# 1
# Brownian Motion

The exposition of Brownian motion is in two parts. Chapter 1 introduces the properties of Brownian motion as a random process, that is, the true technical features of Brownian motion which gave rise to the theory of stochastic integration and stochastic calculus. Annex A presents a number of useful computations with Brownian motion which require no more than its probability distribution, and can be analysed by standard elementary probability techniques.

## 1.1 ORIGINS

In the summer of 1827 Robert Brown, a Scottish medic turned botanist, microscopically observed minute pollen of plants suspended in a fluid and noticed increments[1] that were highly irregular. It was found that finer particles moved more rapidly, and that the motion is stimulated by heat and by a decrease in the viscosity of the liquid. His investigations were published as *A Brief Account of Microscopical Observations Made in the Months of June, July and August 1827*. Later that century it was postulated that the irregular motion is caused by a very large number of collisions between the pollen and the molecules of the liquid (which are microscopically small relative to the pollen). The hits are assumed to occur very frequently in any small interval of time, independently of each other; the effect of a particular hit is thought to be small compared to the total effect. Around 1900 Louis Bachelier, a doctoral student in mathematics at the Sorbonne, was studying the behaviour of stock prices on the Bourse in Paris and observed highly irregular increments. He developed the first mathematical specification of the increment reported by Brown, and used it as a model for the increment of stock prices. In the 1920s Norbert Wiener, a mathematical physicist at MIT, developed the fully rigorous probabilistic framework for this model. This kind of increment is now called a Brownian motion or a Wiener process. The position of the process is commonly denoted



[1] This is meant in the mathematical sense, in that it can be positive or negative.

by $B$ or $W$. Brownian motion is widely used to model randomness in economics and in the physical sciences. It is central in modelling financial options.

## 1.2 BROWNIAN MOTION SPECIFICATION

The physical experiments suggested that:

- the increment is continuous
- the increments of a particle over disjoint time intervals are independent of one another
- each increment is assumed to be caused by independent bombardments of a large number of molecules; by the Central Limit Theorem of probability theory the sum of a large number of independent identically distributed random variables is approximately normal, so each increment is assumed to have a normal probability distribution
- the mean increment is zero as there is no preferred direction
- as the position of a particle spreads out with time, it is assumed that the variance of the increment is proportional to the length of time the Brownian motion has been observed.

Mathematically, the random process called Brownian motion, and denoted here as $B(t)$, is defined for times $t \geq 0$ as follows. With time on the horizontal axis, and $B(t)$ on the vertical axis, at each time $t$, $B(t)$ is the position, in one dimension, of a physical particle. It is a random variable. The collection of these random variables indexed by the continuous time parameter $t$ is a *random process* with the following properties:

(a) The increment is continuous; when recording starts, time and position are set at zero, $B(0) = 0$
(b) Increments over non-overlapping time intervals are independent random variables
(c) The increment over any time interval of length $u$, from any time $t$ to time $(t + u)$, has a normal probability distribution with mean zero and variance equal to the length of this time interval.

As the probability density of a normally distributed random variable with mean $\mu$ and variance $\sigma^2$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

the probability *density* of the position of a Brownian motion at the end of time period $[0, t]$ is obtained by substituting $\mu = 0$ and $\sigma = \sqrt{t}$, giving

$$\frac{1}{\sqrt{t}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x}{\sqrt{t}}\right)^2\right]$$

where $x$ denotes the value of random variable $B(t)$. The probability *distribution* of the increment $B(t+u) - B(t)$ is

$$\mathbb{P}[B(t+u)-B(t) \leq a] = \int_{x=-\infty}^{a} \frac{1}{\sqrt{u}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x}{\sqrt{u}}\right)^2\right] dx$$

Note that the starting time of the interval does not figure in the expression for the probability distribution of the increment. The probability distribution depends only on the time spacing; it is the same for all time intervals that have the same length. As the standard deviation at time $t$ is $\sqrt{t}$, the longer the process has been running, the more spread out is the density, as illustrated in Figure 1.1.

As a reminder of the randomness, one could include the state of nature, denoted $\omega$, in the notation of Brownian motion, which would then be $B(t, \omega)$, but this is not commonly done. For each fixed time $t^*$, $B(t^*, \omega)$ is a function of $\omega$, and thus a random *variable*. For a particular $\omega^*$ over the time period $[0, t]$, $B(t, \omega^*)$ is a function of $t$ which is known as a *sample path* or trajectory. In the technical literature this is often denoted as $t \longmapsto B(t)$. On the left is an element from the domain, on the right the corresponding function value in the range. This is as in ordinary calculus where an expression like $f(x) = x^2$ is nowadays often written as $x \longmapsto x^2$.

As the probability distribution of $B(t)$ is normal with standard deviation $\sqrt{\Delta t}$, it is the same as that of $\sqrt{\Delta t}\, Z$, where $Z$ is a standard normal random variable. When evaluating the probability of an expression involving $B(t)$, it can be convenient to write $B(t)$ as $\sqrt{\Delta t} Z$.

The Brownian motion distribution is also written with the cumulative standard normal notation $N(\text{mean, variance})$ as $B(t+u) - B(t) \sim N(0, u)$, or for any two times $t_2 > t_1$ as $B(t_2) - B(t_1) \sim N(0, t_2 - t_1)$. As $\mathbb{V}\text{ar}[B(t)] = \mathbb{E}[B(t)^2] - \{\mathbb{E}[B(t)]\}^2 = t$, and $\mathbb{E}[B(t)] = 0$, the second moment of Brownian motion is $\mathbb{E}[B(t)^2] = t$. Over a time step $\Delta t$, where $\Delta B(t) \stackrel{\text{def}}{=} B(t + \Delta t) - B(t)$, $\mathbb{E}\{[\Delta B(t)]^2\} = \Delta t$. A normally distributed random variable is also known as a Gaussian random variable, after the German mathematician Gauss.

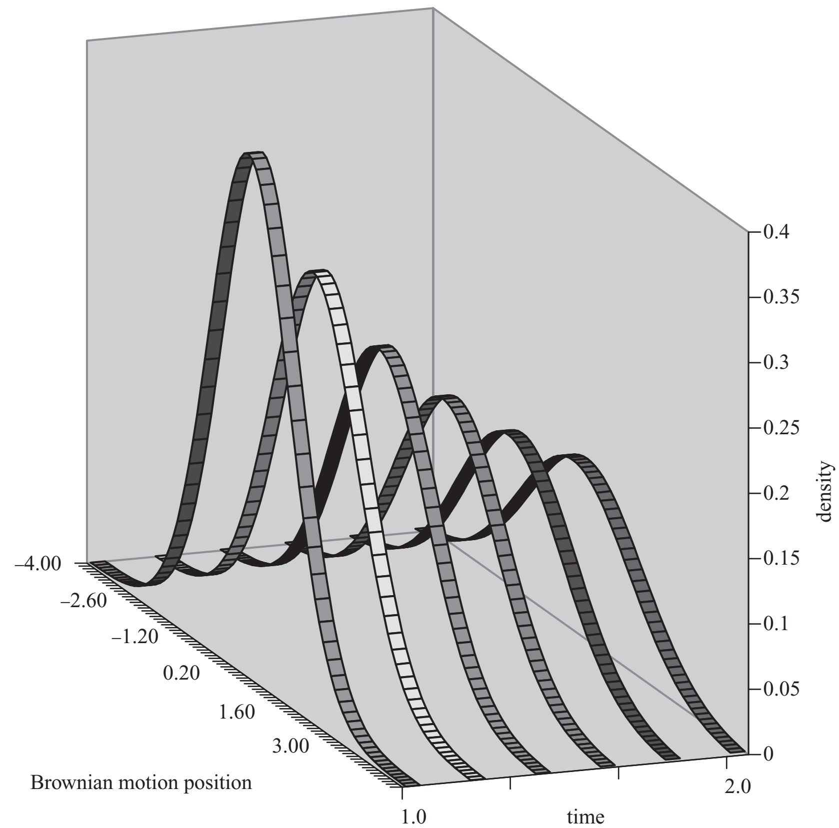


**Figure 1.1** Brownian motion densities

## 1.3 USE OF BROWNIAN MOTION IN STOCK PRICE DYNAMICS

Brownian motion arises in the modelling of the evolution of a stock price (often called the stock price dynamics) in the following way. Let $\Delta t$ be a time interval, $S(t)$ and $S(t+\Delta t)$ the stock prices at current time $t$ and future time $(t+\Delta t)$, and $\Delta B(t)$ the Brownian motion increment over $\Delta t$. A widely adopted model for the stock price dynamics, in a discrete time setting, is

