# 1 Introduction

## 1.1 BACKGROUND

Many physical phenomena in engineering and science can be described in terms of partial differential equations. In general, solving these equations by classical analytical methods for arbitrary shapes is almost impossible. The finite element method (FEM) is a numerical approach by which these partial differential equations can be solved approximately. From an engineering standpoint, the FEM is a method for solving engineering problems such as stress analysis, heat transfer, fluid flow and electromagnetics by computer simulation.

Millions of engineers and scientists worldwide use the FEM to predict the behavior of structural, mechanical, thermal, electrical and chemical systems for both design and performance analyses. Its popularity can be gleaned by the fact that over $1 billion is spent annually in the United States on FEM software and computer time. A 1991 bibliography (Noor, 1991) lists nearly 400 finite element books in English and other languages. A web search (in 2006) for the phrase 'finite element' using the Google search engine yielded over 14 million pages of results. Mackerle (http://ohio.ikp.liu.se/fe) lists 578 finite element books published between 1967 and 2005.

To explain the basic approach of the FEM, consider a plate with a hole as shown in Figure 1.1 for which we wish to find the temperature distribution. It is straightforward to write a heat balance equation for each point in the plate. However, the solution of the resulting partial differential equation for a complicated geometry, such as an engine block, is impossible by classical methods like separation of variables. Numerical methods such as finite difference methods are also quite awkward for arbitrary shapes; software developers have not marketed finite difference programs that can deal with the complicated geometries that are commonplace in engineering. Similarly, stress analysis requires the solution of partial differential equations that are very difficult to solve by analytical methods except for very simple shapes, such as rectangles, and engineering problems seldom have such simple shapes.

The basic idea of FEM is to divide the body into *finite elements*, often just called *elements*, connected by *nodes*, and obtain an approximate solution as shown in Figure 1.1. This is called the finite element *mesh* and the process of making the mesh is called *mesh generation*.

The FEM provides a systematic methodology by which the solution, in the case of our example, the temperature field, can be determined by a computer program. For linear problems, the solution is determined by solving a system of linear equations; the number of unknowns (which are the nodal temperatures) is equal to the number of nodes. To obtain a reasonably accurate solution, thousands of nodes are usually needed, so computers are essential for solving these equations. Generally, the accuracy of the solution improves as the number of elements (and nodes) increases, but the computer time, and hence the cost, also increases. The finite element program determines the temperature at each node and the heat flow through each element. The results are usually presented as computer visualizations, such as contour



*A First Course in Finite Elements* J. Fish and T. Belytschko

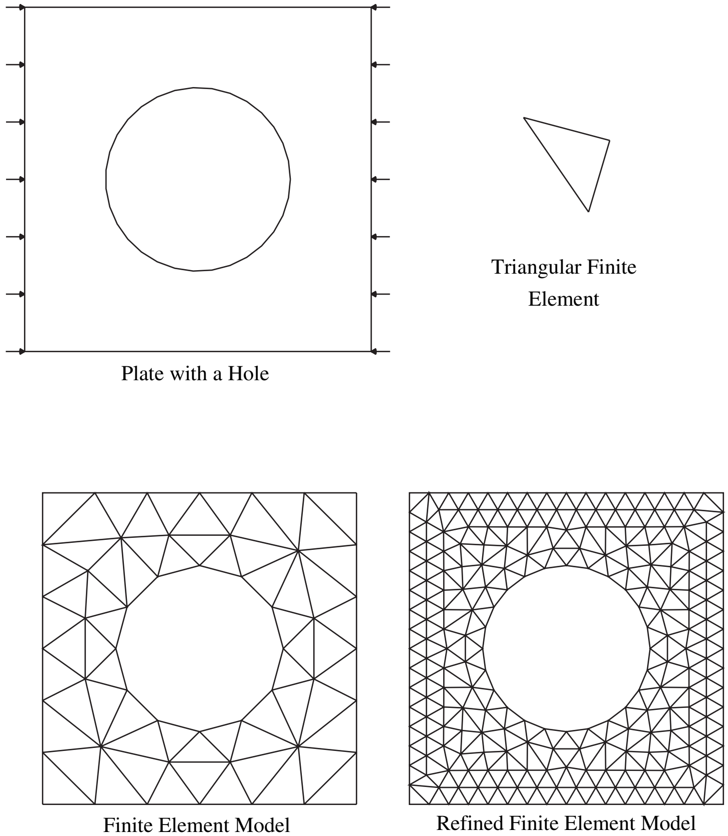


**Figure 1.1** Geometry, loads and finite element meshes.

plots, although selected results are often output on monitors. This information is then used in the engineering design process.

