Part I

Linear Multivariable Control Systems

1

Canonical representations and stability analysis of linear MIMO systems

1.1 INTRODUCTION

In the first section of this chapter, we consider in general the key ideas and concepts concerning canonical representations of linear multi-input multi-output (MIMO) control systems (also called *multivariable* control systems) with the help of the *characteristic transfer functions* (or *characteristic gain functions*) method (MacFarlane and Belletrutti 1970; MacFarlane *et al.* 1977; MacFarlane and Postlethwaite 1977; Postlethwaite and MacFarlane 1979). We shall see how, using simple mathematical tools of the theory of matrices and linear algebraic operators, one can associate a set of N so-called *one-dimensional characteristic systems* acting in the complex space of input and output vector-valued signals along N linearly independent directions (axes of the *canonical basis*) with an N-dimensional (i.e. having N inputs and N outputs) MIMO system. This enables us to reduce the stability analysis of an interconnected MIMO system to the stability analysis of N *independent* characteristic systems, and to formulate the generalized Nyquist criterion. We also consider some notions concerning the *singular value decomposition* (SVD) used in the next chapter for the performance analysis of MIMO systems. In the subsequent sections, we focus on the structural and geometrical features of important classes of MIMO systems – *uniform* and *normal* systems – and derive canonical representations for their transfer function matrices. In the last section, we discuss multivariable root loci. That topic, being immediately related to the stability analysis, is also very significant for the MIMO system design.

1.2 GENERAL LINEAR SQUARE MIMO SYSTEMS

1.2.1 Transfer matrices of general MIMO systems

Consider an N-dimensional controllable and observable *square* (that is having the same number of inputs and outputs) MIMO system, as shown in Figure 1.1. Here, $\varphi(s)$, $f(s)$ *and* $\varepsilon(s)$

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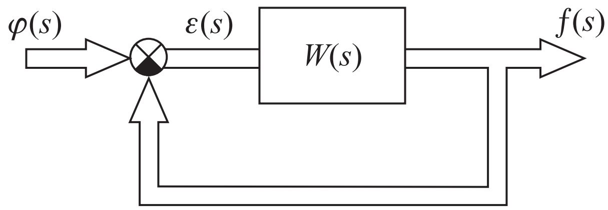

Figure 1.1 Block diagram of a general-type linear MIMO feedback system.

stand for the Laplace transforms of the N-dimensional input, output and error vector signals $\varphi(t)$, $f(t)$ and $\varepsilon(t)$, respectively (we shall regard them as elements of some N-dimensional complex space $\mathbb{C}^N$); $W(s) = \{w_{kr}(s)\}$ denotes the square transfer function matrix of the open-loop system of order $N \times N$ (for simplicity, we shall call this matrix the *open-loop transfer matrix*) with entries $w_{kr}(s)$ $(k, r = 1, 2, \ldots, N)$, which are scalar proper rational functions in complex variable s. The elements $w_{kk}(s)$ on the principal diagonal of $W(s)$ are the transfer functions of the *separate* channels, and the nondiagonal elements $w_{kr}(s)$ $(k \neq r)$ are the transfer functions of *cross-connections* from the rth channel to the kth.

Henceforth, we shall not impose any restrictions on the number N of separate channels, i.e. on the dimension of the MIMO system, and on the *structure* (type) of the matrix $W(s)$. At the same time, so as not to encumber the presentation and to concentrate on the primary ideas, later on, we shall assume that the scalar transfer functions $w_{kr}(s)$ do not have multiple poles (we mean each individual transfer function). Also, we shall refer to the general-type MIMO system of Figure 1.1 as simply the *general* MIMO system (so as not to introduce any ambiguity concerning the *type* of system, which is conventionally defined in the classical control theory as the number of pure integrators in the open-loop system transfer function).

The output $f(s)$ and error $\varepsilon(s)$ vectors, where

$$\varepsilon(s) = \varphi(s) - f(s), \tag{1.1}$$

are related to the input vector $\varphi(s)$ by the following operator equations:

$$f(s) = \Phi(s)\varphi(s), \quad \varepsilon(s) = \Phi_\varepsilon(s)\varphi(s), \tag{1.2}$$

where

$$\Phi(s) = [I + W(s)]^{-1} W(s) = W(s)[I + W(s)]^{-1} \text{ and} \tag{1.3}$$

$$\Phi_\varepsilon(s) = [I + W(s)]^{-1} \tag{1.4}$$

are the transfer function matrices of the closed-loop MIMO system (further, for short, referred to as the *closed-loop transfer matrices*) with respect to output and error signals, and I is the unit matrix. The transfer matrices $\Phi_\varepsilon(s)$ and $\Phi(s)$ are usually called the *sensitivity function matrix* and *complementary sensitivity function matrix*.[1]

By straightforward calculation, it is easy to check that $\Phi_\varepsilon(s)$ and $\Phi(s)$ satisfy the relationship:

$$\Phi(s) + \Phi_\varepsilon(s) = I. \tag{1.5}$$

[1] The terms *sensitivity function* and *complementary sensitivity function* were introduced by Bode (1945).

From here, we come to the important conclusion that it is impossible to bring to zero the system error if the input signal is a sum (mixture) of a reference signal and disturbances, where the latter may be, for example, the measurement or other noises. Indeed, if the system ideally tracks the input reference signal, that is if the matrix $\Phi_\varepsilon(s)$ identically equals the zero matrix, then, due to the superposition principle (Ogata 1970; Kuo 1995), that system also ideally reproduces at the output the input noise [since, if $\Phi_\varepsilon(s) = 0$, then the matrix $\Phi(s)$ in Equation (1.5) is equal to the unit matrix *I*]. A certain trade-off may only be achieved provided the input reference signal and the measurement noise have nonoverlapping (at least, partially) frequency ranges.[2]

1.2.2 MIMO system zeros and poles

1.2.2.1 Open-loop MIMO systems

A single-input single-output (SISO) feedback control system with the open-loop transfer function $W(s)$ is depicted in Figure 1.2. That system may be regarded, if $N = 1$, as a specific case of

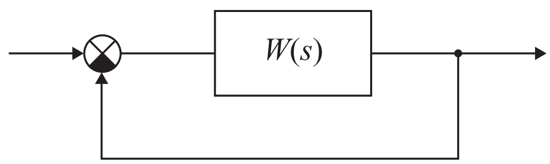

Figure 1.2 Block diagram of a SISO control system ($N = 1$).

the MIMO system of Figure 1.1. The transfer function $W(s)$ is a rational function in complex variable s and can be expressed as a quotient of two polynomials $M(s)$ and $D(s)$ with real coefficients:

$$W(s) = \frac{M(s)}{D(s)}, \tag{1.6}$$

where the order m of $M(s)$ is equal to or less than the order n of $D(s)$, that is we consider only physically feasible systems.

From the classical control theory, we know that the *poles* p_i of $W(s)$ are the roots of the denominator polynomial $D(s)$, and *zeros* z_i are the roots of the numerator polynomial $M(s)$ (Ogata 1970; Kuo 1995). In the case of usual SISO systems with *real* parameters, complex poles and zeros always occur in complex conjugate pairs. Obviously, at the zeros z_i, the transfer function $W(s)$ vanishes and, at the poles p_i, it tends to infinity (or $1/W(s)$ vanishes).

In the multivariable case, the situation is not so simple, and this refers to the MIMO system zeros in particular. This indeed explains the large number of papers in which there are given different definitions and explanations of the MIMO system zeros: from the state-space positions, by means of polynomial matrices and the Smith-McMillan form, etc. (Sain and Schrader 1990; Wonham 1979; Rosenbrock 1970, 1973; Postlethwaite and MacFarlane 1979; Vardulakis 1991).

First, let us consider the *open-loop* MIMO system *poles*. We call any complex number p_i the *pole* of the open-loop transfer matrix $W(s)$ if p_i is the pole of at least one of the entries

[2] The MIMO system accuracy is discussed in Chapter 2.

$w_{kr}(s)$ of the matrix $W(s)$. In fact, if at least one of the entries $w_{kr}(s)$ of $W(s)$ tends to infinity as $s \to p_i$, then $W(s)$ tends (strictly speaking, by norm) to infinity. Therefore, p_i may be regarded as the pole of $W(s)$. As a result, we count the set of the poles of all $w_{kr}(s)$ as the poles of $W(s)$. Such a *prima facie* formal definition of the MIMO system pole seems evident but it leads, as we shall see later, to rather interesting results.

Let the transfer matrix $W(s)$ be expanded, taking into account the above assumption that $w_{kr}(s)$ have no multiple roots, into partial fractions as:

$$W(s) = \sum_{i=1}^{n} \frac{K_i}{s - p_i} + D, \tag{1.7}$$

where n is the total number of simple poles of $W(s)$;

$$K_i = \lim_{s \to p_i} (s - p_i) W(s) \tag{1.8}$$

are the *residue matrices* of $W(s)$ at the finite poles p_i; and the constant matrix D is

$$D = \lim_{s \to \infty} W(s). \tag{1.9}$$

Note that the matrix D differs from the zero matrix if any of $w_{kr}(s)$ have the same degree of the numerator and denominator polynomials.

The *rank* r_i of the ith pole p_i is defined as the rank of the residue matrix K_i, and it is called the *geometric multiplicity* of that pole. Among all poles of the open-loop MIMO system, of special interest are those of rank N, which are also the poles of *all* the nonzero elements $w_{kr}(s)$. In what follows, we shall call such poles the *absolute* poles of the open-loop MIMO system. It is easy to see that if a complex number p_i is an absolute pole of the transfer matrix $W(s)$, then the latter can be represented as

$$W(s) = \frac{1}{s - p_i} W_1(s), \tag{1.10}$$

where the matrix $W_1(p_i)$ is nonsingular [that matrix cannot have entries with poles at the same point p_i owing to the assumption that $w_{kr}(s)$ have no multiple poles].

In a certain sense, it is more complicated to introduce the notion of *zero* of the transfer matrix $W(s)$, as an arbitrary complex number s that brings any of the transfer functions $w_{kr}(s)$ to vanishing, cannot always be regarded as the zero of $W(s)$. We introduce the following two definitions:

1. A complex number z_i is said to be an *absolute* zero of the transfer matrix $W(s)$ if it reduces the latter to the zero matrix.
2. A complex number z_i is said to be a *local* zero of rank k of $W(s)$, if substituting it into $W(s)$ makes the latter singular and of rank $N - k$. The local zero of rank N is, evidently, the absolute zero of $W(s)$.[3]

[3] The notion of MIMO system zero as a complex number z that reduces at $s = z$, the *local rank* of the matrix $W(s)$, is given, for example, in MacFarlane (1975).

Let us discuss these statements. It is clear that if a number z_i is an absolute zero of $W(s)$, then we can express that matrix as

$$W(s) = (s - z_i)W_1(s), \tag{1.11}$$

where $W_1(z_i)$ differs from the zero matrix and has rank N. In other words, the absolute zero must also be the *common zero* of all the nonzero elements $w_{kr}(s)$ of $W(s)$.

We are not quite ready yet for detailed discussion of the notion of the open-loop MIMO system local zero, but, as a simple example, consider the following situation. Let z_i be the common zero of all elements $w_{kr}(s)$ of the kth row or the rth column of $W(s)$, i.e. $w_{kr}(z_i) = 0$ when $k = const$, $r = 1, 2, \ldots, N$, or when $r = const$, $k = 1, 2, \ldots, N$. Then, obviously, if the rank of $W(s)$ is N for almost all values of s [i.e. the *normal* rank of $W(s)$ is N], then the matrix $W(z_i)$ will have at least rank $N - 1$, since, for $s = z_i$, the elements of the kth row or the rth column of $W(z_i)$ are zero. Structurally, the equality to zero of all elements of the kth row of $W(s)$ means that for $s = z_i$, both the direct transfer function $w_{kk}(s)$ of the kth channel and the transfer functions of all cross-connections leading to the kth channel from all the remaining channels become zeros. Analogously, the equality to zero of all elements of the rth column of $W(s)$ means that for $s = z_i$, both the direct transfer function $w_{rr}(s)$ of the rth channel and the transfer functions of all cross-connections leading from the rth channel to all the remaining channels become zeros. This situation may readily be expanded to the case of local zero of rank k. Thus, if, for $s = z_i$, the elements of any k rows or any k columns of $W(s)$ become zeros, then z_i is the local zero of rank k. At this point, however, a natural question arises of whether local zeros of the matrix $W(s)$ exist which reduce its normal rank but do not have the above simple explanation and, if such zeros, then what is their number?

A sufficiently definite answer to that question is obtained in the following subsections, and here we shall try to establish a link between the introduced notions of the open-loop MIMO system poles and zeros, and the determinant of $W(s)$. It is easy to see that both the absolute and local zeros of $W(s)$ make $\det W(s)$ vanishing, since the determinants of the zero matrices as well as of the singular matrices identically equal zero. Besides, from the standard rules of calculating the determinants of matrices (Gantmacher 1964; Bellman 1970), we have that if some elements of $W(s)$ tend to infinity, then the determinant $\det W(s)$ also tends to infinity. In other words, the poles of $W(s)$ are the poles of $\det W(s)$. Based on this, we can represent $\det W(s)$ as a quotient of two polynomials in s:

$$\det W(s) = \frac{Z(s)}{P(s)} \tag{1.12}$$

and call the *zeros* of $W(s)$ the roots of the equation

$$Z(s) = 0 \tag{1.13}$$

and the *poles* of $W(s)$ the roots of the equation

$$P(s) = 0. \tag{1.14}$$

Let us denote the degrees of polynomials $Z(s)$ and $P(s)$ as m and n, respectively, where, in practice, $m \leq n$. We shall call $Z(s)$ the *zeros polynomial* and $P(s)$ the *poles polynomial*, or the *characteristic polynomial*, of the open-loop MIMO system.

Strictly speaking, the given heuristic definition of the zeros and poles of $W(s)$ as the roots of Equations (1.13) and (1.14), which are obtained from Equation (1.12), is only valid if the polynomials $Z(s)$ and $P(s)$ do not have coincident roots.[4] The rigorous determination of the zeros and poles polynomials $Z(s)$ and $P(s)$ can be accomplished via the Smith-McMillan canonical form (Kailath 1980).[5] Besides, there exists another important detail concerning the definition of the MIMO system poles with the help of Equation (1.14), which deserves special attention and will be discussed in Remark 1.6.

Based on the introduced notions of the open-loop MIMO system absolute poles and zeros, we can generally write down for the matrix $W(s)$ the following expression:

$$W(s) = \frac{\alpha(s)}{\beta(s)} W_1(s). \tag{1.15}$$

where $\alpha(s)$ and $\beta(s)$ represent scalar polynomials of order m_0 and n_0, whose roots are absolute zeros and poles of $W(s)$. Here, the matrix $W_1(s)$ in Equation (1.15) must have rank N at the absolute zeros z_i and poles p_i. Substituting Equation (1.15) into Equation (1.12) and taking into account the rule of multiplying the determinant by a scalar number yields

$$\det W(s) = \frac{[\alpha(s)]^N}{[\beta(s)]^N} \det W_1(s) = \frac{[\alpha(s)]^N}{[\beta(s)]^N} \frac{Z_1(s)}{P_1(s)}, \tag{1.16}$$

from which it follows that *all absolute zeros and poles of the open-loop MIMO system have the algebraic multiplicity N* (this is another, equivalent, definition of absolute zeros and poles). Returning to the definition of the open-loop MIMO system poles and summarizing, we can finally formulate the following:

– the absolute pole of W(s) is the pole whose algebraic multiplicity, considering the latter as that of the root of Equation (1.14), and geometric multiplicity, as the rank of the residue matrix in Equation (1.7), coincide and are equal to the number of channels N.

This notion, as well as the notion of absolute zero, will be very useful when studying properties of multivariable root loci and MIMO systems performance (see Section 1.6 and Chapter 2).

1.2.2.2 Closed-loop MIMO systems

Let us first consider, as in the previous part, the SISO system of Figure 1.2. The closed-loop transfer functions of that system with respect to output and error signals are:

$$\Phi(s) = \frac{W(s)}{1 + W(s)} = \frac{M(s)}{D(s) + M(s)} \tag{1.17}$$

$$\Phi_\varepsilon(s) = \frac{1}{1 + W(s)} = \frac{D(s)}{D(s) + M(s)}. \tag{1.18}$$

[4] If polynomials $P(s)$ and $Z(s)$ do have common roots (i.e. coincident *zeros* and *poles*), and they are cancelled in Equation (1.12), then there always exists a danger that the mentioned zeros and poles correspond to *different directions* of the MIMO system (see Subsection 1.2.3).

[5] The Smith-McMillan canonical form is discussed in Remark 1.6.

From here, we can see that the zeros of the open-loop and closed-loop systems [the zeros of the transfer functions $W(s)$ and $\Phi(s)$] coincide, and the zeros of the transfer function with respect to error (the sensitivity function) $\Phi_\varepsilon(s)$ coincide with the poles of the open-loop system. The last fact is very significant when analyzing the accuracy of control systems subjected to slowly changing deterministic signals (see Chapter 2).

The expression $1 + W(s)$ is usually called the *return difference* of the SISO system[6] and the closed-loop poles are zeros of the equation

$$1 + W(s) = \frac{D(s) + M(s)}{D(s)} = 0, \tag{1.19}$$

i.e. are the roots of the equation

$$D(s) + M(s) = 0. \tag{1.20}$$

Now let us proceed to MIMO systems. From the closed-loop transfer matrix with respect to output (complimentary sensitivity function) $\Phi(s)$ [Equation (1.3)]:

$$\Phi(s) = [I + W(s)]^{-1}\, W(s) = W(s)\, [I + W(s)]^{-1}\,, \tag{1.21}$$

we can immediately notice that if a complex number z_i is an *absolute zero* of the open-loop MIMO system, i.e. at $s = z_i$, the matrix $W(s)$ vanishes, then it is also the absolute zero of the closed-loop MIMO system. Further, based on the well known *Silvester's low of degeneracy*,[7] we have that the local zeros of $W(s)$ (which drop the rank of the latter) are also the local zeros of $\Phi(s)$. Hence, quite similarly to the SISO case, the zeros of the open-loop and closed-loop transfer matrices $W(s)$ and $\Phi(s)$ coincide.

The corresponding result for the closed-loop transfer matrix with respect to the error signal (sensitivity function) $\Phi_\varepsilon(s)$ [Equation (1.4)]:

$$\Phi_\varepsilon(s) = [I + W(s)]^{-1} \tag{1.22}$$

may be obtained quite easily if we remember that the inverse matrix in the right-hand part of that expression is found as follows:

$$[I + W(s)]^{-1} = \frac{1}{\det[I + W(s)]} Adj([I + W(s)]), \tag{1.23}$$

where $Adj(\cdot)$ denotes the *adjoint* matrix formed from the initial matrix by replacing all its elements by their cofactors and by subsequent transposing. By analogy with the SISO case, in the MIMO case, the matrix $I + W(s)$ is called the *return difference* matrix, and its determinant is equal to

$$\det[I + W(s)] = \det[I + W(\infty)] \frac{P_{cl}(s)}{P(s)}, \tag{1.24}$$

[6] The origin of the term *return difference*, introduced by Bode, is due to the fact that if we break the closed-loop system with unit negative feedback at an arbitrary point and inject at that point some signal $y(s)$, then the difference between that signal and the signal $-W(s)y(s)$ returning through the feedback loop to the break point is equal to $[1 + W(s)]y(s)$.

[7] That low states that the degeneracy of the product of two matrices is at least as great as the degeneracy of either matrix and, at most, as great as the sum of degeneracies of the matrices (Derusso *et al.* 1965).

where $P(s)$ and $P_{cl}(s)$ are the *characteristic polynomials* of the open-loop and closed-loop MIMO system, respectively (Postlethwaite and MacFarlane 1979). The constant matrix $W(\infty)$ in Equation (1.24) coincides with D [Equations (1.7) and (1.9)], and differs from the zero matrix under indicated conditions there.[8] Substituting Equation (1.24) into Equation (1.23) and then into Equation (1.4), we obtain for $\Phi_\varepsilon(s)$

$$\Phi_\varepsilon(s) = \frac{P(s)}{\det[I + W(\infty)]P_{cl}(s)} Adj([I + W(s)]). \tag{1.25}$$

From Equation (1.25) it ensues that the open-loop MIMO system poles, i.e. the roots of the characteristic polynomial $P(s)$, are, as in the SISO case, the zeros of the closed-loop transfer matrix $\Phi_\varepsilon(s)$, where the absolute poles of $W(s)$ become the absolute zeros of $\Phi_\varepsilon(s)$.[9]

Thus, we have ascertained that zeros of open-loop and closed-loop MIMO systems are related just as zeros of open-loop and closed-loop SISO systems. We can state once more that SISO systems are a special case of MIMO systems, when $N = 1$.

1.2.3 Spectral representation of transfer matrices: characteristic transfer functions and canonical basis

The significant notions of poles and zeros of linear square MIMO systems discussed in the preceding sections form the necessary basis on which the frequency-domain multivariable control theory is built. At the same time, we have not yet succeeded in understanding the geometrical and structural properties of multivariable control systems and in establishing links between frequency-domain representations of SISO and MIMO systems. Now, we proceed to solving that task. Note also that, speaking about structural properties of MIMO systems, we shall mean both the ‘internal’ features ensuing from the geometrical properties of linear algebraic operators acting in finite-dimensional Hilbert spaces and the ‘external’ features depending, first of all, on the technical characteristics of MIMO systems (though, ultimately, all these features and characteristics are inseparably linked).

1.2.3.1 Open-loop MIMO systems

Equations (1.1)–(1.4) describe the behaviour of the closed-loop MIMO system of Figure 1.1 with respect to a *natural* coordinate system (natural basis) formed by a set of N *orthonormal* unit vectors e_i ($i = 1, 2, \ldots, N$), where the kth component of e_k is unity and all other components are zero. The $\varphi_i(s)$, $f_i(s)$ and $\varepsilon_i(s)$ coordinates of the vectors $\varphi(s)$, $f(s)$ and $\varepsilon(s)$ with respect to that basis are just the Laplace transforms of the actual input, output and error signals, respectively, in the ith separate channel of the system. Formally, the transfer matrices $W(s)$, $\Phi(s)$ and $\Phi_\varepsilon(s)$ may be regarded as some linear operators mapping an N-dimensional complex space $\mathbb{C}^N$ of the input vectors $\varphi(s)$ into the corresponding spaces of the output or error vectors $f(s)$ or

[8] Later on, we shall omit the case of the *singular* matrix $[I + W(\infty)]$, which has no practical significance.

[9] Strictly speaking, inherent only in the MIMO systems, situations in which some poles of the open-loop system do not change after closing the feedback loop and coincide with the corresponding poles of the closed-loop system are not included here. In such cases, the coincident roots of $P_{cl}(s)$ and $P(s)$ in Equation (1.25) must be cancelled, and they should not be regarded as zeros of the transfer function matrix $\Phi_\varepsilon(s)$. That issue will be considered in more detail when studying the properties of multivariable root loci in Section 1.5 and Remark 1.6.

$\varepsilon(s)$. This suggests using the mathematical tools of the theory of linear algebraic operators and functional analysis for the study of linear MIMO systems (Strang 1976; Vulich 1967; Danford and Schwarts 1962). It is known that internal, structural properties of linear algebraic operators exhibit much more saliently, not in the natural, but in some other, specially chosen coordinate systems (Porter 1966; Derusso *et al*. 1965; Gantmacher 1964). In particular, that refers to the so-called *canonical* basis formed by the *normalized* (i.e. having unit length) *characteristic vectors* of the matrix operator, with respect to which the latter has diagonal form with the *characteristic values* (*eigenvalues*) on the principal diagonal. Since the $w_{kr}(s)$ elements of the open-loop transfer matrix $W(s)$ are scalar proper rational functions in complex variable s, the eigenvalues $q_i(s)$ of $W(s)$, i.e. the roots of the equation

$$\det[qI - W(s)] = 0 \tag{1.26}$$

are also functions of variable s. These complex functions $q_i(s)$ are called *characteristic transfer functions* (CTF) of the open-loop MIMO system (MacFarlane and Belletrutti 1970; Postlethwaite and MacFarlane 1979).[10]

If we assume, for the sake of simplicity, that all CTFs $q_i(s)$ $(i = 1, 2, \ldots, N)$ are *distinct*,[11] then the corresponding normalized eigenvectors $c_i(s)$ $(|c_i(s)| = 1)$ of $W(s)$ are *linearly independent*, and constitute the basis of the N-dimensional complex space $\mathbb{C}^N$ (recall that $\varphi(s)$, $f(s)$ and $\varepsilon(s)$ belong to that space). We call that basis the *canonical basis* of the open-loop MIMO system.

Having formed from $c_i(s)$, the *modal matrix* $C(s) = [c_1(s)\, c_2(s) \ldots c_N(s)]$ [the latter is nonsingular owing to the assumption of linear independency of c_i(s)], we can represent the matrix $W(s)$ by the *similarity transformation* in the following form:

$$W(s) = C(s)\; diag\{q_i(s)\} C^{-1}(s), \tag{1.27}$$

where $diag\{q_i(s)\}$ denotes the diagonal matrix with the elements $q_i(s)$ on the principal diagonal.

Before proceeding, let us consider briefly some mathematical notions and pre-requisites which are inherent in complex Hilbert spaces and which are necessary for the further exposition. The first of these notions is the *scalar* (or *inner*) product, which is written for two arbitrary N-dimensional column vectors x and y with components x_i and y_i as $<x, y>$, and is defined as

$$<x, y> = \tilde{x}^T y = y^T \tilde{x} = \sum_{i=1}^{N} \tilde{x}_i y_i, \tag{1.28}$$

where ^{T} is the symbol of transposition, and the wavy line above denotes the complex conjugation (Derusso *et al.* 1965). Based on Equation (1.28), we can represent the Euclidean norm (length) of the vector x as

$$|x| = \sqrt{<x, x>} \tag{1.29}$$

[10] We give here rather a simplified interpretation of characteristic transfer functions, which is quite enough for engineering applications. In essence, the reader may assume that $q_i(s)$ are found for fixed values $s = const$, for which *W(s)* reduces to a usual numerical matrix with complex elements (see also Remark 1.2).

[11] When accomplishing an engineering design, this may always be achieved by arbitrary small perturbations of the MIMO system parameters within the accuracy of their specifications.

and, for *real* vectors x and y, we can define the angle θ between them by the equation

$$< x, y >= |x|\,|y| \cos\theta. \tag{1.30}$$

In particular, two vectors x and y are said to be *orthogonal* if $< x, y > = 0$ and *collinear* if $< x, y > = \pm |x|\,|y|$. Note that the complex conjugation of components of x in Equation (1.28) is needed to provide the coincidence of the usual norm and the norm specified by scalar product for complex-valued vectors [Equation (1.29)]. In the mathematical and technical literature, one can encounter a definition of scalar product in which the complex conjugate components of the second, instead of the first, vector in Equation (1.28) are taken. This should not introduce ambiguity, as, in principle, it is an equivalent definition of the scalar product. For our aims, it is more convenient to introduce the scalar product in the form of Equation (1.28), and the reader should remember this in what follows. Now, as the product $< x, y >$ in complex space $\mathbb{C}^N$ is not always real-valued, in general, we are not able to introduce the notion of an angle between two vectors x and y. Nevertheless, the conditions $< x, y > = 0$ and $< x, y > = \pm |x|\,|y|$ determine, as before, orthogonality and collinearity of any two vectors, and these two notions turn out to be as useful as in the real space.

Along with the notion of a basis in $\mathbb{C}^N$ (as such a basis may serve any set of N linearly independent vectors in $\mathbb{C}^N$), we also need the notion of a *dual* (or *reciprocal*) basis, which is defined as follows. Let the set of N normalized vectors c_i constitute a basis of $\mathbb{C}^N$, i.e. any vector y in $\mathbb{C}^N$ may be represented as a linear combination of the form

$$y = \sum_{i=1}^{N} \alpha_i c_i, \tag{1.31}$$

where the α_i scalars are the coordinates of y in the basis $\{c_1, c_2, \ldots, c_N\}$. Let us define a set $\{c_1^+, c_2^+, \ldots, c_N^+\}$ of N vectors such that

$$< c_i, c_j^+ > = \delta_{ij} \quad (i, j = 1, 2, \ldots, N), \tag{1.32}$$

where δ_{ij} is the symbol of Kronecker ($\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$) (Derusso *et al*. 1965). It is easy to show that the vectors $c_1^+, c_2^+, \ldots, c_N^+$ in Equation (1.32) are linearly independent and also constitute a basis in $\mathbb{C}^N$. That basis is said to be dual to the basis $\{c_1, c_2, \ldots, c_N\}$. From the way of finding the dual basis [Equation (1.32)], where the axis c_i^+ dual to c_i must be orthogonal to all other axes c_k ($k \neq i$), it follows that orthonormal bases always coincide with their dual bases. The major benefit of the dual basis is that the α_i coordinates of y in the basis $\{c_1, c_2, \ldots, c_N\}$ in Equation (1.31) are expressed as[12]

$$\alpha_i = < c_i^+, y > \tag{1.33}$$

and this feature of the dual basis proves to be extremely useful when studying geometrical properties of linear MIMO systems. Note that if we denote by C the matrix composed of the normalized vectors $\{c_1, c_2, \ldots, c_N\}$, the rows of the inverse matrix C^{-1} are complex conjugate to the reciprocal basis axes (given the initial basis, this is, in fact, a way to calculate numerically the dual basis) (Derusso *et al*. 1965; Gasparyan 1976).