$$\frac{S(t+\Delta t)-S(t)}{S(t)} = \mu\,\Delta t + \sigma\,\Delta B(t)$$

where $\mu$ and $\sigma$ are constants. This is a stochastic *difference* equation which says that the change in stock price, relative to its current value at time $t$, $[S(t + \Delta t) - S(t)]/S(t)$, grows at a non-random rate of $\mu$ per unit of time, and that there is also a random change which is proportional to the increment of a Brownian motion over $\Delta t$, with proportionality parameter $\sigma$. It models the *rate of return* on the stock, and evolved from the first model for stock price dynamics postulated by Bachelier in 1900, which had the change in the stock price itself proportional to a Brownian motion increment, as

$$\Delta S(t) = \sigma\, \Delta B(t)$$

As Brownian motion can assume negative values it implied that there is a probability for the stock price to become negative. However, the limited liability of shareholders rules this out. When little time has elapsed, the standard deviation of the probability density of Brownian motion, $\sqrt{t}$, is small, and the probability of going negative is very small. But as time progresses the standard deviation increases, the density spreads out, and that probability is no longer negligible. Half a century later, when research in stock price modelling began to take momentum, it was judged that it is not the level of the stock price that matters to investors, but the rate of return on a given investment in stocks.

In a continuous time setting the above discrete time model becomes the stochastic *differential* equation

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dB(t)$$

or equivalently $dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, dB(t)$, which is discussed in Chapter 5. It is shown there that the stock price process $S(t)$ which satisfies this stochastic differential equation is

$$S(t) = S(0) \exp[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma B(t)]$$

which cannot become negative. Writing this as

$$S(t) = S(0) \exp(\mu t) \exp[\sigma B(t) - \tfrac{1}{2}\sigma^2 t]$$

gives a decomposition into the non-random term $S(0) \exp(\mu t)$ and the random term $\exp[\sigma B(t) - \frac{1}{2}\sigma^2 t]$. The term $S(0) \exp(\mu t)$ is $S(0)$ growing at the continuously compounded constant rate of $\mu$ per unit of time, like a savings account. The random term has an expected value of 1. Thus the expected value of the stock price at time $t$, given $S(0)$, equals $S(0)\exp(\mu t)$. The random process $\exp[\sigma B(t) - \frac{1}{2}\sigma^2 t]$ is an example of a martingale, a concept which is the subject of Chapter 2.

## 1.4 CONSTRUCTION OF BROWNIAN MOTION FROM A SYMMETRIC RANDOM WALK

Up to here the reader may feel comfortable with most of the mathematical specification of Brownian motion, but wonder why the variance is proportional to time. That will now be clarified by constructing Brownian motion as the so-called *limit in distribution* of a symmetric random walk, illustrated by computer simulation. Take the time period $[0, T]$ and partition it into $n$ intervals of equal length $\Delta t \stackrel{\text{def}}{=} T/n$. These intervals have endpoints $t_k \stackrel{\text{def}}{=} k\,\Delta t$, $k = 0, \ldots, n$. Now consider a particle which moves along in time as follows. It starts at time 0 with value 0, and moves up or down at each discrete time point with equal probability. The magnitude of the increment is specified as $\sqrt{\Delta t}$. The reason for this choice will be made clear shortly. It is assumed that successive increments are *independent* of one another. This process is known as a symmetric (because of the equal probabilities) random walk. At time-point 1 it is either at level $\sqrt{\Delta t}$ or at level $-\sqrt{\Delta t}$. If at time-point 1 it is at $\sqrt{\Delta t}$, then at time-point 2 it is either at level $\sqrt{\Delta t} + \sqrt{\Delta t} = 2\sqrt{\Delta t}$ or at level $\sqrt{\Delta t} - \sqrt{\Delta t} = 0$. Similarly, if at time-point 1 it is at level $-\sqrt{\Delta t}$, then at time-point 2 it is either at level 0 or at level $-2\sqrt{\Delta t}$, and so on. Connecting these positions by straight lines gives a continuous path. The position at any time between the discrete time points is obtained by linear interpolation between the two adjacent discrete time positions. The complete picture of all possible discrete time positions is given by the nodes in a so-called binomial tree, illustrated in Figure 1.2 for $n = 6$. At time-point $n$, the node which is at the end of a path that has $j$ up-movements is labelled $(n, j)$, which is very convenient for doing tree arithmetic.

When there are $n$ intervals, there are $(n + 1)$ terminal nodes at time $T$, labelled $(n, 0)$ to $(n, n)$, and a total of $2^n$ different paths to these terminal nodes. The number of paths ending at node $(n, j)$ is given by a Pascal triangle. This has the same shape as the binomial tree. The upper and lower edge each have one path at each node. The number of paths going to any intermediate node is the sum of the number of paths going to the preceding nodes. This is shown in Figure 1.3. These numbers are the binomial coefficients from elementary probability theory.

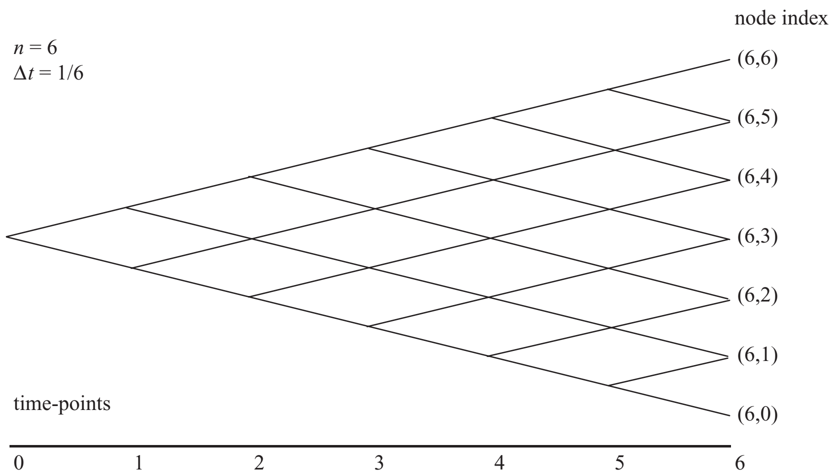


**Figure 1.2** Symmetric binomial tree

Each path has a probability $(\frac{1}{2})^n$ of being realized. The total probability of terminating at a particular node equals the number of different paths to that node, times $(\frac{1}{2})^n$. For $n = 6$ these are shown on the Pascal triangle in Figure 1.2. It is a classical result in probability theory that as $n$ goes to infinity, the terminal probability distribution of the symmetric

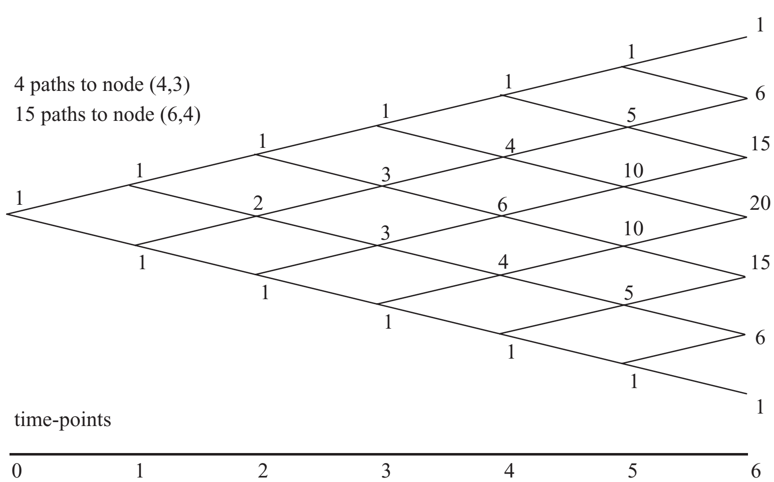


**Figure 1.3** Pascal triangle

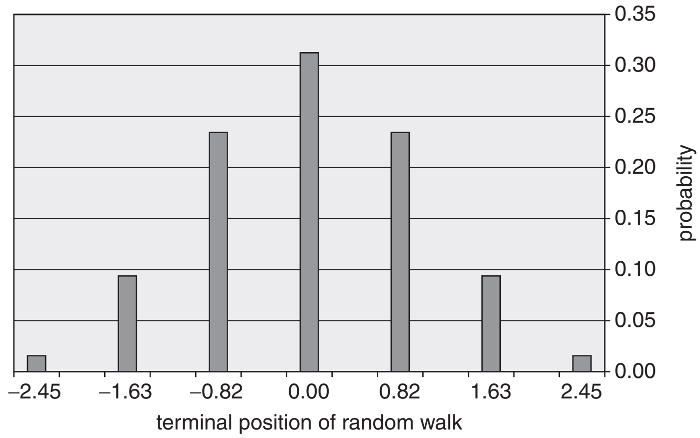


**Figure 1.4** Terminal probabilities

random walk tends to that of a normal distribution. The picture of the terminal probabilities for the case $n = 6$ is shown in Figure 1.4.