The same basic approach is used in other types of problems. In stress analysis, the field variables are the displacements; in chemical systems, the field variables are material concentrations; and in electromagnetics, the potential field. The same type of mesh is used to represent the geometry of the structure or component and to develop the finite element equations, and for a linear system, the nodal values are obtained by solving large systems (from $10^3$ to $10^6$ equations are common today, and in special applications, $10^9$) of linear algebraic equations.

This text is limited to linear finite element analysis (FEA). The preponderance of finite element analyses in engineering design is today still linear FEM. In heat conduction, linearity requires that the conductance be independent of temperature. In stress analysis, linear FEM is applicable only if the material behavior is linear elastic and the displacements are small. These assumptions are discussed in more depth later in the book. In stress analysis, for most analyses of operational loads, linear analysis is adequate as it is usually undesirable to have operational loads that can lead to nonlinear material behavior or large displacements. For the simulation of extreme loads, such as crash loads and drop tests of electronic components, nonlinear analysis is required.

The FEM was developed in the 1950s in the aerospace industry. The major players were Boeing and Bell Aerospace (long vanished) in the United States and Rolls Royce in the United Kingdom. M.J. Turner, R.W. Clough, H.C. Martin and L.J. Topp published one of the first papers that laid out the major ideas in 1956

(Turner *et al.*, 1956). It established the procedures of element matrix assembly and element formulations that you will learn in this book, but did not use the term 'finite elements'. The second author of this paper, Ray Clough, was a professor at Berkeley, who was at Boeing for a summer job. Subsequently, he wrote a paper that first used the term 'finite elements', and he was given much credit as one of the founders of the method. He worked on finite elements only for a few more years, and then turned to experimental methods, but his work ignited a tremendous effort at Berkeley, led by the younger professors, primarily E. Wilson and R.L. Taylor and graduate students such as T.J.R. Hughes, C. Felippa and K.J. Bathe, and Berkeley was the center of finite element research for many years. This research coincided with the rapid growth of computer power, and the method quickly became widely used in the nuclear power, defense, automotive and aeronautics industries.

Much of the academic community first viewed FEM very skeptically, and some of the most prestigious journals refused to publish papers on FEM: the typical resistance of mankind (and particularly academic communities) to the new. Nevertheless, several capable researchers recognized its potential early, most notably O.C. Zienkiewicz and R.H. Gallagher (at Cornell). O.C. Zienkiewicz built a renowned group at Swansea in Wales that included B. Irons, R. Owen and many others who pioneered concepts like the isoparametric element and nonlinear analysis methods. Other important early contributors were J.H. Argyris and J.T. Oden.

Subsequently, mathematicians discovered a 1943 paper by Courant (1943), in which he used triangular elements with variational principles to solve vibration problems. Consequently, many mathematicians have claimed that this was the original discovery of the method (though it is somewhat reminiscent of the claim that the Vikings discovered America instead of Columbus). It is interesting that for many years the FEM lacked a theoretical basis, i.e. there was no mathematical proof that finite element solutions give the right answer. In the late 1960s, the field aroused the interest of many mathematicians, who showed that for linear problems, such as the ones we will deal with in this book, finite element solutions converge to the correct solution of the partial differential equation (provided that certain aspects of the problem are sufficiently smooth). In other words, it has been shown that as the number of elements increases, the solutions improve and tend in the limit to the exact solution of the partial differential equations.

