1

Notation – Matrices and Tensors

In this chapter, we give the definition of matrices and tensors for the purpose of notation used in the book. We summarize the basic relations that are useful for reading the text, without any proofs or in-depth presentation. More details are given on the web (Theory, Chapter 1, Examples, Chapter 1). For further reading the reader should consult books specialized to this topic (e.g. Fung 1965, Malvern 1969, Mase & Mase 1999).

1.1 Matrix representation of mathematical objects

For some physical quantities, a single number is sufficient to define the quantity. For example, we use T to denote temperature at a given material point, and associate a certain value with T (e.g. $T = 20\,^{\circ}C$). The quantity specified by a single number is called a *scalar*. We will denote scalars by italic letters.

However, many quantities need more than one number to be completely defined. For example, in order to define velocity of a particle, we need to know not only the magnitude of the velocity, but also its direction and orientation in space. The spatial direction, magnitude and orientation are defined by, say, three velocity components in a Cartesian coordinate system: $v_1,\ v_2,\ v_3$. We denote the velocity by a bold letter $\mathbf{v}$, associating with it three scalars $v_1,\ v_2,\ v_3$. In general, we use a bold lower case letter for a *vector* $\mathbf{b}$, or a column *matrix* of the order $1 \times n$, defined as

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_{n-1} \\ b_n \end{bmatrix} \tag{1.1.1}$$

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We will be using notation of a transpose of a vector, $\mathbf{b}^T$, which assumes an interchange of the rows and columns, i.e.,

$$\mathbf{b}^T = \left[b_1 \; b_2 \; . \, . \; b_{n-1} \; b_n \right] \tag{1.1.2}$$

For some physical quantities we need more complex representation than a vector. For example, the state of stress at a material point is represented by values of forces per unit area at three orthogonal planes (details are shown on the web – Theory, Chapter 1). Hence we need nine scalars (three for each force), which we order in a two-dimensional matrix form as $\sigma_{11}, \; \sigma_{12}, \; \sigma_{13}; \ldots\ldots; \sigma_{31}, \; \sigma_{32}, \; \sigma_{33}$. In general, a *two-dimensional matrix* **B** (capital bold letter is used for a matrix) of order $m \times n$ is defined as a mathematical object with terms B_{ij} in which the first index denotes the row number, and the second index represents the column number,

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & . \; . & B_{1n-1} & B_{1n} \\ B_{21} & B_{22} & . \; . & B_{2n-1} & B_{2n} \\ . & . & . \; . & . & . \\ . & . & . \; . & . & . \\ B_{m-1,1} & B_{m-1,2} & . \; . & B_{m-1,n-1} & B_{m-1,n} \\ B_{m1} & B_{m2} & . \; . & B_{m,n-1} & B_{m,n} \end{bmatrix} \tag{1.1.3}$$

We will be mainly using *square matrices*, where $m = n$. If the columns and rows are interchanged, then we have the transposed matrix,

$$\left(\mathbf{B}^T \right)_{ij} = B_{ji} \tag{1.1.4}$$

Following this definition of matrix, it is possible to extend the matrix to have more than two dimensions. However, in our presentation throughout the book, by a matrix we assume a *two-dimensional square matrix*, unless otherwise stated. A square matrix is *symmetric* when

$$B_{ji} = B_{ij} \quad i, j = 1, 2, \ldots\ldots, n \tag{1.1.5}$$

1.2 Basic relations in matrix algebra

We list here some of the basic matrix algebra relationships that are used in this book.

The *addition* of vectors **a** and **b** is expressed by

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \quad \text{or} \quad c_i = a_i + b_i \quad i = 1, 2, \ldots\ldots, n \tag{1.2.1}$$

resulting in the vector **c** with components c_i. In the case of subtraction we have a 'minus' instead of 'plus' sign. Summation of matrices **A** and **B** assumes the same order of these matrices, say $m \times n$, and is given as

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{or} \quad C_{ij} = A_{ij} + B_{ij} \quad i = 1, 2, \ldots\ldots, m; \; j = 1, 2, \ldots\ldots, n \tag{1.2.2}$$

The resulting matrix **C** is also of the order $m \times n$, with terms C_{ij}.

The *scalar (matrix) multiplication* of vectors **a** and **b** results in the scalar c according to the relation

$$c = \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} = \sum_{s=1}^{n} a_s b_s = a_s b_s \tag{1.2.3}$$

We will generally omit the summation sign by using the convention that the summation is carried over repeated indices (here the index 's' is also called the dummy index). This is known as the *Einstein summation convention*. On the other hand, a dyadic multiplication $\mathbf{ab}^T$ gives the matrix **C**,

$$\mathbf{C} = \mathbf{ab}^T \quad \text{or} \quad C_{ij} = a_i b_j \tag{1.2.4}$$

with the terms C_{ij}.