Let the increment of the position of the random walk from time-point $t_k$ to $t_{k+1}$ be denoted by discrete two-valued random variable $X_k$. This has an expected value of

$$\mathbb{E}[X_k] = \tfrac{1}{2}\sqrt{\Delta t} + \tfrac{1}{2}(-\sqrt{\Delta t}) = 0$$

and variance

$$\begin{aligned}\mathbb{V}\text{ar}[X_k] &= \mathbb{E}[X_k^2] - \{\mathbb{E}[X_k]\}^2 \\ &= \mathbb{E}[X_k^2] = \tfrac{1}{2}(\sqrt{\Delta t})^2 + \tfrac{1}{2}(-\sqrt{\Delta t})^2 = \Delta t\end{aligned}$$

The position of the particle at terminal time $T$ is the sum of $n$ independent identically distributed random variables $X_k$, $S_n \stackrel{\text{def}}{=} X_1 + X_2 + \cdots + X_n$. The expected terminal position of the path is

$$\begin{aligned}\mathbb{E}[S_n] &= \mathbb{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] = n0 = 0\end{aligned}$$

Its variance is

$$\mathbb{V}\text{ar}[S_n] = \mathbb{V}\text{ar}\left[\sum_{k=1}^{n} X_k\right]$$

As the $X_k$ are independent this can be written as the sum of the variances $\sum_{k=1}^{n} \mathbb{V}\text{ar}[X_k]$, and as the $X_k$ are identically distributed they have the same variance $\Delta t$, so

$$\mathbb{V}\text{ar}[S_n] = n\Delta t = n(T/n) = T$$

For larger $n$, the random walk varies more frequently, but the magnitude of the increment $\sqrt{\Delta t} = \sqrt{T/n}$ gets smaller and smaller. The graph of the probability distribution of

$$Z_n \stackrel{\text{def}}{=} \frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathbb{V}\text{ar}[S_n]}} = \frac{S_n}{\sqrt{T}}$$

looks more and more like that of the standard normal probability distribution.

**Limiting Distribution** The probability distribution of $S_n$ is determined uniquely by its moment generating function.[2] This is $\mathbb{E}[\exp(\theta S_n)]$, which is a function of $\theta$, and will be denoted $m(\theta)$.

$$\begin{aligned} m(\theta) &\stackrel{\text{def}}{=} \mathbb{E}[\exp(\theta\{X_1 + \cdots + X_k + \cdots + X_n\})] \\ &= \mathbb{E}[\exp(\theta X_1) \cdots \exp(\theta X_k) \cdots \exp(\theta X_n)] \end{aligned}$$

As the random variables $X_1, \ldots, X_n$ are independent, the random variables $\exp(\theta X_1), \ldots, \exp(\theta X_n)$ are also independent, so the expected value of the product can be written as the product of the expected values of the individual terms

$$m(\theta) = \prod_{k=1}^{n} \mathbb{E}[\exp(\theta X_k)]$$

As the $X_k$ are identically distributed, all $\mathbb{E}[\exp(\theta X_k)]$ are the same, so

$$m(\theta) = \{\mathbb{E}[\exp(\theta X_k)]\}^n$$

As $X_k$ is a discrete random variable which can take the values $\sqrt{\Delta t}$ and $-\sqrt{\Delta t}$, each with probability $\frac{1}{2}$, it follows that $\mathbb{E}[\exp(\theta X_k)] = \frac{1}{2}\exp(\theta\sqrt{\Delta t}) + \frac{1}{2}\exp(-\theta\sqrt{\Delta t})$. For small $\Delta t$, using the power series expansion of exp and neglecting terms of order higher than $\Delta t$, this can be approximated by

$$\tfrac{1}{2}(1 + \theta\sqrt{\Delta t} + \tfrac{1}{2}\theta^2\Delta t) + \tfrac{1}{2}(1 - \theta\sqrt{\Delta t} + \tfrac{1}{2}\theta^2\Delta t) = 1 + \tfrac{1}{2}\theta^2\Delta t$$

so

$$m(\theta) \approx (1 + \tfrac{1}{2}\theta^2\Delta t)^n$$

[2] See Annex A, *Computations with Brownian motion.*

As $n \to \infty$, the probability distribution of $S_n$ converges to the one determined by the limit of the moment generating function. To determine the limit of $m$ as $n \to \infty$, it is convenient to change to ln.

$$\ln[m(\theta)] \approx n \ln(1 + \tfrac{1}{2}\theta^2 \Delta t)$$

Using the property, $\ln(1 + y) \approx y$ for small $y$, gives

$$\ln[m(\theta)] \approx n \tfrac{1}{2}\theta^2 \Delta t$$

and as $\Delta t = T/n$

$$m(\theta) \approx \exp(\tfrac{1}{2}\theta^2 T)$$

This is the moment generating function of a random variable, $Z$ say, which has a normal distribution with mean 0 and variance $T$, as can be readily checked by using the well-known formula for $\mathbb{E}[\exp(\theta Z)]$. Thus in the continuous-time limit of the discrete-time framework, the probability density of the terminal position is

$$\frac{1}{\sqrt{T}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x}{\sqrt{T}}\right)^2\right]$$

which is the same as that of a Brownian motion that has run an amount of time $T$. The probability distribution of $S_n = \sqrt{T} Z_n$ is then normal with mean 0 and variance $T$.

The full proof of the convergence of the symmetric random walk to Brownian motion requires more than what was shown. Donsker's theorem from advanced probability theory is required, but that is outside the scope of this text; it is covered in *Korn/Korn* Excursion 7, and in *Capasso/Bakstein* Appendix B. The construction of Brownian motion as the limit of a symmetric random walk has the merit of being intuitive. See also *Kuo* Section 1.2, and *Shreve II* Chapter 3. There are several other constructions of Brownian motion in the literature, and they are mathematically demanding; see, for example, *Kuo* Chapter 3. The most accessible is *Lėvy's interpolation method*, which is described in *Kuo* Section 3.4.

**Size of Increment** Why the size of the random walk increment was specified as $\sqrt{\Delta t}$ will now be explained. Let the increment over time-step $\Delta t$ be denoted $y$. So $X_k = y$ or $-y$, each with probability $\frac{1}{2}$, and

$$\mathbb{E}[X_k] = \tfrac{1}{2}y + \tfrac{1}{2}(-y) = 0$$

$$\mathbb{V}\text{ar}[X_k] = \mathbb{E}[X_k^2] - \{\mathbb{E}[X_k]\}^2 = \tfrac{1}{2}y^2 + \tfrac{1}{2}(-y)^2 - 0^2 = y^2$$

Then $\mathbb{V}\text{ar}[S_n] = n\mathbb{V}\text{ar}[X_k]$ as the successive $X_k$ are independent

$$\mathbb{V}\text{ar}[S_n] = ny^2 = \frac{T}{\Delta t}y^2 = T\frac{y^2}{\Delta t}$$

Now let both $\Delta t \to 0$ and $y \to 0$, in such a way that $\mathbb{V}\text{ar}[S_n]$ stays finite. This is achieved by choosing $\frac{y^2}{\Delta t} = c$, a positive constant, so $\mathbb{V}ar[S_n] = Tc$. As time units are arbitrary, there is no advantage in using a $c$ value other than 1.

So if one observes Brown's experiment at equal time intervals, and models this as a symmetric random walk with increment $y$, then the continuous-time limit is what is called Brownian.

This motivates why Brownian motion has a variance equal to the elapsed time. Many books introduce the variance property of Brownian motion without any motivation.

**Simulation of Symmetric Random Walk** To simulate the symmetric random walk, generate a succession of $n$ random variables $X$ with the above specified two-point probabilities and multiply these by $\pm\sqrt{\Delta t}$. The initial position of the walk is set at zero. Three random walks over the time period [0, 1] are shown in Figure 1.5.