E. Wilson developed one of the first finite element programs that was widely used. Its dissemination was hastened by the fact that it was 'freeware', which was very common in the early 1960s, as the commercial value of software was not widely recognized at that time. The program was limited to two-dimensional stress analysis. It was used and modified by many academic research groups and industrial laboratories and proved instrumental in demonstrating the power and versatility of finite elements to many users.

Then in 1965, NASA funded a project to develop a general-purpose finite element program by a group in California led by Dick MacNeal. This program, which came to be known as NASTRAN, included a large array of capabilities, such as two- and three-dimensional stress analyses, beam and shell elements, for analyzing complex structures, such as airframes, and analysis of vibrations and time-dependent response to dynamic loads. NASA funded this project with \$3 000 000 (like \$30 000 000 today). The initial program was put in the public domain, but it had many bugs. Shortly after the completion of the program, Dick MacNeal and Bruce McCormick started a software firm that fixed most of the bugs and marketed the program to industry. By 1990, the program was the workhorse of most large industrial firms and the company, MacNeal-Schwendler, was a \$100 million company.

At about the same time, John Swanson developed a finite element program at Westinghouse Electric Corp. for the analysis of nuclear reactors. In 1969, Swanson left Westinghouse to market a program called ANSYS. The program had both linear and nonlinear capabilities, and it was soon widely adopted by many companies. In 1996, ANSYS went public, and it now (in 2006) has a capitalization of \$1.8 billion.

Another nonlinear software package of more recent vintage is LS-DYNA. This program was first developed at Livermore National Laboratory by John Hallquist. In 1989, John Hallquist left the laboratory to found his own company, Livermore Software and Technology, which markets the program. Intially, the program had nonlinear dynamic capabilities only, which were used primarily for crashworthiness, sheet metal forming and prototype simulations such as drop tests. But Hallquist quickly added a large range of capabilities, such as static analysis. By 2006, the company had almost 60 employees.

ABAQUS was developed by a company called HKS, which was founded in 1978. The program was initially focused on nonlinear applications, but gradually linear capabilities were also added. The program was widely used by researchers because HKS introduced gateways to the program, so that users could add new material models and elements. In 2005, the company was sold to Dassault Systemes for $413 million. As you can see, even a 5% holding in one of these companies provided a very nice nest egg. That is why young people should always consider starting their own companies; generally, it is much more lucrative and exciting than working for a big corporation.

In many industrial projects, the finite element database becomes a key component of product development because it is used for a large number of different analyses, although in many cases, the mesh has to be tailored for specific applications. The finite element database interfaces with the CAD database and is often generated from the CAD database. Unfortunately, in today's environment, the two are substantially different. Therefore, finite element systems contain *translators*, which generate finite element meshes from CAD databases; they can also generate finite element meshes from digitizations of surface data. The need for two databases causes substantial headaches and is one of the major bottlenecks in computerized analysis today, as often the two are not compatible.

The availability of a wide range of analysis capabilities in one program makes possible analyses of many complex real-life problems. For example, the flow around a car and through the engine compartment can be obtained by a fluid solver, called computational fluid dynamics (CFD) solver. This enables the designers to predict the drag factor and the lift of the shape and the flow in the engine compartment. The flow in the engine compartment is then used as a basis for heat transfer calculations on the engine block and radiator. These yield temperature distributions, which are combined with the loads, to obtain a stress analysis of the engine.

Similarly, in the design of a computer or microdevice, the temperatures in the components can be determined through a combination of fluid analysis (for the air flowing around the components) and heat conduction analysis. The resulting temperatures can then be used to determine the stresses in the components, such as at solder joints, that are crucial to the life of the component. The same finite element model, with some modifications, can be used to determine the electromagnetic fields in various situations. These are of importance for assessing operability when the component is exposed to various electromagnetic fields.

In aircraft design, loads from CFD calculations and wind tunnel tests are used to predict loads on the airframe. A finite element model is then used with thousands of load cases, which include loads in various maneuvers such as banking, landing, takeoff and so on, to determine the stresses in the airframe. Almost all of these are linear analyses; only determining the ultimate load capacity of an airframe requires a nonlinear analysis. It is interesting that in the 1980s a famous professor predicted that by 1990 wind tunnels would be used only to store computer output. He was wrong on two counts: Printed computer output almost completely disappeared, but wind tunnels are still needed because turbulent flow is so difficult to compute that complete reliance on computer simulation is not feasible.

