{v}_b(t)$ are the velocities of particles a and b.

Example 1.3.2 *Consider an air-traffic controller tracking the positions of two aircraft. The controller knows the positions and velocities of the aircraft at some instant in time* ($t = 0$). *Assuming that the aircraft maintain their velocities, show that the relative velocity can be used to decide whether there is a danger of a collision at some later time.*

Solution 1.3.2 *The relative position vector at* $t = 0$ *is*

$$\mathbf{R}_0 = \mathbf{R}_{ab}(0) = \mathbf{r}_b(0) - \mathbf{r}_a(0).$$

The relative velocity is computed from the velocities of the aircraft:

$$\mathbf{V}_0 = \mathbf{V}_{ab}(0) = \mathbf{v}_b(0) - \mathbf{v}_a(0).$$

Since the aircraft have constant velocities the relative velocity is also constant and it can be integrated with respect to time to obtain

$$\mathbf{R}_{ab}(t) = \mathbf{R}_0 + \mathbf{V}_0 t.$$

The aircraft will collide if at some time t, $\mathbf{R}_{ab}(t) = \mathbf{0}$, *i.e. when* $\mathbf{R}_0 = -\mathbf{V}_0 t$. *This is a vector equation and it can only be satisfied if both the directions and magnitudes of both sides of the equation are the same. Clearly we can only obtain a solution for* $t > 0$ *if* $\mathbf{R}_0$ *and* $\mathbf{V}_0$ *are* anti-parallel *i.e. if* $\mathbf{R}_0 = R_0\hat{\mathbf{n}}$ *and* $\mathbf{V}_0 = -V_0\hat{\mathbf{n}}$, *where* R_0 *and* V_0 *are positive magnitudes and* $\hat{\mathbf{n}}$ *is a unit vector. If the vectors are anti-parallel, the collision time is* R_0/V_0.

In the previous example, we worked entirely in the frame of reference in which the air traffic controller is at rest. It is tempting to identify the relative velocity $\mathbf{V}_{ab}$ also as the velocity of the aircraft b relative to the pilot of aircraft a. Strictly speaking we have not proved this: $\mathbf{V}_{ab}$ is the velocity of b relative to a as determined by the air traffic controller, not by the pilot of aircraft a. In classical mechanics, where time is universal, the two are equivalent and specifying the relative velocity between two bodies does not need us to further specify who is doing the observing. That the assumption of universal time enters into this matter can be seen by exploring the expression $\mathbf{V}_{ab}(t) = \frac{\mathrm{d}\mathbf{R}_{ab}(t)}{\mathrm{d}t}$. Whose time is represented by t? That of the air-traffic controller or that of the pilot in aircraft a? If we accept the concept of absolute time then it doesn't matter and both record the same relative velocity. But we really ought to recognise that the assumption of universal time is just that: an assumption. This is not an irrelevant matter for, as we shall see in Part II, the universality of time breaks down, becoming most apparent when relative velocities start to approach the speed of light.

1.3.3 Uniform acceleration

In many physical situations the acceleration does not change with time. Integration of Eq. (1.13) then gives

$$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \mathbf{v}_0 + \mathbf{a}\,t, \tag{1.16}$$

where $\mathbf{v}_0$ is the velocity at time $t = 0$. Since $\mathbf{v}_0$ is a constant vector, integration again yields

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}\,t^2. \tag{1.17}$$

In general the vectors $\mathbf{r}_0$, $\mathbf{v}_0$ and $\mathbf{a}$ will have different directions and each of the vector equations, Eq. (1.16) and Eq. (1.17), is shorthand for three different scalar equations, one for each of the three spatial components. An important simplification occurs in situations where the velocity, acceleration and displacement are all collinear (i.e. all in the same direction). Then we need only consider the components of the vectors along the direction of motion, i.e.

$$v = v_0 + at \tag{1.18}$$

and

$$r = r_0 + v_0 t + \frac{1}{2}at^2. \tag{1.19}$$

Squaring Eq. (1.18) and substituting using Eq. (1.19) yields a third equation that is often useful in solving problems that don't deal explicitly with time:

$$v^2 = v_0^2 + 2a(r - r_0). \tag{1.20}$$

Even if $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ are not collinear then Eq. (1.18), Eq. (1.19) and Eq. (1.20) can still be applied to each of the Cartesian components of the vectors since the basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are independent of time.