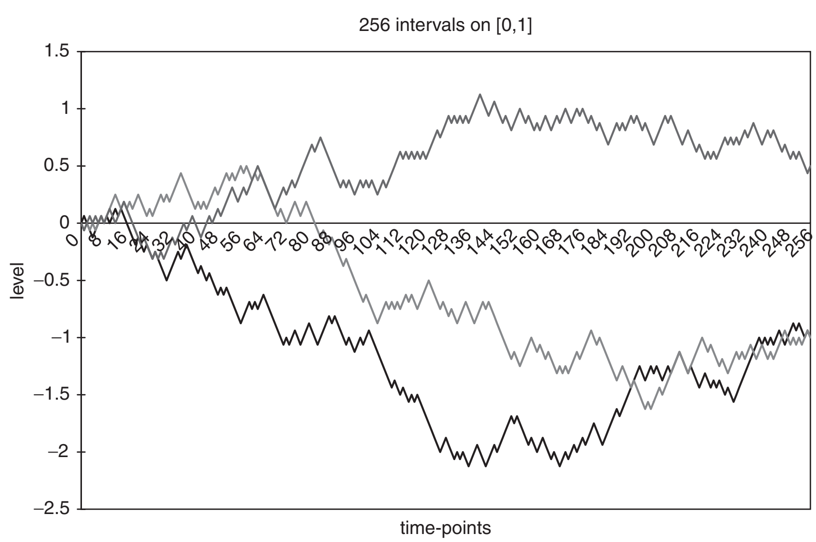


**Figure 1.5** Simulated symmetric random walks

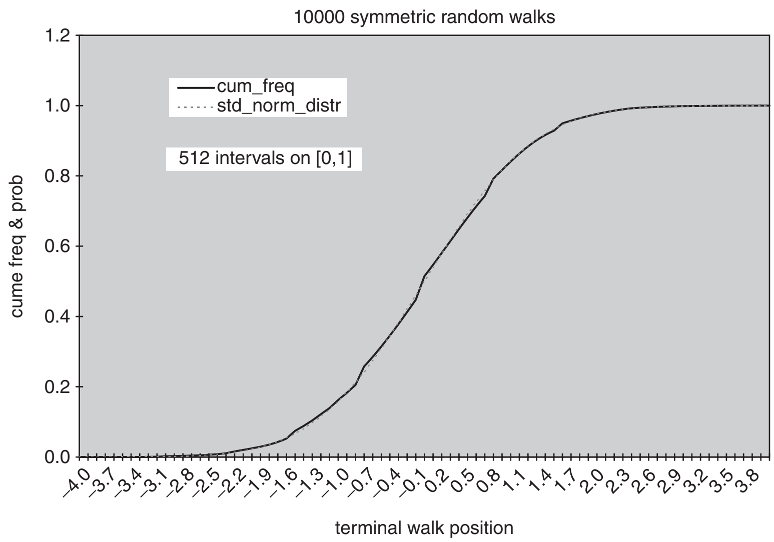


**Figure 1.6** Simulated versus exact

For a batch of 100 simulated symmetric walks of 512 steps, the cumulative frequency of the terminal positions is shown in Figure 1.6, together with the limiting standard normal probability distribution.

The larger the number of simulations, the closer the cumulative frequency resembles the limiting distribution. For 10 000 simulated walks the difference is not graphically distinguishable. The simulation statistics for the position at time 1 are shown in Figure 1.7.

## 1.5 COVARIANCE OF BROWNIAN MOTION

A *Gaussian process* is a collection of normal random variables such that any finite number of them have a multivariate normal distribution. Thus Brownian motion increments are a Gaussian process. Consider the covariance between Brownian motion positions at any times $s$ and $t$, where $s < t$. This is the expected value of the product of the deviations

| | sample | exact |
|---|---|---|
| mean | 0.000495 | 0 |
| variance | 1.024860 | 1 |

**Figure 1.7** Simulation statistics

of these random variables from their respective means

$$\mathbb{C}\text{ov}[B(s), B(t)] = \mathbb{E}[\{B(s) - \mathbb{E}[B(s)]\}\{B(t) - \mathbb{E}[B(t)]\}]$$

As $\mathbb{E}[B(s)]$ and $\mathbb{E}[B(t)]$ are zero, $\mathbb{C}\text{ov}[B(s), B(t)] = \mathbb{E}[B(s)B(t)]$. Note that the corresponding time intervals $[0, s]$ and $[0, t]$ are overlapping. Express $B(t)$ as the sum of independent random variables $B(s)$ and the subsequent increment $\{B(t) - B(s)\}$, $B(t) = B(s) + \{B(t) - B(s)\}$. Then

$$\begin{aligned}\mathbb{E}[B(s)B(t)] &= \mathbb{E}[B(s)^2 + B(s)\{B(t) - B(s)\}] \\ &= \mathbb{E}[B(s)^2] + \mathbb{E}[B(s)\{B(t) - B(s)\}]\end{aligned}$$

Due to independence, the second term can be written as the product of $\mathbb{E}$s, and

$$\begin{aligned}\mathbb{E}[B(s)B(t)] &= \mathbb{E}[B(s)^2] + \mathbb{E}[B(s)]\mathbb{E}[B(t) - B(s)] \\ &= s + 0\,0 = s\end{aligned}$$

If the time notation was $t < s$ then $\mathbb{E}[B(s)B(t)] = t$. Generally for any times $s$ and $t$

$$\mathbb{E}[B(s)B(t)] = \min(s, t)$$

For increments during any two non-overlapping time intervals $[t_1, t_2]$ and $[t_3, t_4]$, $\Delta B(t_1)$ is independent of $\Delta B(t_3)$, so the expected value of the product of the Brownian increments over these non-overlapping time intervals (Figure 1.8) equals the product of the expected values

$$\begin{aligned}&\mathbb{E}[\{B(t_2) - B(t_1)\}\{B(t_4) - B(t_3)\}] \\ &\quad = \mathbb{E}[B(t_2) - B(t_1)]\mathbb{E}[B(t_4) - B(t_3)] = 0\,0 = 0\end{aligned}$$

whereas $\mathbb{E}[B(t_1)B(t_3)] = t_1 \neq \mathbb{E}[B(t_1)]\mathbb{E}[B(t_3)]$.

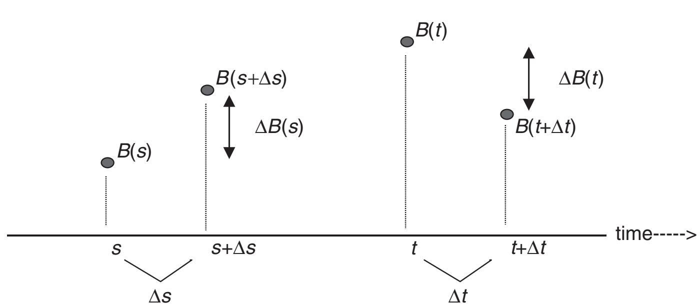


**Figure 1.8** Non-overlapping time intervals

## 1.6 CORRELATED BROWNIAN MOTIONS

Let $B$ and $B^*$ be two *independent* Brownian motions. Let $-1 \leq \rho \leq 1$ be a given number. For $0 \leq t \leq T$ define a new process

$$Z(t) \stackrel{\text{def}}{=} \rho B(t) + \sqrt{1-\rho^2}\,B^*(t)$$

At each $t$, this is a linear combination of independent normals, so $Z(t)$ is normally distributed. It will first be shown that $Z$ is a Brownian motion by verifying its expected value and variance at time $t$, and the variance over an arbitrary time interval. It will then be shown that $Z$ and $B$ are correlated.