Manufacturing processes are also simulated by finite elements. Thus, the solidification of castings is simulated to ensure good quality of the product. In the design of sheet metal for applications such as cars and washing machines, the forming process is simulated to insure that the part can be formed and to check that after springback (when the part is released from the die) the part still conforms to specifications.

Similar procedures apply in most other industries. Indeed, it is amazing how the FEM has transformed the engineering workplace in the past 40 years. In the 1960s, most engineering design departments consisted of a room of 1.5 m × 3 m tables on which engineers drew their design with T-squares and other drafting instruments. Stresses in the design were estimated by simple formulas, such as those that you learn in strength of materials for beam stretching, bending and torsion (these formulas are still useful, particularly for checking finite element solutions, because if the finite element differs from these formulas by an order of magnitude, the finite element solution is usually wrong). To verify the soundness of a design, prototypes were made and tested. Of course, prototypes are still used today, but primarily in the last stages of a design. Thus, FEA has led to tremendous reductions in design cycle time, and effective use of this tool is crucial to remaining competitive in many industries.

A question that may occur to you is: Why has this tremendous change taken place? Undoubtedly, the major contributor has been the exponential growth in the speed of computers and the even greater decline in the cost of computational resources. Figure 1.2 shows the speed of computers, beginning with the first electronic computer, the ENIAC in 1945. Computer speed here is measured in megaflops, a rather archaic term that means millions of floating point operations per second (in the 1960s, real number multiplies were called floating point operations).

The ENIAC was developed in 1945 to provide ballistic tables. It occupied 1800 $ft^2$ and employed 17468 vacuum tubes. Yet its computational power was a small fraction of a $20 calculator. It was not until the 1960s that computers had sufficient power to do reasonably sized finite element computations. For example, the 1966 Control Data 6600, the most powerful computer of its time, could handle about 10 000 elements in several hours; today, a PC does this calculation in a matter of minutes. Not only were these computers slow, but they also had very little memory: the CDC 6600 had 32k words of random access memory, which had to accommodate the operating system, the compiler and the program.

As can be seen from Figure 1.2, the increase in computational power has been linear on a log scale, indicating a geometric progression in speed. This geometric progression was first publicized by Moore, a founder of Intel, in the 1990s. He noticed that the number of transistors that could be packed on a chip, and hence the speed of computers, doubled every 18 months. This came to be known as Moore's law, and remarkably, it still holds.

From the chart you can see that the speed of computers has increased by about eight orders of magnitude in the last 40 years. However, the improvement is even more dramatic if viewed in terms of cost in inflation-adjusted currency. This can be seen from Table 1.1, which shows the costs of several computers in 1968 and 2005, along with the tuition at Northwestern, various salaries, the price of an average car and the price of a