As an example, let us consider the problem of projectile motion in a uniform gravitational field. Close to the Earth's surface any object accelerates towards the centre of the Earth. This acceleration has magnitude

$$g \approx 9.81\,\mathrm{ms}^{-2}$$

although the exact value depends on where you are on the surface of the Earth. The fact that all objects fall at the same rate is rather amazing, but we will defer a discussion of that until the next chapter. Here we only want to use the result that the acceleration is uniform, which is true so long as we stick to low altitudes and ignore the effects of air resistance.

Example 1.3.3 *Determine the path of a projectile fired with speed u at an angle θ to the horizontal. Neglect air resistance. Use the path to determine the range of the projectile.*

Solution 1.3.3 *The choice the co-ordinate system is up to us. Since we want to separate the description of the motion into Cartesian components, we choose the* y*-axis to be upwards and the* x*-axis to be horizontal and in the same plane as the initial velocity. We then have*

$$\mathbf{a} = -g\mathbf{j}, \quad \text{and}$$

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} = u\cos\theta\mathbf{i} + u\sin\theta\mathbf{j}.$$

We write the position of the projectile as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j},$$

where x *and* y *depend on time. For convenience we let* $\mathbf{r} = \mathbf{0}$ *at* $t = 0$*. Our choice of co-ordinate system means that there is no acceleration in the* x*-direction. Thus we have,*

$$x = u_x t = ut\cos\theta.$$

In the y*-direction we have*

$$y = u_y t - \frac{1}{2}gt^2 = ut\sin\theta - \frac{1}{2}gt^2.$$

These are parametric equations for x *and* y *(with time as the parameter). To obtain the path of the projectile we eliminate* t *to get* y *as a function of* x*:*

$$y = \frac{u_y}{u_x}x - \frac{g}{2u_x^2}x^2 = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2.$$

This is the equation of a parabola (see Figure 1.8). To obtain the range of the projectile we need to find values of x *such that* $y = 0$*. These are* $x = 0$*, the launch position, and* $x = (2u^2\sin\theta\cos\theta)/g = (u^2\sin 2\theta)/g$*, the horizontal range of the projectile. Notice that the range is maximal for* $\theta = 45°$*.*

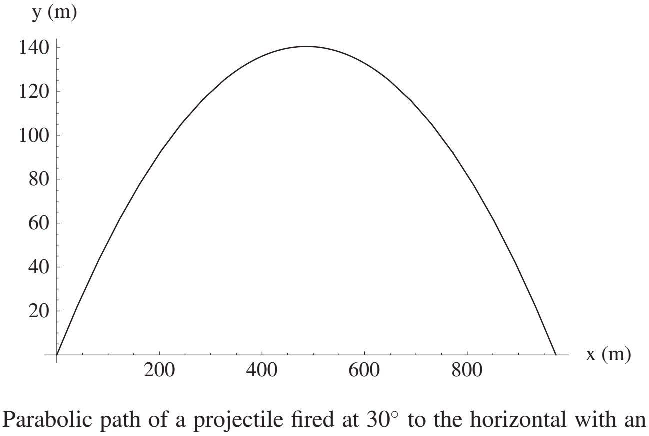

Figure 1.8 Parabolic path of a projectile fired at 30° to the horizontal with an initial speed of 105 ms^{-1}. Note that the distance scales are different on the horizontal and vertical axes.

1.3.4 Velocity and acceleration in plane-polar co-ordinates: uniform circular motion

Circular motion arises frequently in physics. Examples may be as simple as a mass whirled on a string, but also include the orbits of satellites around the Earth, and the motion of charged particles in a magnetic field. Where circular motion is concerned, problems are often most easily solved in polar co-ordinates. In this section we determine the equations for the velocity and acceleration in polar co-ordinates.