The *matrix multiplication* between a matrix and a vector, or between two matrices, is defined as follows:

$$\mathbf{c} = \mathbf{Ab}, \quad c_i = A_{ik} b_k \tag{1.2.5}$$

$$\mathbf{C} = \mathbf{AB}, \quad C_{ij} = A_{ik} B_{kj} \tag{1.2.6}$$

Note that the following relation can be proved

$$\mathbf{C}^T = (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T, \quad \left(\mathbf{C}^T\right)_{ij} = B_{ki} A_{jk} \tag{1.2.7}$$

The scalar multiplication of two matrices is given as

$$c = \mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij} \tag{1.2.8}$$

The inverse of a matrix **A**, denoted as $\mathbf{A}^{-1}$, is the matrix which satisfies the relation

$$\mathbf{AA}^{-1} = \mathbf{I}, \quad A_{ik} A^{-1}_{kj} = \delta_{ij} \tag{1.2.9}$$

where **I** is the identity matrix (diagonal terms equal to one, all other terms equal to zero), and δ_{ij} are the Kronecker delta symbols: $\delta_{ij} = 1$ for $i = j$; $\delta_{ij} = 0$ for $i \neq j$.

The determinant of a matrix **A**, denoted as *det***A** or $\|\mathbf{A}\|$, is defined as

$$\det \mathbf{A} \equiv \|\mathbf{A}\| = e_{ijk} A_{1i} A_{2j} A_{3k}, \quad i, j, k = 1, 2, 3 \tag{1.2.10}$$

where e_{ijk} is the permutation symbol, with values: $e_{ijk} = 0$ for $i = j$ or $j = k$ or $i = k$ or $i = j = k$; $e_{ijk} = 1$ for even permutation of 1,2,3; $e_{ijk} = -1$ for odd permutation of 1,2,3. Calculation of the matrix determinant is needed for the matrix inversion, therefore the inverse matrix exists if the determinant of the matrix is not equal to zero.

Two matrices **A** and **B** are orthogonal if the following relationship is satisfied:

$$\mathbf{AB} = \mathbf{I}, \quad \text{or} \quad A_{ik} B_{kj} = \delta_{ij}, \quad i, j = 1, 2, \ldots\ldots, m; \; k = 1, 2, \ldots\ldots, n \tag{1.2.11}$$

Note that the identity matrix is of dimension $m \times m$. The matrix **A** is orthogonal if

$$\mathbf{A}^T\mathbf{A} = \mathbf{I}, \quad \text{or} \quad A_{ki} A_{kj} = \delta_{ij}, \quad i, j = 1, 2, \ldots\ldots, m; \; k = 1, 2, \ldots\ldots, n \tag{1.2.12}$$

1.3 Definition of tensors and some basic tensorial relations

Tensors are mathematical objects defined by components associated with a coordinate system. These components change when the coordinate system is changed, according to certain rules (tensorial transformation rules). Note, however, that tensors (as well as vectors) do not change with the change of the coordinate system, only their components change. We will use tensors to represent physical quantities for which the transformation rules have the physical background. Throughout the book we will be using the Cartesian coordinate system with unit vectors (triad) $\mathbf{i}_1,\ \mathbf{i}_2,\ \mathbf{i}_3$, shown in Fig. 1.3.1.

A first-order tensor $\mathbf{b}$ is represented in two coordinate systems (with unit vectors $\mathbf{i}_k$ and $\bar{\mathbf{i}}_k$) as

$$\mathbf{b} = b_k\mathbf{i}_k = b_1\mathbf{i}_1 + b_2\mathbf{i}_2 + b_3\mathbf{i}_3 = \bar{b}_k\bar{\mathbf{i}}_k = \bar{b}_1\bar{\mathbf{i}}_1 + \bar{b}_2\bar{\mathbf{i}}_2 + \bar{b}_3\bar{\mathbf{i}}_3 \tag{1.3.1}$$

where b_k and $\bar{b}_k$ are the vector components in the two coordinate systems. The relationships between the components in the two systems are given by

$$\bar{b}_j = T_{jk} b_k \tag{1.3.2}$$

where T_{jk} are the cosines of angles between the unit vectors $\bar{\mathbf{i}}_j$ and $\mathbf{i}_k$ of the two coordinate systems, $T_{jk} = \cos(\bar{\mathbf{i}}_j,\ \mathbf{i}_k)$. Note that this equation represents the matrix multiplication of the form (1.2.5), involving the 3×3 transformation matrix $\mathbf{T}$ and the 3×1 vector of the form (1.1.1).