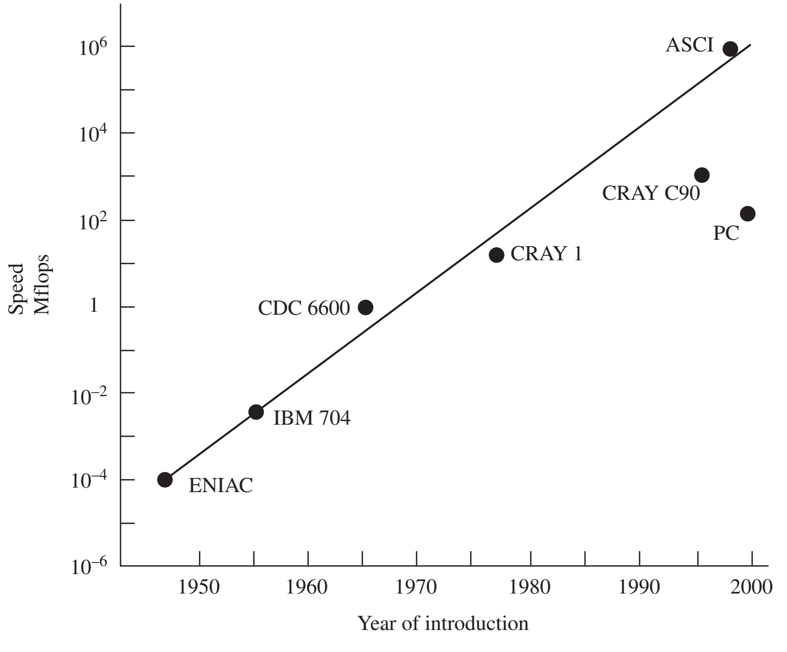


**Figure 1.2** Historical evolution of speed of computers.

**Table 1.1** Costs of some computers and costs of selected items for an estimate of uninflated dollars (from Hughes–Belytschko Nonlinear FEM Short Course).

| | Costs | |
|---|---|---|
| | **1968** | **2005** |
| CDC 6600 (0.5–1 Mflops) | \$8 000 000 | |
| 512 Beowulf cluster (2003) 1 Tflop | | \$500 000 |
| Personal computer (200–1600 Mflops) | | \$500–3000 |
| B.S. Engineer (starting salary, Mech Eng) | \$9000 | \$51 000 |
| Assistant Professor of Engineering (9 mo start salary) | \$11 000 | \$75 000 |
| 1 year tuition at Northwestern | \$1800 | \$31 789 |
| GM, Ford or Chrysler sedan | \$3000 | \$22 000 |
| Mercedes SL | \$7000 | \$90–120 K |
| Decrease in real cost of computations | | $10^7$ to $10^8$ |

Some figues are approximate.

decent car (in the bottom line). It can be seen that the price of computational power has decreased by a factor of over a hundred from 1968 to 2006. During that time, the value of our currency has diminished by a factor of about 10, so the cost of computer power has decreased by a factor of a billion! A widely circulated joke, originated by Microsoft, was that if the automobile industry had made the same progress as the computer industry over the past 40 years, a car would cost less than a penny. The auto industry countered that if computer industry designed and manufactured cars, they would lock up several times a day and you would need to press start to stop the car (and many other ridiculous things). Nevertheless, electronic chips are an area where tremendous improvements in price and performance have been made, and this has changed our lives and engineering practice.

The price of finite element software has also decreased, but only a little. In the 1980s, the software fees for corporate use of NASTRAN were on the order of \$200 000–1 000 000. Even a small firm would have to pay on the order of \$100 000. Today, NASTRAN still costs about \$65 000 per installation, the cost of ABAQUS starts at \$10 000 and LS-DYNA costs \$12 000. Fortunately, all of these companies make student versions available for much less. The student version of ABAQUS comes free with the purchase of this book; a university license for LS-DYNA costs \$500. So today you can solve finite element problems as large as those solved on supercomputers in the 1990s on your PC.

As people became aware of the rapidly increasing possibilities in engineering brought about by computers in the 1980s, many fanciful predictions evolved. One common story on the West Coast was that by the next century, in which we are now, when an engineer came to work he would don a headgear, which would read his thoughts. He would then pick up his design assignment and picture the solution. The computer would generate a database and a visual display, which he would then modify with a few strokes of his laser pen and some thoughts. Once he considered the design visually satisfactory, he would then think of ‘FEM analysis’, which would lead the computer to generate a mesh and visual displays of the stresses. He would then massage the design in a few places, with a laser pen or his mind, and do some reanalyses until the design met the specs. Then he would push a button, and a prototype would drop out in front of him and he could go surfing.

Well, this has not come to pass. In fact, making meshes consumes a significant part of engineering time today, and it is often tedious and causes many delays in the design process. But the quality of products that can be designed with the help of CAD and FEM is quite amazing, and it can be done much quicker than before. The next decade will probably see some major changes, and in view of the hazards of predictions, we will not make any, but undoubtedly FEM will play a role in your life whatever you do.