The expected value of $Z(t)$ is

$$\begin{aligned}\mathbb{E}[Z(t)] &= \mathbb{E}[\rho B(t) + \sqrt{1-\rho^2}\,B^*(t)]\\ &= \rho\mathbb{E}[B(t)] + \sqrt{1-\rho^2}\,\mathbb{E}[B^*(t)]\\ &= \rho 0 + \sqrt{1-\rho^2}\,0 = 0\end{aligned}$$

The variance of $Z(t)$ is

$$\begin{aligned}\mathbb{V}\text{ar}[Z(t)] &= \mathbb{V}\text{ar}[\rho B(t) + \sqrt{1-\rho^2}\,B^*(t)]\\ &= \mathbb{V}\text{ar}[\rho B(t)] + \mathbb{V}\text{ar}[\sqrt{1-\rho^2}\,B^*(t)]\end{aligned}$$

as the random variables $\rho B(t)$ and $\sqrt{1-\rho^2}\,B^*(t)$ are independent. This can be written as

$$\rho^2\mathbb{V}\text{ar}[B(t)] + \left(\sqrt{1-\rho^2}\right)^2\mathbb{V}\text{ar}[B^*(t)] = \rho^2 t + \left(1-\rho^2\right)t = t$$

Now consider the increment

$$\begin{aligned}Z(t+u) - Z(t) &= [\rho B(t+u) + \sqrt{1-\rho^2}\,B^*(t+u)]\\ &\quad - [\rho B(t) + \sqrt{1-\rho^2}\,B^*(t)]\\ &= \rho[B(t+u) - B(t)]\\ &\quad + \sqrt{1-\rho^2}\,[B^*(t+u) - B^*(t)]\end{aligned}$$

$B(t+u) - B(t)$ is the random increment of Brownian motion $B$ over time interval $u$ and $B^*(t+u) - B^*(t)$ is the random increment of Brownian motion $B^*$ over time interval $u$. These two random quantities are independent, also if multiplied by constants, so the $\mathbb{V}$ar of the sum is the sum of $\mathbb{V}$ar

$$\begin{aligned}\mathbb{V}\text{ar}[Z(t+u) - Z(t)] &= \mathbb{V}\text{ar}\{\rho[B(t+u) - B(t)] \\ &\quad + \sqrt{1-\rho^2}\,[B^*(t+u) - B^*(t)]\} \\ &= \mathbb{V}\text{ar}\{\rho[B(t+u) - B(t)]\} \\ &\quad + \mathbb{V}\text{ar}\{\sqrt{1-\rho^2}\,[B^*(t+u) - B^*(t)]\} \\ &= \rho^2 u + \left(\sqrt{1-\rho^2}\right)^2 u = u\end{aligned}$$

This variance does not depend on the starting time $t$ of the interval $u$, and equals the length of the interval. Hence $Z$ has the properties of a Brownian motion. Note that since $B(t+u)$ and $B(t)$ are not independent

$$\begin{aligned}\mathbb{V}\text{ar}[B(t+u) - B(t)] &\neq \mathbb{V}\text{ar}[B(t+u)] + \mathbb{V}\text{ar}[B(t)] \\ &= t+u+t = 2t+u\end{aligned}$$

but

$$\begin{aligned}\mathbb{V}\text{ar}[B(t+u) - B(t)] &= \mathbb{V}\text{ar}[B(t+u)] + \mathbb{V}\text{ar}[B(t)] \\ &\quad - 2\mathbb{C}\text{ov}[B(t+u), B(t)] \\ &= (t+u) + t - 2\min(t+u, t) \\ &= (t+u) + t - 2t = u\end{aligned}$$

Now analyze the correlation between the processes $Z$ and $B$ at time $t$. This is defined as the covariance between $Z(t)$ and $B(t)$ scaled by the product of the standard deviations of $Z(t)$ and $B(t)$:

$$\mathbb{C}\text{orr}[Z(t), B(t)] = \frac{\mathbb{C}\text{ov}[Z(t), B(t)]}{\sqrt{\mathbb{V}\text{ar}[Z(t)]}\sqrt{\mathbb{V}\text{ar}[B(t)]}}$$

The numerator evaluates to

$$\begin{aligned}\mathbb{C}\text{ov}[Z(t), B(t)] &= \mathbb{C}\text{ov}[\rho B(t) + \sqrt{1-\rho^2}\,B^*(t), B(t)] \\ &= \mathbb{C}\text{ov}[\rho B(t), B(t)] + \mathbb{C}\text{ov}[\sqrt{1-\rho^2}\,B^*(t), B(t)] \\ &\quad \text{due to independence} \\ &= \rho\mathbb{C}\text{ov}[B(t), B(t)] + \sqrt{1-\rho^2}\,\mathbb{C}\text{ov}[B^*(t), B(t)] \\ &= \rho\mathbb{V}\text{ar}[B(t), B(t)] + \sqrt{1-\rho^2}\,0 \\ &= \rho t\end{aligned}$$

Using the known standard deviations in the denominator gives

$$\mathbb{C}\text{orr}[Z(t), B(t)] = \frac{\rho t}{\sqrt{t}\sqrt{t}} = \rho$$

Brownian motions $B$ and $Z$ have correlation $\rho$ at all times $t$. Thus if two correlated Brownian motions are needed, the first one can be $B$ and the second one $Z$, constructed as above. Brownian motion $B^*$ only serves as an intermediary in this construction.

## 1.7 SUCCESSIVE BROWNIAN MOTION INCREMENTS

The increments over non-overlapping time intervals are independent random variables. They all have a normal distribution, but because the time intervals are not necessarily of equal lengths, their variances differ. The joint probability distribution of the positions at times $t_1$ and $t_2$ is

$$\begin{aligned}
&\mathbb{P}[B(t_1) \le a_1, B(t_2) \le a_2] \\
&\quad = \int_{x_1=-\infty}^{a_1} \int_{x_2=-\infty}^{a_2} \frac{1}{\sqrt{t_1}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_1 - 0}{\sqrt{t_1}}\right)^2\right] \\
&\quad\quad \times \frac{1}{\sqrt{t_2 - t_1}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_2 - x_1}{\sqrt{t_2 - t_1}}\right)^2\right] dx_1\, dx_2
\end{aligned}$$

This expression is intuitive. The first increment is from position 0 to $x_1$, an increment of $(x_1 - 0)$ over time interval $(t_1 - 0)$. The second increment starts at $x_1$ and ends at $x_2$, an increment of $(x_2 - x_1)$ over time interval $(t_2 - t_1)$. Because of independence, the integrand in the above expression is the product of conditional probability densities. This generalizes to any number of intervals. Note the difference between the increment of the motion and the position of the motion. The increment over any time interval $[t_{k-1}, t_k]$ has a normal distribution with mean zero and variance equal to the interval length, $(t_k - t_{k-1})$. This distribution is not dependent on how the motion got to the starting position at time $t_{k-1}$. For a known position $B(t_{k-1}) = x$, the position of the motion at time $t_k$ , $B(t_k)$, has a normal density with mean $x$ and variance as above. While this distribution is not dependent on how the motion got to the starting position, it is dependent on the position of the starting point via its mean.

### 1.7.1 Numerical Illustration

A further understanding of the theoretical expressions is obtained by carrying out numerical computations. This was done in the mathematical software Mathematica. The probability density function of a increment was specified as

```
[f[u_,w_]: = (1/(Sqrt[u] * Sqrt[2 * Pi])) * Exp[−0.5 * (w/Sqrt[u]) ∧ 2]
```

A time interval of arbitrary length uNow = 2.3472 was specified. The expectation of the increment over this time interval, starting at time 1, was then specified as

```
NIntegrate[(x2−x1) * f[1, x1] * f[uNow, x2−x1],
  {x1, −10, 10},{x2, −10, 10}]
```

Note that the joint density is multiplied by (x2−x1). The normal densities were integrated from −10 to 10, as this contains nearly all the probability mass under the two-dimensional density surface. The result was 0, in accordance with the theory. The variance of the increment over time interval uNow was computed as the expected value of the second moment

```
NIntegrate[((x2−x1)∧ 2) * f[1, x1] * f[uNow, x2−x1],
  {x1, −10, 10},{x2, −10, 10}]
```

Note that the joint density is multiplied by (x2−x1) ∧ 2. The result was 2.3472, exactly equal to the length of the time interval, in accordance with the theory.

**Example 1.7.1** This example (based on *Klebaner* example 3.1) gives the computation of $\mathbb{P}[B(1) \leq 0, B(2) \leq 0]$. It is the probability that both the position at time 1 and the position at time 2 are not positive. The position at all other times does not matter. This was specified in Mathematica as

```
NIntegrate[f[1, x1] * f[1, x2−x1],{x1, −10, 0},{x2, −10, 0}]
```

To visualize the joint density of the increment (Figure 1.9), a plot was specified as

```
Plot3D[f[1, x1] * f[1, x2−x1],{x1, −4, 4},{x2, −4, 4}]
```

The section of the probability density surface pertaining to this example is plotted in Figure 1.10.