[12] We suggest that the reader checks this as an exercise.

The last notion, needed in the future, is based on the so-called *dyadic* designations (Porter 1966). To proceed, we have to impart some additional shading to what we mean by the symbols $<$ and $>$ in Equation (1.28). In the following, we shall formally attribute the symbol $>$ to a *column* vector, i.e. the designation $y>$ we shall consider identical to the designation y. Similarly, the symbol $<$ will be attributed to a complex conjugate *row* vector, i.e. the designation $< x$ will be regarded as equivalent to $\tilde{x}^T$. Then, preserving the usual definition of scalar product [Equation (1.28)], we can impart a certain sense to the expression $y >< x$. Now, in accordance with the conventional rules, multiplying the column vector $y>$ by the complex conjugate row vector $<x$ yields a square matrix S:

$$S = y >< x. \tag{1.34}$$

Multiplying that matrix by an arbitrary vector z (or by $z>$, which is the same) yields

$$v = Sz = (y><x)z> = y> \underbrace{<x, z>}_{\alpha} = \alpha y. \tag{1.35}$$

From Equation (1.35), it is clear that the range of values of S is one-dimensional, and is a linear subspace of $\mathbb{C}^N$ generated (spanned) by the vector y. In other words, the S matrix transforms any vector z in $\mathbb{C}^N$ to a vector which is always directed along y. Also, those vectors z that are perpendicular to x are transformed to a zero vector (because, in that case, the scalar $\alpha = <x, z>$ equals zero). The matrix S represented in Equation (1.34) is said to be *a dyad of rank one*.

Given a normalized basis $\{c_1, c_2, \ldots, c_N\}$ of $\mathbb{C}^N$ and a set $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ of N linearly independent (but not necessarily normalized) vectors, the matrix S represented as

$$S = \sum_{i=1}^{N} c_i >< \gamma_i, \tag{1.36}$$

i.e. as a sum of N dyads of rank one, is said to be *a dyad of rank N*. Its range of values is, obviously, the whole space $\mathbb{C}^N$. Among all possible dyads composed via the basis $\{c_1, c_2, \ldots, c_N\}$ [Equation (1.36)], the most significant is the dyad for which the vectors $\gamma_1, \gamma_2, \ldots, \gamma_N$ coincide with the dual basis $c_1^+, c_2^+, \ldots, c_N^+$. It is easy to show, in that case, that the *identity operator I* in $\mathbb{C}^N$ may be represented through the dyads $c_i >< c_i^+$ of rank one as a sum:

$$I = \sum_{i=1}^{N} c_i >< c_i^+. \tag{1.37}$$

Further, which is far more important, if a square matrix A is a matrix of *simple structure*, i.e. possesses the full set of N linearly independent eigenvectors c_i, then it can be represented not only by the similarity transformation, but also in the form of a *spectral decomposition*, as a sum of N dyads:

$$A = \sum_{i=1}^{N} c_i > \lambda_i < c_i^+, \tag{1.38}$$

where λ_i are the eigenvalues of the matrix A.

Now, we have proceeded enough in the understanding of the geometrical structure of linear operators in finite-dimensional Hilbert spaces and can use the corresponding apparatus for investigating properties of the open-loop MIMO system. Based on Equation (1.38) and remembering that all CTFs $q_i(s)$ are assumed distinct,[13] we can write down the open-loop transfer matrix $W(s)$ of the MIMO system in the form of the spectral decomposition:

$$W(s) = \sum_{i=1}^{N} c_i(s) > q_i(s) < c_i^+(s). \tag{1.39}$$

The spectral representation in Equation (1.39) permits a visual geometric interpretation of the internal structure of the open-loop and, as will be seen later, closed-loop MIMO systems. The CTFs $q_i(s)$ may be regarded as transfer functions of some abstract, fictitious one-dimensional 'characteristic' systems, in which each characteristic system 'acts' in $\mathbb{C}^N$ along one axis $c_i(s)$ of the open-loop MIMO system canonical basis. For any s, the vector $f(s)$ is represented, based on Equation (1.39) and Figure 1.1, as

$$f(s) = W(s)\varepsilon(s) = \left[\sum_{i=1}^{N} c_i(s) > q_i(s) < c_i^+(s)\right]\varepsilon(s) = \sum_{i=1}^{N}[q_i(s)\beta_i(s)]c_i(s), \tag{1.40}$$

where

$$\beta_i(s) = < c_i^+(s), \varepsilon(s) >, \quad i = 1, 2, \ldots, N \tag{1.41}$$

are the projections of $\varepsilon(s)$ on $c_i(s)$, i.e. as a linear combination of 'responses' of the open-loop MIMO system along the canonical basis axes.

On the other hand, having Equation (1.37) for the identity operator I in $\mathbb{C}^N$, we are able to write down the expansion of the output vector $f(s)$ along the canonical axes in the form

$$f(s) = If(s) = \left[\sum_{i=1}^{N} c_i >< c_i^+\right] f(s) = \sum_{i=1}^{N} \alpha_i(s)c_i(s), \tag{1.42}$$

where

$$\alpha_i(s) = < c_i^+(s), f(s) >, \quad i = 1, 2, \ldots, N \tag{1.43}$$

are the coordinates of $f(s)$ along the axes $c_i(s)$. Comparing Equations (1.40)–(1.43) yields

$$\alpha_i(s) = q_i(s)\beta_i(s), \quad i = 1, 2, \ldots, N, \tag{1.44}$$

from which we come to the conclusion that the projection $\alpha_i(s)$ of the output vector $f(s)$ on any canonical basis axis $c_i(s)$ is equal to the corresponding projection $\beta_i(s)$ of the error vector $\varepsilon(s)$ multiplied by the CTF $q_i(s)$ of the one-dimensional characteristic system acting along that very axis $c_i(s)$. If the error vector $\varepsilon(s)$ is directed along one, say the kth, canonical basis axis,

[13] The matrices of the *simple structure* can always be brought to diagonal form via the similarity transformation. The assumption of distinct CTFs guarantees this possibility, but is not necessary. For example, *normal* matrices with multiple eigenvalues can always be brought to diagonal form. More strictly, a square matrix can be brought to diagonal form in a certain basis if the *algebraic* and *geometric* multiplicities of all eigenvalues coincide.

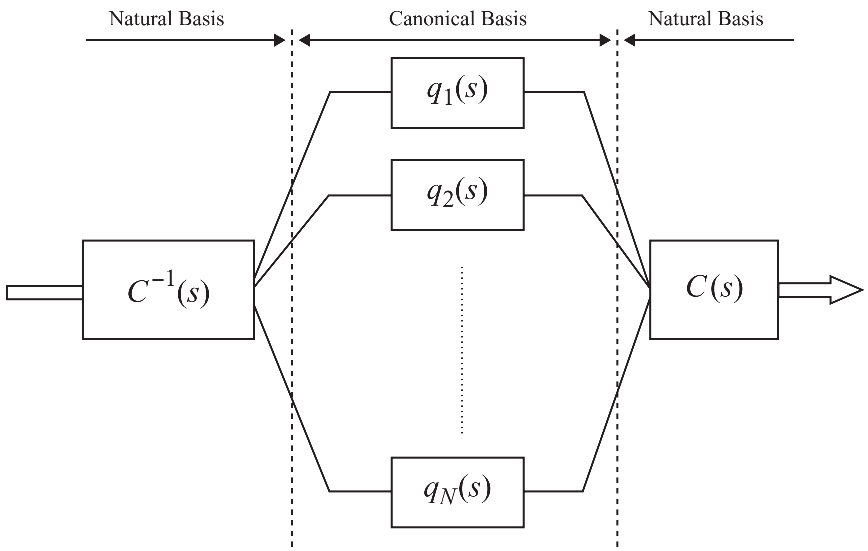

Figure 1.3 Representation of the open-loop MIMO system via the similarity transformation.

i.e. if $\varepsilon(s) = \beta_k(s)c_k(s)$, then, based on the properties of the dual basis [Equation (1.32)], all $\beta_i(s)$ $(i \neq k)$ in Equation (1.41) are identically equal to zero and the output vector

$$f(s) = q_k(s)\beta_k(s)c_k(s) \tag{1.45}$$

is also directed along the same axis $c_k(s)$. In other words, the axes of the open-loop MIMO system canonical basis determine those N directions in complex space $\mathbb{C}^N$ along which the system acts as a certain fictitious one-dimensional system, and only 'stretches' or 'squeezes' the input signals.[14] The block diagrams in Figures 1.3 and 1.4, which represent in graphical form Equations (1.27) and (1.39)–(1.44), schematically illustrate all these statements.

It must be noted that both forms of the canonical representation of transfer matrices – by means of similarity transformation [Equation (1.27)] and dyadic notation [Equation (1.39)] – are equivalent, and the reader should always remember this. The first representation seems to be useful and convenient due to its compactness for some general derivations and proofs, and the second one, possibly, gives the reader a better intuitive or physical feeling for the geometrical structure and behaviour of the MIMO system.

We already have enough knowledge to gain a deeper insight into the notions of poles and zeros of MIMO control systems. Based on Equation (1.27), the determinant of the open-loop transfer matrix $W(s)$ is equal to

$$\begin{aligned}\det W(s) &= \frac{Z(s)}{P(s)} = \det\left[C(s)diag\{q_i(s)\}C^{-1}(s)\right]\\ &= \underbrace{\det C(s)\det C^{-1}(s)}_{I}\det[diag\{q_i(s)\}] = \prod_{i=1}^{N} q_i(s),\end{aligned} \tag{1.46}$$

[14] It is worth noting here that we have not yet spoken about such important geometrical properties of the canonical basis as *orthogonality* or *skewness* of axes; as we shall see in the next chapters, it is difficult to overestimate, from the MIMO system performance viewpoint, the significance of these features.

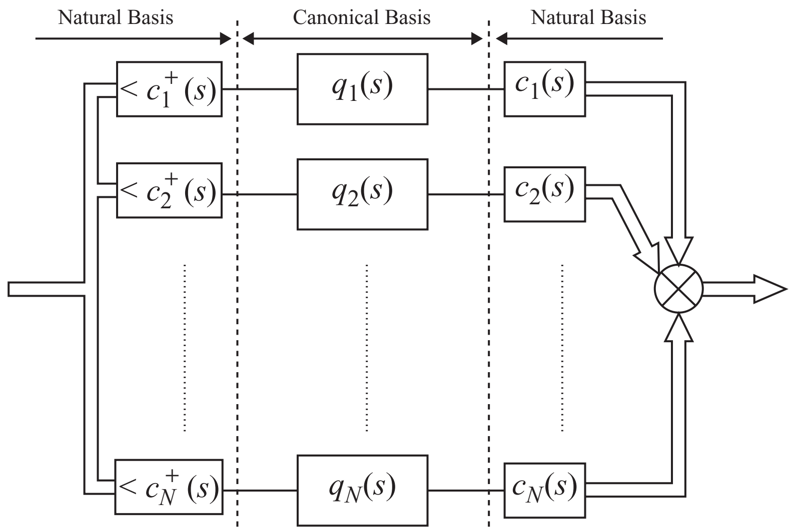

Figure 1.4 Spectral representation of the open-loop MIMO system by means of dyads.

where it is taken into account that the product of determinants of a matrix and its inverse is equal to unity, and the determinant of a diagonal matrix is equal to the product of diagonal elements.

Thus, we have obtained that poles and zeros of the open-loop MIMO system coincide with poles and zeros of all open-loop characteristic systems associated with the MIMO system. Moreover, starting from Equation (1.15) and recalling that if a matrix is multiplied by a scalar, then all eigenvalues of that matrix are multiplied by the same scalar, we come to the important conclusion that the *absolute zeros and poles of the open-loop MIMO system are common poles and zeros of all CTFs* $q_i(s)$. Further, if a complex number z is a local zero of rank k $(1 \leq k \leq N)$, then that number must be the zero of any k CTFs $q_i(s)$. It is easy to ascertain that at the local zero of rank k, the rank of $W(s)$ is equal to $N - k$, since the rank of each dyad in spectral representation [Equation (1.39)] is unity and each dyad projects complex space $\mathbb{C}^N$ onto one of the canonical basis axes. Therefore, if k terms in Equation (1.39) are zero, then the remaining $N - k$ terms define $N - k$ basis axes, and the matrix $W(s)|_{s=z}$ itself is of rank $N - k$. Eventually, we realize that evaluating zeros and poles of the open-loop MIMO system with the help of Equations (1.12)–(1.14), i.e. based on the computation of the determinant det $W(s)$, may bring about the incorrect cancellation of poles and zeros of characteristic systems which, in fact, act along *different* axes of the canonical basis.

In conclusion, note that, for many classes of MIMO systems widespread in practice, the CTFs $q_i(s)$ can be expressed as a quotient of two polynomials $M_i(s)$ and $D_i(s)$, i.e. in the form

$$q_i(s) = \frac{M_i(s)}{D_i(s)}, \quad i = 1, 2, \ldots, N, \tag{1.47}$$

where $M_i(s)$ and $D_i(s)$ are defined analytically for any number N of separate channels. This fact plays an extremely important role, as it allows us to rigorously select zeros and poles [the roots of the equations $M_i(s) = 0$ and $D_i(s) = 0$] of each *separate* one-dimensional characteristic system. We accentuate here that Equation (1.46) indicates only that the roots of zeros and poles polynomials $Z(s)$ and $P(s)$ taken together are equal to zeros and poles of the CTFs $q_i(s)$. Generally, we do not have any specific information (except for the conclusion concerning absolute zeros and poles) about the distribution of the MIMO system poles and zeros among individual characteristic systems. As for the above-mentioned prevailing classes of MIMO systems, and to those classes belong the so-called uniform, circulant, anticirculant and some other systems, we can immediately write down for their zeros and poles polynomials

$$Z(s) = \prod_{i=1}^{N} M_i(s) = 0, \quad P(s) = \prod_{i=1}^{N} D_i(s) = 0, \tag{1.48}$$

which, as will be seen in the following, considerably simplifies the analysis of such systems.

Although representation of the CTFs $q_i(s)$ as a quotient of two polynomials [Equation (1.47)] in the general case is, unfortunately, impossible, sometimes, we shall formally suppose that the CTFs $q_i(s)$ are of the form in Equation (1.47). It will allow us, without encumbering the presentation, to make a number of statements indicating a close relationship between the stability and performance of SISO and MIMO systems.

1.2.3.2 Closed-loop MIMO systems

In this section, we bring to a logical completion the task of describing linear MIMO control systems by means of the CTFs. Substituting the representation of the open-loop transfer matrix $W(s)$ via the similarity transformation [Equation (1.27)] into the complementary sensitivity function matrix $\Phi(s)$ [Equation (1.3)] and the sensitivity function matrix $\Phi_\varepsilon(s)$ [Equation (1.4)], after a number of simple transformations, yields

$$\Phi(s) = C(s) diag \left\{ \frac{q_i(s)}{1 + q_i(s)} \right\} C^{-1}(s), \quad \Phi_\varepsilon(s) = C(s) diag \left\{ \frac{1}{1 + q_i(s)} \right\} C^{-1}(s). \tag{1.49}$$

Analogously, substituting the spectral decomposition in Equation (1.39) into Equations (1.3) and (1.4) gives

$$\Phi(s) = \sum_{i=1}^{N} c_i(s) > \frac{q_i(s)}{1 + q_i(s)} < c_i^+(s), \quad \Phi_\varepsilon(s) = \sum_{i=1}^{N} c_i(s) > \frac{1}{1 + q_i(s)} < c_i^+(s). \tag{1.50}$$

The examination of these expressions enables us to draw the important conclusion that the modal matrix $C(s)$ and, consequently, the canonical basis of the closed-loop MIMO system coincide with the modal matrix and canonical basis of the open-loop system. In other words, introducing the unit negative feedback does not change the canonical basis of the system. Moreover, if $q_i(s) (i = 1, 2, \ldots, N)$ are the CTFs of the open-loop MIMO system, then the corresponding CTFs of the closed-loop MIMO system with respect to the output and error

$$\Phi_i(s) = \frac{q_i(s)}{1 + q_i(s)}, \quad \Phi_{\varepsilon i}(s) = \frac{1}{1 + q_i(s)}, \quad i = 1, 2, \ldots, N \tag{1.51}$$

are related to $q_i(s)$ by the very same relationships as the usual transfer functions of open- and closed-loop SISO feedback systems.

As for the geometrical interpretation of the closed-loop MIMO system internal structure, it is entirely inherited, based on Equations (1.50) and (1.39), from the open-loop system. In particular, the output $f(s)$ and error $\varepsilon(s)$ vectors of the closed-loop system can be represented as linear combinations of the system 'responses' along the canonical basis axes:

$$f(s) = \sum_{i=1}^{N} \left[\Phi_i(s) < c_i^+(s), \varphi(s) >\right] c_i(s), \quad \varepsilon(s) = \sum_{i=1}^{N} \left[\Phi_{\varepsilon i}(s) < c_i^+(s), \varphi(s) >\right] c_i(s) \tag{1.52}$$

where the CTFs $\Phi_i(s)$ and $\Phi_{\varepsilon i}(s)$ are defined by Equation (1.51). Analogously, if the input vector $\varphi(s)$ is directed along one of the canonical basis axes, only the corresponding closed-loop characteristic system will take part in the MIMO system response, and the output and error vectors will be directed along the same axis. All this is illustrated schematically in Figures 1.5 and 1.6, and the reader can compare these diagrams with those of the open-loop MIMO system in Figures 1.3 and 1.4.

Before proceeding to the stability analysis of the general MIMO system of Figure 1.1, let us summarize some results. We have ascertained, based on the theory of linear algebraic operators, that a set of N one-dimensional so-called characteristic systems may be associated with an N-dimensional linear MIMO system.

Each of the characteristic systems acts in $\mathbb{C}^N$ along one specified direction – the canonical basis axis – and, in $\mathbb{C}^N$, in all, there are, assuming no multiple CTFs, just N such linearly independent directions. If we formally accept that the CTFs $q_i(s)$ may be represented as a quotient of two polynomials in the form of Equation (1.47), and this is always possible for a great number of practical multivariable systems, then, instead of Equation (1.49), we can write:

$$\Phi(s) = C(s) diag \left\{ \frac{M_i(s)}{D_i(s) + M_i(s)} \right\} C^{-1}(s), \quad \Phi_\varepsilon(s) = C(s) diag \left\{ \frac{D_i(s)}{D_i(s) + M_i(s)} \right\} C^{-1}(s), \tag{1.53}$$

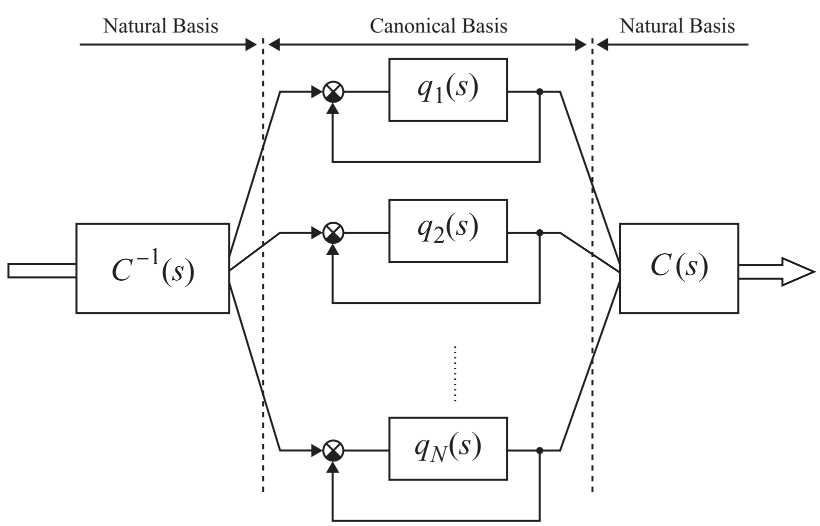

Figure 1.5 Representation of the closed-loop MIMO system via the similarity transformation.

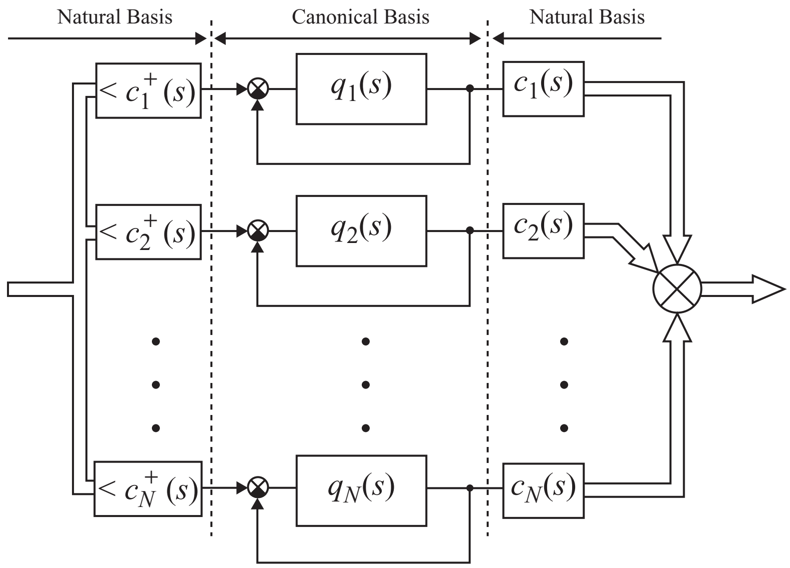

Figure 1.6 Spectral representation of the closed-loop MIMO system by means of dyads.

from which it immediately ensues that the zeros of the complementary sensitivity function matrix $\Phi(s)$ coincide with *zeros* of the open-loop characteristic systems, i.e. with the roots of the equations $M_i(s) = 0$, and the zeros of the sensitivity function matrix $\Phi_\varepsilon(s)$ coincide with *poles* of the open-loop characteristic systems, i.e. with the roots of the equations $D_i(s) = 0$, where, in both cases, $i = 1, 2, \ldots, N$. All this indicates a deep internal relationship between dynamical and other properties of a real MIMO system and the corresponding properties of N fictitious isolated SISO characteristic systems associated with the given MIMO system. We thereby have a necessary basis for extending the fundamental principles of the classical control theory to the multivariable case, and the rest of the textbook is devoted to that task. As we shall see later, many results of the classical theory concerning linear and nonlinear SISO feedback systems may indeed directly be generalized to the MIMO case (i.e. to N-channel MIMO systems) in such a manner that if we assume $N = 1$, then the 'multidimensional' methods simply coincide with their conventional 'one-dimensional' counterparts. At the same time, in the multivariable case, the *structural features* of MIMO systems are of considerable interest and, in a number of cases, the mentioned features allow a reduction in the analysis and design of a MIMO system to the analysis and design of a single SISO characteristic system, regardless of the actual number N of separate channels.

1.2.4 Stability analysis of general MIMO systems

Earlier, considering the properties of the MIMO system zeros and poles, we used Equation (1.24), which is basic for analyzing the stability of the closed-loop MIMO system. Let us write that equation once again:

$$\det[I + W(s)] = \det[I + W(\infty)]\frac{P_{cl}(s)}{P(s)}, \tag{1.54}$$

where $P(s)$ and $P_{cl}(s)$ are the characteristic polynomials of the open-loop and closed-loop MIMO system. As follows from Equation (1.54), the poles of the closed-loop MIMO system, i.e. the roots of the characteristic polynomial $P_{cl}(s)$, coincide with zeros of the determinant of the return difference matrix $I + W(s)$. Therefore, for the stability of the linear MIMO system in Figure 1.1, which, recall, is assumed controllable and observable, it is necessary and sufficient that the roots of the equation

$$\det[I + W(s)] = 0 \tag{1.55}$$

lie in the open left half-plane of the complex plane (Postlethwaite and MacFarlane 1979). Further, for simplicity, we shall call Equation (1.55) the characteristic equation of the closed-loop MIMO system. Using the canonical representation of the transfer matrix $W(s)$ via similarity transformation [Equation (1.27)], Equation (1.55) may be reduced to the following form:

$$\det\left[C(s)diag\{1 + q_i(s)\}C^{-1}(s)\right] = 0, \tag{1.56}$$

which immediately yields

$$\det[I + W(s)] = \underbrace{\det C(s) \det C^{-1}(s)}_{I} \det[diag\{1 + q_i(s)\}] = \prod_{i=1}^{N} [1 + q_i(s)] = 0. \tag{1.57}$$

Assuming that the CTFs $q_i(s)$ may be represented as a quotient of two polynomials $M_i(s)$ and $D_i(s)$ [Equation (1.47)], instead of Equation (1.57), we can also write

$$\det[I + W(s)] = \prod_{i=1}^{N} [1 + q_i(s)] = \frac{\prod_{i=1}^{N} [D_i(s) + M_i(s)]}{\prod_{i=1}^{N} D_i(s)} = 0. \tag{1.58}$$

Equations (1.57) and (1.58) show that the characteristic equation of the N-dimensional closed-loop MIMO system splits into N corresponding equations of the one-dimensional characteristic systems

$$1 + q_i(s) = 0, \quad \text{or} \quad D_i(s) + M_i(s) = 0, \quad i = 1, 2, \ldots, N. \tag{1.59}$$

This means that the complex plane of the closed-loop MIMO system roots can be regarded as superpositions of N complex planes of the closed-loop characteristic systems roots.[15] Hence, for the stability of a linear MIMO system, it is necessary and sufficient that all closed-loop characteristic systems be stable. Even if only one of the characteristic systems is on the stability boundary or is unstable, the corresponding equation in Equation (1.59) will have roots on the imaginary axis or in the right half-plane, and then, owing to Equations (1.57) and (1.58), just the same roots will have the closed-loop MIMO system. And, vice versa, in the case of an unstable closed-loop MIMO system, there always exists such an unstable characteristic system

[15] More rigorously, in general, these roots lie on N different sheets of the Riemann surface (Postlethwaite and MacFarlane 1979).

whose right half-plane roots coincide with the corresponding roots of Equation (1.55). So, we can state that the described approach enables replacing the stability analysis of an *N*-channel linear MIMO system by the stability analysis of *N* SISO characteristic systems, or, in other words, it reduces an *N*-dimensional task to *N* one-dimensional tasks.

At this point, before proceeding with our study of the linear MIMO system stability issues and running, to a certain extent, ahead, note a specific and very significant feature of characteristic systems, which has no analogues in the classical control theory. As the CTFs $q_i(s)$ are transfer functions of abstract SISO systems with, in general, complex parameters, the location of zeros and poles of each single open- and closed-loop characteristic system is not necessarily symmetric with respect to the real axis of the complex plane, which takes place in the case of usual SISO systems. At the same time, the entire set of the open- and closed-loop MIMO system zeros and poles must be symmetric with respect to the real axis, as all coefficients of the entries of $W(s)$, $\Phi(s)$ and $\Phi_\varepsilon(s)$ are real-valued numbers. This means that for any characteristic system with a nonsymmetrical zeros and poles distribution, there always exists a 'complex conjugate' characteristic system whose distribution of zeros and poles can be obtained from the previous by mirror mapping with respect to the real axis.

Example 1.1 When introducing any theory, the well chosen examples are very important, and it is difficult to overestimate the significance of such examples. According to a prominent mathematician, (note, the mathematician!) I. M. Gelfand, 'the theories come and go, but the examples stay'. We shall try to explain various theoretical concepts with the help of different examples, but some of the central examples in the textbook will be the indirect guidance (tracking) systems of orbital astronomic telescopes. This is not only because the author worked for many years in the area of development and manufacture of such systems. First of all, this is because the diversity and 'flexibility' of structures and schemes of indirect guidance systems allow us to visually and readily illustrate, by means of a few examples taken from the practice systems, the very miscellaneous topics in the theory of linear and nonlinear MIMO control systems. The kinematic scheme of an indirect guidance system of the telescope (i.e. of the system in which guiding is accomplished with the help of two reference stars[16]) is depicted in Figure 1.7. In such systems, the sensitivity axes of the measuring devices (in this case, the stellar sensors) do not coincide with the telescope tracking axes (the axes of the gimbal mount). As a result, in the system, there appear rigid cross-connections among the separate channels (sometimes called *kinematical*), which depend on angles between the explored and reference stars and on some construction factors. These cross-connections are constant for the given mutual location of stars and they change only on passing to another explored star, for which a new pair of suitable reference stars is usually chosen.

The expanded block diagram of that system, for the case of tracking with respect to the telescope's two transverse axes, is given in Figure 1.8.[17] The matrix R of the rigid cross-connections in Figure 1.8 has the form

$$R = \begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\sin\alpha_2 & \cos\alpha_2 \end{pmatrix}, \tag{1.60}$$

[16] The *reference stars* are stars which are noticeably brighter than the surrounding stars and therefore may be used for tracking purposes.

[17] If, for solving scientific (in particular, astronomical) tasks, the tracking around the telescope optical axis is also needed, in the system, the third channel also exists, generally cross-connected with the two previous channels (Gasparyan 1986).