A second-order tensor $\mathbf{B}$ is defined as

$$\begin{aligned}\mathbf{B} &= B_{jk}\mathbf{i}_j\mathbf{i}_k = B_{11}\mathbf{i}_1\mathbf{i}_1 + B_{12}\mathbf{i}_1\mathbf{i}_2 + B_{13}\mathbf{i}_1\mathbf{i}_3 + \ldots\ldots + B_{31}\mathbf{i}_3\mathbf{i}_1 + B_{32}\mathbf{i}_3\mathbf{i}_2 + B_{33}\mathbf{i}_3\mathbf{i}_3 = \\ &= \bar{B}_{jk}\bar{\mathbf{i}}_j\bar{\mathbf{i}}_k = \bar{B}_{11}\bar{\mathbf{i}}_1\bar{\mathbf{i}}_1 + \bar{B}_{12}\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2 + \bar{B}_{13}\bar{\mathbf{i}}_1\bar{\mathbf{i}}_3 + \ldots\ldots + \bar{B}_{31}\bar{\mathbf{i}}_3\bar{\mathbf{i}}_1 + \bar{B}_{32}\bar{\mathbf{i}}_3\bar{\mathbf{i}}_2 + \bar{B}_{33}\bar{\mathbf{i}}_3\bar{\mathbf{i}}_3\end{aligned} \tag{1.3.3}$$

with components B_{jk} and $\bar{B}_{jk}$ in the coordinate systems $\mathbf{i}_k$ and $\bar{\mathbf{i}}_k$, respectively. These components can be represented in the matrix form (1.1.3). The transformation of the tensorial components due to change of the coordinate system is

$$\bar{B}_{jm} = T_{jk} B_{ks} T_{ms} \tag{1.3.4a}$$

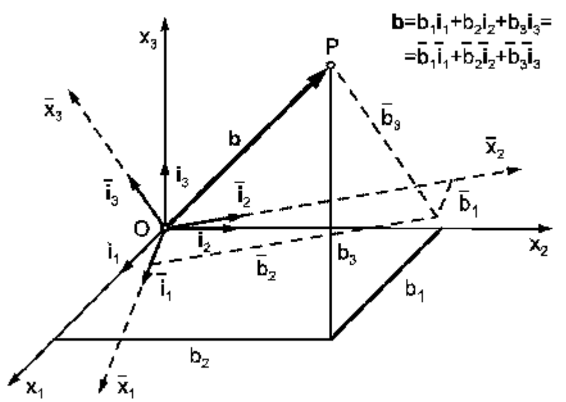

Fig. 1.3.1 Graphical representation of a vector $\mathbf{b}$ in two Cartesian systems

which corresponds to the matrix multiplication (1.2.6),

$$\overline{\mathbf{B}} = \mathbf{T}\mathbf{B}\mathbf{T}^T \tag{1.3.4b}$$

Tensors of higher order can be defined, following (1.3.3), but we will use the second-order tensors and will call them tensors.

We further cite the tensorial relations used in the book. The *dot product (multiplication)* of two vectors, tensor and vector, and two tensors, are consecutively defined as follows:

$$c = \mathbf{a} \cdot \mathbf{b} = a_k \mathbf{i}_k \cdot b_m \mathbf{i}_m = a_k b_k \tag{1.3.5}$$

$$\begin{gathered} \mathbf{c} = \mathbf{A}\mathbf{b} = A_{jk}\mathbf{i}_j\mathbf{i}_k \cdot b_m \mathbf{i}_m = A_{jk} b_m \mathbf{i}_j \mathbf{i}_k \cdot \mathbf{i}_m = A_{jk} b_m \mathbf{i}_j \delta_{km} = A_{jk} b_k \mathbf{i}_j \\ c_j = A_{jk} b_k \end{gathered} \tag{1.3.6}$$

$$\begin{gathered} \mathbf{C} = \mathbf{A}\mathbf{B} = A_{jk}\mathbf{i}_j\mathbf{i}_k \cdot B_{ms}\mathbf{i}_m\mathbf{i}_s = A_{jk}B_{ms}\mathbf{i}_j\mathbf{i}_k \cdot \mathbf{i}_m\mathbf{i}_s = A_{jk}B_{ms}\delta_{km}\mathbf{i}_j\mathbf{i}_s = A_{jk}B_{ks}\mathbf{i}_j\mathbf{i}_s \\ C_{js} = A_{jk}B_{ks} \end{gathered} \tag{1.3.7}$$

Here we have employed the orthogonality of the unit vectors $\mathbf{i}_k$ and $\mathbf{i}_m$, i.e., $\mathbf{i}_k \cdot \mathbf{i}_m = \delta_{km}$. The dot product of two vectors is also called the scalar product. It can be seen that the dot product of two vectors gives a scalar, the dot product of tensor and vector gives vector, and the dot product of two tensors gives a tensor.