The position of a particle moving in a plane is

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{1.21}$$

As the particle moves, both r and $\hat{\mathbf{r}}$ may change, i.e. they are both implicitly time-dependent. The velocity of the particle is calculated by differentiation of the product $r\hat{\mathbf{r}}$.

$$\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{\mathbf{r}}) = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\mathbf{r}} + r\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}. \tag{1.22}$$

Since $\hat{\mathbf{r}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$,

$$\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = -\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{i} + \cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{j} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\boldsymbol{\theta}}, \tag{1.23}$$

where we have used the definition Eq. (1.8) for $\hat{\boldsymbol{\theta}}$. Thus,

$$\mathbf{v} = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\mathbf{r}} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\boldsymbol{\theta}}. \tag{1.24}$$

The tangential contribution $r\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\boldsymbol{\theta}}$ is zero if the particle moves radially (constant θ) whereas the radial velocity $\frac{\mathrm{d}r}{\mathrm{d}t}\hat{\mathbf{r}}$ is zero for motion in a circle (constant r). We introduce the angular speed $\omega = \mathrm{d}\theta/\mathrm{d}t$, to simplify the notation. The velocity is then

$$\mathbf{v} = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\mathbf{r}} + r\omega\hat{\boldsymbol{\theta}}. \tag{1.25}$$

The general expression for acceleration can be obtained by differentiation of Eq. (1.24) and further application of Eq. (1.8). However, at this point we will concern ourselves with the case of uniform circular motion, i.e. $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$ and ω constant. In which case, we only need worry about the tangential term in (1.24) and

$$\mathbf{a} = \frac{\mathrm{d}}{\mathrm{d}t}(r\omega\hat{\boldsymbol{\theta}}) = \frac{\mathrm{d}r}{\mathrm{d}t}\omega\hat{\boldsymbol{\theta}} + r\frac{\mathrm{d}\omega}{\mathrm{d}t}\hat{\boldsymbol{\theta}} - r\omega^2\hat{\mathbf{r}}, \tag{1.26}$$

where we have used

$$\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\omega\hat{\mathbf{r}}.$$

The first two terms in Eq. (1.26) are zero for uniform circular motion, so we obtain

$$\mathbf{a}(\text{uniform circular motion}) = -r\omega^2\hat{\mathbf{r}}. \tag{1.27}$$

Notice that the acceleration here is not a result of a change in the magnitude of $\mathbf{v}$; this is constant. Rather, the direction of $\hat{\boldsymbol{\theta}}$ (and hence that of $\mathbf{v}$) is constantly changing and this gives rise to the acceleration in the radial direction. Notice also that the acceleration in Eq. (1.27) points towards the centre of the circular orbit, i.e. in the direction of $-\hat{\mathbf{r}}$.

We have derived Eq. (1.27) using the formal differentiation of the time-dependent position vector $r\hat{\mathbf{r}}$. We can also understand the result geometrically. We begin by sketching the important vectors in Figure 1.9. We show the position of the particle at times t and $t + \Delta t$ (points A and B respectively) as well as the corresponding velocity vectors. The velocity vectors are tangential to the path of the particle and have equal magnitudes ($v = r\omega$). Let's construct the velocity difference $\Delta\mathbf{v} = \mathbf{v}(t + \Delta t) - \mathbf{v}(t)$: you should be able to see from the diagram that $\Delta\mathbf{v}$ points approximately towards the centre of the circle from the midpoint of the circular arc between A and B. In the triangle of velocity vectors formed by $\mathbf{v}(t + \Delta t)$, $\mathbf{v}(t)$ and $\Delta\mathbf{v}$ we can approximate the magnitude of $\Delta\mathbf{v}$ by a circular arc, and write $\Delta v \approx v\Delta\theta = v\omega\Delta t = r\omega^2\Delta t$. In the limit $\Delta t \to 0$ the approximation becomes exact, $a = \Delta v/\Delta t \to r\omega^2$, and the acceleration points exactly in the direction $-\hat{\mathbf{r}}$. We are therefore led to Eq. (1.27).