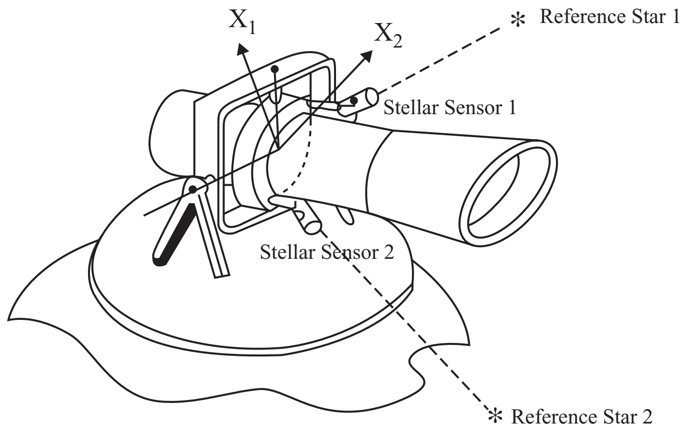

Figure 1.7 Kinematic scheme of the indirect guidance system of the space telescope.

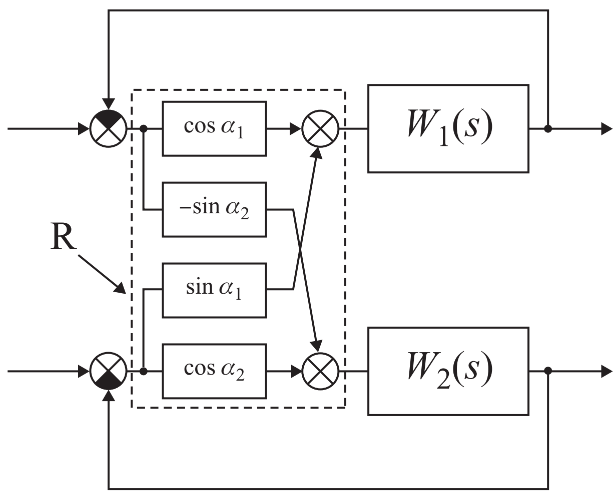

Figure 1.8 Expanded block diagram of the indirect guidance system of Figure 1.7.

where constant angles α_1 and α_2 depend on the above-mentioned factors; $W_1(s)$ and $W_2(s)$ are the transfer functions of separate channels of the system.

Note that for $\alpha_1 = 0$, $\alpha_2 = 0$, which geometrically corresponds to using the explored star as a reference one, the matrix R [Equation (1.60)] becomes the unit matrix I, and the MIMO system of Figure 1.8 splits into two *independent* channels. The above statements about the distribution of the MIMO system zeros and poles are illustrated by the examples of a common SISO system and the two-dimensional system of Figure 1.8 in Figure 1.9. In Figure 1.9, the crosses and circles denote poles and zeros of the open-loop systems, and black squares denote poles of the closed-loop systems. The transfer function of the open-loop SISO system is taken as

$$W(s) = \frac{50(s+3)}{s(s+5)(s+10)}. \tag{1.61}$$

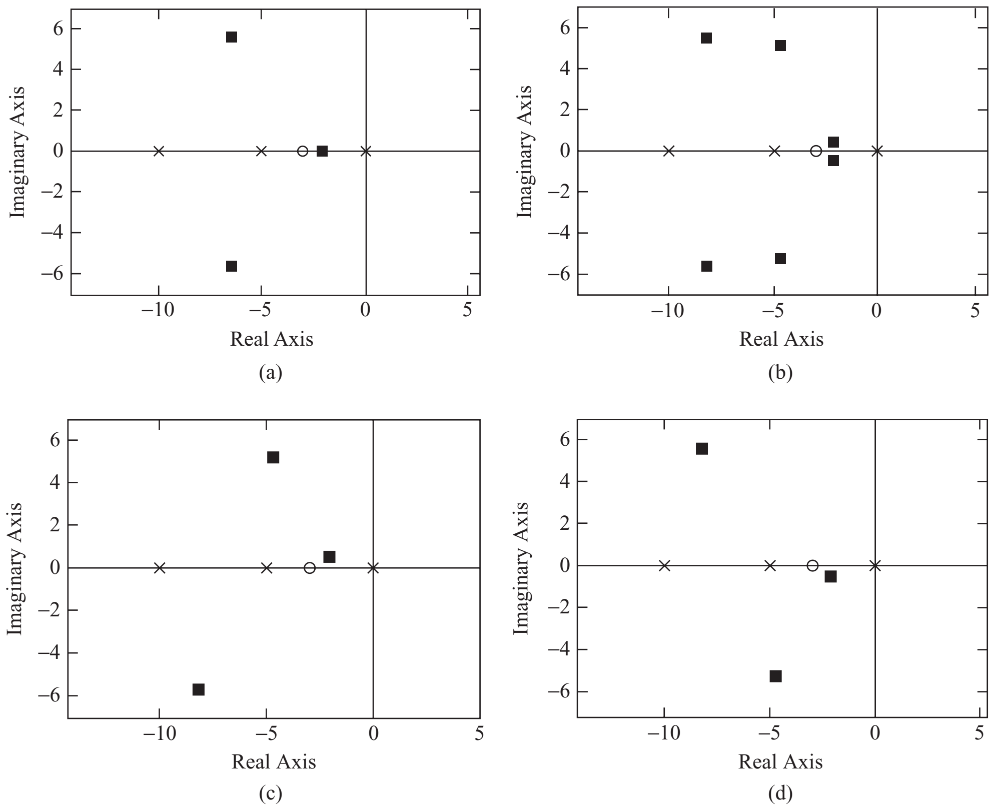

Figure 1.9 Typical distribution of zeros and poles of SISO and MIMO control systems. (a) One-dimensional system ($N = 1$); (b) two-dimensional system ($N = 2$); (c) $N = 2$, the first characteristic system; (d) $N = 2$, the second characteristic system.

As for the two-dimensional system, we assume that it has identical transfer functions in the separate channels, which are the same as the transfer function in Equation (1.61), i.e. $W_1(s) = W_2(s) = W(s)$, and rigid cross-connections described by a numerical matrix R [Equation (1.60)] (such MIMO systems, called *uniform*, are considered at length in the next section), where $\alpha_1 = 30^\circ$, $\alpha_2 = 20^\circ$. The matrix R [Equation (1.60)] for these values of angles α_1 and α_2 is equal to

$$R = \begin{pmatrix} 0.866 & 0.50 \\ -0.342 & 0.94 \end{pmatrix}. \tag{1.62}$$

Note that in this example, the poles and zeros of the open-loop characteristic systems are located symmetrically with respect to the real axes and coincide with the poles and zeros of the transfer function $W(s)$ [Equation (1.61)]. Nevertheless, the roots of each closed-loop characteristic system are nonsymmetrical.

In principle, for the stability analysis of characteristic systems, any of the well known stability criteria used for common SISO systems can be applied, after some modifications

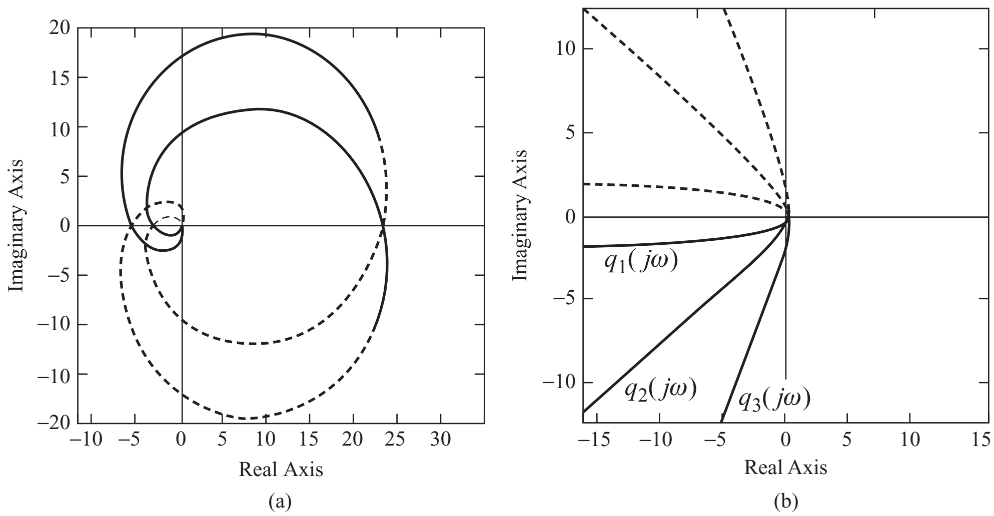

Figure 1.10 Characteristic gain loci of stable MIMO systems; (a) stable two-dimensional system; (b) stable three-dimensional system. (a) $N = 2$, $k = 2$; (b) $N = 3$, $k = 0$.

(Ogata 1970; Kuo 1995). However, for practical applications, the most convenient is possibly the Nyquist criterion, of which various generalizations to the multivariable case are given, for example, in (Postlethwaite and MacFarlane 1979; Desoer and Wang 1980; Stevens 1981). For the objectives of this book, we shall formulate that criterion, sacrificing to a certain extent the mathematical rigour, in the following way. Define as the Nyquist plots (or the *characteristic gain loci*) of the open-loop CTFs $q_i(s)$ $(i = 1, 2, \ldots, N)$ the curves in the complex plane which correspond to the CTFs $q_i(j\omega)$ as angular frequency ω changes from $-\infty$ to $+\infty$.[18] Then, *if the open-loop MIMO characteristic equation has* k *poles in the right half-plane, for the stability of the closed-loop system, it is necessary and sufficient that the total sum of anticlockwise encirclements of the critical point (*-1, j0*) by the characteristic gain loci* $q_i(j\omega)$ *be equal to* k.

If the open-loop MIMO system is stable, none of the $q_i(j\omega)$ loci must encircle the point $(-1, j0)$. Note that in the formulation of the generalized Nyquist criterion, we speak about the *total sum* of encirclements of $(-1, j0)$, and no restrictions or conditions are imposed on the number of encirclements of that point by each isolated $q_i(j\omega)$ locus. The characteristic gain loci of a stable two-dimensional system, which is unstable in the open-loop state and has, in that state, two right half-plane poles, are shown in Figure 1.10(a). The gain loci of a stable three-dimensional system, in the case of stable $W(s)$, are depicted in Figure 1.10(b). The solid lines in these plots correspond to the $q_i(j\omega)$ loci for positive frequencies $\omega \geq 0$ and the dotted lines to negative frequencies $\omega < 0$.

One of the primary advantages of the above-stated Nyquist criterion is that it allows, just as in the SISO case, judging about stability of the closed-loop MIMO system by frequency characteristics, namely the characteristic gain loci $q_i(j\omega)$ of the open-loop MIMO system.

[18] Strictly speaking, the so-called *Nyquist contour*, which we have simplistically replaced by the imaginary axis, must pass around from the right, moving along the semicircle of an infinitesimal radius, all poles and zeros of the open-loop MIMO system, as well as the *branch points*, located on the imaginary axis (Postlethwaite and MacFarlane 1979).

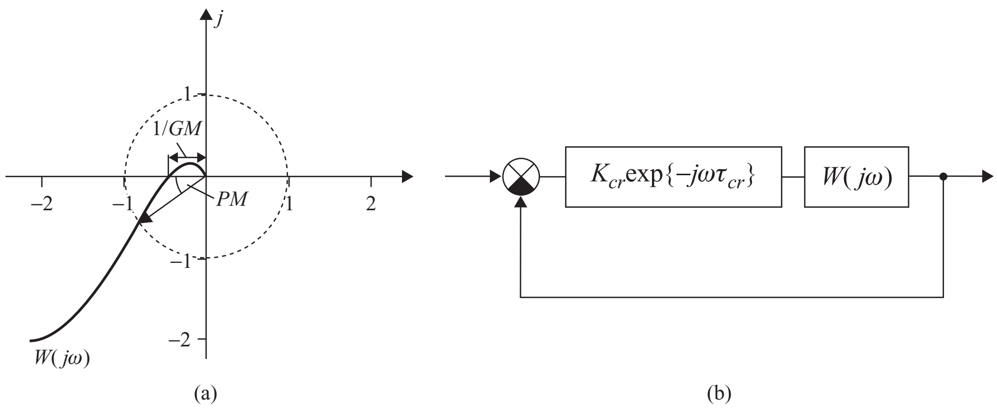

Figure 1.11 Gain and phase margins of SISO systems.

Besides, on substituting $s = j\omega$, the matrix $W(s)$ becomes a usual numerical matrix $W(j\omega)$ with complex entries, and finding the eigenvalues $q_i(j\omega)$ of the latter for the fixed values $\omega = \omega_k$ does not now present any difficulty. In particular, the modern application programs and packages (for example, those in MATLAB) enable the user to calculate the eigenvalues of complex-valued square matrices of practically any size, and, consequently, to create tools for the computer-aided stability analysis of linear MIMO systems with practically any number N of separate channels.

Note that on substituting $s = -j\omega$, the matrix $W(-j\omega)$ is complex conjugate to the matrix $W(+j\omega)$. Therefore, the eigenvalues $q_i(-j\omega)$ of $W(-j\omega)$ are complex conjugate to those of $W(+j\omega)$. This implies that the branches of loci $q_i(j\omega)$ $(i = 1, 2, \ldots, N)$ for negative frequencies $\omega < 0$ are the mirror mapping with respect to the real axis of a set of the $q_i(j\omega)$ loci branches for positive frequencies $\omega > 0$ (generally, each of these loci may not possess the indicated symmetry). That is why, in the analysis of the MIMO system stability, it is enough to consider only positive frequencies $\omega \geq 0$. Accordingly, the formulation of the Nyquist criterion is changed to a certain extent, and now it states that *if the open-loop MIMO system is unstable, then the closed-loop system will be stable, provided the total sum of anticlockwise encirclements of the critical point* $(-1, \mathrm{j}0)$ *by all* $q_i(j\omega)$ *loci is equal to* $\mathrm{k}/2$, *where* k *is the number of unstable poles of* $W(s)$.

The discussion of the issues related to the MIMO system stability analysis by means of the generalized Nyquist criterion would be incomplete if we did not dwell on such classical notions as *gain and phase margins*, which originated in the control theory on the basis of Nyquist plots (Ogata 1970; Kuo 1995).[19] For common SISO systems, these notions are clarified in Figure 1.11.

The *phase margin* (PM) is defined as the angle between the real negative axis and the line to the point at which $W(j\omega)$ intersects the circle of unit radius.[20] The *gain margin* (GM)

[19] Further, for brevity, we shall confine our presentation to systems with 'usual' $W(j\omega)$ loci. Therefore, we shall not consider the *conditionally stable* systems, which have the 'beak-shaped' Nyquist plots, and the systems *unstable* in the open-loop state. The conditionally stable systems become unstable *on both increasing and decreasing* the open-loop gain *K*. For systems unstable in the open-loop state, the zero value *K = 0* always belongs to the unstable region, and the stable closed-loop system may become unstable *on decreasing* the gain *K*.

[20] At that point, $|W(j\omega_c)| = 1$, and the corresponding frequency $\omega = \omega_c$ is called *crossover frequency*.

is defined as the reciprocal to the magnitude $|W(j\omega)|$ at the point of intersection of $W(j\omega)$ with the real negative axis [at that point, $\arg\{W(j\omega)\} = -180^\circ$], and is usually expressed in decibels. For stable systems, both stability margins are reckoned positive. In fact, the phase margin indicates which additional negative phase shift should be introduced in the open-loop system at the crossover frequency ω_c to bring the closed-loop system to the stability boundary. Analogously, the gain margin is equal to the additional gain, which should be introduced in the open-loop system to bring the closed-loop system to the stability boundary.

Recall that the ideal time delay element

$$W_{TD} = \exp\{-j\omega\tau\}, \tag{1.63}$$

has unitary magnitude at all frequencies, and introduces a negative linear phase shift $-\omega\tau$. That allows us to define the *critical delay* as $\tau_{cr} = PM/\omega_c$[21] and to explain visually the notions of gain and phase margins with the help of the block diagram in Figure 1.11(b), in which the entered conventional element $K_{cr}\exp\{-j\omega\tau_{cr}\}$ illustrates the discussed notions.

Taking into account the significance of these notions in the classical control theory, it would be natural to introduce analogous notions for MIMO systems. A great number of papers in the scientific and technical literature are devoted to that issue, among which the works of Safonov (1980, 1981), in which gain and phase margins of MIMO systems are introduced from the position of the robustness theory, should be especially noted. As for us, we shall pursue here, for us, the simpler and more natural way, based on the observation that the utilization of the generalized Nyquist criterion to a MIMO system is equivalent, roughly speaking, to the utilization of the classical Nyquist criterion to each of characteristic systems associated with the MIMO system. As the gain and phase margins of *each* characteristic gain locus $q_i(j\omega)$ may readily be evaluated [just as shown in Figure 1.11(a)], then we can, in such a way, associate two sets of N gain margins $\{GM_i\}$ and N phase margins $\{PM_i\}$ with an N-dimensional MIMO system. Now, it seems quite logical to count as MIMO system stability margins the *leasts* of values GM_i and $PM_i (i = 1, 2, \ldots, N)$, i.e. to define the gain GM and phase GM margins of a MIMO system as:

$$GM = \min\{GM_i\}, \quad PM = \min\{PM_i\}. \tag{1.64}$$

Here, however, the specific features of MIMO systems should be taken into account. First, we should realize that the quantities GM and PM determined by Equation (1.64) may, in general, correspond to *different* characteristic systems, which is evident from Figure 1.12.

Further, and far more importantly, on trying to build a matrix analogue to the block diagram in Figure 1.11, we *have no right* to introduce in the MIMO system a *diagonal* conventional element $diag\{K_{icr}\exp\{-j\omega\tau_{icr}\}\}$ with different critical gains K_{icr} and time delays τ_{icr}, and must introduce a *scalar* matrix $K_{cr}\exp\{-j\omega\tau_{cr}\}I$, i.e. introduce the *same* elements $K_{cr}\exp\{-j\omega\tau_{cr}\}$ in all channels (Figure 1.13). This is because introducing a diagonal matrix $diag\{K_{icr}\exp\{-j\omega\tau_{icr}\}\}$ violates the canonical representation of $W(s)$, since the multiplication of a diagonal matrix by a square matrix changes both the eigenvalues and the eigenvectors of the latter. As for the multiplication of $W(j\omega)$ by a scalar matrix $K_{cr}\exp\{-j\omega\tau_{cr}\}I$, which corresponds to the multiplication of $W(j\omega)$ by a *scalar* $K_{cr}\exp\{-j\omega\tau_{cr}\}$, the canonical basis

[21] That quantity is sometimes called *delay margin* (Bosgra *et al.* 2004).

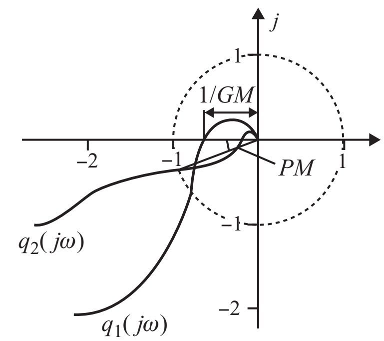

Figure 1.12 MIMO system stability margins ($N = 2$).

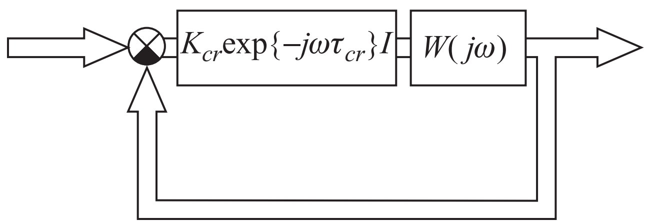

Figure 1.13 Definition of the MIMO system stability margins.

of the MIMO system does not change, and all CTFs $q_i(j\omega)$ multiply by the same scalar.[22] Only in such a case will the changes in the coefficient K_{cr} bring about the proportional changes of magnitudes of *all* characteristic gain loci $q_i(j\omega)$, and changes in the time delay τ_{cr} bring about the rotation about the origin of the complex plane of *all* $q_i(j\omega)$ by the same angle. This preserves the geometrical and physical sense of the above notions of the MIMO system gain and phase margins as the least gain and phase margins of characteristic systems.

Hence, when using the quantities GM and PM [Equation (1.64)] as measures of the stability margins, we should always remember which sense is given to these values; namely, they define the gain and phase margins of the MIMO system only for *simultaneous and identical* changes of gains and time delays in all channels. Note, finally, the evident fact that the MIMO system stability analysis on the base of the generalized Nyquist criterion may be accomplished, as in the SISO case, with the help of the logarithmic characteristics (the Bode diagrams) and the Nichols plots.

Remark 1.1 Of course, the introduced notions of the MIMO system stability margins may seem somewhat narrow from the point of view of their practical application for the MIMO system analysis and design. Indeed, this does in fact happen in some situations, as we shall see in the example given below. In these situations, we have to deal with the MIMO systems that are in a certain sense not *robust*, i.e. in which even slight perturbations of the system parameters may bring about significant changes in the dynamics, and even instability of the

[22] This ensues from the well known in the theory of matrices rule of multiplication of a square matrix by a scalar (Gantmacher 1964).

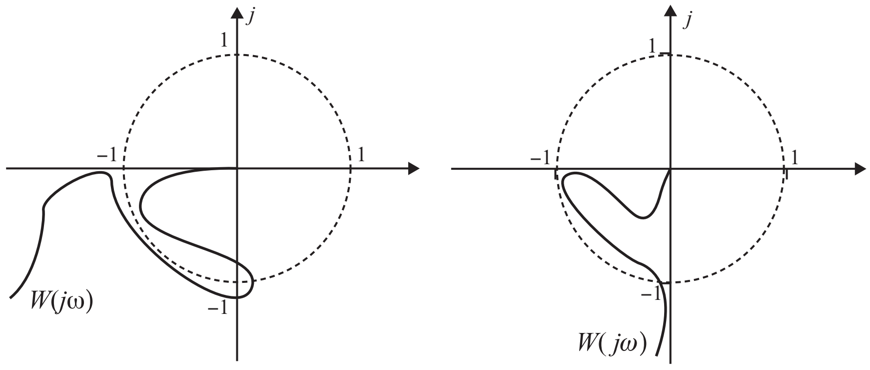

Figure 1.14 Nyquist plots of 'non-robust' SISO systems.

system. Recall, however, that the notions of gain and phase margins are subjected to some 'criticism' (mainly by the specialists on the robustness theory), even in the case of SISO systems. In Figure 1.14, the Nyquist plots of the SISO systems are shown, which, having excellent gain and phase margins, are not robust nonetheless, because their plots are too close to the critical point $(-1, j0)$ (Bosgra *et al.* 2004). On the other hand, for common control systems – and such systems are by far more numerous than the systems with the exotic characteristics of Figure 1.14 (it would probably be very difficult, if possible at all, to achieve such characteristics for real systems) – the gain and phase margins have been, and are now, quite good, convenient and intuitively comprehensible notions, generally recognized by practicing engineers. In this connection, the introduction of the analogous, that is having the same clear and visual physical interpretation, characteristics for MIMO systems may be considered as a quite grounded and necessary step, though those characteristics should be treated with certain caution.

Example 1.2 Below, we consider the stability analysis of a general two-axis guidance system of the telescope, for two different variants of the separate channel transfer functions and rigid cross-connections described by the matrix in Equation (1.60). In the first variant, we have a simple system with the following transfer functions:

$$W_1(s) = \frac{1}{s}, \quad W_2(s) = \frac{0.5}{s+0.2} \tag{1.65}$$

The Nyquist plots of the CTFs $q_1(j\omega)$ and $q_2(j\omega)$ of the guidance system with transfer functions as in Equation (1.65), for two different combinations of angles α_1 and α_2: $\alpha_1 = 30°$, $\alpha_2 = 20°$ and $\alpha_1 = 30°$, $\alpha_2 = -20°$, are shown in Figure 1.15. The inspection of these plots shows that the system is stable for both combinations of α_1 and α_2, and the Nyquist plots of $q_1(j\omega)$ and $q_2(j\omega)$ are quite close to the corresponding plots of the isolated separate channels [Equation (1.65)].

Consider now the same system with more complicated transfer functions of separate channels:

$$W_1(s) = \frac{50(s+3)}{s(s+5)(s+10)}, \quad W_2(s) = \frac{100}{s(s+2)(s+4)(s+8)}. \tag{1.66}$$

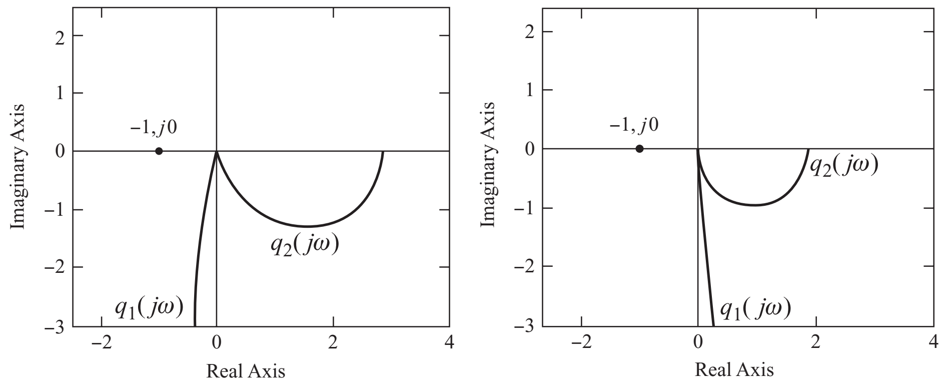

Figure 1.15 Stability analysis of the general two-dimensional guidance system with transfer functions [Equation (1.65)] for different angles α_1 and α_2 (the Nyquist plots). (a) $\alpha_1 = 30°$, $\alpha_2 = 20°$; (b) $\alpha_1 = 30°$, $\alpha_2 = -20°$.

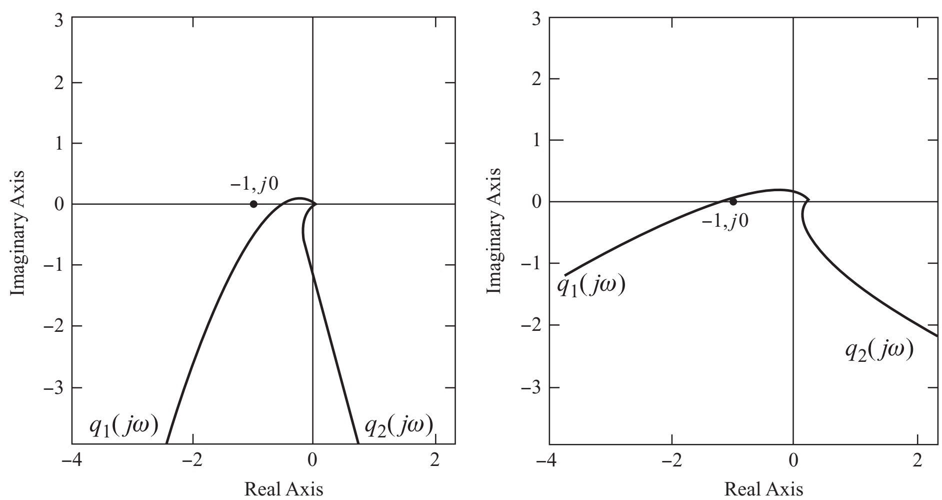

Figure 1.16 Stability analysis in the case of transfer functions [Equation (1.66)]. (a) $\alpha_1 = 30°$, $\alpha_2 = 20°$; (b) $\alpha_1 = 60°$, $\alpha_2 = 70°$.

The characteristic gain loci of that system for different angles α_1 and α_2 are shown in Figure 1.16. The plots in Figure 1.16 show that cross-connections may considerably affect the stability of the guidance system, and even bring about instability (the case of $\alpha_1 = 60°$, $\alpha_2 = 70°$).

Example 1.3 As another example of the MIMO system stability analysis which illustrates the need to exercise caution when estimating the stability margins on the base of Equation (1.64), consider the celebrated example attributed usually to J. Doyle (Doyle and Stein 1981). As a result of that example, many authors have drawn conclusions about the certain inefficiency, not to say unreliability, of the approaches based on the characteristic gain loci method.

Given the open-loop system with the transfer function matrix

$$W(s) = \frac{1}{(s+1)(s+2)} \begin{pmatrix} -47s+2 & 56s \\ -42s & 50s+2 \end{pmatrix}. \tag{1.67}$$

In Doyle and Stein (1981), it is shown that this matrix can be represented in the form

$$W(s) = \begin{pmatrix} 7 & -8 \\ -6 & 7 \end{pmatrix} \begin{pmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{2}{s+2} \end{pmatrix} \begin{pmatrix} 7 & 8 \\ 6 & 7 \end{pmatrix}, \tag{1.68}$$

i.e. for this system, the CTFs $q_1(s)$ and $q_2(s)$ have, assuming the unit regulator $K = I$, the form of the first order transfer functions:

$$q_1(s) = \frac{1}{s+1}, \quad q_2(s) = \frac{2}{s+2}, \tag{1.69}$$

which indicate an infinite gain margin, and the phase margin of 180°. The loci $q_1(j\omega)$ and $q_2(j\omega)$ of the system corresponding to Equation (1.69) are shown in Figure 1.17(a) and justify that conclusion {though both these loci are identical in form, their frequency numberings along the loci differ due to the different time constants of $q_1(s)$ and $q_2(s)$ [Equation (1.69)]}.