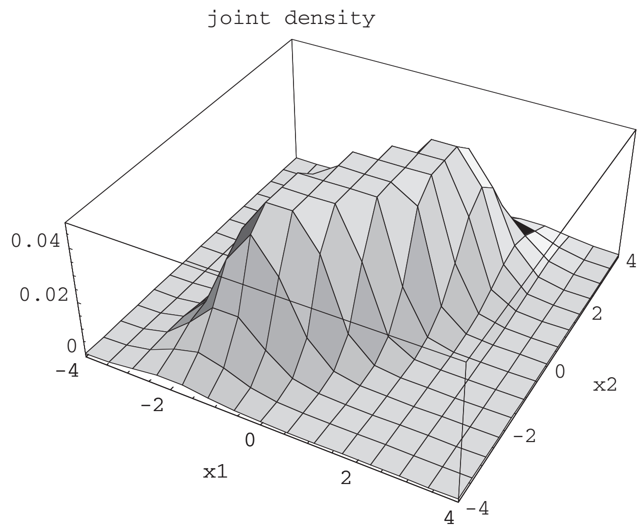


**Figure 1.9** Joint density

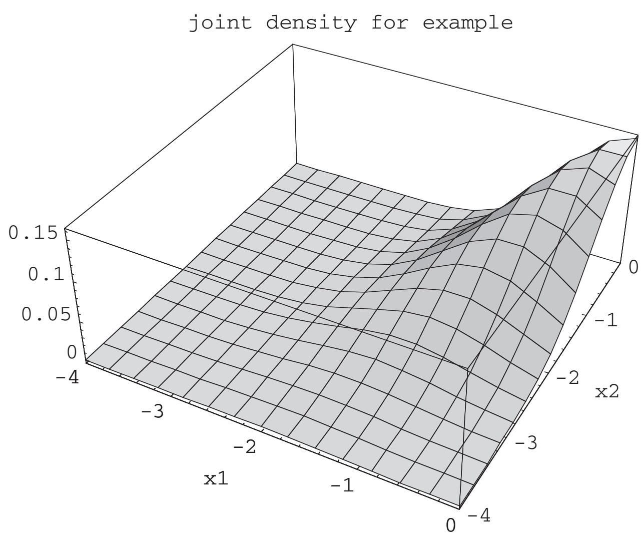


**Figure 1.10** Joint density for example

The result of the numerical integration was 0.375, which agrees with Klebaner's answer derived analytically. It is the volume under the joint density surface shown below, for $x_1 \leq 0$ and $x_2 \leq 0$. $\mathbb{P}[B(1) \leq 0] = 0.5$ and $\mathbb{P}[B(2) \leq 0] = 0.5$. Multiplying these probabilities gives 0.25, but that is not the required probability because random variables $B(1)$ and $B(2)$ are not independent.

## 1.8 FEATURES OF A BROWNIAN MOTION PATH

The properties shown thus far are simply manipulations of a normal random variable, and anyone with a knowledge of elementary probability should feel comfortable. But now a highly unusual property comes on the scene. In what follows, the time interval is again $0 \leq t \leq T$, partitioned as before.

### 1.8.1 Simulation of Brownian Motion Paths

The path of a Brownian motion can be simulated by generating at each time-point in the partition a normally distributed random variable with mean zero and standard deviation $\sqrt{\Delta t}$. The time grid is discrete but the values of the position of the Brownian motion are now on a continuous scale. Sample paths are shown in Figure 1.11.

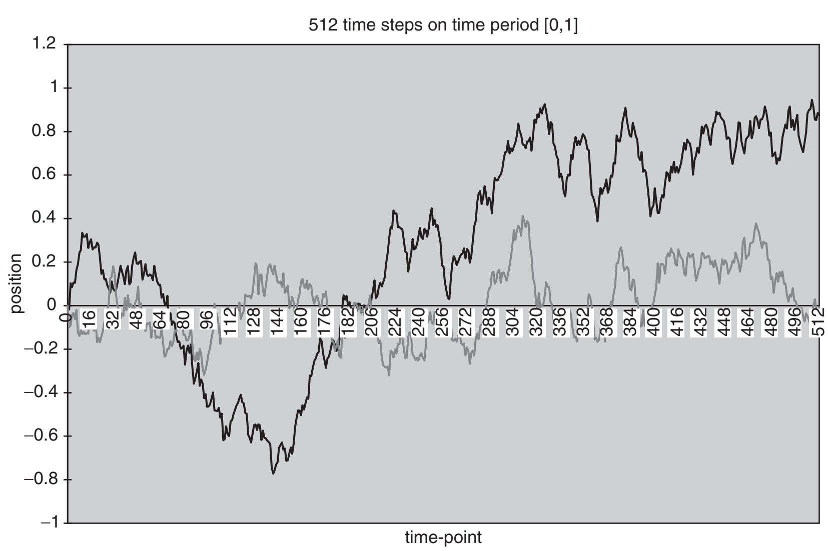


**Figure 1.11** Simulated Brownian motion paths

| | sample | exact |
|---|---|---|
| mean | 0.037785 | 0 |
| variance | 1.023773 | 1 |

**Figure 1.12** Brownian motion path simulation statistics

A batch of 1000 simulations of a standard Brownian motion over time period [0, 1] gave the statistics shown in Figure 1.12 for the position at time 1. The cumulative frequency of the sample path position at time 1 is very close to the exact probability distribution, as shown in Figure 1.13. For visual convenience cume_freq is plotted as continuous.

### 1.8.2 Slope of Path

For the symmetric random walk, the magnitude of the slope of the path is

$$\frac{|S_{k+1} - S_k|}{\Delta t} = \frac{\sqrt{\Delta t}}{\Delta t} = \frac{1}{\sqrt{\Delta t}}$$

This becomes infinite as $\Delta t \to 0$. As the symmetric random walk converges to Brownian motion, this puts in question the differentiability

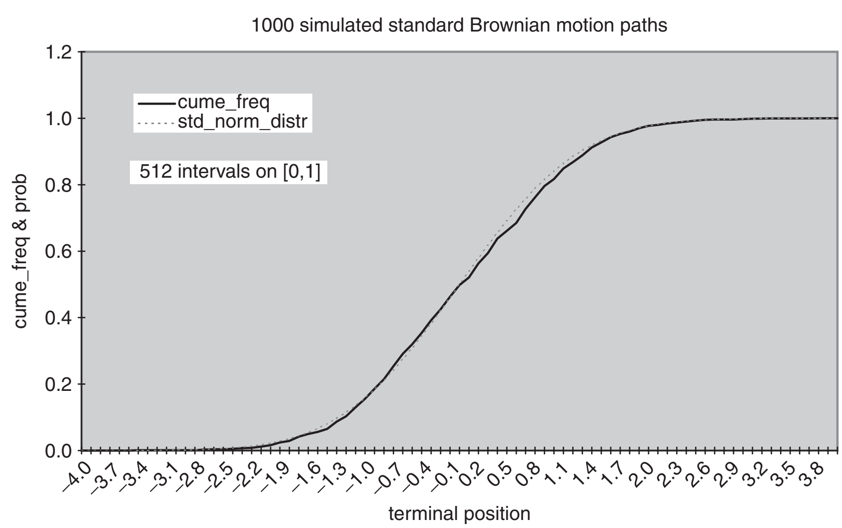


**Figure 1.13** Simulated frequency versus exact Brownian motion distribution

of a Brownian motion path. It has already been seen that a simulated Brownian motion path fluctuates very wildly due to the independence of the increments over successive small time intervals. This will now be discussed further.

### 1.8.3 Non-Differentiability of Brownian Motion Path

First, non-differentiability is illustrated in the absence of randomness. In ordinary calculus, consider a continuous function $f$ and the expression $[f(x+h)-f(x)]/h$. Let $h$ approach 0 from above and take the limit $\lim_{h\downarrow 0}\{[f(x+h)-f(x)]/h\}$. Similarly take the limit when $h$ approaches 0 from below, $\lim_{h\uparrow 0}\{[f(x+h)-f(x)]/h\}$. If both limits exist, and if they are equal, then function $f$ is said to be differentiable at $x$. This limit is called the derivative (or slope) at $x$, denoted $f'(x)$.

**Example 1.8.1**

$$f(x) \stackrel{\text{def}}{=} x^2$$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-x^2}{h} = \frac{x^2+2xh+h^2-x^2}{h} = \frac{2xh+h^2}{h}$$

Numerator and denominator can be divided by $h$, since $h$ is not equal to zero but approaches zero, giving $(2x+h)$, and

$$\lim_{h\downarrow 0}(2x+h) = 2x \qquad \lim_{h\uparrow 0}(2x+h) = 2x$$

Both limits exist and are equal. The function is differentiable for all $x$, $f'(x) = 2x$.