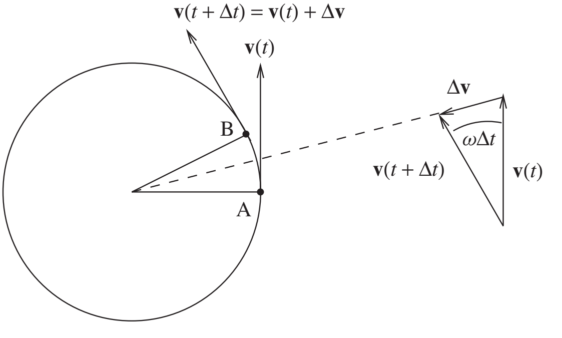

Figure 1.9 Uniform circular motion. Notice that the changing direction of the velocity vector results in a vector $\Delta\mathbf{v}$ that points approximately towards the centre of the circle. In the limit of vanishingly-small Δt this vector corresponds to the acceleration and points exactly towards the centre.

1.4 STANDARDS AND UNITS

In this chapter we have introduced the concepts of space and time without saying too much about measurement. Measurement of a physical quantity consists of making a comparison of that quantity, either directly or indirectly, with a standard. A standard is something on which we must all be able to agree and which defines the unit in which the measurement will be expressed. We will illustrate the idea by considering the legendary origins of the yard as a unit of length. Legend has it that the yard was originally defined to be the distance from tip of King Henry I of England's nose to the end of his thumb. Clearly the direct use of this standard of measurement would have been a little inconvenient; you can be sure that pretty soon a rod would have been cut to the correct length and used as a substitute for the King's own person. The use of this rod for measurement is an example of indirect measurement, though still using the same standard yard it doesn't require the King to be present. Desirable characteristics of a standard are reproducibility and precision. Reproducibility means that the standard can be used over and over again to give a consistent definition of the unit, one which doesn't vary with time. If the English people had reason to suspect that the King had grown or shrunk (perhaps by later comparisons with the rod) then they might have faced a dilemma: reject the King as the means to define the standard yard (in favour of the rod) or keep the definition using the King and face the problems associated with their not choosing a reproducible standard of length. Furthermore, the distance from the tip of the King's nose to the end of his thumb is not a terribly precise standard. Just consider the question of how he should hold his head while the measurement is taking place. The yard defined in this way clearly can only be expected to be accurate at the level of a few percent. It is easy to think of standards for length that are both more reproducible and more precise than this legendary definition.

Units are either fundamental, as is the case with the second (s), the kilogram (kg) and the metre (m), or they are derived units, such as the unit of velocity ($\mathrm{m\ s^{-1}}$). For each of the fundamental units, there must be a precise and reproducible laboratory standard. In the case of the S.I. unit of mass, the kilogram, the standard is a lump of platinum-iridium alloy kept at the International Bureau of Weights and Measures (BIPM), at Sèvres in France. The SI unit of time, the second, was originally 1/86,400 of the mean solar day, and then later defined as a fraction of the mean tropical year. Neither of these standards could approach the accuracy of those based on the frequency of radiation emitted by certain atoms and in 1967 the second was redefined as exactly 9,192,631,770 cycles of the transition between two hyperfine levels in ^{133}Cs. In practice this standard uses a cavity filled with an ionised vapour of ^{133}Cs. Standing electromagnetic waves are created in the cavity using a radio-frequency oscillator circuit. When the frequency of the oscillator matches that of the atomic transition a resonance is observed. At resonance the oscillator circuit will then, by definition, make precisely 9,192,631,770 cycles in one second. Clocks based on sophisticated versions of this technique, such as those at the National Institute of Standards and Technology in the USA, are capable of measuring time to an accuracy of better than one nanosecond in a day.

The standard unit of length, the metre was once defined as one ten-millionth of the distance on the Meridian through Paris from the pole to the equator. This standard was replaced in 1874 and 1889 by standards based on the length, at zero degrees centigrade, of a prototype platinum-iridium bar. In 1984, standards based on prototype bars were superseded by the current standard distance that light travels in vacuum during a time interval of exactly 1/299,792,458 of a second. The effect of this definition is to fix the speed of light in vacuum at exactly 299,792,458 $\mathrm{ms^{-1}}$. The justification for this choice of standard relies on our belief in the constancy of the speed of light in vacuum, a phenomenon that will be discussed in later chapters.