However, as established in Doyle and Stein (1981), introducing in the system the diagonal static regulator of the form

$$K = \begin{pmatrix} 1.13 & 0 \\ 0 & 0.88 \end{pmatrix} \tag{1.70}$$

i.e. *simultaneously* increasing the gain of the first channel by 0.13 and decreasing that of the second channel by 0.12, makes the closed-loop system unstable! The given example is rather instructive and, from another point of view, it clearly discloses how unpredictable and unexpected changes in the MIMO system's characteristics may be, even in the case of slight perturbations of their parameters.

This, at first sight, irrefutable demonstration that the stability margins calculated through the CTFs may lead to an erroneous conclusion in fact demands more careful and deeper treatment. We have already stated before that gain and phase margins [Equation (1.64)] are valid only for *simultaneous and identical* changes in the separate channel gains. In this regard, the discussed MIMO system indeed remains stable for arbitrary large but the *same* gains in separate channels, since, in that case, the MIMO system modal decomposition is not violated and the canonical basis is unchanged. However, if we multiply together the matrices $W(s)$ [Equation (1.67)] and K [Equation (1.70)], then the resulting canonical basis and the CTFs will be completely different.[23] The characteristic gain loci of the new perturbed system with regulator K [Equation (1.70)] are depicted in Figure 1.17(b), from which it is quite clear that the MIMO system is unstable and its new CTFs have nothing in common with those in Figure 1.17(a).

[23] We shall return to this example in Example 1.13 and Chapter 2, with more detailed explanation.

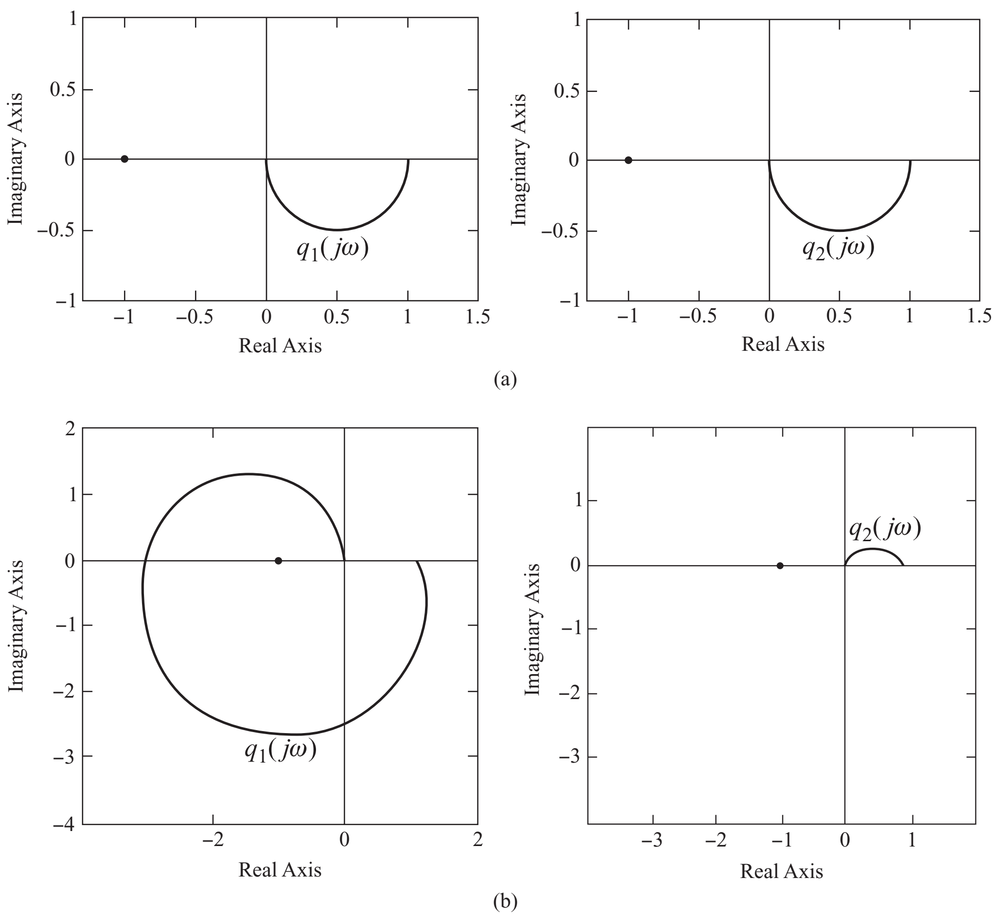

Figure 1.17 Stability analysis of a two-dimensional system [Equation (1.67)]. (a) Characteristic gain loci in the case of the unit regulator $K = I$; (b) characteristic gain loci of the system with the perturbed regulator K [Equation (1.70)].

1.2.5 Singular value decomposition of transfer matrices

As we already know, to any N-dimensional linear MIMO system of *simple structure* correspond N linearly independent 'canonical' directions (canonical basis axes) in complex space $\mathbb{C}^N$, along which the MIMO system acts as a certain SISO system. The canonical bases of the open- and closed-loop MIMO systems are the same. This feature is very remarkable, since it assures that all transfer matrices $W(s)$, $\Phi(s)$ and $\Phi_\varepsilon(s)$ of the open- and closed-loop MIMO systems are brought to diagonal form by a similarity transformation via a *unique* transformation (modal) matrix. Moreover, all CTFs of the closed- and open-loop one-dimensional characteristic systems associated with the MIMO system are related in the canonical basis by the very same expressions as usual transfer functions of common SISO systems. Thus, the $|q_i(s)|$, $|q_i(s)|/|1 + q_i(s)|$ and $1/|1 + q_i(s)|$ magnitudes of the corresponding CTFs give, for any fixed value of complex variable s, the 'gain magnitude' of the MIMO system along the corresponding ith axis of the canonical basis.[24]

Together with the canonical basis axes, another set of directions in $\mathbb{C}^N$ which are uniquely defined by the MIMO system transfer matrices may be associated with a linear MIMO system. These directions are no less (and many specialists on robust control quite soundly believe that more) important for the understanding of the internal structure and performance of multivariable systems than the directions of the canonical basis axes. The matter concerns the so-called *singular value decomposition* of the transfer function matrices.

Recall in brief that any square matrix A of order $N \times N$ allows decomposing in the form[25]

$$A = U diag\{\sigma_i\} V^*, \tag{1.71}$$

where the asterisk $*$ denotes the operation of complex conjugation and transposition (from here on, we call this operation the *conjugation* of a matrix); U and V are *unitary* matrices, for which the inverse matrices coincide with conjugated (i.e. $U^{-1} = U^*$, $V^{-1} = V^*$); $diag\{\sigma_i\}$ is the diagonal matrix of nonnegative real numbers,[26] arranged in descending order, i.e. $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N$ (Marcus and Minc 1992; Voevodin and Kuznetsov 1984).

Decomposition [Equation (1.71)] is said to be a *singular value decomposition* and numbers σ_i are called the *singular values of A*. These numbers are equal to positive values of the square roots taken of the eigenvalues of the Hermitian matrix AA^* (or A^*A).[27] The columns of the matrix U in Equation (1.71) constitute the orthonormal *left* (or *output*) *singular basis* of the matrix A in $\mathbb{C}^N$ and are the eigenvectors of the matrix AA^*. The columns of the matrix V constitute, accordingly, the orthonormal *right* (or *input*) *singular basis* and are the eigenvectors of the matrix A^*A. Denoting the columns of U by u_i and the columns of V by v_i, Equation (1.71) can be written down as a sum of N dyads of rank one:

$$A = \sum_{i=1}^{N} u_i > \sigma_i < v_i. \tag{1.72}$$

From here, it is evident that A transforms the orthonormal right singular basis into a mutually orthogonal set of vectors directed along the axes of the left singular basis, and real numbers σ_i serve as 'gains' in the corresponding directions. This can also be written as follows:

$$Av_i = \sigma_i u_i, \quad i = 1, 2, \ldots, N, \tag{1.73}$$

where it is more clearly accentuated that numbers σ_i relate *different* directions in $\mathbb{C}^N$.

The singular values of square matrices possess a number of features relating them to eigenvalues λ_i and determinants of matrices. Thus, in particular, a square matrix is nonsingular if

[24] For arbitrary values of variable s with a nonzero real part, the listed magnitudes have no definite sense, but, when considering the steady-state oscillations in stable MIMO systems, i.e. when we accept $s = j\omega$, they obtain a clear and simple physical interpretation and may be viewed as 'gains' for the given frequency ω.

[25] The singular value decomposition is applicable to any *rectangular* matrix, but, here, for simplicity, we consider only the case of square matrices, which are interesting for us.

[26] For zero singular values, which are, in principle, possible, see Remark 1.7.

[27] A matrix A is said to be Hermitian if it coincides with its conjugate, i.e. if $A = A^*$. The eigenvalues of a Hermitian matrix are always real and, for a Hermitian matrix B, written in the form $B = AA^*$, they are always nonnegative (Marcus and Minc 1992).

and only if all its singular values are nonzero. This ensues from the following, well known in the theory of matrices rule:

$$|\det A| = \prod_{i=1}^{N} \sigma_i \tag{1.74}$$

from which, in addition, it follows (as $\det A = \prod_{i=1}^{N} \lambda_i$) that the product of the eigenvalues magnitudes (or the magnitude of the eigenvalues product) is also equal to the product of the singular values. Among the singular values, of primary interest are the largest and smallest of them, where the largest number $\sigma_{\max}$ is called the *spectral norm* of the matrix A, and is usually denoted by $||A||$ (Gantmacher 1964).[28] This norm possesses the property that for any two vectors x and y in $\mathbb{C}^N$, where $y = Ax$, the following inequality holds:

$$|y| \leq \|A\| \, |x| \, , \tag{1.75}$$

where the (Euclidian) norms of vectors x and y are defined by Equation (1.29). According to the inequality in Equation (1.75), the spectral norm $||A||$ may be interpreted as the maximum 'gain' of the matrix A with respect to the *magnitude* of the 'input' vectors x in $\mathbb{C}^N$. Analogously, the least singular value σ_{min} guarantees that the following inequality holds:

$$|y| \geq \sigma_{min} \, |x| \, , \tag{1.76}$$

i.e. it gives the lower boundary of the A matrix gain (by magnitudes) for any vectors x.

Recalling that, by the definition of eigenvalues λ_i and eigenvectors c_i, we have, for any i ($i = 1, 2, \ldots, N$), the equality $y = Ac_i = \lambda_i c_i$, it is easy to understand that the magnitudes of all eigenvalues of any square matrix are bounded from above and below by singular values $\sigma_{\max}$ and σ_{min}:

$$\sigma_{min} \leq |\lambda_i| \leq \sigma_{\max}, \quad i = 1, 2, \ldots, N. \tag{1.77}$$

All that has been stated about singular values and their relationship with eigenvalues may be summarized as follows. With any square matrix A, there are related two orthonormal coordinate systems (input and output singular bases) and one more, generally nonorthogonal, coordinate system formed by the eigenvectors of the matrix A. The two specific axes of singular bases give the directions of the largest and smallest of the A matrix 'gains', and these two directions are always mutually orthogonal, both in the 'input' and in the 'output'. In the directions of the eigenvectors, the matrix A acts as scalar multipliers with intermediate, owing to Equation (1.77), 'gains'.

Now, we can readily use the introduced notions to transfer matrices $W(s)$, $\Phi(s)$ and $\Phi_\varepsilon(s)$ of the MIMO system. Here, we emphasize immediately that, in the general case, as opposed to the unique set of the canonical basis axes, all listed transfer matrices have their own pairs of input and output singular bases, i.e. two *distinct* orthonormal coordinate systems are associated with each matrix, as well as their own sets of singular values. Another essential feature of the singular value decomposition of transfer matrices is that nothing can generally be said, for

[28] We shall consider this matter in more detail in Chapter 2.

example, about the relationship between the corresponding characteristics of the open-loop and closed-loop MIMO system, and each transfer matrix must be considered individually and independently of other matrices. Taking this into account, we shall consider only the case of the transfer matrix $\Phi(s)$ with respect to the MIMO system output. That matrix can be expressed by the singular value decomposition in the form

$$\Phi(s) = [I + W(s)]^{-1} W(s) = U_{\Phi}(s) diag\{\sigma_{\Phi i}(s)\} V_{\Phi}(s) = \sum_{i=1}^{N} u_{\Phi i}(s) > \sigma_{\Phi i}(s) < v_{\Phi i}(s), \tag{1.78}$$

where $U_{\Phi}(s)$, $V_{\Phi}(s)$ are the unitary matrices composed of the orthonormal vectors $u_{\Phi i}(s)$ and $v_{\Phi i}(s)$; $\sigma_{\Phi i}(s)$ are real positive singular values arranged in descending order, where the subscripts in Equation (1.78) explicitly indicate that all these characteristics belong *only* to the transfer matrix $\Phi(s)$.

The output singular basis of $\Phi(s)$ consists of the orthonormal eigenvectors $u_{\Phi i}(s)$ of the Hermitian matrix $\Phi(s)\Phi^*(s)$, and the *input singular basis* consists of the eigenvectors $v_{\Phi i}(s)$ of $\Phi^*(s)\Phi(s)$. The singular values $\sigma_{\Phi i}(s)$ are square roots of the eigenvalues of $\Phi(s)\Phi^*(s)$ [or $\Phi^*(s)\Phi(s)$]. The largest singular value $\sigma_{\Phi\max}(s)$ is equal to the spectral norm $||\Phi(s)||$ of the transfer matrix $\Phi(s)$ and defines the maximum gain with respect to the input vectors magnitude, and the smallest singular value $\sigma_{\Phi\min}(s)$ yields the lower boundary of the gain. The axes of the largest and the smallest 'gains' are always mutually orthogonal, both in the input and in the output spaces. It is worth especially emphasizing that if the input vector $\varphi(s)$ of the MIMO system is directed along any one axis of the input singular basis, then, in forming the output vector $f(s)$, which is also directed along the corresponding axis of the output singular basis, all N one-dimensional characteristic systems generally participate.

The singular value decomposition of transfer matrices does not allow judgment about the system stability (due to the reality of singular values), but it is very important when analysing the performance and especially the robustness of the system. We shall return to that point in Chapter 2. Note finally that, in Section 1.4, we shall consider the so-called *normal* MIMO systems, for which both input and output singular bases coincide with the canonical one. In conclusion, we make some general remarks, which are quite important for proper comprehension of the essence of the presented material, as well as for the sensible assessment of those assumptions and suppositions on which we rely.

Remark 1.2 In most textbooks and monographs on multivariable control, the open-loop transfer matrix $W(s)$ is usually represented as a series connection of the plant and the controller (also frequently called the regulator, compensator, etc.). Moreover, in the practical tasks, the matrix block diagram in Figure 1.1 may have much more complicated form and consist of different matrix elements connected in series, in parallel or forming inner feedback loops. In the mentioned cases, however, all such schemes can readily be brought, based upon the well known rules of transformation of matrix block diagrams (Morozovski 1970), to the form of Figure 1.1. We have intentionally chosen in this section such a structure, to emphasize the significant fact that the CTFs and canonical basis depend on the *resultant* transfer matrix $W(s)$ of the open-loop MIMO system, and, in the general case, *nothing can be said about the relationship of the canonical representations of* $W(s)$ *and those of the individual matrix elements constituting the system*. This circumstance seemingly explains the situation that in the scientific and technical literature, there are actually no efficient *engineering* techniques presented for the MIMO systems design based on the CTFs method, and the latter is mainly viewed as a sophisticated method applicable only for the MIMO systems stability analysis.

As for the singular value decomposition, here, we generally cannot even speak about analytical relationships between the corresponding decompositions of the open-loop and closed-loop MIMO systems, which, in the context of the CTFs approach, are very simple and visual.

Remark 1.3 We have considered the principal ideas of the CTFs method in a somewhat simplified fashion, sacrificing in many respects the mathematical rigour. As shown in Postlethwaite and MacFarlane (1979), the CTFs, being the roots of Equation (1.26), generally belong to the class of multi-valued *algebraic functions* and constitute one mathematical entity. The *branches* of the characteristic gain loci are situated on different sheets of a Riemann surface, and transition from one sheet to another is accomplished through the 'cuts' connecting the point in infinity with the *branch points* of the algebraic function, at which two or more CTFs are equal. The presence of these branch points explains many of the not quite usual features of, say, root loci of multivariable systems, i.e. the features that are difficult or even impossible to explain from the position of the classical control theory. At the same time, if, when solving practical engineering tasks, we neglect the behaviour of the CTFs in the neighborhood of branch points, then, for any fixed value $s = const$ not coinciding with the branch points and poles of $W(s)$, the latter may be viewed as a usual numerical matrix with complex entries. Then, as was already noted in Section 1.2.3, the problem of finding the CTFs is simply reduced to computing the eigenvalues of the matrix $W(s)$.

It is also worth noting that in the case of two-dimensional systems which frequently occur in practice, the roots of Equation (1.26), i.e. the CTFs of the open-loop system, can be written in the analytical form

$$q_{1,2}(s) = \frac{tr\{W(s)\}}{2} \pm \sqrt{\frac{tr\{W(s)\}^2}{4} - \det W(s)}, \tag{1.79}$$

where $tr\{W(s)\}$ denotes the *trace* (the sum of diagonal elements) of $W(s)$, from which it is clear that, in general, the CTFs of the linear MIMO system cannot always be represented as a quotient of two rational polynomials in s.[29] If all CTFs could be represented as proper rational functions in s, then many MIMO system analysis and design issues would be solved far more simply and effectively.

Incidentally, the above-mentioned branch points are determined by equating the *discriminant* (the radical expression) in Equation (1.79) to zero.

Remark 1.4 The assumption of no repeated (multiple) CTFs of the matrix $W(s)$ is not very restrictive. First, in the following text, we shall consider some MIMO systems which have multiple CTFs but, at the same time, are of simple structure and can be brought to diagonal form. Further, as is known from the theory of matrices (Gantmacher 1964), in the case of multiple eigenvalues, it is always possible to reduce a matrix in a specially chosen *Jordan basis* to a *Jordan canonical form*, which can be viewed as the best approach to diagonal form, and in which, with the multiple eigenvalues, are related *Jordan blocks*. In any Jordan block, whose order is determined by the algebraic multiplicity of the given repeated eigenvalue, the principal diagonal is formed of these identical eigenvalues. The first subdiagonal (immediately

[29] On replacing $W(s)$ by $W(s)W^*(s)$ in Equation (1.79), we obtain the analogous equation for calculating the singular values of matrix $W(s)$.

above and to the right of the principal diagonal) consists of ones and zeros, in the proportions determined by the algebraic and geometric multiplicities of eigenvalues.

Structurally, the Jordan blocks will determine, in the canonical representations of the MIMO system transfer matrices in Figures 1.3–1.6, the blocks with identical elements in the direct channels and *one-sided* unit connections from the lower channel to the upper one (if the numbering of channels is from the top down). It is interesting to note here that the presence of unit subdiagonals in the Jordan blocks does not alter the characteristic equation expressed via the CTFs $q_i(s)$ [Equation (1.59)], i.e. does not affect the stability of the MIMO system.

Broadly speaking, the problem of multiple eigenvalues, as well as the adjacent problem of determining the corresponding Jordan blocks, belongs to complicated problems of the theory of matrices. However, for us, it represents more of a theoretical rather than a practical interest, since the presence of multiple eigenvalues is, for the most part, the exception but not the rule (at least for the general type of matrices). And, above all, in engineering practice, all linear models are mathematical idealizations of real objects, in which all the model parameters are specified with certain accuracy. Therefore, as was already noted in Section 1.2.3, we are always able to choose the parameters of any MIMO system model, within the given specification accuracy, in a way that excludes the presence of multiple roots. All this explains why the general MIMO systems with repeated CTFs are not considered in the following.

Remark 1.5 It is not out of place to put, what is at first sight, a trivial question: what is a multivariable control system? Formally, any set of N independent SISO systems, regarded as an integrated unit, may serve as a paradigm of a MIMO system. However, such an approach intuitively gives rise to some doubts about its appropriateness; in fact, these independent SISO systems can readily be handled by means of conventional methods of the classical control theory. Note that if we classify MIMO systems by the type of their transfer matrices, then even the set of independent SISO systems may be divided into two classes – *scalar* MIMO systems, i.e. a set of N identical SISO systems, and *diagonal* MIMO systems, consisting of N different SISO subsystems. The next class of MIMO systems, from the point of view of complexity, includes *triangular* systems, i.e. MIMO systems with zero entries above or below the principal diagonal. Such systems do not differ much in complexity from the scalar or diagonal systems, since the triangularity of the transfer matrices implies the presence in the system of one-sided connections from the previous channels to the subsequent ones, or in the reverse order. Such one-sided connections, which are equivalent to the external disturbances, cannot affect the stability of the system. All listed systems have a common feature, namely all their CTFs coincide with the transfer functions of direct channels of the open-loop MIMO system [with the diagonal elements of $W(s)$]. From here, we come to the following rule: *if the determinant of the open-loop transfer matrix* W(s) *coincides with the product of diagonal elements* $w_{ii}(s)$ *of that matrix, i.e. if*

$$\det W(s) = \prod_{i=1}^{N} w_{ii}(s), \tag{1.80}$$

then the cross-connections do not affect the stability of separate channels, and the stability of the MIMO system can be investigated exploiting conventional methods of the classical control theory.

Of course, this rule is heuristic, but if it holds, then we can reduce the system, from the stability analysis viewpoint, to a set of N usual SISO systems, and thereby do not resort to more complicated techniques of the multivariable control theory. Incidentally, in practice, when checking the condition in Equation (1.80), one can use a numerical matrix composed of the gains of nonzero entries in $W(s)$.

Proceeding in the same way, it would be possible to speak about *block-scalar*, *block-diagonal* and *block-triangular* MIMO systems, which are structurally divided into a number of blocks. To each of these blocks corresponds an individual set of cross-connected SISO systems. From the stability viewpoint, all diagonal blocks of such MIMO systems are independent, and the characteristic gain loci method may be applied to each of them. We will not, however, encumber the following presentation with such systems. The interested reader will be able to extend, without any difficulty, all of the above results to the case of block MIMO systems.

Note also that triangular systems represent a plain and simple example of MIMO systems, for which the introduction of feedback does not alter the location of some closed-loop poles, i.e. for which some of the open-loop system poles coincide with the closed-loop poles. This concerns the poles of the *nondiagonal* elements of a triangular MIMO system.

Further, we shall mainly suppose that the condition in Equation (1.80) does not hold, which is equivalent to irreducibility of the MIMO system characteristic equation (i.e. to impossibility of representing it as a product of two or more equations of a lesser degree).

Remark 1.6 Recall in brief the main concepts concerning the representation of proper rational matrices via *the Smith-McMillan canonical form*. Note immediately that this canonical form is generally applicable to rectangular matrices, but we shall only consider the interesting case for us of square matrices having normal rank N. As is well known, any square proper rational transfer matrix $W(s)$ of order $N \times N$ can be represented in the following canonical form (Kailath 1980):

$$W(s) = H(s) diag \left\{ \frac{\varepsilon_i(s)}{\psi_i(s)} \right\} M(s), \tag{1.81}$$

where both square polynomial matrices $H(s)$ and $M(s)$ are *unimodular*, i.e. their determinants do not depend on complex variable s and are equal to constant, nonzero values. The polynomials $\varepsilon_i(s)$ and $\psi_i(s)$ in Equation (1.81) possess the property that each $\varepsilon_i(s)$ divides all *subsequent* $\varepsilon_{i+j}(s)$, and each $\psi_i(s)$ divides all *preceding* $\psi_{i-j}(s)$. Strictly speaking, the zeros and poles polynomials $N(s)$ and $P(s)$ of the open-loop MIMO system, which we heuristically defined in Section 1.2.2 via the determinant $\det W(s)$ [Equation (1.11)], must in fact be expressed by polynomials $\varepsilon_i(s)$ and $\psi_i(s)$ in the form (Postlethwaite and MacFarlane 1979):

$$N(s) = \prod_{i=1}^{N} \varepsilon_i(s) = 0, \quad P(s) = \prod_{i=1}^{N} \psi_i(s) = 0, \tag{1.82}$$

The point is that when determining poles and zeros of the open-loop MIMO system by means of the determinant in Equation (1.11), mutual cancellation of poles and zeros, which actually correspond to *different* diagonal entries of the Smith-McMillan form [Equation (1.81)] (i.e. correspond to *different* directions) and therefore are not to be cancelled, is not excluded. Only in the case of no coincident roots of polynomials $N(s)$ and $P(s)$ in Equation (1.82) can we avoid such situations.

There exists, however, another significant feature inherent only in multivariable systems, which should be taken into account when dealing with them. Recall that the determinant of a square matrix A of order $N \times N$ represents an algebraic sum of $N!$ terms, where each term is formed in such a way that it contains the product of N entries of A, taken one by one from each row and column (Marcus and Minc 1992). If there are any *zero* entries in A, then it may turn out that some *nonzero* entries of A will not be presented in the determinant $\det A$. To explain more clearly the possible consequence of such a situation, consider the following variants of the transfer matrix $W(s)$, for $N = 3$:

$$W(s) = \begin{pmatrix} w_{11}(s) & 0 & 0 \\ w_{21}(s) & w_{22}(s) & w_{23}(s) \\ w_{31}(s) & 0 & w_{33}(s) \end{pmatrix}, \quad W(s) = \begin{pmatrix} w_{11}(s) & w_{12}(s) & 0 \\ w_{21}(s) & w_{22}(s) & w_{23}(s) \\ 0 & 0 & w_{33}(s) \end{pmatrix}. \tag{1.83}$$

It is easy to see that all nondiagonal scalar transfer functions are not presented in the determinant of the first matrix $W(s)$, and the transfer function $w_{23}(s)$ is not presented in the determinant of the second matrix. This implies that the corresponding poles of nonzero transfer functions will be *absent* in the poles polynomial $P(s)$ determined by means of $\det W(s)$ [in Equation (1.82), those poles are always present owing to the way of finding the polynomials $\psi_i(s)$]. The simplest instance of such structures is the class of *triangular* MIMO systems discussed in the previous remark.

Hence, in the case of zero entries in the open-loop transfer matrix $W(s)$, it makes sense to check whether all transfer functions of cross-connections are present in the expression for the determinant $\det W(s)$, and, accordingly, in the poles polynomial $P(s)$. The importance of such a test lies in the fact that all poles of the 'hidden' transfer functions *do not change on closing the feedback loop, i.e. they become the poles of the closed-loop system* (we shall always suppose such poles are stable).

Remark 1.7 Considering the *gain matrix* of the MIMO system, or, more correctly, the open-loop transfer matrix $W(s)$ for $s = 0$, we should pay proper attention to another significant aspect. Below, to present more visually the essence of the matter, we discuss a simplified situation and assume that all zero poles of the open-loop MIMO system are *absolute* poles, i.e. $W(s)$ can be represented as

$$W(s) = \frac{1}{s^r} W_1(s), \tag{1.84}$$

where the r integer defines the closed-loop system *type*.[30]

Recall that in Section 1.2.2, we set the requirement that the matrix $W_1(s)$ be nonsingular at any absolute pole, i.e. be of rank N. Now, we approach this question from the position of the CTFs method, and impose some additional restrictions on the 'gain matrix' $W_1(0)$. Since the CTFs [Equation (1.84)] of the open-loop system can be represented in the form

$$q_i(s) = \frac{1}{s^r} q_{1i}(s), \quad i = 1, 2, \ldots, N, \tag{1.85}$$

it becomes clear that if the 'gain' of the ith CTF $K_i = q_{1i}(0)$ is real and equal to zero, which corresponds to singularity of the matrix $W_1(0)$, this physically means *breaking the feedback*

[30] See Chapter 2, in which the issue of the MIMO system *type* is considered in more detail.

loop in the ith SISO characteristic system in Figures 1.5 and 1.6. Actually, this involves *loss of control* along the ith axis of the MIMO system canonical basis. Further, if $K_i = q_{1i}(0)$ is real and negative, then this implies the *change from negative feedback to positive in the ith characteristic system*, which generally (but not necessarily) can also bring about nonoperability of the overall MIMO system. Therefore, the case of negative $q_{1i}(0)$ requires especially careful consideration.