**Example 1.8.2 (see Figure 1.14)**

$$f(x) \stackrel{\text{def}}{=} |x|$$

For $x > 0$, $f(x) = x$ and if $h$ is also $> 0$ then $f(x+h) = x+h$

$$\lim_{h\downarrow 0}\frac{f(x+h)-f(x)}{h} = \frac{x+h-x}{h} = 1$$

For $x < 0$, $f(x) = -x$, and if $h$ is also $< 0$, then $f(x+h) = -(x+h)$

$$\lim_{h\uparrow 0}\frac{f(x+h)-f(x)}{h} = \frac{-(x+h)-(-x)}{h} = -1$$

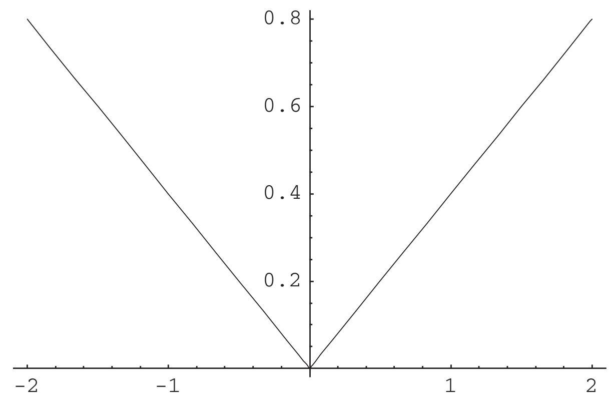


**Figure 1.14** Function modulus x

Here both limits exist but they are not equal, so $f'(x)$ does not exist. This function is not differentiable at $x = 0$. There is not one single slope at $x = 0$.

**Example 1.8.3 (see Figure 1.15)**

$$f(x) = c_1 \mid x - x_1 \mid + c_2 \mid x - x_2 \mid + c_3 \mid x - x_3 \mid$$

This function is not differentiable at $x_1, x_2, x_3$, a *finite* number of points.

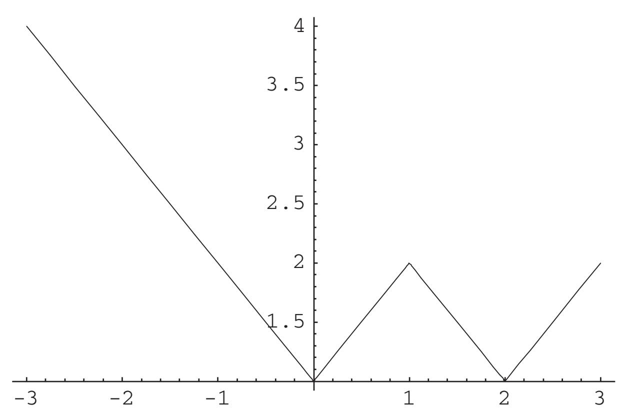


**Figure 1.15** Linear combination of functions modulus x

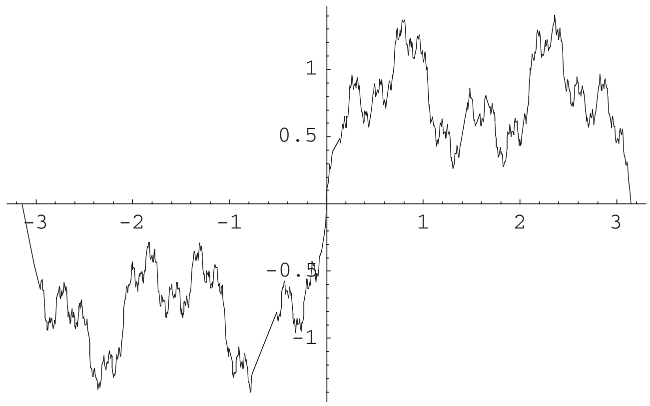


**Figure 1.16** Approximation of non-differentiable function

**Example 1.8.4**

$$f(x) = \sum_{i=0}^{\infty} \frac{\sin(3^i x)}{2^i}$$

It can be shown that this function is non-differentiable at *any* point $x$. This, of course, cannot be shown for $i = \infty$, so the variability is illustrated for $\sum_{i=0}^{10}$ in Figure 1.16.

**Brownian Motion** Now use the same framework for analyzing differentiability of a Brownian motion path. Consider a time interval of length $\Delta t = 1/n$ starting at $t$. The rate of change over time interval $[t, t + \Delta t]$ is

$$X_n \stackrel{\text{def}}{=} \frac{B(t + \Delta t) - B(t)}{\Delta t} = \frac{B(t + 1/n) - B(t)}{1/n}$$

which can be rewritten as $X_n = n[B(t + 1/n) - B(t)]$. So $X_n$ is a normally distributed random variable with parameters

$$\mathbb{E}[X_n] = n^2 \left[ B\left(t + \frac{1}{n}\right) - B(t) \right] = n0 = 0$$

$$\mathbb{V}ar[X_n] = n^2 \mathbb{V}ar \left[ B\left(t + \frac{1}{n}\right) - B(t) \right]$$

$$= n^2 \frac{1}{n} = n$$

$$\mathbb{S}tdev[X_n] = \sqrt{n}$$

$X_n$ has the same probability distribution as $\sqrt{n}\,Z$, where $Z$ is standard normal. Differentiability is about what happens to $X_n$ as $\Delta t \to 0$, that is, as $n \to \infty$. Take any positive number $K$ and write $X_n$ as $\sqrt{n}\,Z$. Then

$$\mathbb{P}[|X_n| > K] = \mathbb{P}[|\sqrt{n}Z| > K] = \mathbb{P}[\sqrt{n}|Z| > K] = \mathbb{P}\left[|Z| > \frac{K}{\sqrt{n}}\right]$$

As $n \to \infty$, $K/\sqrt{n} \to 0$ so

$$\mathbb{P}[|X_n| > K] = \mathbb{P}\left[|Z| > \frac{K}{\sqrt{n}}\right] \to \mathbb{P}[|Z| > 0]$$

which equals 1. As $K$ can be chosen arbitrarily large, the rate of change at time $t$ is not finite, and the Brownian motion path is not differentiable at $t$. Since $t$ is an arbitrary time, the *Brownian motion path is nowhere differentiable*. It is impossible to say at any time $t$ in which direction the path is heading.

The above method is based on the expositions in *Epps* and *Klebaner*. This is more intuitive than the ‘standard proof’ of which a version is given in *Capasso/Bakstein*.

### 1.8.4 Measuring Variability

The variability of Brownian motion will now be quantified. From $t_k$ to $t_{k+1}$ the absolute Brownian motion increment is $|B(t_{k+1}) - B(t_k)|$. The sum over the entire Brownian motion path is $\sum_{k=0}^{n-1} |B(t_{k+1}) - B(t_k)|$. This is a random variable which is known as the *first variation* of Brownian motion. It measures the length of the Brownian motion path, and thus its variability. Another measure is the sum of the square increments, $\sum_{k=0}^{n-1}[B(t_{k+1}) - B(t_k)]^2$. This random second-order quantity is known as the *quadratic* variation (or second variation). Now consider successive refinements of the partition. This keeps the original time-points and creates additional ones. Since for each partition the corresponding variation is a random variable, a sequence of random variables is produced. The question is then whether this sequence converges to a limit in some sense. There are several types of convergence of sequences of random variables that can be considered.[3] As the time intervals in the composition of the variation get smaller and smaller, one may be inclined to think that the variation will tend to zero. But it turns out that regardless of the size of an interval, the increment over

[3] See Annex E, *Convergence Concepts.*

| steps | dt | first_var | quadr_var | third_var |
|---|---|---|---|---|
| 2000 | 0.00050000 | 16.01369606 | 0.2016280759 | 0.0031830910 |
| 4000 | 0.00025000 | 19.39443203 | 0.1480559146 | 0.0014367543 |
| 8000 | 0.00012500 | 25.84539243 | 0.1298319380 | 0.0008117586 |
| 16000 | 0.00006250 | 32.61941799 | 0.1055395009 | 0.0004334750 |
| 32000 | 0.00003125 | 40.56883140 | 0.0795839944 | 0.0001946600 |
| 64000 | 0.00001563 | 43.36481866 | 0.0448674991 | 0.0000574874 |
| 128000 | 0.00000781 | 44.12445062 | 0.0231364852 | 0.0000149981 |
| 256000 | 0.00000391 | 44.31454677 | 0.0116583498 | 0.0000037899 |
| 512000 | 0.00000195 | 44.36273548 | 0.0058405102 | 0.0000009500 |
| 1024000 | 0.00000098 | 44.37481932 | 0.0029216742 | 0.0000002377 |
| | *limit* | about 44.3 | 0 | 0 |

**Figure 1.17** Variation of function which has a continuous derivative

that interval can still be infinite. It is shown in Annex C that as *n* tends to infinity, the first variation is not finite, and the quadratic variation is positive. This has fundamental consequences for the way in which a stochastic integral may be constructed, as will be explained in Chapter 3. In contrast to Brownian motion, a function in ordinary calculus which has a derivative that is continuous, has positive first variation and zero quadratic variation. This is shown in *Shreve II*. To support the derivation in Annex C, variability can be verified numerically. This is the object of Exercise [1.9.12] of which the results are shown in Figure 1.17 and 1.18.