We will also use the *cross-product* of two vectors defined as

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}, \quad \text{or} \quad c_i = e_{ijk} a_j b_k \tag{1.3.8}$$

The *scalar product* of two tensors gives a scalar, and is defined as

$$c = \mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij} \tag{1.3.9}$$

The *Euclidean norms* of a vector and a tensor are

$$\|\mathbf{b}\| = \left(b_j b_j\right)^{1/2}, \quad \|\mathbf{A}\|_2 = \left(A_{ij} A_{ij}\right)^{1/2} \tag{1.3.10}$$

The *rotation tensor* $\mathbf{R}$ corresponding to two coordinate systems with unit vectors $\mathbf{i}_k$ and $\bar{\mathbf{i}}_k$ is defined by the components

$$R_{km} = \cos(\mathbf{i}_k, \bar{\mathbf{i}}_m) \tag{1.3.11}$$

It can be shown that the following relationship holds (see web – Theory, Chapter 1)

$$\bar{\mathbf{i}}_m = \mathbf{R}\mathbf{i}_m \tag{1.3.12}$$

leading to a rotation of vector $\mathbf{i}_m$. Multiplication of any vector $\mathbf{b}$ by the rotation tensor $\mathbf{R}$ rotates this vector as it rotates the vector $\mathbf{i}_m$. The following relation is valid $\mathbf{R} = \mathbf{T}^T$. It is important to emphasize that multiplication of a vector $\mathbf{b}$ by the transformation matrix $\mathbf{T}$ gives the vector components in a rotated coordinate system of the same vector (see (1.3.2) and Fig. 1.3.1; also see web – Theory, Chapter 1). On the other hand, multiplication of a vector **b** by the rotation tensor **R** produces another vector $\bar{\mathbf{b}}$, rotated with respect to **b** (see web – Theory, Chapter 1).

Finally, we define the principal values and principal directions of a tensor. In order to introduce these quantities, consider the following equation

$$\mathbf{A}\mathbf{p} = \lambda\mathbf{p} \quad \text{or} \quad (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{p} = \mathbf{0} \tag{1.3.13}$$

where λ is a scalar, and **p** is a unit vector. The equation has a nontrivial solution ($\mathbf{p} \neq \mathbf{0}$) if determinant of the system matrix $\mathbf{A} - \lambda\mathbf{I}$ is equal to zero,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0, \quad \text{or} \quad \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{1.3.14}$$

where $I_1,\ I_2,\ I_3$ are the first, second and third invariants of the tensor **A**,

$$\begin{aligned} &I_1 = trA = A_{ii} = A_{11} + A_{22} + A_{33}, \quad I_2 = \frac{1}{2}\left(A_{ii}A_{jj} - A_{ij}A_{ji}\right), \\ &I_3 = \det\mathbf{A} = e_{ijk}A_{1i}A_{2j}A_{3k}, \quad i, j, k = 1, 2, 3 \end{aligned} \tag{1.3.15}$$

If the matrix **A** is symmetric and its terms are real numbers, then there are three real solutions $\lambda_1,\ \lambda_2,\ \lambda_3$ which are the *principal values*, or *eigenvalues* of matrix **A**. To each principal value λ_k there corresponds the *principal vector*, or *eigenvector* $\mathbf{p}_k$. It can be shown that principal vectors $\mathbf{p}_k$ are orthogonal (or orthonormal if eigenvectors are unit vectors), forming the *principal basis* of the tensor **A** (see Example 1.5-4). Therefore, the tensor **A** in the principal basis, written in a matrix and tensorial form, is:

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad \text{or} \quad \mathbf{A} = \lambda_1\mathbf{p}_1\mathbf{p}_1 + \lambda_2\mathbf{p}_2\mathbf{p}_2 + \lambda_3\mathbf{p}_3\mathbf{p}_3 \tag{1.3.16}$$

In practical calculations of the principal vectors, we use (for a given principal value λ_k) two of the equations (1.3.13) and one representing the condition that $\mathbf{p}_k$ is a unit vector. Therefore the following system of equations is solved:

$$\begin{aligned} &(A_{11} - \lambda_k)\,p_{(k)1} + A_{12}p_{(k)2} + A_{13}p_{(k)3} = 0 \\ &A_{21}p_{(k)1} + (A_{22} - \lambda_k)\,p_{(k)2} + A_{23}p_{(k)3} = 0, \quad \text{no sum on } k \\ &\left(p_{(k)1}\right)^2 + \left(p_{(k)2}\right)^2 + \left(p_{(k)3}\right)^2 = 1 \end{aligned} \tag{1.3.17}$$

Further details about the eigenvalue problem can be seen in Examples 1.5-4; 2.1-4 and 2.1-6 on the web (Examples, Chapter 2).

1.4 Vector and tensor differential operations and integral theorems

Here, the vector and tensor differential operations and integral theorems used in this book are summarized. Throughout the text, we mainly refer to the Cartesian coordinate system.