Since the matrix $W_1(0)$ in Equation (1.84) is always real-valued, then, together with the real coefficients $K_i = q_{1i}(0)$, the CTFs $q_i(s)$ may as well have (for $s = 0$) complex conjugate pairs of coefficients $K_i = \alpha_i + j\beta_i$, $K_{i+1} = \alpha_i - j\beta_i$, to which correspond two complex conjugate eigenvectors $c_i(0)$, $c_{i+1}(0) = \tilde{c}_i(0)$ [columns of the modal matrix $C(0)$]. In such cases, to understand the physical nature of the complex conjugate coefficients K_i and K_{i+1}, it is worthwhile passing to the *real-valued canonical form* of the matrix $W_1(0)$. It can be shown (Sobolev 1973) that by replacing in $C(0)$ the complex conjugate columns $c_i(0)$ and $c_{i+1}(0)$ by a pair of real ones, which are the real and complex parts of the replaced ones (this operation yields a new *real-valued* transformation matrix T), the expression

$$W_1(0) = C(0) diag\{K_i\} C^{-1}(0) \tag{1.86}$$

can be reduced to a *real-valued* form

$$W_1(0) = T \Lambda T^{-1}. \tag{1.87}$$

Here, the *block-diagonal* matrix Λ is composed as follows: all real coefficients K_i in the diagonal matrix $diag\{K_i\}$ in Equation (1.86) become the corresponding diagonal entries of the matrix Λ, and the complex cells

$$\begin{pmatrix} \alpha_i + j\beta_i & 0 \\ 0 & \alpha_i - j\beta_i \end{pmatrix} \tag{1.88}$$

are replaced in Λ by real cells[31]

$$\begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}. \tag{1.89}$$

Now, it becomes evident that the complex conjugate 'gains' K_i and K_{i+1} of the matrix $W_1(0)$ physically correspond in the real-valued space to *two* interconnected characteristic systems with the same gains α_i in the *direct channels* and *antisymmetrical* cross-connections with the coefficients β_i.

Summarizing, the following conclusion can be drawn: *to preserve the operability of the MIMO system, all real parts of the characteristic system 'gains' $K_i = q_{1i}(0)$ $(i = 1, 2, \ldots, N)$ must be nonzero*. Otherwise, in some characteristic systems, breaking of the feedback loop occurs. If the real parts of some coefficients $q_{1i}(0)$ are negative, which corresponds to replacing the negative feedback by positive, then it can impose essential restrictions on the stability and performance of the MIMO system.

[31] We assume here, for simplicity, that the complex conjugate eigenvalues in Equation (1.84) are arranged in pairs.

Note, finally, that since the product of singular values of any square matrix is equal to the product of the magnitudes of its eigenvalues (see Section 1.2.4), the equality to zero of any singular value of $W(s)$ [or $\Phi(s)$] also implies breaking the feedback loop in some characteristic system.

Remark 1.8 Obviously, the theoretical statements of this section do not impose any restrictions or conditions on the number of separate channels of MIMO systems, and serve for making up an intuitive insight into such significant concepts as poles and zeros, as well as intrinsic geometrical and structural features of the systems in question. As a result, the necessary foundation is created, based on which we can proceed to the issues of the MIMO system performance analysis and design. Of course, everyone realizes that together with the increase in the MIMO system dimension and/or that in the order of scalar transfer functions forming the matrix $W(s)$, the computational difficulties increase. And, here, it turns out that within the frame of frequency-domain analysis and design methods for MIMO systems, the considered approach actually *does not impose any restrictions on the number of separate channels, and on the order of individual transfer functions*. Such well known issues in the modern control theory as model reduction or approximation [these issues constitute an integral part of the state-of-the-art design methods due to a number of factors (Skogestad and Postlethwaite 2005)] simply lose their significance, if not sense, when using the ideas and concepts of the CTFs method! Recall that, in Section 1.2.4, it was indicated that the practical analysis of the MIMO system stability based on the generalized Nyquist criterion in fact reduces to the computation of eigenvalues of complex numerical matrices of order $N \times N$, which result by substituting $s = j\omega$ into $W(s)$. This is clear, since, for $s = j\omega$, all transfer functions $w_{kr}(s)$ $(k, r = 1, 2, \ldots, N)$ in $W(s)$ become usual complex scalars, regardless of the orders of their numerator and denominator polynomials, from which $W(j\omega)$ is composed. Concerning the computation of eigenvalues, for example, the MATLAB software readily performs it in a few seconds (or a few fractions of a second) for matrices whose order may be of several tens and even hundreds – this is quite adequate for the existing needs in MIMO systems practical design. In this respect, the CTFs method has an indisputable advantage over the methods that are more or less based on the MIMO systems representation in state space. Finally, another essential merit of the considered approach consists in its applicability to the MIMO systems with time delays, i.e. the elements $w_{kr}(s)$ can contain transcendental transfer functions $\exp\{-\tau_{kr}s\}$. Of course, here, the MIMO system transfer matrices become ones with transcendental entries, and standard notions of poles and zeros lose their sense. However, the generalized Nyquist criterion and the geometrical features of the MIMO systems hold.

1.3 UNIFORM MIMO SYSTEMS

We proceed now to the study of some special *types* (or *classes*) of linear MIMO systems, which possess, owing to their specific structures, quite peculiar features and characteristics. In various technical applications, such as aerospace engineering, chemical industry and many others, the so-called *uniform* MIMO systems (Sobolev 1973; Gasparyan 1976, 1986) very often occur. The separate channels of uniform MIMO systems have identical transfer functions, and the cross-connections are rigid, i.e. are characterized by a real-valued numerical matrix, or are described, up to the values of gain, by a common (scalar) transfer function.

1.3.1 Characteristic transfer functions and canonical representations of uniform MIMO systems

The matrix block diagram of a linear uniform MIMO system is shown in Figure 1.18, where

$$w(s) = \frac{M(s)}{D(s)} \tag{1.90}$$

is the scalar transfer function of identical separate channels, which is a proper rational function in complex variable s, and R is a numerical matrix of order $N \times N$ describing rigid cross-connections.

It is easy to notice that the transfer matrix $W(s)$ of the open-loop uniform system in Figure 1.18

$$W(s) = w(s)R \tag{1.91}$$

coincides, up to complex scalar multiplier $w(s)$, with the numerical matrix of cross-connections R. This leads to interesting structural and dynamic properties of uniform MIMO systems and allows separating them into an individual class of multivariable control systems.

The comparison of Equation (1.91) with Equation (1.15) immediately shows that all poles and zeros of the transfer function $w(s)$ [Equation (1.13)] are *absolute poles and zeros of the open-loop uniform system*, and, as a consequence, must be common poles and zeros of all open-loop CTFs $q_i(s)$. Consider the canonical representations of the uniform MIMO system transfer matrices. From the theory of matrices, we know that the multiplication of numerical matrices by a scalar multiplier does not change the eigenvectors, and results in the multiplication of all eigenvalues by the same multiplier (Gantmacher 1964). Hence, if we denote by $\lambda_i (i = 1, 2, \ldots, N)$ the eigenvalues of R, which, by analogy with the assumptions made when discussing general MIMO systems, we shall suppose *distinct*, and by C, the modal matrix composed of the linearly independent normalized eigenvectors c_i of R, then canonical representations [Equations (1.27) and (1.39)] of the open-loop uniform system have the following form:

$$W(s) = C diag\{\lambda_i w(s)\} C^{-1} \tag{1.92}$$

$$W(s) = \sum_{i=1}^{N} c_i > \lambda_i w(s) < c_i^+, \tag{1.93}$$

where c_i^+ are dual to c_i. Note here that the eigenvalues λ_i of the real-valued matrix R can be real or complex numbers, and the complex eigenvalues must always occur in complex conjugate pairs (Gelfand 1966).

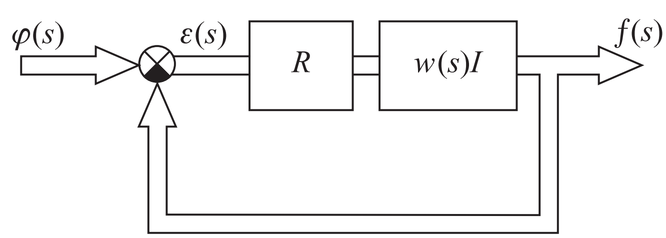

Figure 1.18 Matrix block diagram of a linear uniform MIMO system.

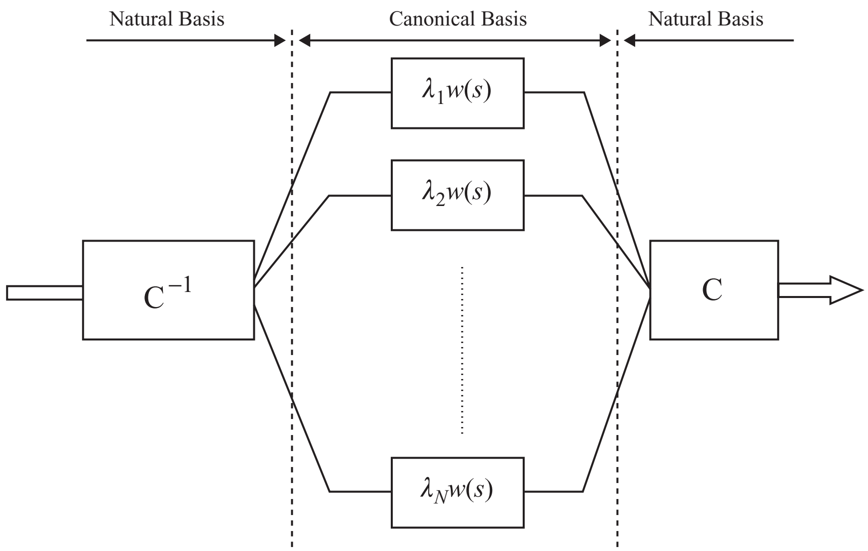

Figure 1.19 Canonical representation of the open-loop uniform MIMO system via the similarity transformation.

Based upon Equations (1.92) and (1.93), the following evident conclusions can be drawn. First, the canonical basis of the linear uniform MIMO system is completely defined by the numerical matrix of cross-connections R, and does not depend on the transfer function $w(s)$ of separate channels. As a consequence, that basis is *constant*, whereas in the general case, the canonical basis of the MIMO system is a function in complex variable s. Secondly, all CTFs

$$q_i(s) = \lambda_i w(s) = \lambda_i \frac{M(s)}{D(s)}, \quad i = 1, 2, \ldots, N \tag{1.94}$$

coincide, up to constant 'gains' λ_i, with transfer function $w(s)$ (Figure 1.19), i.e. we verify the above conclusion that in the case of uniform systems, the poles and zeros of transfer function $w(s)$ are common poles and zeros of all CTFs $q_i(s)$.

The listed properties, especially the latter, enable us to bring very closely the methods of investigating uniform systems to the corresponding classical methods for common SISO systems, and the reader will comprehend that in the following sections.

It is easy to see that having the singular value decomposition of the real matrix R in the form

$$R = U diag\{\sigma_{Ri}\} V^T, \tag{1.95}$$

where U and V are real *orthogonal* matrices (i.e. $U^{-1} = U^T$ and $V^{-1} = V^T$),[32] for the corresponding decomposition of the open-loop transfer matrix $W(s)$ [Equation (1.91)], we have

$$W(s) = \exp\{\arg w(s)\} U diag\{\sigma_{Ri} \, |w(s)|\} V^T, \tag{1.96}$$

[32] Recall that, for real matrices, the operations of conjugation and transposition coincide.

where $w(s) = |w(s)| \exp\{j \arg w(s)\}$, and from which it is obvious that for any $s = const$, the singular values of $W(s)$ are equal to the products of singular values σ_{Ri} of R and the magnitude of transfer function $w(s)$. The left and right singular bases of the matrix $W(s)$ coincide, up to complex scalar multiplier $\exp\{j \arg w(s)\}$, with the singular bases of the matrix R.[33] Note also that the ratio of the largest singular value of $W(s)$ to the smallest one does not depend on $w(s)$ and is equal to the ratio of the corresponding singular values of R.

Taking into account Equations (1.92)–(1.94), we have, for the canonical representations of the closed-loop uniform system, transfer matrices with respect to output and error:

$$\Phi(s) = C diag\left\{\frac{\lambda_i w(s)}{1+\lambda_i w(s)}\right\} C^{-1} = \sum_{i=1}^{N} c_i > \frac{\lambda_i w(s)}{1+\lambda_i w(s)} < c_i^+ \tag{1.97}$$

$$\Phi_\varepsilon(s) = C diag\left\{\frac{1}{1+\lambda_i w(s)}\right\} C^{-1} = \sum_{i=1}^{N} c_i > \frac{1}{1+\lambda_i w(s)} < c_i^+. \tag{1.98}$$

All that has been stated in the previous section about the geometrical interpretation of the internal structure of general MIMO systems remains, naturally, valid, and also for uniform systems. Unfortunately, for the closed-loop uniform system transfer matrices $\Phi(s)$ [Equation (1.97)] and $\Phi_\varepsilon(s)$ [Equation (1.98)], there is no well defined relationship between the singular values and singular bases of these matrices, and the corresponding values and bases of the matrix R, as it is in the case of the open-loop transfer matrix $W(s)$. Moreover, the singular bases of the matrices $\Phi(s)$ and $\Phi_\varepsilon(s)$ depend generally on the transfer function $w(s)$ of separate channels. However, from the point of view of practical computing, this fact is not very burdensome and just emphasizes the less intimate connections of the singular values, as compared with the CTFs, with the open-loop and closed-loop MIMO system transfer matrices.

1.3.2 Stability analysis of uniform MIMO systems

The stability of the linear closed-loop uniform system is determined by the distribution of roots of the characteristic equation

$$\det[I + w(s)R] = \prod_{i=1}^{N} [1 + \lambda_i w(s)] = 0, \tag{1.99}$$

which, apparently, is equivalent to a set of N equations

$$1 + \lambda_i w(s) = 0, \quad i = 1, 2, \ldots, N \tag{1.100}$$

or, taking into account Equation (1.90), to a set of equations

$$D(s) + \lambda_i M(s) = 0, \quad i = 1, 2, \ldots, N \tag{1.101}$$

coinciding with characteristic equations of the closed-loop one-dimensional characteristic systems (Sobolev 1973; Gasparyan 1976).

[33] That multiplier with the unity magnitude can be referred to either U or V.

The fact that the CTFs $q_i(s)$ [Equation (1.94)] differ from the common transfer function $w(s)$ of separate channels only by numerical coefficients λ_i considerably simplifies the stability analysis of uniform systems. Here, two formulations of the generalized Nyquist criterion are possible, leading to different graphical techniques and procedures (for simplicity, we shall call them the *direct* and *inverse*). The first of them (the 'direct' formulation) essentially coincides with that presented in Section 1.2.4. The only difference is that for the stability of the closed-loop uniform system, it is necessary that the sum of anticlockwise encirclements of the critical point $(-1, j0)$ by *each* characteristic gain locus (the Nyquist plot) $\lambda_i w(j\omega)$, as the angular frequency ω changes from $-\infty$ to $+\infty$, be equal to k_0, where k_0 is the number of the right half-plane poles of transfer function $w(s)$. If the characteristic gain loci $\lambda_i w(j\omega)$ are plotted only for positive frequencies $\omega \geq 0$, then the mentioned sum must be $k_0/2$. According to this formulation, N Nyquist plots of $\lambda_i w(j\omega)$ are plotted in the complex plane. Each of them is obtained from $w(j\omega)$ by multiplying the magnitude of the latter by $|\lambda_i|$, and subsequently rotating about the origin by angle $\arg \lambda_i$. Next, the location of all Nyquist plots of $\lambda_i w(j\omega)$ with respect to the point $(-1, j0)$ is analysed.

In accordance with the second 'inverse' formulation, Equation (1.100) should be rewritten in the form

$$w(s) = -\frac{1}{\lambda_i}, \quad i = 1, 2, \ldots, N. \tag{1.102}$$

In this case, in the complex plane, we have a *single* locus $w(j\omega)$ (i.e. the Nyquist plot) of identical separate channels and N *critical points* $-1/\lambda_i$. For the stability of the closed-loop uniform system, it is necessary and sufficient that the Nyquist plot of $w(j\omega)$ encircles *each* of N points $-1/\lambda_i$ in an anticlockwise direction k_0 times (Sobolev 1973). In the case of stable $w(s)$, i.e. for $k_0 = 0$, both variants of using the generalized Nyquist criterion are qualitatively illustrated in Figure 1.20 for $N = 4$, in which the characteristic gain loci are plotted for positive frequencies $\omega \geq 0$.

The simple form of CTFs of the open-loop uniform system makes it very convenient using the logarithmical form of the Nyquist criterion, since the Bode magnitude and phase plots of characteristic systems are obtained from the corresponding plots of transfer function $w(s)$

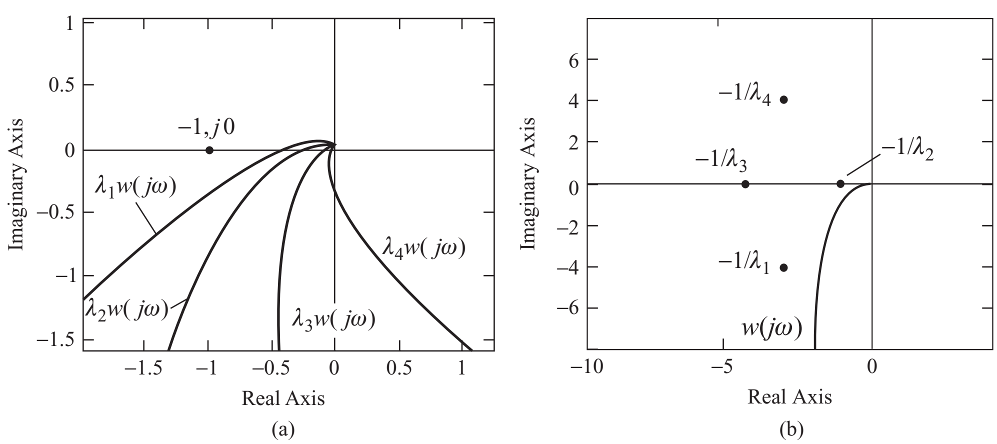

Figure 1.20 Stability analysis of the uniform system with the help of the Nyquist criterion ($N = 4$). (a) 'direct' form; (b) 'inverse' form.

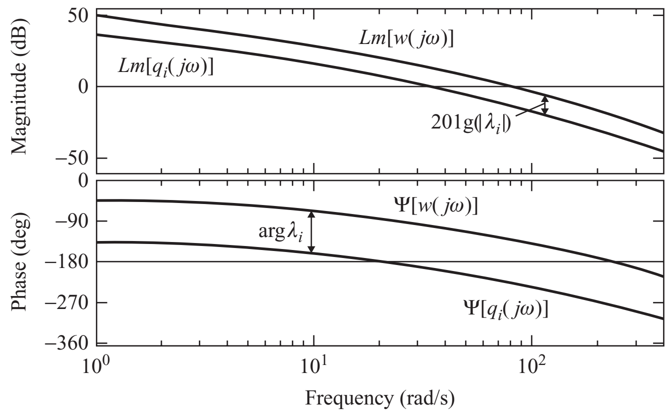

Figure 1.21 Stability analysis of the uniform system by means of the Bode diagram.

by simple shifts along the ordinate axis. The Bode magnitude plot $Lm[w(j\omega)]$ is shifted by magnitudes $Lm[\lambda_i] = 20\,\mathrm{lg}(|\lambda_i|)$, and the Bode phase plot $\Psi[w(j\omega)]$ is shifted by arg λ_i $(i = 1, 2, \ldots, N)$ (Figure 1.21).[34] With respect to the shifted plots, the Nyquist criterion has the standard form known from the classical control theory (Ogata 1970; Kuo 1995).

Example 1.4 Consider the two-axis indirect guidance system described in Example 1.1. Let the transfer functions of both channels be identical, i.e. $W_1(s) = W_2(s) = W_0(s)$, and have the form

$$W_0(s) = \frac{15000000000(s+3)}{s(s+0.33)(s+400)^2(s+500)}. \tag{1.103}$$

With such transfer functions, the two-axis system in Figures 1.7 and 1.8 is uniform. Consider two cases, with different combinations of angles α_1 and α_2: $\alpha_1 = 40^\circ$, $\alpha_2 = 35^\circ$ and $\alpha_1 = 40^\circ$, $\alpha_2 = -35^\circ$, i.e. the magnitudes of α_1 and α_2 in the second case do not change, but the angle α_2 has a negative sign. The matrix of cross-connections [Equation (1.60)]:

$$R = \begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\sin\alpha_2 & \cos\alpha_2 \end{pmatrix} \tag{1.104}$$

for the given combinations of α_1 and α_2 is

$$R_+ = \begin{pmatrix} 0.766 & 0.643 \\ -0.574 & 0.819 \end{pmatrix}, \quad R_- = \begin{pmatrix} 0.766 & 0.643 \\ 0.574 & 0.819 \end{pmatrix}, \tag{1.105}$$

where the matrix R_+ corresponds to the positive angle α_2, and R_- to the negative one. The eigenvalues of R are defined by the following expression:

$$\lambda_{1,2} = \frac{\cos\alpha_1 + \cos\alpha_2}{2} \pm \sqrt{\frac{(\cos\alpha_1 + \cos\alpha_2)^2}{4} - \cos(\alpha_1 - \alpha_2)} \tag{1.106}$$

[34] Further, everywhere, $Lm[\cdot]$ and $\Psi[\cdot]$ denote the Bode magnitude and phase plots of the corresponding transfer functions.

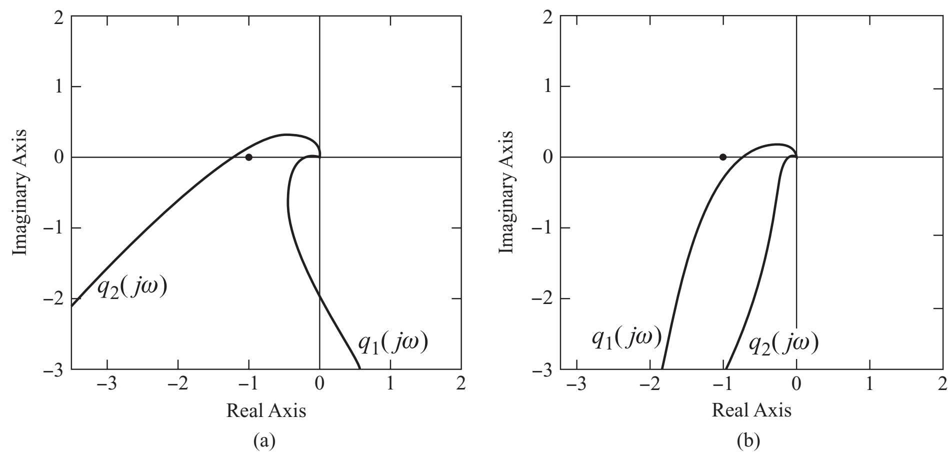

Figure 1.22 Stability analysis of the uniform guidance system ('direct' form). (a) $\alpha_1 = 40^\circ$, $\alpha_2 = 35^\circ$; (b) $\alpha_1 = 40^\circ$, $\alpha_2 = -35^\circ$.

and, correspondingly, are equal to

$$\lambda_1 = 0.7926 + j0.6066, \quad \lambda_2 = 0.7926 - j0.6066, \quad \text{for } R_+, \tag{1.107}$$

and

$$\lambda_1 = 1.4004, \quad \lambda_2 = 0.1848, \quad \text{for } R_-, \tag{1.108}$$

i.e. in the case of positive angle α_2, the eigenvalues are complex conjugate and, in the second case, they are real. The Nyquist plots of the uniform guidance system with the given combinations of α_1 and α_2 are shown in Figures 1.22 and 1.23, where Figure 1.22 corresponds to the

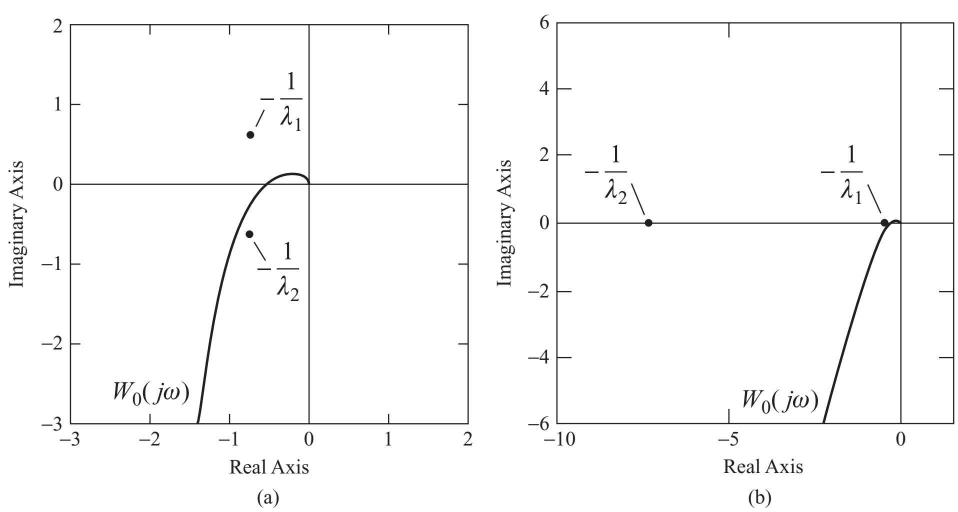

Figure 1.23 Stability analysis of the uniform guidance system ('inverse' form). (a) $\alpha_1 = 40^\circ$, $\alpha_2 = 35^\circ$; (b) $\alpha_1 = 40^\circ$, $\alpha_2 = -35^\circ$.

first ('direct') formulation of the Nyquist criterion and Figure 1.23 to the second ('inverse') formulation. These figures show that the stability of the guidance system depends considerably on the sign of α_2. For the positive value of α_2 and complex conjugate eigenvalues [Equation (1.106)], the system is unstable and, for the negative α_2 and real eigenvalues, it is stable. That fact is not a mandatory rule but, in practice, the complex conjugate eigenvalues of the matrix R are usually more 'dangerous' from the stability viewpoint. In conclusion, we point out that the gain and phase margins of the uniform guidance system for $\alpha_2 = -35^\circ$ are determined by the first characteristic system ($\lambda_1 = 1.4004$) and are equal, respectively, to $GM = 3.16$ dB and $PM = 15.36^\circ$.

Example 1.5 The second example in this section can serve as a visual illustration that the CTFs method allows not only the obtaining of quantitative data and estimates, but also the drawing of qualitative conclusions about stability margins of MIMO systems. Consider the same uniform guidance system as in the previous example, assuming angles α_1 and α_2 are equal, i.e. $\alpha_1 = \alpha_2 = \alpha$, and find the value of angle α for which the system reaches the stability boundary.

Geometrically, the equal values of angles α_1 and α_2 mean that the 'measurement' coordinate system OX_1X_2, composed of the sensitivity axes of the stellar sensors, is orthogonal, and is rotated by angle α with respect to the telescope-fixed orthogonal coordinate system ('guidance' coordinate system) OY_1Y_2 (Figure 1.24).

The matrix R [Equation (1.60)] in this case has the form

$$R = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{1.109}$$

and is antisymmetrical (Gasparyan 1976) and, more significantly for us at this point, orthogonal, i.e. $\det R = 1$ and $R^{-1} = R^T$ (Bellman 1970). Eigenvalues [Equation (1.106)] of R have a simple form and are expressed as

$$\lambda_{1,2} = \cos\alpha \pm j\sin\alpha = \exp\{\pm j\alpha\} \tag{1.110}$$

i.e. they are located on the unit circle and form angles $\pm\alpha$ with the positive direction of the real axis. For $\alpha = 0^\circ$, the coordinate systems OX_1X_2 and OY_1Y_2 in Figure 1.24 coincide, the matrix R [Equation (1.109)] becomes the unit matrix I, and both eigenvalues $\lambda_{1,2}$ [Equation (1.110)] are equal to unity. Structurally, this means that the two-axis guidance system splits

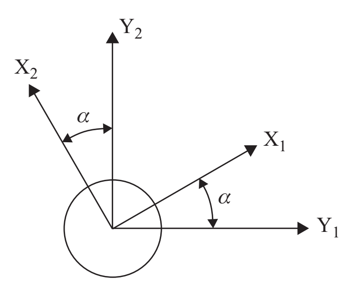

Figure 1.24 Reciprocal location of the coordinate systems for $\alpha_1 = \alpha_2 = \alpha$.