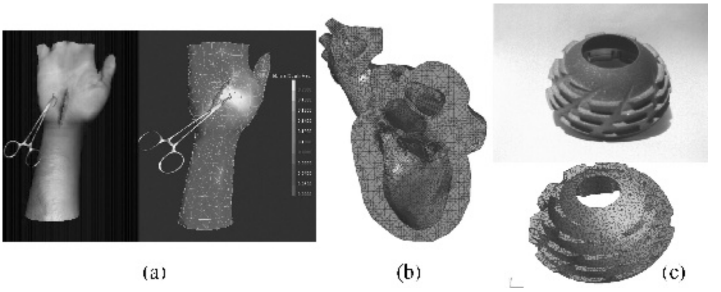


**Figure 1.3** Applications in predictive medicine. (a) Overlying mesh of a hand model near the wound.[1] (b) Cross-section of a heart model.[2] (c) Portion of hip replacement: physical object and finite element model.[3]

## 1.2 APPLICATIONS OF FINITE ELEMENTS

In the following, we will give some examples of finite element applications. The range of applications of finite elements is too large to list, but to provide an idea of its versatility we list the following:

a. stress and thermal analyses of industrial parts such as electronic chips, electric devices, valves, pipes, pressure vessels, automotive engines and aircraft;
b. seismic analysis of dams, power plants, cities and high-rise buildings;
c. crash analysis of cars, trains and aircraft;
d. fluid flow analysis of coolant ponds, pollutants and contaminants, and air in ventilation systems;
e. electromagnetic analysis of antennas, transistors and aircraft signatures;
f. analysis of surgical procedures such as plastic surgery, jaw reconstruction, correction of scoliosis and many others.

This is a very short list that is just intended to give you an idea of the breadth of application areas for the method. New areas of application are constantly emerging. Thus, in the past few years, the medial community has become very excited with the possibilities of predictive, patient-specific medicine.

One approach in predictive medicine aims to use medical imaging and monitoring data to construct a model of a part of an individual's anatomy and physiology. The model is then used to predict the patient's response to alternative treatments, such as surgical procedures. For example, Figure 1.3(a) shows a hand wound and a finite element model. The finite element model can be used to plan the surgical procedure to optimize the stitches.

Heart models, such as shown in Figure 1.3(b), are still primarily topics of research, but it is envisaged that they will be used to design valve replacements and many other surgical procedures. Another area in which finite elements have been used for a long time is in the design of prosthesis, such as shown in Figure 1.3(c). Most prosthesis designs are still generic, i.e. a single prosthesis is designed for all patients with some variations in sizes. However, with predictive medicine, it will be possible to analyze characteristics of a particular patient such as gait, bone structure and musculature and custom-design an optimal prosthesis.

FEA of structural components has substantially reduced design cycle times and enhanced overall product quality. For example in the auto industry, linear FEA is used for acoustic analysis to reduce interior noise, for analysis of vibrations, for improving comfort, for optimizing the stiffness of the chassis and for increasing the fatigue life of suspension components, design of the engine so that temperatures and stresses are acceptable, and many other tasks. We have already mentioned CFD analyses of the body and engine

[1]With permission from Mimic Technologies.
[2]Courtesy of Chandrajit Bajaj, University of Texas at Austin.

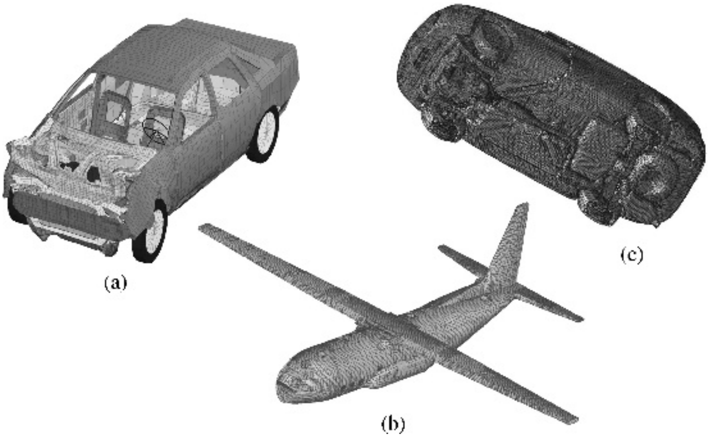


**Figure 1.4** Application to aircraft design and vehicle crash safety: (a) finite element model of Ford Taurus crash;[3] (b) finite element model of C-130 fuselage, empennage and center wing[4] and (c) flow around a car.[5]