Differential Operations

We start with the *differential operator* '*nabla*' or '*del*', defined as

$$\nabla = \frac{\partial}{\partial x_1}\mathbf{i}_1 + \frac{\partial}{\partial x_2}\mathbf{i}_2 + \frac{\partial}{\partial x_3}\mathbf{i}_3 \equiv \frac{\partial}{\partial x_k}\mathbf{i}_k \tag{1.4.1}$$

which is a vector-operator with components $\partial/\partial x_k,\ k = 1, 2, 3$. When applied to a scalar function $\phi\,(x_1, x_2, x_3)$, the *gradient of the function* ϕ is obtained,

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{i}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{i}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{i}_3 \equiv \frac{\partial\phi}{\partial x_k}\mathbf{i}_k \tag{1.4.2}$$

as a vector, with the components $\partial\phi/\partial x_k$. The *gradient of a vector field* $\mathbf{b}\,(x_1, x_2, x_3)$ is

$$\nabla\mathbf{b} = \frac{\partial}{\partial x_k}\left(b_j\mathbf{i}_j\right)\mathbf{i}_k = \frac{\partial b_j}{\partial x_k}\mathbf{i}_j\mathbf{i}_k \tag{1.4.3}$$

and represents the second-order tensor with components $(\nabla\mathbf{b})_{kj} = \partial b_j/\partial x_k$. The $\nabla\mathbf{b}$ is called the *dyadic product* of the vectors ∇ and $\mathbf{b}$.

The *divergence of a vector field* $\mathbf{b}\,(x_1, x_2, x_3)$ is defined as

$$\nabla\cdot\mathbf{b} \equiv div\mathbf{b} = \frac{\partial}{\partial x_k}\left(b_j\mathbf{i}_j\right)\cdot\mathbf{i}_k = \frac{\partial b_j}{\partial x_k}\mathbf{i}_j\cdot\mathbf{i}_k = \frac{\partial b_j}{\partial x_k}\delta_{jk} = \frac{\partial b_k}{\partial x_k} \tag{1.4.4}$$

and represents a scalar. The *divergence of a tensor field* $\mathbf{A}\,(x_1,\ x_2,\ x_3)$ is

$$\nabla\cdot\mathbf{A} \equiv div\mathbf{A} = \mathbf{i}_k\frac{\partial}{\partial x_k}\cdot\left(A_{jm}\mathbf{i}_j\mathbf{i}_m\right) = \frac{\partial A_{jm}}{\partial x_k}\mathbf{i}_k\cdot\mathbf{i}_j\mathbf{i}_m = \frac{\partial A_{km}}{\partial x_k}\mathbf{i}_m \tag{1.4.5}$$

Therefore, $\nabla\cdot\mathbf{A}$ is a vector with components $(\nabla\cdot\mathbf{A})_m = \partial A_{km}/\partial x_k$.

The *curl of a vector field* is (see the definition of cross-product (1.3.8)) is

$$\nabla\times\mathbf{b} = e_{mjk}\frac{\partial b_k}{\partial x_j}\mathbf{i}_m = \left(\frac{\partial b_3}{\partial x_2} - \frac{\partial b_2}{\partial x_3}\right)\mathbf{i}_1 + \left(\frac{\partial b_1}{\partial x_3} - \frac{\partial b_3}{\partial x_1}\right)\mathbf{i}_2 + \left(\frac{\partial b_2}{\partial x_1} - \frac{\partial b_1}{\partial x_2}\right)\mathbf{i}_3 \tag{1.4.6}$$

representing a vector with components shown here.

The *Laplacian operator* is defined as

$$\nabla\cdot\nabla \equiv \Delta = \mathbf{i}_k\frac{\partial}{\partial x_k}\cdot\mathbf{i}_m\frac{\partial}{\partial x_m} = \frac{\partial^2}{\partial x_k\partial x_k} \equiv \frac{\partial^2}{(\partial x_1)^2} + \frac{\partial^2}{(\partial x_2)^2} + \frac{\partial^2}{(\partial x_3)^2} \tag{1.4.7}$$

which is a scalar differential operator. Hence, the *Laplacian of a scalar field* is the scalar

$$\nabla\cdot\nabla\phi \equiv \Delta\phi = \frac{\partial^2\phi}{\partial x_k\partial x_k} \equiv \frac{\partial^2\phi}{(\partial x_1)^2} + \frac{\partial^2\phi}{(\partial x_2)^2} + \frac{\partial^2\phi}{(\partial x_3)^2} \tag{1.4.8}$$

while the *Laplacian of a vector field* is the vector

$$\nabla\cdot\nabla\mathbf{b} \equiv \Delta\mathbf{b} = \frac{\partial^2 b_j}{\partial x_k\partial x_k}\mathbf{i}_j = \left(\Delta b_j\right)\mathbf{i}_j \tag{1.4.9}$$

with components Δb_j.