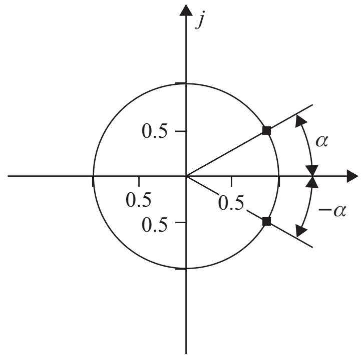

Figure 1.25 Eigenvalues location on the unit circle.

into two independent channels. For $\alpha \neq 0$, there appear cross-connections, and eigenvalues $\lambda_{1,2}$ move from the point $(+1, j0)$ in the opposite directions along the unit circle (Figure 1.25). The CTFs $q_1(j\omega)$ and $q_2(j\omega)$ of the open-loop system are

$$q_1(j\omega) = \exp\{j\alpha\}W_0(j\omega), \quad q_2(j\omega) = \exp\{-j\alpha\}W_0(j\omega), \tag{1.111}$$

i.e. the first characteristic gain locus $q_1(j\omega)$ is obtained from the Nyquist plot of $W_0(j\omega)$ by the anticlockwise rotation of $W_0(j\omega)$ about the origin through the angle α, and the second locus $q_2(j\omega)$ is obtained by the clockwise rotation of $W_0(j\omega)$ through the same angle. In the case of 'inverse' formulation of the Nyquist criterion, graphical plotting is carried out on the plane of the single Nyquist plot of $W_0(j\omega)$, where critical points $-1/\lambda_{1,2}$ coincide with the point $(-1, j0)$ for $\alpha = 0°$ and, for $\alpha \neq 0$, they move along the unit circle, forming with the negative real axis angles $-\alpha$ and $+\alpha$. All this is illustrated in Figure 1.26, from which it is finally clear that the critical value of angle α, for which the cross-connected guidance system reaches the

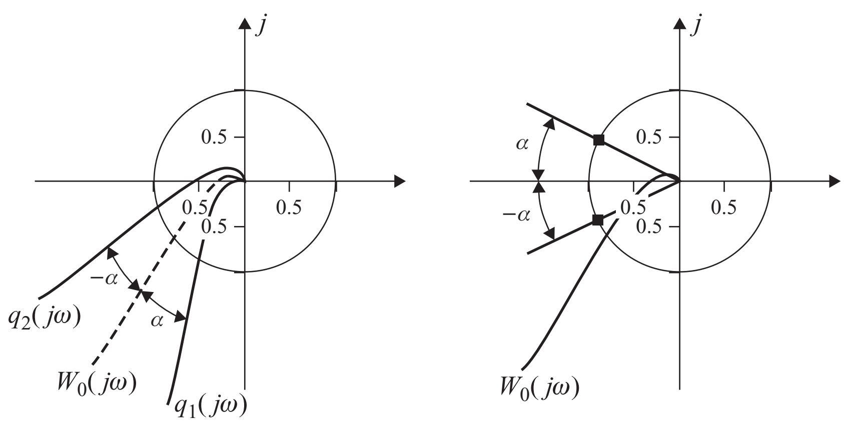

Figure 1.26 Stability analysis of the guidance system for $\alpha_1 = \alpha_2$. (a) 'direct' form; (b) 'inverse' form.

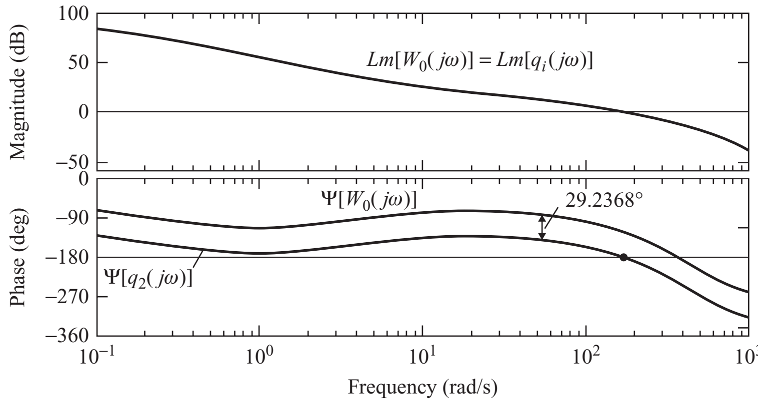

Figure 1.27 Bode diagram of the guidance system for $\alpha = 29.2368°$.

stability boundary, is equal to *the phase margin* of identical isolated separate channels of the system. This inference is general for two-dimensional uniform systems with the orthogonal matrix of cross-connections R [Equation (1.109)] and is valid for any transfer function $W_0(s)$ of separate channels. For our system with transfer function $W_0(j\omega)$ [Equation (1.103)], the phase margin of the separate channel is $PM = 29.2368°$. Letting the angle α be equal to that value, the guidance system is on the stability boundary. The Bode diagram given in Figure 1.27 verifies that conclusion. In this figure, the bold dot at the intersection of the line $-180°$ with the Bode phase plot of the second characteristic system is mapping of the point $(-1, j0)$. Note that the Bode magnitude plots of the CTFs $q_1(j\omega)$ and $q_2(j\omega)$ coincide with the corresponding plot of $W_0(j\omega)$ (since $|\lambda_1| = |\lambda_2| = 1$), and the Bode phase plots of $q_1(j\omega)$ and $q_2(j\omega)$ are obtained from the phase plot of $W_0(j\omega)$ by the parallel shifts $\pm\alpha$ along the ordinate axis.

Thus, in the considered example, we have obtained not only the *quantitative*, but also the *qualitative* information about the stability margins of the two-dimensional uniform system with an orthogonal matrix of cross-connections. This confirms the effectiveness of approaches based on the CTFs.

Remark 1.9 As we know, in the case of general linear MIMO systems, the CTFs are *algebraic functions* and are situated on different sheets of a unique Riemann surface, and the transition from one branch of the characteristic gain loci to another is related to the so-called *branch points*.[35] A natural question arises of whether the specific structural features of uniform MIMO systems lead, from that point of view, to some peculiarities, if we take into account that the CTFs of these systems have quite a simple form [Equation (1.105)] and can readily be found analytically. To answer this question from the general position of the CTFs method, we substitute the transfer matrix of the open-loop uniform system [Equation (1.102)] into Equation (1.26), from which, after some elementary transformations, we obtain

$$\det[qI - C diag\{\lambda_i w(s)\}C^{-1}] = \det[C diag\{q - \lambda_i w(s)\}C^{-1}] = \prod_{i=1}^{N}[q - \lambda_i w(s)] = 0. \tag{1.112}$$

[35] See Remark 1.3.

This expression shows that Equation (1.26), the solution to which gives the CTFs of linear MIMO systems, in the case of uniform systems, splits into N first-order equations, linear with respect to q. Then, based on the general theory (Postlethwaite and MacFarlane 1979), we can state that in the case of uniform systems, we have N *isolated* algebraic functions, and each of them is situated on an individual *one-sheeted* Riemann surface. In other words, the CTFs of uniform systems, being proper rational functions in complex variable s, *do not have branch points* and thereby can be treated, not in the least disregarding the mathematical rigour, as a set of N independent SISO systems.

Remark 1.10 The matrix block diagram of Figure 1.18 is, broadly speaking, typical for uniform MIMO systems. At the same time, in different technical applications, systems having far more complicated structures, with several contours of feedforward or/and feedback connections occur; these systems may formally be viewed as uniform, since all their separate channels are identical, and different blocks of cross-connections are described by some numerical matrices (Sobolev 1973). For example, one such structure is depicted in Figure 1.28, where R_1 and R_2 are numerical matrices, and $w_1(s)$ and $w_2(s)$ are scalar transfer functions. Unfortunately, not all such MIMO systems with identical channels and several matrices of rigid cross-connections can be referred to as uniform systems, and the point is that different numerical matrices of cross-connections are generally brought to diagonal form in different canonical bases. However, it is also possible to select among such MIMO systems those that satisfy the uniformity requirements and can be studied on the base of the techniques developed in this section. The fact is that, as is well known from the theory of matrices (Marcus and Minc 1992), different square matrices (in our case, R_1 and R_2) can be brought simultaneously to diagonal form in a certain basis if they are *commutative*, i.e. $R_1R_2 = R_2R_1$.

From here, we have the following main rule determining when a MIMO system belongs to the class of uniform systems:

A MIMO system is uniform if all separate channels of the system are identical, i.e. all dynamical blocks constituting the system are described by some scalar transfer matrices, and all numerical matrices of cross-connections are commutative.

The indicated condition of commutativity actually means that all matrix blocks of the uniform MIMO system can be interchanged, and that condition reinforces the analogy between methods of study of uniform systems and common SISO systems. Accordingly, if the mentioned condition is not satisfied, then the system must be regarded as a general MIMO system, and its study can be accomplished only by the corresponding methods and computational procedures.

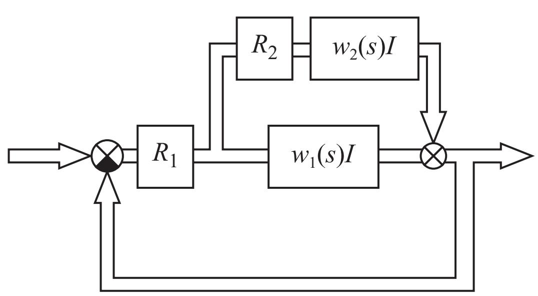

Figure 1.28 Definition of the notion ‘uniform MIMO system’.

1.4 NORMAL MIMO SYSTEMS

So far, we have considered general and uniform MIMO systems, paying no special attention to geometrical features or peculiarities of their canonical bases. In this section, we discuss the so-called *normal* systems,[36] which constitute a significant class of multivariable control systems with individual attributes and characteristics inherent only to that class.

1.4.1 Canonical representations of normal MIMO systems

A linear MIMO system is said to be *normal* if the open-loop transfer matrix of the system is normal, i.e. commutes with its conjugate (complex conjugate and transposed) matrix:

$$W(s)W^*(s) = W^*(s)W(s) \tag{1.113}$$

It is easy to show that normality of $W(s)$ implies normality of the closed-loop MIMO system transfer matrices $\Phi(s)$ and $\Phi_\varepsilon(s)$, and, vice versa, normality of $\Phi(s)$ or $\Phi_\varepsilon(s)$ implies normality of $W(s)$. In other words, instead of Equation (1.113), we can write two equivalent conditions of the MIMO system normality:

$$\Phi(s)\Phi^*(s) = \Phi^*(s)\Phi(s) \tag{1.114}$$

and

$$\Phi_\varepsilon(s)\Phi_\varepsilon^*(s) = \Phi_\varepsilon^*(s)\Phi_\varepsilon(s). \tag{1.115}$$

The primary, principal property of normal MIMO systems consists in the orthogonality of their canonical bases composed of the normalized eigenvectors $c_i(s)(|c_i(s)| = 1)$ of $W(s)$. Owing to the conditions in Equation (1.32), the dual basis of a normal MIMO system is also orthogonal and identically coincides with the canonical basis (Derusso *et al.* 1965; Gasparyan 1976, 1986). As will be shown in the following chapters, all this predetermines a number of remarkable features of normal MIMO systems, essentially distinguishing them from all other types of MIMO systems.

Since the canonical basis of normal MIMO systems is orthonormal, the modal matrix $C(s)$ belongs to the class of *unitary* matrices for which $C^{-1}(s) = C^*(s)$, i.e. the inverse matrix coincides with conjugate. From that, and taking into account the coincidence of the dual and canonical bases ($c_i^+(s) = c_i(s)$), we obtain the following canonical representations of the transfer matrices of normal MIMO systems via the similarity transformation and dyadic designations:

$$W(s) = C(s)diag\{q_i(s)\}C^*(s) = \sum_{i=1}^{N} c_i(s) > q_i(s) < c_i(s) \tag{1.116}$$

$$\Phi(s) = C(s)diag\left\{\frac{q_i(s)}{1+q_i(s)}\right\}C^*(s) = \sum_{i=1}^{N} c_i(s) > \frac{q_i(s)}{1+q_i(s)} < c_i(s) \tag{1.117}$$

$$\Phi_\varepsilon(s) = C(s)diag\left\{\frac{1}{1+q_i(s)}\right\}C^*(s) = \sum_{i=1}^{N} c_i(s) > \frac{1}{1+q_i(s)} < c_i(s). \tag{1.118}$$

[36] The term *normal* MIMO system originates from the theory of matrices (Bellman 1970; Gantmacher 1964).

To normal systems belong circulant, anticirculant, simple symmetrical and antisymmetrical MIMO systems (Gasparyan 1976, 1981, 1986), including two-dimensional systems with symmetrical and antisymmetrical cross-connections (Barski 1966; Kazamarov *et al.* 1967; Krassovski 1957) frequently occurring in various technical applications.

Consider now the singular value decomposition of the normal MIMO system. In Section 1.2.5, it was indicated that with each transfer matrix of the open- or closed-loop MIMO system, there are two associated orthonormal bases called the *left* and *right* (or *input* and *output*) *singular bases* of the MIMO system, as well as a set of N real numbers called the *singular values*. Generally, each transfer matrix $W(s)$, $\Phi(s)$ or $\Phi_\varepsilon(s)$ has its own set of singular bases and singular values, and there is no explicit relationship among the sets belonging to different matrices. As was pointed out in Section 1.2.5, the left singular basis of a square matrix A consists of eigenvectors of the Hermitian matrix AA^*, and the right singular basis consists of eigenvectors of the Hermitian matrix A^*A; the singular values of A are equal to positive values of the square roots taken of the eigenvalues of AA^* (or A^*A). Let us compose the corresponding Hermitian matrices for the transfer matrix $W(s)$ of the open-loop normal MIMO system. Taking into account that for unitary matrices, the inverse matrix coincides with its conjugate yields

$$\begin{aligned} W(s)W^*(s) &= C(s)diag\{q_i(s)\}\underbrace{C^*(s)C(s)}_{I}diag\{\tilde{q}_i(s)\}C^*(s) \\ &= C(s)diag\{|q_i(s)|^2\}C^*(s), \end{aligned} \tag{1.119}$$

from which we come to an extremely important conclusion that for the open-loop normal system, *the left and right singular bases coincide with each other and coincide with the canonical basis, and singular values are equal to the magnitudes of the CTFs* $\mathrm{q_i(s)}$ *of the open-loop characteristic systems*. Proceeding in the same manner, we find for the transfer matrices $\Phi(s)$ and $\Phi_\varepsilon(s)$ of the closed-loop normal system:

$$\Phi(s)\Phi^*(s) = C(s)diag\left\{\left|\frac{q_i(s)}{1+q_i(s)}\right|^2\right\}C^*(s) \tag{1.120}$$

$$\Phi_\varepsilon(s)\Phi_\varepsilon^*(s) = C(s)diag\left\{\frac{1}{|1+q_i(s)|^2}\right\}C^*(s). \tag{1.121}$$

Hence, *all singular bases of the closed-loop normal MIMO system coincide with the canonical basis, and the singular values are equal to magnitudes of the corresponding transfer functions of the closed-loop SISO characteristic systems*. The spectral norms of the transfer matrices $W(s)$, $\Phi(s)$ and $\Phi_\varepsilon(s)$ are equal to the *largest of the CTFs magnitudes*, i.e.

$$\|W(s)\| = \max_i(|q_i(s)|),\ \|\Phi(s)\| = \max_i\left(\left|\frac{q_i(s)}{1+q_i(s)}\right|\right),\ \|\Phi_\varepsilon(s)\| = \max_i\left(\frac{1}{|1+q_i(s)|}\right). \tag{1.122}$$

Thus, in the case of the normal MIMO system, all singular bases coincide with the canonical basis, and the singular values are directly expressed through the transfer functions of the corresponding characteristic systems, i.e. here, we have a unified and visual internal geometrical structure of the system.

We proceed now to the case of normal uniform systems. We show first that the normality condition of the uniform system is entirely determined by properties of the numerical matrix of cross-connections R (Figure 1.18) and does not depend on the transfer function $w(s)$ of identical separate channels. Let the uniform system be normal. Then, from Equations (1.113) and (1.91), taking into account the equality $w(s)w^*(s) = |w(s)|^2$ and knowing that for real matrices, the operations of conjugation and transposing are equivalent, instead of Equation (1.113), we have

$$|w(s)|^2 RR^T = |w(s)|^2 R^T R. \tag{1.123}$$

From here, canceling the common factor $|w(s)|^2$ on both sides, we finally obtain the following condition of normality of the uniform system:

$$RR^T = R^T R, \tag{1.124}$$

which has the form of the condition of normality of the real matrix R (Gantmacher 1964), well known in the theory of matrices. We will not write out the expressions for transfer matrices of normal uniform systems – they evidently ensue from Equations (1.119)–(1.122), and the reader is able to do it without difficulty on his own.

To normal systems belong the uniform systems with circulant and anticirculant, symmetrical and antisymmetrical, as well as with orthogonal matrices of cross-connections, that is the types of uniform systems described most in the scientific and technical literature (Gasparyan 1976, 1986). By antisymmetrical matrices, here, we mean, as it is accepted in the multivariable control theory (Chorol *et al.* 1976), such matrices R that can be represented in the form

$$R = rI + R_\circ, \tag{1.125}$$

where r is a scalar, and $R_\circ$ is a usual skew-symmetrical matrix (Gantmacher 1964) satisfying the condition $R_\circ = -R_\circ^T$.[37]

In conclusion, note that in the case of normal MIMO systems, the Euclidian norms ('lengths') of the input, output and error vectors are, by pairs, the same in both the natural and canonical bases. This is explained by the fact that magnitudes of the vectors are invariant under the similarity transformations via the unitary modal matrix $C(s)$ (or C for uniform systems) (Derusso *et al.* 1965).

1.4.2 Circulant MIMO systems

In the last part of this section, we consider two special subclasses of normal systems, namely the circulant and anticirculant MIMO systems. The chief distinctive feature of these systems is independence of their canonical bases from s. Generally, in contrast to normal uniform systems, these systems can have different *dynamical* cross-connections between the separate channels. Moreover, the class of normal uniform systems includes the corresponding subclasses

[37] To such uniform systems belong the two-axis guidance system with the orthogonal (and, at the same time, antisymmetrical) matrix of cross-connections R [Equation (1.109)] discussed in Example 1.5.

of systems which, by the structure of the numerical matrix of cross-connections, also belong to circulant or anticirculant systems.

The distinctive feature of *circulant* MIMO systems is that their transfer matrices $W(s)$, $\Phi(s)$ and $\Phi_\varepsilon(s)$are circulant. Recall that in a circulant matrix, each subsequent row is obtained from the preceding row by shifting all elements (except for the Nth) by one position to the right; the Nth element of the preceding row then becomes the first element of the following row (Davis 1979; Voevodin and Kuznetsov 1984). For the matrix $W(s)$, this looks like this:

$$W(s) = \begin{pmatrix} w_0(s) & w_1(s) & w_2(s) & \ldots\ldots\ldots & w_{N-1}(s) \\ w_{N-1}(s) & w_0(s) & w_1(s) & \ldots\ldots & w_{N-2}(s) \\ w_{N-2}(s) & w_{N-1}(s) & w_0(s) & \ldots\ldots & w_{N-3}(s) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ w_1(s) & w_2(s) & w_3(s) & \ldots\ldots\ldots & w_0(s) \end{pmatrix}. \tag{1.126}$$

Each diagonal of a circulant matrix consists of the same elements, and the diagonals located at the same distance from the lower left corner and from the principal diagonal consist of identical elements.[38] Physically, this means that in the circulant MIMO system, it is possible to single out some groups of subsystems with identical transfer functions of all cross-connections, i.e. having internal symmetry. The MIMO systems described by circulant matrices constitute a significant class of multivariable control systems especially widespread in process control. Industrial and other examples of circulant systems include cross-directional control of paper machines if the edge effects are neglected, multizone crystal growth furnaces, dyes for plastic films, burner furnaces, some gyroscopic platforms (Hovd and Skogestad 1992, 1994a; Chorol *et al.* 1976; Sobolev 1973) and many others.

It is easy to see that any circulant matrix is completely defined by the first (or any other) row. Using the designations $w_0(s)$, $w_i(s)(i = 1, 2, \ldots, N-1)$[39] [Equation (1.126)] for the first row of the circulant matrix $W(s)$, the latter can be represented in the matrix polynomial form

$$W(s) = w_0(s)I + \sum_{k=1}^{N-1} w_k(s)U^k, \tag{1.127}$$

where I is the unit matrix and

$$U = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{1.128}$$

is the orthogonal *permutation matrix* (Marcus and Minc 1992).[40] Since the permutation matrix U plays a crucial role in the theory of circulant MIMO systems, let us consider it at more

[38] From the general positions of the theory of matrices, the circulant matrices belong to the so-called *Toeplitz* matrices (Voevodin and Kuznetsov 1984).

[39] These designations differ from those used before for describing general MIMO systems, but are more convenient in this case.

[40] In essence, the permutation matrix U can be regarded as the simplest circulant matrix.

length. Like all orthogonal matrices (Gantmacher 1964), the matrix U satisfies the following conditions: $U^{-1} = U^T$, $\det U = -1$ (the so-called *improper* orthogonality); all eigenvalues of U have unit magnitudes. If we multiply U by a vector x, then the first component x_1 becomes the Nth, the second component becomes the first, etc. On increasing the power of U, both nonzero diagonals shift to the place of the next diagonal on the right, and the powers of U satisfy the following conditions that can be readily checked by direct calculations:

$$U^k = (U^{N-k})^T = (U^{N-k})^{-1}, \quad U^N = I, \quad k = 1, 2, \ldots, N-1. \tag{1.129}$$

The eigenvalues β_i ($i = 1, 2, \ldots, N$) of the permutation matrix U are the roots of the characteristic equation

$$\det[\beta I - U] = \beta^N - 1 = 0 \tag{1.130}$$

and, for any N, are expressed in the analytical form

$$\beta_i = \exp\left\{ j\frac{2\pi(i-1)}{N} \right\}, \quad i = 1, 2, \ldots, N. \tag{1.131}$$

Geometrically, the roots β_i are situated in the complex plane at the vertices of a regular N-sided polygon inscribed in the unit circle, and the first root β_1 is always real and equal to unity (Figure 1.29). The normalized eigenvectors c_i of U have a very simple form (Bellman 1970):

$$c_i = \frac{1}{\sqrt{N}}\left[1 \;\; \beta_i \;\; \beta_i^2 \ldots . \beta_i^{N-1}\right]^T, \quad i = 1, 2, \ldots ., N. \tag{1.132}$$

Note that since the permutation matrix U belongs to normal matrices, the modal matrix C of U composed of c_i [Equation (1.132)] is unitary, i.e. $C^{-1} = C^*$. It is worth emphasizing that all components of the vectors c_i [Equation (1.132)] have the same magnitudes, equal to $1/\sqrt{N}$. This fact plays an important role in analyzing self-oscillations in nonlinear circulant systems (see Chapter 3).

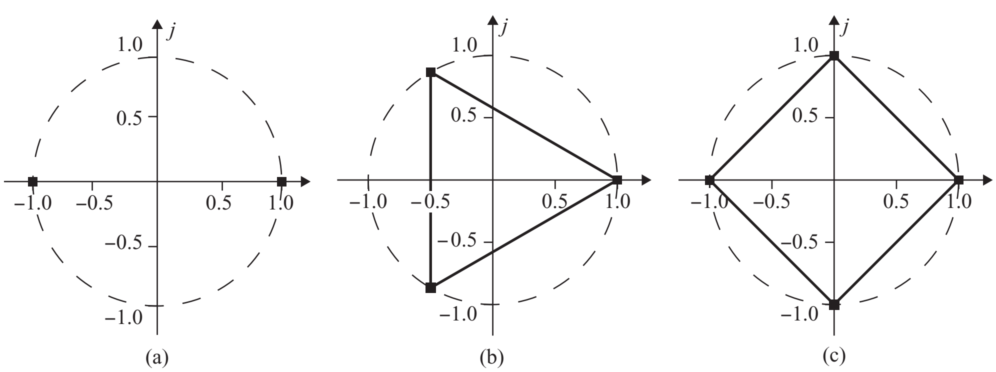

Figure 1.29 The eigenvalues of the permutation matrix U [Equation (1.128)]. (a) $N = 2$; (b) $N = 3$; (c) $N = 4$.

In the theory of matrices, it is proved that if a square matrix A is represented as the matrix polynomial in some matrix B, then the eigenvalues of A are equal to the values of the corresponding scalar polynomials that are obtained from the matrix polynomial by replacing B with the eigenvalues of the latter. Also, the eigenvectors of the matrices A and B, corresponding to the associated eigenvalues, coincide (Marcus and Minc 1992). Applying this to the circulant matrix $W(s)$ [Equation (1.127)] yields that the CTFs $q_i(s)$ can be represented, for any number N of separate channels, in the analytical form

$$q_i(s) = w_0(s) + \sum_{k=1}^{N-1} w_k(s) \exp\left\{ j \frac{2\pi(i-1)}{N} k \right\}, \quad i = 1, 2, \ldots, N. \tag{1.133}$$

Further, from the above statement concerning eigenvectors of matrix polynomials, it follows that the canonical basis of the circulant matrix $W(s)$, i.e. the canonical basis of the circulant system (and, naturally, the modal matrix C), is inherited from the permutation matrix U [Equation (1.128)].

The possibility of representing the CTFs $q_i(s)$ in analytical form for any N considerably simplifies the study of circulant systems. It is interesting and important to note that canonical bases of circulant systems do not depend on complex variable s and are the same for all circulant systems with the same number of channels N, independently of the specific form of the transfer functions $w_0(s)$ and $w_k(s)(k = 1, 2, \ldots, N-1)$.

Consider briefly the uniform circulant MIMO systems. It is easy to understand that for a uniform system to be circulant, it is necessary and sufficient that the matrix of cross-connections R be circulant. Denoting by $r_\circ$ and r_k $(k = 1, 2, \ldots, N-1)$ the elements of the first row of R, we obtain for the matrix $W(s)$ and the CTFs $q_i(s)$ of the circulant uniform system the following expressions:

$$W(s) = w(s)[r_\circ I + \sum_{k=1}^{N-1} r_k U^k] \tag{1.134}$$

and

$$q_i(s) = \left[r_\circ + \sum_{k=1}^{N-1} r_k \exp\left\{ j \frac{2\pi(i-1)}{N} k \right\} \right] w(s), \quad i = 1, 2, \ldots, N, \tag{1.135}$$

where $w(s)$ is the scalar transfer function of identical separate channels. The canonical basis and modal matrix of the circulant uniform system are also inherited, evidently, from the permutation matrix U.

Circulant MIMO systems can be symmetrical and, for odd N, antisymmetrical. A circulant system is said to be *symmetrical* if the open-loop transfer matrix $W(s)$ [together with $\Phi(s)$ and $\Phi_\varepsilon(s)$] is symmetrical, i.e. if the condition $W(s) = W^T(s)$ holds. In view of the structural cyclicity of circulant matrices, for a circulant system to be symmetrical, it is enough that the following equalities take place:

$$w_k(s) = w_{N-k}(s), \quad k = 1, 2, \ldots, N-1. \tag{1.136}$$

1.4.2.1 Simple symmetrical systems

Among symmetrical circulant systems, the so-called *simple symmetrical* MIMO systems demand special attention, for which the transfer functions of all cross-connections coincide (Sobolev 1973; Gasparyan 1981, 1986; Hovd and Skogestad 1992, 1994a), i.e.

$$w_k(s) = w_1(s), \quad k = 2, \ldots, N-1. \tag{1.137}$$

A significant feature of simple symmetrical MIMO systems is that they have only *two distinct CTFs*. Substituting Equation (1.137) into Equation (1.133) and carrying out some simple transformations yields

$$q_1(s) = w_0(s) + (N-1)w_1(s) \tag{1.138}$$

and

$$q_2(s) = q_3(s) = \ldots\ldots = q_N(s) = w_0(s) - w_1(s). \tag{1.139}$$

Hence, in the case of simple symmetrical MIMO systems, there exist, independently of the number of separate channels N, *only two* transfer functions of the characteristic systems *differing from each other*. In multivariable control theory, the first function $q_1(s)$ [Equation (1.138)] is frequently called the transfer function of the *average motion*, and all the others [Equation (1.139)] are called the transfer functions of the *relative motions*. These terms have a simple explanation. Let the vectors y and x be related by a simple symmetrical matrix $W(s)$, i.e.

$$y = W(s)x \tag{1.140}$$

or, in the expanded form,

$$y_i = w_0(s)x_i + w_1(s)\sum_{\substack{k=1\\k\neq i}}^{N} x_k, \qquad i = 1, 2, \ldots, N. \tag{1.141}$$

Let us introduce the so-called *average* scalar coordinates

$$\bar{y} = \frac{1}{N}\sum_{k=1}^{N} y_k, \qquad \bar{x} = \frac{1}{N}\sum_{k=1}^{N} x_k, \tag{1.142}$$

having an obvious physical sense. Then, summing N equations [Equation (1.141)] term by term and performing simple transformations, we obtain the relationship between the average coordinates $\bar{y}$ and $\bar{x}$ in the form

$$\bar{y} = [w_0(s) + (N-1)w_1(s)]\,\bar{x}. \tag{1.143}$$

The comparison of this expression with Equation (1.138) completely explains the name for $q_1(s)$: *the transfer function of the average motion.*

Let us determine now the difference between any two (say, the rth and the kth) equations in Equation (1.141). We obtain

$$y_r - y_k = [w_0(s) - w_1(s)](x_r - x_k). \tag{1.144}$$

The comparison of this equation with Equation (1.139) shows that *the transfer functions of the relative motions* relate to each other the differences between the output and input variables of any two channels of the simple symmetrical system. Besides, if we introduce $N-1$ *relative* coordinates y_k^r and x_k^r for the components y_k and x_k ($k = 2, \ldots, N$), defining them as the deviation of each component y_k, x_k from the corresponding *average* values of $\bar{y}$ and $\bar{x}$: $y_k^r = y_k - \bar{y}$, $x_k^r = x_k - \bar{x}$, then it is easy to show that these variables are also related by the CTFs [Equation (1.139)]. In other words, the CTFs of relative motions can also be viewed as the functions relating the *deviations* from the corresponding *average motion* of the output and input variables of separate channels.