PROBLEMS 1

1.1 For the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, find:

(a) the vectors $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} - \mathbf{b}$;
(b) the magnitudes of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$;
(c) a unit vector in the same direction as $\mathbf{a}$.

1.2 A bird leaves its nest and flies 100 m NE (i.e. at a bearing of 45°) then 150 m at a bearing of 150° and finally 50 m due W, where it lands in a tree. How far is the tree from the nest? In what direction is the tree from the nest?

1.3 The vertices of a triangle, A, B and C, have position vectors $\mathbf{r}_A$, $\mathbf{r}_B$ and $\mathbf{r}_C$. Write down $\overrightarrow{AB}$, the position of B relative to A, in terms of $\mathbf{r}_A$ and $\mathbf{r}_B$. Hence show that the vector sum of the successive sides of the triangle ($\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$) is zero. Draw a sketch to demonstrate this result geometrically.

1.4 Hubble found that distant galaxies are receding from us with a speed proportional to their distance from the Earth. The velocity of the i-th galaxy is given by

$$\mathbf{v}_i = H_0 \mathbf{r}_i,$$

where $\mathbf{r}_i$ is the position vector of that galaxy with respect to us at the origin, and H_0 is a constant (known as Hubble's constant). Show that this recession of the galaxies does *not* imply that we are at the centre of the Universe.

1.5 For the vectors

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k},$$

$$\mathbf{B} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k},$$

$$\mathbf{C} = 4\mathbf{i} - \mathbf{j} - 3\mathbf{k},$$

$$\mathbf{D} = -\mathbf{i} + \mathbf{j} + \mathbf{k},$$

find their magnitudes and the scalar products $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \cdot \mathbf{C}$, $\mathbf{A} \cdot \mathbf{D}$ and $\mathbf{B} \cdot \mathbf{D}$. Hence find the angles between $\mathbf{A}$ and each of $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ and that between $\mathbf{B}$ and $\mathbf{D}$. Evaluate the vector products $\mathbf{A} \times \mathbf{B}$, $\mathbf{A} \times \mathbf{D}$ and $\mathbf{B} \times \mathbf{D}$. Check that the magnitudes of these agree with the corresponding geometrical expressions ($|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin\theta$ etc.).

1.6 A charged particle is accelerated uniformly from rest in an electric field. If after 1.0 nanoseconds the particle has travelled 10 μm, work out its acceleration.

1.7 A coin is dropped from the top of a tall building. If an observer on the ground measures the speed of the coin immediately before impact to be $65.0\,\mathrm{ms}^{-1}$, how tall is the building? For how long was the coin falling? Neglect effects due to air resistance.

1.8 A missile malfunctions in flight and has a subsequent trajectory described by the position vector ($\mathbf{s}$) at time (t), given by,

$$\mathbf{s} = 0.3t\,\mathbf{i} + 0.5t\,\mathbf{j} - 0.005t^2\,\mathbf{k},$$

where t is measured in seconds and the magnitude of **s** is measured in km.

(a) What is the speed of the missile at $t = 0$? In which plane is the velocity at this time?
(b) What is the speed of the missile at $t = 30$ s? What is the angle between the velocity vector and the (positive) z axis at this time?

1.9 An airport travelator of length 50.0 m moves at a speed of $1.0\,\mathrm{ms}^{-1}$. An athlete capable of running at a speed of $10.0\,\mathrm{ms}^{-1}$ bets a friend that he can run to the end of the travelator and back again in exactly 10.0 s, as long as the time to change direction and restart is not included. The athlete loses the bet. What mistake has the athlete made? (Assume that at the start of each leg, the athlete is already running at full speed.)

1.10 A ferryman crosses a fast-flowing river. The ferryman knows that her boat travels at a speed v in still water, and that with the engine off the boat will drift at a speed u in a direction parallel to the bank, where u is less than v. If the line joining the two ferry stations makes a right-angle with the bank, and the stations are separated by a distance d, derive an expression for the time taken to cross the river. What happens if u is greater than v?