compartments previously. The FEMs used in these analyses are exactly like the ones described in this book. Nonlinear FEA is used for crash analysis with both models of the car and occupants; a finite element model for crash analysis is shown in Figure 1.4(a) and a finite element model for stiffness prediction is shown in Figure 1.4(c). Notice the tremendous detail in the latter; these models still require hundreds of man-hours to develop. The payoff for such a modeling is that the number of prototypes required in the design process can be reduced significantly.

Figure 1.4(b) shows a finite element model of an aircraft. In the design of aircraft, it is imperative that the stresses incurred from thousands of loads, some very rare, some repetitive, do not lead to catastrophic failure or fatigue failure. Prior to the availability of FEA, such a design relied heavily on an evolutionary

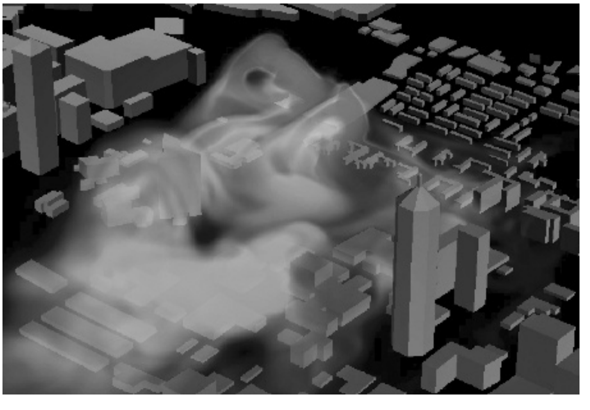

**Figure 1.5** Dispersion of chemical and biological agents in Atlanta. The red and blue colors represent the highest and lowest levels of contaminant concentration.[6]

[3]Courtesy of the Engineering Directorate, Lawrence Livermore National Laboratory.
[4]Courtesy of Mercer Engineering Research Center.
[5]Courtesy of Mark Shephard, Rensselaer.
[6]Courtesy of Shahrouz Aliabadi.

process (basing new designs on old designs), as tests for all of the loads are not practical. With FEA, it has become possible to make much larger changes in airframe design, such as the shift to composites.

In a completely different vein, finite elements also play a large role in environmental decision making and hazard mitigation. For example, Figure 1.5 is a visualization of the dispersal of a chemical aerosol in the middle of Atlanta obtained by FEA; the aerosol concentration is depicted by color, with the highest concentration in red. Note that the complex topography of this area due the high-rise buildings, which is crucial to determining the dispersal, can be treated in great detail by this analysis. Other areas of hazard mitigation in which FEA offers great possibilities are the modeling of earthquakes and seismic building response, which is being used to improve their seismic resistance, the modeling of wind effects on structures and the dispersal of heat from power plant discharges. The latter, as the aerosol dispersal, involves the advection–diffusion equation, which is one of the topics of this book. The advection–diffusion equation can also be used to model drug dispersal in the human body. Of course, the application of these equations to these different topics involves extensive modeling, which is the value added by engineers with experience and knowledge, and constitutes the topic of validation, which is treated in Chapters 8 and 9.

### *Matrix Algebra and Computer Programs*

It is highly recommended that students familiarize themselves with matrix algebra and programming prior to proceeding with the book. An introduction to matrix algebra and applications in MATLAB is given in a Web chapter (Chapter 12) which is available on www.wileyeurope/college/Fish.

This webpage also includes the MATLAB programs which are referred to in this book and other MATLAB programs for finite element analysis. We have chosen to use a web chapter for this material to provide an option for updating this material as MATLAB and the programs change. We invite readers who develop other finite element programs in MATLAB to contact the first author (Jacob Fish) about including their programs. We have also created a blog where students and instructors can exchange ideas and place alternative finite element programs. This forum is hosted at http://1coursefem.blogspot.com/

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