1.4.2.2 Antisymmetrical circulant systems

A circulant system with an *odd* number of separate channels N is called *antisymmetrical* if elements of the first row of the transfer matrix $W(s)$ satisfy the conditions:

$$w_k(s) = -w_{N-k}(s), \quad k = 1, 2, \ldots, N-1 \tag{1.145}$$

and the matrix $W(s)$ can be represented in the form

$$W(s) = w_0(s)I + \underbrace{\left\{\sum_{k=1}^{(N-1)/2} w_k(s)[U^k - (U^k)^{-1}]\right\}}_{A}, \tag{1.146}$$

where the matrix A is skew-symmetrical, i.e. $A = -A^T$.

The CTFs $q_i(s)$ of antisymmetrical circulant systems have the form

$$q_i(s) = w_0(s) + j2\sum_{k=1}^{(N-1)/2} w_k(s)\sin\left(\frac{2\pi(i-1)}{N}k\right), \quad i = 1, 2, \ldots, N \tag{1.147}$$

and it is worth mentioning that for any odd N, the first CTF $q_1(s)$ always coincides with the transfer function of direct channels $w_0(s)$. At last, note that if all transfer functions of cross-connection are the same, i.e. together with the condition of 'correct signs' [Equation (1.145)], the condition in Equation (1.137) holds, then such systems are called *simple antisymmetrical* circulant systems. Simple antisymmetrical systems of odd order N are described by Equations (1.145) and (1.147), assuming that $w_k(s) = w_1(s)$ for all $k = 2, \ldots, N-1$.

In conclusion, circulant systems of even order N can be antisymmetrical only if the transfer function $w_{N/2}(s)$ is identically equal to zero, which is unacceptable in most practical tasks. In particular, the above expressions do not allow describing such important in practice systems as two-dimensional systems with identical channels and antisymmetrical cross-connections. In the case of an even number of channels, to study the simple antisymmetrical systems, we must apply the theory of anticirculant MIMO systems discussed in the next section.

Example 1.6 In Chorol *et al.* (1976), a three-axis stabilized platform in which the measurements of angular velocities are accomplished by single-channel two-dimensional gyroscopic devices with amplitude-phase modulation is considered. The open-loop transfer matrix of that system has the form

$$W(s) = w(s)\begin{pmatrix} w_0(s) & w_1(s) & -w_1(s) \\ -w_1(s) & w_0(s) & w_1(s) \\ w_1(s) & -w_1(s) & w_0(s) \end{pmatrix}, \tag{1.148}$$

where the scalar multiplier $w(s)$ corresponds to the transfer functions of identical channels in the system power part. As evident from Equation (1.148), the control system belongs to simple antisymmetrical systems and can be handled by the above methods. Substituting the elements of $W(s)$ [Equation (1.148)] into Equation (1.147) immediately yields

$$q_1(s) = w(s)w_0(s), \quad q_{2,3}(s) = w(s)[w_0(s) \pm j\sqrt{3}w_1(s)]. \tag{1.149}$$

This completely agrees with the results developed in Chorol *et al.* (1976) based on the *complex coordinates and complex transfer functions method*.

Example 1.7 For the last two to three decades, the so-called *hexapods* have found wide application in various technical branches, such as active vibration isolation, control of the secondary mirror for large telescopes, laboratory testing devices, etc. (Geng and Haynes 1994; Pernechele *et al*. 1998; Joshi and Kim 2004, 2005). Physically, a hexapod consists of a movable payload platform connected to the fixed base by six variable-length struts. The length of each strut can be controlled independently by six linear actuators and sensors, to achieve independent translational and rotational motions along the *X*-, *Y*- and *Z*-axes. The simplified kinematic scheme of the hexapod is shown in Figure 1.30.

In general, hexapods provide six degrees of freedom to the platform. If each pair of struts works synchronously to increase or decrease their length, then the hexapod has three degrees of freedom. In view of the constructional symmetry of hexapods, the corresponding control systems are described by the matrices of order 3 × 3 and 6 × 6, having the following form

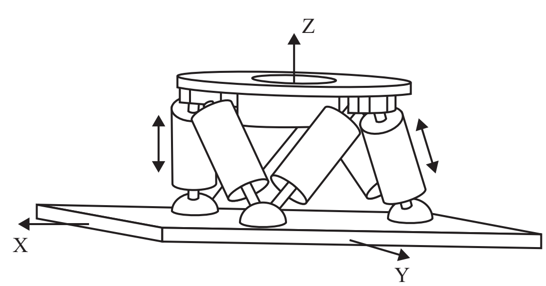

Figure 1.30 The kinematics of a hexapod.

(Joshi and Kim 2004):

$$W_{H3}(s) = \begin{pmatrix} w_0(s) & w_1(s) & w_1(s) \\ w_1(s) & w_0(s) & w_1(s) \\ w_1(s) & w_1(s) & w_0(s) \end{pmatrix},$$

$$W_{H6}(s) = \begin{pmatrix} w_0(s) & w_1(s) & w_1(s) & w_1(s) & w_1(s) & w_1(s) \\ w_1(s) & w_0(s) & w_1(s) & w_1(s) & w_1(s) & w_1(s) \\ w_1(s) & w_1(s) & w_0(s) & w_1(s) & w_1(s) & w_1(s) \\ w_1(s) & w_1(s) & w_1(s) & w_0(s) & w_1(s) & w_1(s) \\ w_1(s) & w_1(s) & w_1(s) & w_1(s) & w_0(s) & w_1(s) \\ w_1(s) & w_1(s) & w_1(s) & w_1(s) & w_1(s) & w_0(s) \end{pmatrix}. \tag{1.150}$$

As we can see from Equation (1.150), the control systems for hexapods belong to simple symmetrical systems. Let the transfer functions $w_0(s)$ and $w_1(s)$ in Equation (1.150) be

$$w_0(s) = \frac{1}{s(0.2s+1)}, \quad w_1(s) = \frac{0.4}{0.1s+1}. \tag{1.151}$$

Then, the CTFs of average motion [Equation (1.138)] are

$$\begin{aligned} q_1^{H3}(s) &= w_0(s) + 2w_1(s) = \frac{0.16s^2 + 0.9s + 1}{0.02s^3 + 0.3s^2 + s} \\ q_1^{H6}(s) &= w_0(s) + 5w_1(s) = \frac{0.4s^2 + 2.1s + 1}{0.02s^3 + 0.3s^2 + s} \end{aligned} \tag{1.152}$$

and all the CTFs of relative motion associated with both matrices $W_{H3}(s)$ and $W_{H6}(s)$ are identical and have the form [Equation (1.139)]

$$q_2(s) = w_0(s) - w_1(s) = \frac{-0.08s^2 - 0.3s + 1}{0.02s^3 + 0.3s^2 + s}. \tag{1.153}$$

The characteristic gain loci of the CTFs [Equations (1.152) and (1.153)] are shown in Figure 1.31, from which it is evident that the stability of both systems is determined by the CTFs of relative motion [Equation (1.153)] which do not depend on N and coincide for all $2 \le i \le N$. The identical gain and phase margins of the discussed systems are equal to $GM = 6.76$ dB and $PM = 53.92°$.

Example 1.8 As an example of a uniform circulant system, consider a hypothetic 16-channel ($N = 16$) system with the transfer function of separate channels given by Equation (1.103), in which we decrease the gain by a factor of 25, and with a circulant matrix of rigid cross-connections R having the form

$$R = I + 0.6\,U, \tag{1.154}$$

where I is the unit matrix and U is the permutation matrix [Equation (1.128)] (both matrices have order 16×16). Physically, such cross-connections mean that in each channel of the system

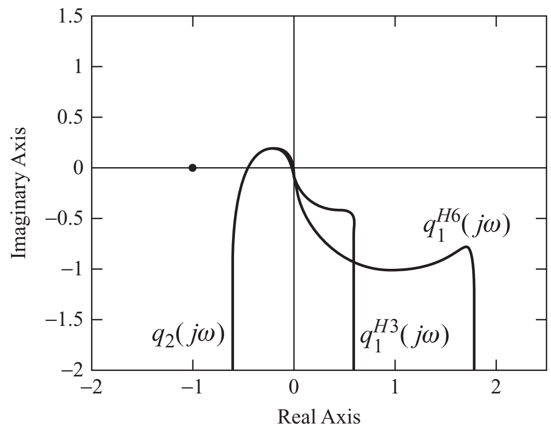

Figure 1.31 Characteristic gain loci of the hexapods, for $N = 3$ and $N = 6$.

enters the signal from the next channel multiplied by 0.6, and in the last channel enters the corresponding signal from the first channel. In other words, all channels of the given uniform circulant system are connected anticlockwise by a 'ring scheme' with the factor of 0.6. The characteristic gain loci of the system in the 'direct' and 'inverse' forms are given in Figure 1.32. These graphs show that the system is stable. Note that the indicated above decrease of the gain by a factor of 25 was caused by the desire to obtain a stable system, since, for the initial gain, the system would be unstable.

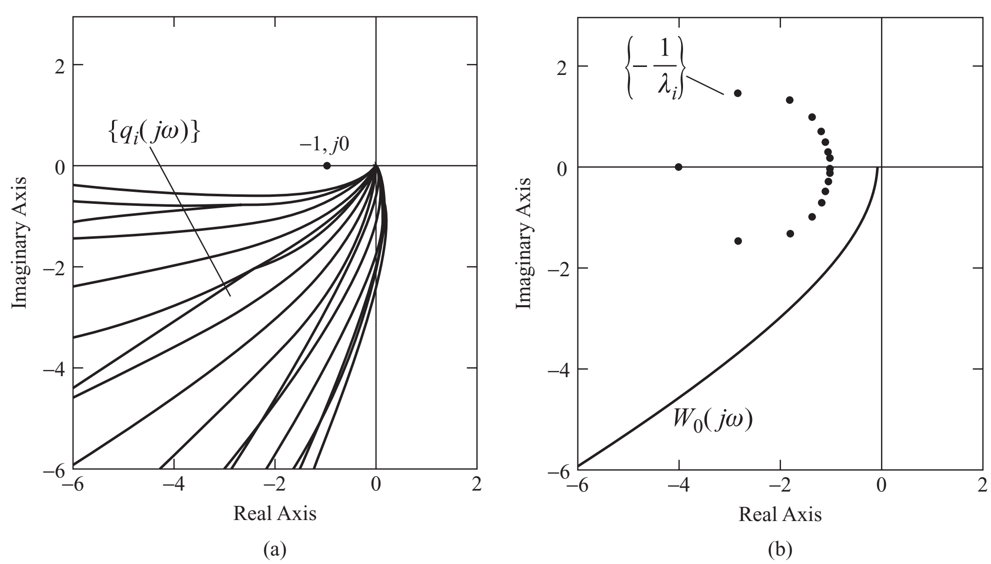

Figure 1.32 Characteristic gain loci of the uniform circulant system ($N = 16$). (a) 'direct' form; (b) 'inverse' form.

1.4.3 Anticirculant MIMO systems

Below, we discuss a specific subclass of normal MIMO systems referred to as *anticirculant*[41] systems, i.e. systems with anticirculant transfer matrices $W(s)$, $\Phi(s)$ and $\Phi_\varepsilon(s)$. The major distinction of anticirculant matrices, as compared with common circulant matrices, is that the elements located on both sides of the principal diagonal have opposite signs. This means that each subsequent row of an anticirculant matrix is obtained from the preceding row by shifting all elements (except for the Nth) by one position to the right; the Nth element of the preceding row becomes the first element of the following, with an opposite sign. Thus, the transfer matrix $W(s)$ of the open-loop anticirculant system can be written as

$$W(s) = \begin{pmatrix} w_0(s) & w_1(s) & w_2(s) & \ldots\ldots & w_{N-1}(s) \\ -w_{N-1}(s) & w_0(s) & w_1(s) & \ldots\ldots & w_{N-2}(s) \\ -w_{N-2}(s) & -w_{N-1}(s) & w_0(s) & \ldots\ldots & w_{N-3}(s) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -w_1(s) & -w_2(s) & -w_3(s) & \ldots\ldots & w_0(s) \end{pmatrix} \tag{1.155}$$

We know that the simplest of circulant matrices is the permutation matrix U [Equation (1.128)], and any circulant matrix can be represented as a matrix polynomial in U. In the case of anticirculant matrices, the *anticirculant permutation matrix* plays a similar role, which we denote by U_-:

$$U_- = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{1.156}$$

If we multiply U_- by a vector x, then the second component x_2 becomes the first, the third becomes the second, etc.; the first component x_1 becomes the Nth, with the opposite sign. Like U [Equation (1.128)], the matrix U_- is orthogonal, i.e. satisfies the conditions $U_-^{-1} = U_-^T$, $\det U_- = 1$ (the *proper* orthogonality), and all eigenvalues of U_- have unit magnitudes. On increasing the power of U_-, both nonzero diagonals shift to the place of the next diagonal on the right; the powers of U_- satisfy the following conditions:

$$U_-^{N-k} = -(U_-^k)^T = -(U_-^k)^{-1}, \quad U_-^N = I, \quad k = 1, 2, \ldots, N-1. \tag{1.157}$$

The characteristic equation of U_- [Equation (1.156)] is

$$\det[\beta I - U_-] = \beta^N + 1 = 0. \tag{1.158}$$

The roots β_i of Equation (1.158) are

$$\beta_i = \exp\left\{ j\frac{[2(i-1)+1]\pi}{N} \right\}, \quad i = 1, 2, \ldots, N \tag{1.159}$$

[41] Seemingly, the term *anticirculant matrix* was introduced in Gasparyan (1981).

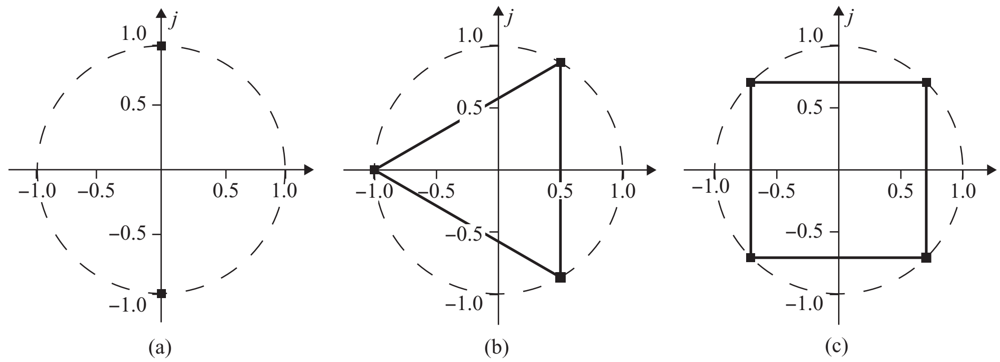

Figure 1.33 The eigenvalues of the anticirculant permutation matrix U_-. (a) $N = 2$; (b) $N = 3$; (c) $N = 4$.

and, like the eigenvalues of the permutation matrix U [Equation (1.128)], are situated in the complex plane at the vertices of a regular N-sided polygon inscribed in the unit circle. For $N = 2, 3, 4$, this is illustrated in Figure 1.33 (it is interesting to compare this figure with Figure 1.29). Recall that the first eigenvalue β_1 of U is always real and equal to unity. Unlike this, for *even* N, all eigenvalues of U_- [Equation (1.156)] are complex conjugate numbers.[42] The orthonormal eigenvectors c_i of U_- are expressed through the eigenvalues β_i [Equation (1.159)] by the same expressions [Equation (1.132)].

As follows from what has been stated, any anticirculant matrix in the form of Equation (1.155) can be represented as a matrix polynomial in U_- of degree $(N-1)$:

$$W(s) = w_0(s)I + \sum_{k=1}^{N-1} w_k(s)U_-^k. \tag{1.160}$$

For anticirculant MIMO systems of even order N, which are of primary interest to us, it is usually more convenient to transform Equation (1.160) to another form:

$$W(s) = w_0(s)I + w_{N/2}(s)U_-^{N/2} + \sum_{k=1}^{(N/2)-1} [w_k(s)U_-^k - w_{N-k}(s)(U_-^k)^{-1}]. \tag{1.161}$$

Based on the properties of matrix polynomials given in Section 1.4.2, the CTFs $q_i(s)$ of an anticirculant system can be represented for any N in the analytical form

$$q_i(s) = w_0(s) + \sum_{k=1}^{N-1} w_k(s)\exp\left\{ j\frac{[2(i-1)+1]\pi}{N}k \right\}, \quad i = 1, 2, \ldots, N. \tag{1.162}$$

The modal matrix C and the orthonormal canonical basis of that system do not depend on complex variable s and coincide with the modal matrix and canonical basis of the anticirculant permutation matrix U_-.

[42] In the case of odd N and $i = (N+1)/2$, the eigenvalue β_i is always equal to minus one.

From the practical viewpoint, the most significant are anticirculant systems with an even number of channels N. The basic statement of this section, completely clarifying the necessity of introducing the concept of anticirculant MIMO systems, can be formulated as follows:

If the number of separate channels N is even and the conditions

$$w_k(s) = w_{N-k}(s), \qquad k = 1, 2, \ldots, N-1 \tag{1.163}$$

hold, then the anticirculant MIMO system is antisymmetrical.

Substituting Equation (1.163) in Equations (1.161) and (1.162) yields, for even-order antisymmetrical MIMO systems:

$$W(s) = w_0(s)I + \left\{ w_{N/2}(s)U_{-}^{N/2} + \sum_{k=1}^{(N/2)-1} w_k(s)[U_{-}^{k} - (U_{-}^{k})^{-1}] \right\} \tag{1.164}$$

and

$$q_i(s) = w_0(s) + j(-1)^{i+1} w_{N/2}(s) + 2j \sum_{k=1}^{(N/2)-1} w_k(s) \sin\left\{ \frac{[2(i-1)+1]\pi}{N} k \right\}$$
$$i = 1, 2, \ldots, N, \tag{1.165}$$

where the braces in Equation (1.164) encompass a skew-symmetrical matrix.

Hence, anticirculant matrices enable us to describe analytically a certain class of antisymmetrical MIMO systems of even order, which was impossible by means of circulant matrices. First of all, this concerns the *simple antisymmetrical* MIMO systems (and, naturally, the simple antisymmetrical uniform systems) of even order, for which, in Equations (1.164) and (1.165), $w_k(s) = w_1(s)$ $(k = 2, \ldots N-1)$ should be assumed. In the specific, but very important, case of two-dimensional systems with antisymmetrical cross-connections, from Equation (1.165), the well known expression follows (Krassovski 1957):[43]

$$q_{1,2}(s) = w_0(s) \pm j w_1(s). \tag{1.166}$$

This indicates that two-dimensional antisymmetrical systems with constant parameters, studied at great length by the method of *complex coordinates* and *complex transfer functions*, constitute a specific case of even-order anticirculant MIMO systems. In Example 1.9, we shall return to that issue in more detail.

Remark 1.11 As shown above, the CTFs $q_i(s)$ of circulant and anticirculant systems can be written in analytical form for any number of separate channels N. From Equations (1.133) and (1.162), it is evident that these CTFs can be represented, after reducing to a common denominator, as a quotient of two rational polynomials in complex variable s. From Equations (1.133) and (1.162), it ensues that for all $q_i(s)$ $(i = 1, 2, \ldots, N)$, the denominator polynomials are the same, have real coefficients and are equal to the product of denominator polynomials of the elements of the first row of $W(s)$. In other words, all poles of the CTFs $q_i(s)$ of circulant or anticirculant systems are the same, and are *absolute*. As for the numerator polynomials of the CTFs $q_i(s)$, they are generally different and have complex coefficients. The only exception from the last rule constitutes *simple symmetrical* MIMO systems for which the numerator

[43] Eigenvalues [Equation (1.110)] of the antisymmetrical matrix R [Equation (1.109)] of the uniform system in Example 1.5 can be directly obtained from this expression.

polynomials of $q_i(s)$ have real coefficients and $N-1$ CTFs *of relative motions* [Equation (1.139)] coincide. A common feature of circulant and anticirculant systems is that their orthonormal canonical bases do not depend on s and on the specific form of the transfer functions $w_0(s)$ and $w_k(s)$ $(k=1,2,\ldots,N-1)$, and coincide, respectively, with the canonical bases of the permutation matrix U [Equation (1.128)] and the anticirculant permutation matrix U_- [Equation (1.156)].

Remark 1.12 In a sense, circulant and anticirculant systems are rather similar in their structural properties to normal uniform systems with orthogonal canonical bases, but the significant feature of the former systems is the presence of *dynamical* cross-connections, which is excluded for uniform systems. At the same time, all that was stated in Remark 1.9 about the CTFs of uniform systems is completely valid, taking into account the preceding remark, for circulant and anticirculant systems, i.e. here, we also have N *isolated, one-sheeted algebraic functions not possessing any branch points*. The last fact allows handling the CTFs $q_i(s)$ of the systems in question, preserving the mathematical rigour, as a set of N-independent SISO systems.

Example 1.9 Below, we outline those significant practical tasks that have stimulated the formation and evolution of the *complex coordinates and complex transfer functions method*, and obtain the CTFs for an axially symmetrical spinning body.

As is well known, the angular motion of a rigid body about the principal axes of inertia is described by Euler's nonlinear dynamical equations (Goldstein 1959):

$$J_X\frac{d\omega_x}{dt}+(J_Z-J_Y)\,\omega_z\omega_y=M_x \tag{1.167}$$

$$J_Y\frac{d\omega_y}{dt}+(J_X-J_Z)\,\omega_x\omega_z=M_y \tag{1.168}$$

$$J_Z\frac{d\omega_z}{dt}+(J_X-J_Y)\,\omega_x\omega_y=M_z, \tag{1.169}$$

where J_X, J_Y and J_Z denote the principal moments of inertia; ω_x, ω_y and ω_z the angular velocities; and M_x, M_y and M_z the external torques. Generally, these torques represent the sum of control torques and disturbances, but, for simplicity, we shall assume them be control torques.

In many technical applications, such as satellite spin stabilization, some guidance systems of rockets and torpedoes, gyroscopic systems, etc. (Kazamarov *et al.* 1967), engineers have to deal with an *axially-symmetrical body* spinning about the symmetry axis with constant angular velocity (usually, that velocity is chosen high enough to impart the gyroscopic, i.e. stabilizing, effect to the body). Assuming for certainty the symmetry axis to be the Z-axis, i.e. assuming $J_X=J_Y=J$, and denoting by $\Omega=const$ the constant angular velocity about that axis[44] yields, instead of Equation (1.168), a set of two equations about the transversal axes X and Y:

$$J\frac{d\omega_x}{dt}+a\omega_y=M_x \tag{1.170}$$

$$J\frac{d\omega_x}{dt}-a\omega_y=M_y, \tag{1.171}$$

[44] That velocity is imparted to the body at the beginning (at zero time), and, as a first approximation, can be accepted as constant if $M_z=0$, i.e. in the case of neglecting the influence of various external disturbances and applying no control torques about the symmetry axis.

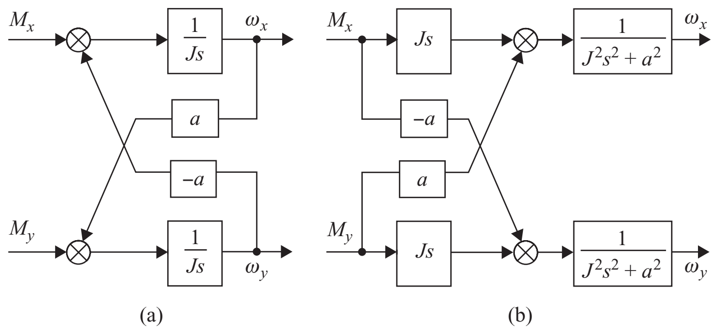

Figure 1.34 Block diagrams of a spinning axially symmetrical body. (a) initial block diagram; (b) equivalent block diagram with direct cross-connections.

where

$$a = (J_Z - J)\Omega = const. \tag{1.172}$$

Hence, we obtain a set of two linear cross-connected differential equations with respect to the angular velocities ω_x and ω_y, with the constant antisymmetrical cross-connections a [Equation (1.172)]. If we pass to operator form, then the block diagram with *inverse* antisymmetrical connections shown in Figure 1.34(a) corresponds to these equations. An equivalent block diagram with *direct* cross-connections is given in Figure 1.34(b). This block diagram represents the following matrix operator equation, which is the result of transformation of initial Equations (1.170) and (1.171):

$$\begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} = \frac{1}{J^2s^2 + a^2} \begin{pmatrix} Js & a \\ -a & Js \end{pmatrix} \begin{bmatrix} M_x \\ M_y \end{bmatrix}. \tag{1.173}$$

The specific pattern of the cross-terms in Equations (1.170), (1.171) and (1.173) has led to an idea of using some duly chosen *complex-valued coordinates* instead of the system coordinates. If we introduce the complex angular velocities and control torques:

$$\bar{\omega} = \omega_x + j\omega_y, \quad \bar{M} = M_x + jM_y, \tag{1.174}$$

then, multiplying Equation (1.171) by the imaginary unit j and adding to Equation (1.170) yields

$$J\frac{d\bar{\omega}}{dt} + ja = \bar{M} \tag{1.175}$$

or, in the operator form, $\bar{\omega} = W_c(s)\bar{M}$, where the function

$$W_c(s) = \frac{1}{Js + ja} \tag{1.176}$$

is called the *complex transfer function* (Krassovski 1957).

Thus, the conversion to complex coordinates [Equation (1.174)] allows decreasing of twice the order of initial Equations (1.170) and (1.171), although the resulting equation has complex coefficients. At first, this approach was mainly used in the engineering mechanics and theory of gyroscopes, and, afterwards, it was adopted in the control theory.[45] It should be noted that owing to complex coefficients, the complex transfer functions generally lose the symmetry with respect to the ordinate axis in the plane of Bode diagrams, and distribution with respect to the real axis in the plane of Nyquist plots – with respect to the real axis, as well as the symmetry of poles and zeros.[46]

Apply now to Equation (1.173) the results of the present section, noticing that the system is antisymmetrical. Based upon Equation (1.166), we immediately obtain

$$q_{1,2}(s) = \frac{1}{J^2 s^2 + a^2}(Js \pm ja) = \frac{1}{Js \mp ja}, \tag{1.177}$$

i.e. one of the CTFs associated with the two-dimensional antisymmetrical system coincides with the complex transfer function $W_c(s)$ [Equation (1.176)]. This once more verifies the conclusion that for systems with fixed parameters, the approach in terms of the CTFs embraces as a special case many results of the complex coordinates and transfer functions method.[47]

Example 1.10 One of the classical instances of nonrobust MIMO systems is the two-dimensional system discussed by J. K. Doyle and others (Doyle 1984; Packard and Doyle 1993). The transfer matrix of that system has the form

$$W(s) = \frac{1}{s^2 + a^2} \underbrace{\begin{pmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{pmatrix}}_{W_1(s)}, \quad a = 10 \tag{1.178}$$

and is antisymmetrical. Introducing into this system the unit negative feedback and static regulator $K = diag\{K_i\} = I$, we obtain a closed-loop system, both roots of which are equal to -1. In Doyle (1984) and Packard and Doyle (1993), it is indicated that breaking by turns one loop at a time, one can erroneously infer that the system has an infinite gain margin and a phase margin of 90°. However, simultaneously changing the unit gains in the separate channels to $K_1 = 1.1$ and $K_2 = 0.9$ makes the system unstable! Consider now the system in Equation (1.178) in terms of the CTFs. Note first that this system, being formally very similar to the control system for the spinning axially-symmetrical body of Example 1.9, possesses rather original features. Thus, despite the scalar multiplier $1/(s^2 + a^2)$ in Equation (1.178), the poles $p_{1,2} = \pm ja$ of the latter *are not the absolute poles of the open-loop system*. This can readily be checked by determining that the matrix $W_1(s)$ in Equation (1.178) is singular for $s_{1,2} = \pm ja$ [this is due to the fact that the values $s_{1,2} = \pm ja$ are *zeros* of the matrix $W_1(s)$ belonging to

[45] Seemingly, the first works in the feedback control theory on the complex coordinates and transfer functions method belong to Krassovski (1957).
[46] See also Section 1.1.4.
[47] The complex coordinates and transfer functions method allows the united handling of two- and three-dimensional antisymmetrical systems having single-channel alternating current sections with amplitude-phase modulation of signals (Kazamarov *et al.* 1967). Strictly speaking, such systems are described by differential equations with periodic coefficients.

different CTFs and each of these zeros compensates *one* of the poles $p_{1,2} = \pm ja$]. Besides, at the zero frequency $\omega = 0$, the eigenvalues of the matrix $W(s)$ are

$$q_{1,2}(j0) = -1 \pm j\frac{1}{a}, \tag{1.179}$$

which, in accordance with Remark 1.7, implies *positive* feedback. These features also appear in the frequency characteristics. It is easy to show that the CTFs $q_{1,2}(s)$ of the system [Equation (1.178)] are

$$q_1(s) = \frac{1 + ja}{s - ja}, \quad q_2(s) = \tilde{q}_1(s) = \frac{1 - ja}{s + ja}. \tag{1.180}$$

The characteristic gain loci $q_1(j\omega)$ and $q_2(j\omega)$ shown in Figure 1.35(a) and (b), in which the dashed lines correspond to $q_2(j\omega)$, do not encircle the critical point $(-1, j0)$, i.e. the system is stable. The enlarged area around the origin is shown in Figure 1.35(b), in which the bold dots indicate the zero frequency $\omega = 0$.[48] These points are quite close to $(-1, j0)$, which suggests the need to study in detail their behaviour under small perturbations of gains of the static regulator K.