Integral Theorems

We list here the integral theorems that are used subsequently. The mostly used is the *Gauss Theorem* (Fung 1965, Bird *et al*. 2002). If a closed volume V in space is bounded by a surface S, then for a vector field we have

$$\int_V \nabla \cdot \mathbf{b} dV = \int_S \mathbf{n} \cdot \mathbf{b} dS, \quad \text{or} \quad \int_V \frac{\partial b_k}{\partial x_k} dV = \int_S b_j n_j dS \tag{1.4.10}$$

where $\mathbf{n}$ is the unit normal of the surface element dS. This relation is also known as the Gauss–Ostrogradskii or the Divergence Theorem. In the case of a scalar field $\phi(x_k)$ and a tensor field $\mathbf{A}(x_k)$, the Gauss theorem is

$$\int_V \nabla \phi dV = \int_S \phi \mathbf{n} dS, \quad \text{or} \quad \int_V \frac{\partial \phi}{\partial x_i} dV = \int_S \phi n_i dS \tag{1.4.11}$$

and

$$\int_V \nabla \cdot \mathbf{A} dV = \int_S \mathbf{n} \cdot \mathbf{A} dS, \quad \text{or} \quad \int_V \frac{\partial A_{ki}}{\partial x_k} dV = \int_S n_k A_{ki} dS \tag{1.4.12}$$

Note that in case of a two-dimensional domain, the volume V becomes the surface S and the surface S in the above integrals becomes the two-dimensional domain contour line L. Proof of the Gauss theorem and additional details are given on the web – Theory, Chapter 1.

Next, we give the expression for the so-called *material derivative of volume integral*. Assume that a continuum is moving in space. Considering the continuum as a set of material particles, we have that physical quantities, such as mass density, temperature, velocity, stresses, are associated with each material particle, changing over time while particles move. Let ψ_V be the total value of a quantity ψ (it is a scalar: temperature, density, . . . , or a component of a vector or tensor) over all material particles occupying currently a fixed closed space volume V_{space}, i.e.

$$\psi_V = \int_{V_{space}} \psi dV \tag{1.4.13}$$

If we want to find the rate of change of ψ_V, we obtain

$$\begin{aligned} \frac{D\psi_V}{Dt} &= \int_{V_{space}} \frac{\partial \psi}{\partial t} dV + \int_{S_{space}} \psi \mathbf{v} \cdot \mathbf{n} dS = \\ &= \int_{V_{space}} \left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x_k} v_k + \psi \frac{\partial v_k}{\partial x_k} \right) dV = \int_{V_{space}} \left(\frac{D\psi}{Dt} + \psi \frac{\partial v_k}{\partial x_k} \right) dV \end{aligned} \tag{1.4.14}$$

The integral over the surface S_{space}, which encloses the volume V_{space}, represents the flux of ψ through the surface. We have used the Gauss theorem (1.4.11) to transform the surface integral to the volume integral. Also we use the notation $D\psi_V/Dt$ to indicate that the time derivative is evaluated assuming the same material particles. Consequently, the derivative

$$\frac{D\psi}{Dt} \equiv \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x_k} v_k \tag{1.4.15}$$

is called the *material (or substantial) derivative* of ψ. If a spatial field of a physical quantity ψ which changes with time is defined, $\psi(x_k, t)$, then the derivative $\partial\psi/\partial t$ is the *local derivative* assuming constant spatial coordinates x_k; while the term $(\partial\psi/\partial x_k)\, \upsilon_k$ is the *convective derivative* which takes into account motion of the material particle. Therefore, for a given spatial field ψ, the sum of these last two derivatives gives the rate of change $D\psi/Dt$ for a material particle (material point) at a given space position. The material derivatives are used when transport phenomena are studied (e.g. mass and heat transport).

Additional details about the relations presented in this section are given on the web – Theory, Chapter 1.

1.5 Examples

Example 1.5-1. Prove the e-δ identity
The following relations between the permutation symbol e_{ijk} and the Kronecker-delta symbol δ_{ij} can be proved:

$$e_{ijk}e_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{E1.5-1.1}$$

Details of the poof of these relations are given on the web – Examples, Section 1.5.

Example 1.5-2. Derive the procedure for calculation of the inverse matrix
For simplicity, consider a 3×3 matrix $\mathbf{A}$. The matrix $\mathbf{A}^{-1}$ is the matrix which satisfies the relation (1.2.9). We write the matrix $\mathbf{A}^{-1}$ as

$$\mathbf{A}^{-1} = \left[A_{ij}^{-1}\right] = \left[\mathbf{x}_{(1)}\mathbf{x}_{(2)}\mathbf{x}_{(3)}\right] \tag{E1.5-2.1}$$

where the vector $\mathbf{x}_{(i)}$ is the i-th column of the matrix $\mathbf{A}^{-1}$, i.e. we have that $x_{(i)j} = A_{ji}^{-1}$. The equations (1.2.9) can be written as a system of three equations:

$$\mathbf{A}\mathbf{x}_{(i)} = \boldsymbol{\delta}_{(i)}, \;\; i = 1, 2, 3 \tag{E1.5-2.2}$$

where the vectors $\boldsymbol{\delta}_{(1)}, \boldsymbol{\delta}_{(2)}, \boldsymbol{\delta}_{(3)}$ have the components $\delta_{(i)j} = \delta_{ij}$.

By solving this system of equations, we obtain that the terms A_{ij}^{-1} of the matrix $\mathbf{A}^{-1}$ are:

$$A_{ij}^{-1} = \frac{1}{D}D^{ji}, \quad \text{or} \quad \mathbf{A}^{-1} = \frac{1}{D}\mathbf{D}^{T} \tag{E1.5-2.3}$$

where $\mathbf{D} = \left[D^{ij}\right]$ is the matrix of cofactors of the matrix $\mathbf{A}$. Details of this derivation and problems for exercise are given on the web – Examples, Section 1.5.