Time period [0,1]

| steps | dt | first_var | quadr_var | third_var |
|---|---|---|---|---|
| 2000 | 0.00050000 | 36.33550078 | 1.0448863386 | 0.0388983241 |
| 4000 | 0.00025000 | 50.47005112 | 1.0002651290 | 0.0253513781 |
| 8000 | 0.00012500 | 71.85800329 | 1.0190467736 | 0.0184259646 |
| 16000 | 0.00006250 | 101.65329098 | 1.0155967391 | 0.0129358213 |
| 32000 | 0.00003125 | 142.19694118 | 0.9987482348 | 0.0089475369 |
| 64000 | 0.00001563 | 202.67088291 | 1.0085537303 | 0.0063915246 |
| 128000 | 0.00000781 | 285.91679729 | 1.0043769437 | 0.0045014163 |
| 256000 | 0.00000391 | 403.18920472 | 0.9969064552 | 0.0031386827 |
| 512000 | 0.00000195 | 571.17487195 | 1.0005573262 | 0.0022306000 |
| 1024000 | 0.00000098 | 807.41653827 | 1.0006685086 | 0.0015800861 |
| | *limit* | not finite | time period | 0 |

**Figure 1.18** Variation of Brownian motion

## 1.9 EXERCISES

The numerical exercises can be carried out in Excel/VBA, Mathematica, MatLab, or any other mathematical software or programming language.

**[1.9.1]** *Scaled Brownian motion* Consider the process $X(t) \stackrel{\text{def}}{=} \sqrt{\gamma}B(t/\gamma)$ where $B$ denotes standard Brownian motion, and $\gamma$ is an arbitrary positive constant. This process is known as scaled Brownian motion. The time scale of the Brownian motion is reduced by a factor $\gamma$, and the magnitude of the Brownian motion is multiplied by a factor $\sqrt{\gamma}$. This can be interpreted as taking snapshots of the position of a Brownian motion with a shutter speed that is $\gamma$ times as fast as that used for recording a standard Brownian motion, and magnifying the results by a factor $\sqrt{\gamma}$.

(a) Derive the expected value of $X(t)$
(b) Derive the variance of $X(t)$
(c) Derive the probability distribution of $X(t)$
(d) Derive the probability density of $X(t)$
(e) Derive $\mathbb{V}\text{ar}[X(t+u) - X(t)]$, where $u$ is an arbitrary positive constant
(f) Argue whether $X(t)$ is a Brownian motion

*Note*: By employing the properties of the distribution of Brownian motion this exercise can be done without elaborate integrations.

**[1.9.2]** *Seemingly Brownian motion* Consider the process $X(t) \stackrel{\text{def}}{=} \sqrt{t}Z$, where $Z \sim N(0, 1)$.

(a) Derive the expected value of $X(t)$
(b) Derive the variance of $X(t)$
(c) Derive the probability distribution of $X(t)$
(d) Derive the probability density of $X(t)$
(e) Derive $\mathbb{V}\text{ar}[X(t+u) - X(t)]$ where $u$ is an arbitrary positive constant
(f) Argue whether $X(t)$ is a Brownian motion

**[1.9.3]** *Combination of Brownian motions* The random process $Z(t)$ is defined as $Z(t) \stackrel{\text{def}}{=} \alpha B(t) - \sqrt{\beta}\, B^*(t)$, where $B$ and $B^*$ are independent standard Brownian motions, and $\alpha$ and $\beta$ are arbitrary positive constants. Determine the relationship between $\alpha$ and $\beta$ for which $Z(t)$ is a Brownian motion.

**[1.9.4]** *Correlation* Derive the correlation coefficient between $B(t)$ and $B(t+u)$.

**[1.9.5]** *Successive Brownian motions* Consider a standard Brownian motion which runs from time $t = 0$ to time $t = 4$.

(a) Give the expression for the probability that its path position is positive at time 4. Give the numerical value of this probability
(b) For the Brownian motion described above, give the expression for the joint probability that its path position is positive at time 1 as well as positive at time 4. No numerical answer is requested.
(c) Give the expression for the expected value at time 4 of the position of the path described in (a). No numerical answer is requested.

**[1.9.6]** *Brownian motion through gates* Consider a Brownian motion path that passes through two gates situated at times $t_1$ and $t_2$.

(a) Derive the expected value of $B(t_1)$ of all paths that pass through gate 1.
(b) Derive the expected value of $B(t_2)$ of all paths that pass through gate 1 and gate 2.
(c) Derive an expression for the expected value of the increment over time interval $[t_1, t_2]$ for paths that pass through both gates.
(d) Design a simulation program for Brownian motion through gates, and verify the answers to (a), (b), and (c) by simulation.

**[1.9.7]** *Simulation of symmetric random walk*

(a) Construct the simulation of three symmetric random walks for $t \in [0, 1]$ on a spreadsheet.
(b) Design a program for simulating the terminal position of thousands of symmetric random walks. Compare the mean and the variance of this sample with the theoretical values.

(c) Derive the probability distribution of the terminal position. Construct a frequency distribution of the terminal positions of the paths in (b) and compare this with the probability distribution.

**[1.9.8]** *Simulation of Brownian motion*

(a) Construct the simulation of three Brownian motion paths for $t \in [0, 1]$ on a spreadsheet.
(b) Construct a simulation of two Brownian motion paths that have a user specified correlation for $t \in [0, 1]$ on a spreadsheet, and display them in a chart.

**[1.9.9]** *Brownian bridge* Random process $X$ is specified on $t \in [0, 1]$ as $X(t) \stackrel{\text{def}}{=} B(t) - tB(1)$. This process is known as a Brownian bridge.

(a) Verify that the terminal position of $X$ equals the initial position.
(b) Derive the covariance between $X(t)$ and $X(t+u)$.
(c) Construct the simulation of two paths of $X$ on a spreadsheet.

**[1.9.10]** *First passage of a barrier* Annex A gives the expression for the probability distribution and the probability density of the time of first passage, $T_L$. Design a simulation program for this, and simulate $\mathbb{E}[T_L]$.

**[1.9.11]** *Reflected Brownian motion* Construct a simulation of a reflected Brownian motion on a spreadsheet, and show this in a chart together with the path of the corresponding Brownian motion.

**[1.9.12]** *Brownian motion variation*

(a) Design a program to compute the *first variation*, *quadratic variation*, and *third variation* of the differentiable ordinary function in Figure 1.16 over $x \in [0, 1]$, initially partitioned into $n = 2000$ steps, with successive doubling to 1024000 steps
(b) Copy the program of (a) and save it under another name. Adapt it to simulate the *first variation*, *quadratic variation*, and *third variation* of Brownian motion

## 1.10 SUMMARY

Brownian motion is the most widely used process for modelling randomness in the world of finance. This chapter gave the mathematical specification, motivated by a symmetric random walk. While this looks innocent enough as first sight, it turns out that Brownian motion has highly unusual properties. The independence of subsequent increments produces a path that does not have the smoothness of functions in ordinary calculus, and is not differentiable at any point. This feature is difficult to comprehend coming from an ordinary calculus culture. It leads to the definition of the stochastic integral in Chapter 3 and its corresponding calculus in Chapter 4.

More on Robert Brown is in the *Dictionary of Scientific Biography*, Vol. II, pp. 516–522. An overview of the life and work of Bachelier can be found in the conference proceedings *Mathematical Finance Bachelier Congress 2000*, and on the Internet, for example in *Wikipedia*. Also on the Internet is the original thesis of Bachelier, and a file named *Bachelier 100 Years*.