Example 1.5-3. Determine the rotation tensor using the relation (1.3.12)
The component form of this relation is

$$\bar{i}_{(\alpha)k} = R_{kj}i_{(\alpha)j} \tag{E1.5-3.1}$$

We multiply this equation by $i_{(\alpha)s}$ and sum such three equations on α to obtain

$$\sum_{\alpha=1}^{3} \bar{i}_{(\alpha)k} i_{(\alpha)s} = \sum_{\alpha=1}^{3} R_{kj} i_{(\alpha)j} i_{(\alpha)s} = R_{kj} \sum_{\alpha=1}^{3} i_{(\alpha)j} i_{(\alpha)s} = R_{kj} \delta_{js} = R_{ks} \tag{E1.5-3.2}$$

We have used here the orthogonality property of the base vectors $\mathbf{i}_{(\alpha)}$. Namely, we have that $i_{(\alpha)k} = i_{(k)\alpha}$ and then

$$\sum_{\alpha=1}^{3} i_{(j)\alpha} i_{(s)\alpha} = \mathbf{i}_{(j)} \cdot \mathbf{i}_{(s)} = \delta_{js} \tag{E1.5-3.3}$$

The relation (E1.5-3.1) can be written in a dyadic (direct notation) as

$$\mathbf{R} = \sum_{\alpha=1}^{3} \bar{\mathbf{i}}_{(\alpha)} \mathbf{i}_{(\alpha)} = \bar{\mathbf{i}}_{(1)} \mathbf{i}_{(1)} + \bar{\mathbf{i}}_{(2)} \mathbf{i}_{(2)} + \bar{\mathbf{i}}_{(3)} \mathbf{i}_{(3)} \tag{E1.5-3.4}$$

The tensor $\mathbf{R}$ written in a matrix form is

$$\mathbf{R} = \begin{array}{c|c|c|c|} & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \hline x_1 & l_1 & l_2 & l_3 \\ \hline x_2 & m_1 & m_2 & m_3 \\ \hline x_3 & n_1 & n_2 & n_3 \\ \hline \end{array} \tag{E1.5-3.5}$$

where the coefficients l_i, m_i, n_i are the cosines of the angles between, respectively, the axes x_1, x_2, x_3 and $\bar{x}_1, \bar{x}_2, \bar{x}_3$. Note that the transformation matrix $\mathbf{T}$ in (1.3.2) is

$$\mathbf{T} = \mathbf{R}^{\mathrm{T}} \quad \begin{array}{c|c|c|c|} & x_1 & x_2 & x_3 \\ \hline \bar{x}_1 & l_1 & m_1 & n_1 \\ \hline \bar{x}_2 & l_2 & m_2 & n_2 \\ \hline \bar{x}_3 & l_3 & m_3 & n_3 \\ \hline \end{array} \tag{E1.5-3.6}$$

EXERCISE

(a) Determine the rotation tensor $\mathbf{R}$ using the relation (1.3.12) and two orthogonal bases $\mathbf{p}_{(\alpha)}$ and $\bar{\mathbf{p}}_{(\alpha)}$ which do not coincide. The solution is (see web – Examples, Section 1.5):

$$\mathbf{R} = \sum_{\alpha}^{3} \bar{\mathbf{p}}_{(\alpha)} \mathbf{p}_{(\alpha)} \tag{E1.5-3.7}$$

(b) Write the rotation tensor in the bases $\mathbf{p}_{(\alpha)}$ and $\bar{\mathbf{p}}_{(\alpha)}$.

(c) Prove that the rotation tensor is orthogonal, i.e. $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Example 1.5-4. Show that that the principal vectors are orthogonal and that the symmetric tensor in the principal directions is diagonal

Consider two principal vectors $\mathbf{p}_m$ and $\mathbf{p}_n$ of the symmetric matrix $\mathbf{A}$, corresponding to the eigenvalues $\lambda_m \neq \lambda_n$. Then, according to (1.3.13) we have

$$\begin{aligned} \mathbf{A}\mathbf{p}_m &= \lambda_m \mathbf{p}_m \\ \mathbf{A}\mathbf{p}_n &= \lambda_n \mathbf{p}_n, \quad \text{no sum on } m \text{ and } n \end{aligned} \tag{E1.5-4.1}$$

If we multiply the first equation by $\mathbf{p}_n^T$ and the second equation by $\mathbf{p}_m^T$ and subtract the resulting scalar equations, we obtain

$$\mathbf{p}_n^T \mathbf{A} \mathbf{p}_m - \mathbf{p}_m^T \mathbf{A} \mathbf{p}_n = 0 = (\lambda_m - \lambda_n)\, \mathbf{p}_n^T \mathbf{p}_m \tag{E1.5-4.2}$$

since the matrix $\mathbf{A}$ is symmetric, and $\mathbf{p}_n^T \mathbf{p}_m = \mathbf{p}_m^T \mathbf{p}_n$. The vectors $\mathbf{p}_m$ and $\mathbf{p}_n$ are the unit vectors, therefore we have that

$$\mathbf{p}_n^T \mathbf{p}_m = \delta_{mn} \tag{E1.5-4.3}$$

which shows that the principal vectors are orthogonal (or orthonormal). The orthogonality of the principal vectors is also applicable in the case when some principal values are equal. Details of the proof for this case can be found elsewhere, e.g. in Bathe (1996).

To prove that $\mathbf{A}$ is diagonal in the coordinate base $\mathbf{p}_m$, we write the system of equations (1.3.13) in a matrix form as

$$\mathbf{A}\mathbf{P} = \mathbf{P}\Lambda \tag{E1.5-4.4}$$

where the matrices $\mathbf{P}$ and $\mathbf{\Lambda}$ are

$$\mathbf{P} = [\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3], \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{E1.5-4.5}$$

Note that $\mathbf{P}^T\mathbf{P} = \mathbf{I}$ due to the othogonality condition (E1.5-4.3). From multiplication of (E1.5-4.4) from the left by $\mathbf{P}^T$ it follows

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{P}\Lambda = \Lambda, \qquad \text{hence} \quad \overline{\mathbf{A}} = \Lambda \tag{E1.5-4.6}$$

where $\overline{\mathbf{A}}$ is the matrix in the coordinate system with the base vectors $\mathbf{p}_m$ (see also the transformation rule (1.3.4a,b)).

EXERCISE

Express the matrix $\mathbf{A}$ in terms of matrices $\mathbf{P}$ and $\mathbf{\Lambda}$ (spectral decomposition of A) and determine $\mathbf{A}^{-1}$. Solution is given on the web – Examples, Section 1.5.

Example 1.5-5. Determine the differential operator 'nabla' in cylindrical coordinate system

The relations between Cartesian x, y, z and cylindrical system r, θ, z are (see Fig. E1.5-5, where P is a point in space and P′ is its projection onto $x - y$ or $r - x$ plane),

$$x = r\cos\theta, \;\; y = r\sin\theta, \;\; z = z \tag{E1.5-5.1}$$

The relations between the unit vectors $\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$ and $\mathbf{r}_0, \mathbf{c}_0, \mathbf{i}_z$ are

$$\mathbf{r}_0 = \cos\theta \mathbf{i}_x + \sin\theta \mathbf{i}_y, \quad \mathbf{c}_0 = -\sin\theta \mathbf{i}_x + \cos\theta \mathbf{i}_y, \quad \mathbf{i}_z = \mathbf{i}_z \tag{E1.5-5.2}$$

Relations between the partial derivatives in the two coordinate systems are

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}, \;\; \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta} \tag{E1.5-5.3}$$

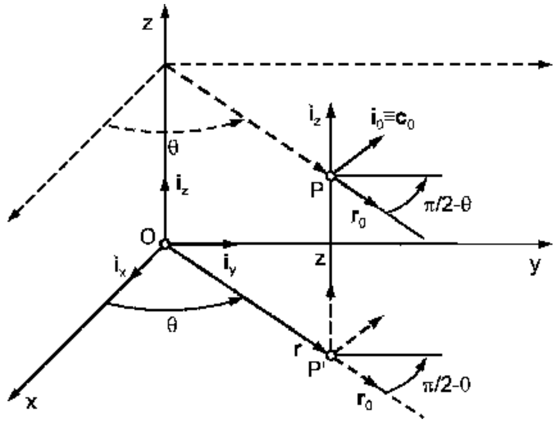

Fig. E1.5-5 Cartesian and cylindrical coordinate systems

Using the relations (E1.5-5.2) and (E1.5-5.3) we obtain that the operator ∇ in the cylindrical coordinate system is

$$\nabla = \mathbf{r}_0 \frac{\partial}{\partial r} + \mathbf{c}_0 \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{i}_z \frac{\partial}{\partial z} \tag{E1.5-5.4}$$

Note that the following relations are valid:

$$\begin{aligned} \frac{\partial \mathbf{r}_0}{\partial \theta} &= -\sin\theta\, \mathbf{i}_x + \cos\theta\, \mathbf{i}_y = \mathbf{c}_0 \\ \frac{\partial \mathbf{c}_0}{\partial \theta} &= -\cos\theta\, \mathbf{i}_x - \sin\theta\, \mathbf{i}_y = -\mathbf{r}_0 \end{aligned} \tag{E1.5-5.5}$$

which are obtained from (E1.5-5.2). Derivatives of $\mathbf{r}_0, \mathbf{c}_0, \mathbf{i}_z$ with respect to r and z are equal to zero.

Detailed derivations of the above relations and problems for exercise are given on the web – Examples, Section 1.5.

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