If we denote the perturbations of gains by ΔK_1 and ΔK_2, then it can be shown that the starting points $q_{1,2}(j0)$ of the characteristic gain loci are described by the expression

$$q_{1,2}(j0) = -\frac{2 + \Delta K_1 + \Delta K_2}{2} \pm \sqrt{\frac{(2 + \Delta K_1 + \Delta K_2)^2}{4} - \frac{(1 + \Delta K_1)(1 + \Delta K_2)(a^2 + 1)}{a^2}}. \tag{1.181}$$

For zero perturbations $\Delta K_1 = \Delta K_2 = 0$, Equation (1.181) coincides, naturally, with Equation (1.179). For nonzero but equal by magnitude and sign deviations $\Delta K_1 = \Delta K_2 = \Delta K$, we have, from Equation (1.181):

$$q_{1,2}(j0) = (1 + \Delta K)\left(-1 \pm j\frac{1}{a}\right) \tag{1.182}$$

i.e. the starting points $q_{1,2}(j0)$ shift proportionally to perturbations ΔK, approaching or moving away (depending on the sign of ΔK) from the origin. Since, for identical perturbations ΔK, both loci $q_1(j\omega)$ and $q_2(j\omega)$ are multiplied by a real factor $(1 + \Delta K)$, it is easy to understand that the form of these loci does not change. This means that the system in Equation (1.178) cannot become unstable under arbitrary large but *identical* ΔK_1 and ΔK_2. The picture changes drastically if the perturbations ΔK_1 and ΔK_2 have different signs. Assuming, for simplicity, that these perturbations are the same by magnitude, i.e. assuming in Equation (1.182) that $\Delta K_1 = -\Delta K_2 = \Delta K$, yields

$$q_{1,2}(j0) = -1 \pm j\frac{1}{a}\sqrt{(a^2 + 1)\Delta K^2 - 1}, \tag{1.183}$$

[48] The characteristic gain loci $q_1(j\omega)$ and $q_2(j\omega)$ considered in the range $-\infty \le \omega \le +\infty$ are complex conjugate, i.e. symmetrical with respect to the real axis.

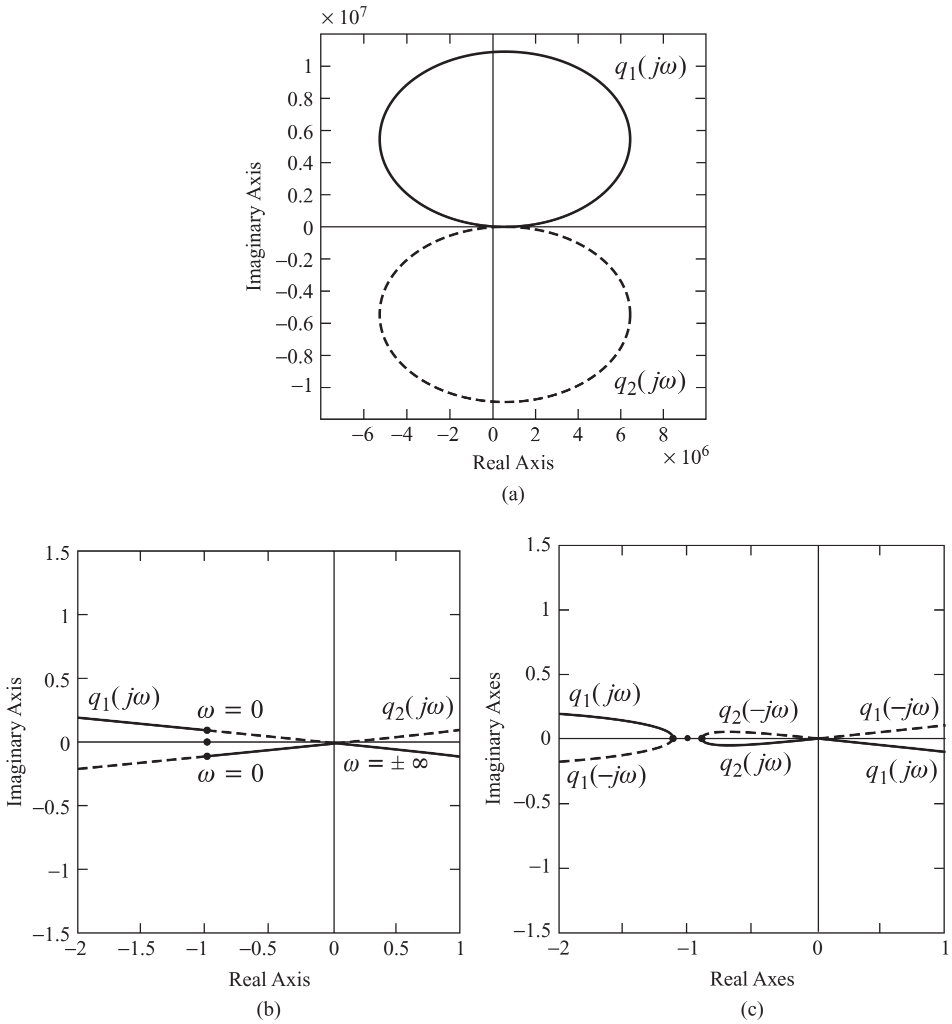

Figure 1.35 Characteristic gain loci of antisymmetrical system [Equation (1.178)]. (a) general view; (b) area around the origin; (c) the CTFs $q_1(j\omega)$ and $q_2(j\omega)$ for $\Delta K_1 = -\Delta K_2 = 0.15$.

from which it is evident that for[49]

$$\Delta K = \frac{1}{\sqrt{a^2 + 1}} \approx 0.0995, \tag{1.184}$$

both starting points of the characteristic gain loci $q_1(j\omega)$ and $q_2(j\omega)$ coincide with the critical point $(-1, j0)$. If we continue increasing ΔK by magnitude, the starting points move from the critical point along the real axis in opposite directions; as a result, one of the characteristic

[49] Equation (1.184) was obtained in (Skogestad and Postlethwaite 1995), based on the characteristic equation of the closed-loop system.

gain loci will necessarily encircle the point $(-1, j0)$ and the system will become unstable. The gain loci $q_1(j\omega)$ and $q_2(j\omega)$ in the area around the origin for $\Delta K_1 = -\Delta K_2 = 0.15$ are shown in Figure 1.35(c), confirming that the locus $q_1(j\omega)$ encircles $(-1, j0)$ [slightly larger perturbations are taken here to obtain more visual results, as, for $\Delta K_1 = -\Delta K_2 = 0.1$, the starting points of $q_1(j\omega)$ and $q_2(j\omega)$ actually merge with $(-1, j0)$]. It is interesting to note that the shapes of the loci have essentially changed in the discussed case. Here, we have, for $-\infty \leq \omega \leq +\infty$, one 'large' closed contour corresponding to $q_1(j\omega)$ and one 'small', corresponding to $q_2(j\omega)$, i.e. now, the loci are not complex conjugate. The point is that for different signs of ΔK_1 and ΔK_2, the system loses the property of antisymmetry and belongs to the class of general MIMO systems.

1.4.4 Characteristic transfer functions of complex circulant and anticirculant systems

The fact that the canonical basis of any circulant or anticirculant transfer matrix $W(s)$ does not depend on the specific form of its elements $w_0(s)$, $w_i(s)$ $(i = 1, 2, \ldots, N-1)$ and coincides with the orthogonal basis composed of the normalized eigenvectors of the permutation matrix U [Equation (1.113)] or anticirculant permutation matrix U_- [Equation (1.137)] plays a crucial role in determining the CTFs of complex systems belonging to the discussed classes. In Section 1.2, it was noted[50] that if the block diagram of an open-loop general MIMO system consists of different matrix elements connected in series, in parallel or forming inner feedback loops, then, for determining the CTFs, it is necessary to find the *resultant* transfer matrix $W(s)$, based upon standard rules of matrix block diagram transformation (Morozovski 1970). In such cases, nothing can be said about the relationship of the CTFs $q_i(s)$ of $W(s)$ with the corresponding CTFs of the separate matrices constituting $W(s)$, since, in general, all these elements are brought to diagonal form in different canonical bases.[51]

The situation is quite different for MIMO systems with circulant or anticirculant transfer matrices. In the following, analyzing complex circulant and anticirculant systems, we assume that all matrix elements constituting the open-loop system are circulant or anticirculant, i.e. we exclude the situations in which the individual elements do not belong, but their sum, product or any other combination belong, to the mentioned classes. Besides, since the structures of the CTFs and eigenvectors (canonical basis axes) of circulant and anticirculant transfer matrices are actually identical, we discuss only circulant systems. Consider, first, the series connection of m matrix elements described by the circulant matrices $W_k(s)\,(k = 1, 2, \ldots, m)$ [Figure 1.36(a)].

In accordance with the general rules (Morozovski 1970), the transfer matrix $W(s)$ of such a connection is equal to the product of transfer matrices of the individual elements arranged in the reversed order

$$W(s) = W_m(s)W_{m-1}(s)\ldots.W_2(s)W_1(s). \tag{1.185}$$

Since the product of any number of circulant matrices yields a matrix of the same type (Voevodin and Kuznetsov 1984), the matrix $W(s)$ [Equation (1.185)] is circulant for any m.

[50] See Remark 1.2.
[51] All the indicated matrix blocks of a general MIMO system are brought to diagonal form in the same canonical basis if all the corresponding transfer matrices are commutative (see also Remark 1.10).

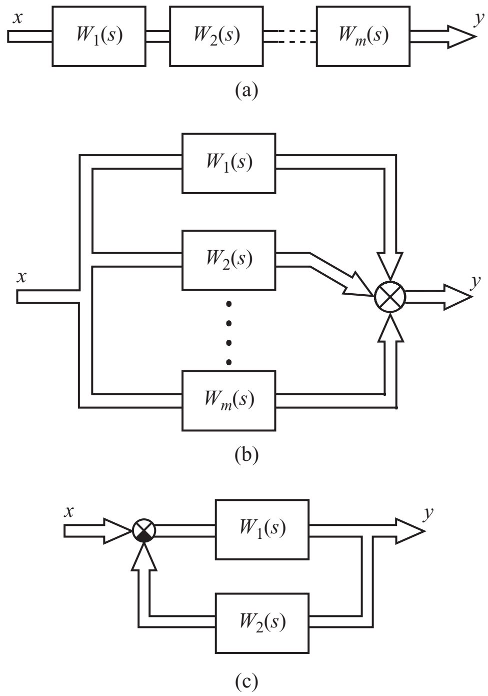

Figure 1.36 Basic structural connections of circulant elements. (a) series connection; (b) parallel connection; (c) feedback connection.

Further, taking into account that each matrix $W_k(s)$ in Equation (1.185) has the canonical representation of the form

$$W_k(s) = C diag\{q_i^k(s)\}C^*, \tag{1.186}$$

where C is the unitary matrix formed by the eigenvectors c_i of the permutation matrix U [Equation (1.113)], and the same representation has the resultant matrix $W(s)$, instead of Equation (1.185), we can write

$$W(s) = C diag\{q_i(s)\}C^* = C diag\{\prod_{k=1}^{m} q_i^k(s)\}C^*, \tag{1.187}$$

from which there immediately follow N equalities

$$q_i(s) = \prod_{k=1}^{m} q_i^k(s), \quad i = 1, 2, \ldots, N. \tag{1.188}$$

Thus, the ith CTF $q_i(s)$ of the series connection of m circulant elements *is equal to the product of the corresponding* i*th CTFs of the individual elements*. It should be emphasized that the order of connection does not play any role, i.e. these matrix elements can be interchanged just as in the case of usual scalar elements in SISO systems (Kuo 1995).

Let us find out now the properties of the parallel connection of m circulant elements [Figure 1.36(b)]. In this case, the resultant matrix $W(s)$ is also circulant and equal to the sum of the individual matrices, i.e.

$$W(s) = \sum_{k=1}^{m} W_k(s). \tag{1.189}$$

From Equation (1.189), taking into account that $W(s)$ is circulant, we obtain

$$W(s) = Cdiag\{q_i(s)\}C^* = Cdiag\left\{\sum_{k=1}^{m} q_i^k(s)\right\}C^* \tag{1.190}$$

and

$$q_i(s) = \sum_{k=1}^{m} q_i^k(s), \quad i = 1, 2, \ldots, N. \tag{1.191}$$

Hence, the ith CTF $q_i(s)$ of the parallel connection of m circulant elements *is equal to the sum of the corresponding* i*th CTFs of the individual elements*.

Consider, finally, the feedback connection of two circulant elements [Figure 1.36(c)], assuming for certainty the negative feedback. The transfer matrix $W(s)$ of such a connection, relating the input and output vectors x and y, is

$$W(s) = [I + W_1(s)W_2(s)]^{-1}W_1(s). \tag{1.192}$$

Substituting the canonical representations in Equation (1.186) into this expression yields

$$W(s) = Cdiag\{q_i(s)\}C^* = Cdiag\left\{\frac{q_i^1(s)}{1 + q_i^1(s)q_i^2(s)}\right\}C^*. \tag{1.193}$$

From Equation (1.193), it is evident that the matrix $W(s)$ is circulant (as it is brought to diagonal form in the canonical basis of the permutation matrix U), and the CTFs of $W(s)$

$$\{q_i(s)\} = \frac{q_i^1(s)}{1 + q_i^1(s)q_i^2(s)} \qquad i = 1, 2, \ldots, N \tag{1.194}$$

are expressed through the ith CTFs $q_i^1(s)$ and $q_i^2(s)$ of $W_1(s)$ and $W_2(s)$ by the same formulae as usual SISO transfer functions in the case of nonunit negative feedback.

Now, it is clear that any arbitrary complex connection pattern of circulant elements forming the open-loop MIMO system can be described by a certain resultant circulant matrix $W(s)$. For determining the CTFs $q_i(s)$ of the latter, we have to replace the matrix block diagram of the open-loop circulant system with a scalar block diagram analogous by form, in which, instead of the matrix elements $W_k(s)$, the CTFs $q_i^k(s)$ are shown.[52] The obtained block diagram can be

[52] Recall that these CTFs can be found in analytical form for any number N of separate channels.

handled in just the same way as a common diagram of a SISO system. It must be emphasized once more that it is possible because all circulant (and anticirculant) matrices are brought to diagonal form in the same canonical basis.

Example 1.11 Consider the three-axis gyrostabilized platform described in Chorol *et al.* (1976), in which angular velocities are measured by special devices with amplitude-phase modulation of signals. The expanded block diagram of the system is depicted in Figure 1.37. If the reference voltage has a nonzero phase shift γ, then, among the system channels, there appear direct antisymmetrical connections. Besides, the noncoincidence of the resonance frequency in the measuring devices with the carrier frequency results in reverse antisymmetrical connections with coefficients K_r. The corresponding matrix block diagram of the gyrostabilized platform is shown in Figure 1.38 (Sobolev 1973), in which U denotes the permutation matrix of order 3×3. The inspection of that block diagram shows that the system belongs to uniform antisymmetrical systems. Therefore, based on the results of

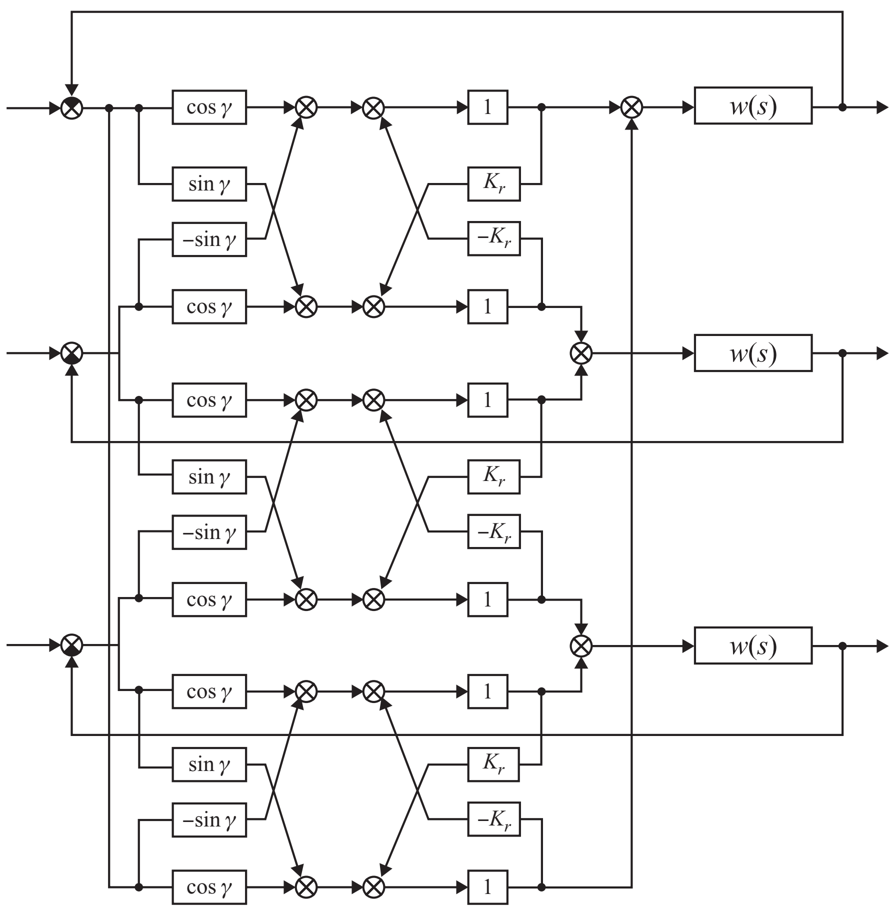

Figure 1.37 Expanded block diagram of the three-axis gyrostabilized platform.

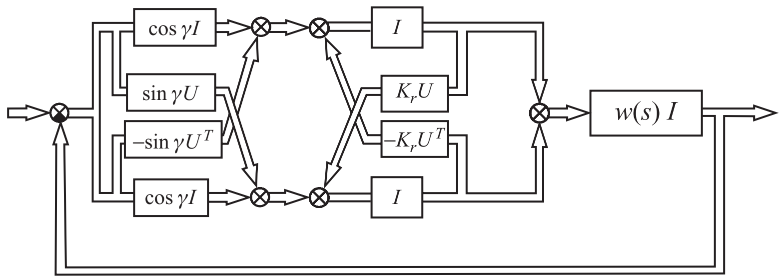

Figure 1.38 Matrix block diagram of the gyrostabilized platform.

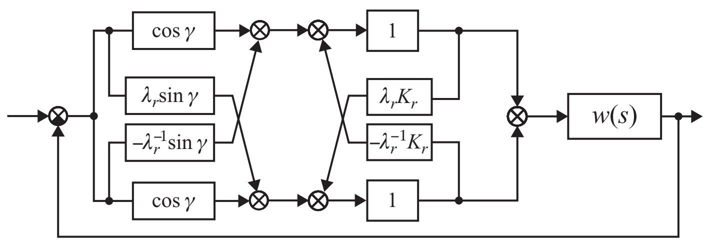

Figure 1.39 Block diagram of one-dimensional characteristic systems ($i = 1, 2, 3$).

Section 1.3,[53] we can draw equivalent block diagrams of one-dimensional characteristic systems. For this purpose, we must replace all vector connections in Figure 1.38 with scalar ones. Then, we must replace the permutation and unit matrices U and I with, respectively, the eigenvalues λ_i and ones (Figure 1.39), where, based on Equation (1.131), the eigenvalues of U are

$$\lambda_1 = 1, \quad \lambda_{2,3} = \cos 120^\circ \pm j \sin 120^\circ = -0.5 \pm j0.866. \tag{1.195}$$

Note that in Figure 1.39, we take into account that the matrix U is orthogonal, i.e. its inverse matrix coincides with the transposed, and that the eigenvalues of any matrix of simple structure and of its inverse matrix are mutually inverse. As an exercise, the reader can check that the CTFs of the *open-loop* characteristic systems determined by the block diagram in Figure 1.39 have the following form:

$$q_1(s) = \frac{2(\cos\gamma - K_r \sin\gamma)}{1 + K_r^2} w(s),$$
$$q_{2,3}(s) = \frac{2[\cos\gamma - K_r \sin\gamma \pm j0.866(\sin\gamma + K_r \cos\gamma)]}{1 + K_r^2} w(s). \tag{1.196}$$

1.5 MULTIVARIABLE ROOT LOCI

The *root locus method* proposed by Evans in 1948 is, together with the frequency-domain approaches, one of the key and most effective methods of the linear SISO systems analysis

[53] See Remark 1.10.

and design (Evans 1948, 1950, 1954; Horowitz 1963; Krall 1970; Kuo 1995; Ogata 1970). Based upon a number of simple graphical rules and procedures, the root locus technique allows predicting, given the distribution of zeros and poles of an *open-loop* SISO system, the behaviour of the roots of the *closed-loop system*, as the total gain (or any other parameter) changes from zero to infinity. Simplicity and clearness of that method predetermined its exceptional popularity among practising engineers, which, in turn, stimulated the strong interest of scientists and researchers. Later, there appeared other variants of root locus techniques, such as the logarithmical root locus method (Kuzovkov 1959; Glaria *et al.* 1994), the analytical method of Bendrikov and Teodorchik (1964), Uderman's method (1963), the method of generalized root loci (Rimski 1969), etc.

With progress of the multivariable control theory, that topic became the object of researchers' close attention. Frequently, in this connection, systems with two inputs and outputs or some special classes of multivariable systems were considered. Thus, the properties of root loci of two- and three-dimensional systems with identical channels and rigid antisymmetrical cross-connections were discussed in Bendrikov and Ogorodnikova (1967, 1968) and Fortescue (1976). The problem of *general properties* of multivariable root loci (called also root trajectories) was studied at great length in Owens (1976), Shaked (1976), Kouvaritakis and Shaked (1976), Postlethwaite and MacFarlane (1979), Hung and MacFarlane (1981), etc.[54] Some specific features of multivariable root loci, having no analogues in the one-dimensional case, were discovered and theoretically justified in Kouvaritakis and Shaked (1976) and Kouvaritakis and Edmunds (1979) on the basis of state-space representation of MIMO systems. Thus, in these and some other works, it was established that the root loci of MIMO systems may intersect or superimpose; in certain situations, the roots of the MIMO system may move along some trajectory, and then change their movement to the opposite direction along the same trajectory, etc. It was also shown that in the case of 'infinite' zeros, i.e. if the total number of poles exceeds the number of 'finite' zeros (which is a usual situation), the MIMO system may have several *groups of asymptotes* (and not one group, as in the SISO case) constituting the so-called *Butterworth configuration* (or *pattern*).[55] Besides, generally, the centres (*pivots*) of these groups are not necessarily located on the real axis.

The task of constructing multivariable root loci in terms of the CTFs was discussed in the pioneering works of MacFarlane and Postlethwaite (1977, 1979). The theory of algebraic functions and concept of Riemann surfaces used by them proved to be very fruitful. In particular, this approach allowed establishing, with necessary mathematical rigour, the basic properties (as well as explaining some 'strange' features) of multivariable root loci, attributing these properties and features to the presence of *branch points* and to the location of root loci branches on *different sheets* of a unique Riemann surface.

Unfortunately, many theoretical results of the cited works considerably yield in simplicity to their classical counterparts. As a result, it is very difficult to apply them in practice. A paradoxical situation has arisen. Many modern mathematical and application program packages enable the user to find without difficulty the roots of closed-loop MIMO systems of, actually, any dimension, as well as to draw the root loci corresponding to changes of any parameter of

[54] Another essential and extensive branch of research concerns the asymptotic behavior of linear *optimal* systems roots, as a scalar weight coefficient in the quadratic cost index changes from 0 to ∞ (Kwakernaak 1976; Shaked and Kouvaritakis 1977; Kouvaritakis 1978).

[55] An r-order Butterworth pattern consists of r straight, evenly spaced lines starting from a common centre (pivot), and forming with positive direction of the real axis angles, equal to $(2k+1)180^\circ/r$ $(k = 0, 1, \ldots, r-1)$ (Butterworth 1930).

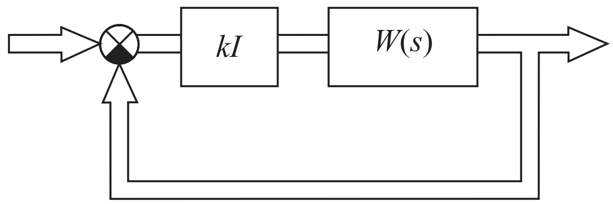

Figure 1.40 Block diagram used for the study of multivariable root loci.

the system. On the other hand, the algorithmical procedures for finding the pivots of MIMO systems asymptotes described, for example, in Kouvaritakis and Shaked (1976) considerably exceed in complexity the mentioned straightforward procedures of finding the multivariable root loci. This should not be interpreted as a call to reject research in that direction; we should just clearly apprehend that theoretical results are needed, chiefly to *explain, understand* and *justify* the character and behaviour of multivariable root loci, and not for immediate utilization in numerical computations. It is worth noting that even in the case of SISO systems, an engineer is now free from the burden of calculating root loci 'by hand'. This task may readily be solved using the application packages of many companies and firms. Thus, the function *rlocus* available in *The Control System Toolbox* in MATLAB is destined for constructing the root loci of SISO systems given the transfer function of the open-loop system. Similar functions exist and in many other packages.

Concluding our introduction, note that, below, we consider properties of multivariable root loci in the context of their relation to analogous properties of usual root loci known from the classical control theory (Evans 1954; Horowitz 1963). At that, we base on the MIMO system representation in terms of the CTFs. Therefore, many significant results demanding the state-space representation of MIMO systems or using the mathematical apparatus that is beyond the scope of this book are omitted or replaced by a heuristic interpretation from the standpoint of the CTFs method.

1.5.1 Root loci of general MIMO systems

The matrix block diagram of a general linear MIMO system used for the study of multivariable root loci is shown in Figure 1.40, in which $W(s)$ is the transfer matrix of the open-loop system of order $N \times N$, I is the unit matrix and k is a real scalar multiplier.

The problem is to construct the root loci of the MIMO system as the 'gain' k changes from zero to infinity.[56] Recalling the common assumption of no repeated CTFs $q_i(s)$, the characteristic equation of the closed-loop MIMO system can be represented as

$$\det[I + kW(s)] = \prod_{i=1}^{N} [1 + kq_i(s)] = 0, \tag{1.197}$$

where it is taken into account that multiplication of a matrix by a scalar results in the multiplication of all the matrix eigenvalues by the same scalar. Evidently, Equation (1.197) is equivalent

[56] For brevity, we consider only the case of positive k. The case of negative k is left for the reader as an exercise.

to the following system of N equations:

$$1 + kq_i(s) = 0 \quad \text{or} \quad kq_i(s) = -1, \quad i = 1, 2, \ldots, N, \tag{1.198}$$

from which it is clear that the roots of the closed-loop MIMO system must satisfy, for any $k = const$ and some i, the following two conditions:

$$|kq_i(s)| = 1, \quad \arg q_i(s) = \pm(2r+1)180^\circ, \quad r = 0, 1, 2, \ldots. \tag{1.199}$$

In other words, the roots of the closed-loop MIMO system are those values of complex variable s for which some CTFs $q_i(s)$ become real negative numbers.

An apparent similarity of conditions in Equation (1.199) to the corresponding conditions for SISO systems (Evans 1954; Horowitz 1963) suggests that no essential problems should arise in extending to the multivariable case the standard 'one-dimensional' root locus techniques. Unfortunately, everything is far from being so simple. As was indicated in Section 1.2, in general, the CTFs of MIMO systems are algebraic functions located on different sheets of a Riemann surface and constituting a unique mathematical entity. Based upon Equation (1.46),

$$\det W(s) = \frac{Z(s)}{P(s)} = \prod_{i=1}^{N} q_i(s), \tag{1.200}$$

where $Z(s)$ and $P(s)$ are the zeros and poles polynomials of the open-loop MIMO system, we know that zeros and poles of the open-loop system coincide with all zeros and poles of *all* the CTFs $q_i(s)$.[57] However, in the general case, we do not know how the zeros and poles of the MIMO system are distributed among different characteristic systems, or, more strictly, how these zeros and poles participate, taking into account the branch points, in forming the branches of the root loci on different sheets of the Riemann surface. If we knew this information, then, really, it would not be difficult to find the multivariable root loci based upon Equations (1.197)–(1.199) and using common classical procedures. At the same time, based upon Equations (1.197)–(1.200), we can draw some conclusions and formulate a number of rules which, as was accentuated before, serve rather for the understanding of the properties of multivariable root loci already constructed, by means of the computer aids, than for using them for calculations 'by hand' (Gasparyan *et al.* 2006). We shall list these rules following the order existing in the classical control theory (Evans 1954).

Rule 1: *Number of root loci.* The number of root loci is equal to the total number n of poles of the open-loop MIMO system, i.e. to the degree n of the poles polynomial $P(s)$; the multivariable root loci are the combination of the set of root loci of the SISO characteristic systems. This rule immediately follows from Equations (1.58) and (1.59), and does not require any special comments.

Rule 2: *Multivariable root loci are continuous curves.* The root loci are continuous because the roots of Equation (1.198) are continuous functions of k, i.e. the arbitrarily small changes of k result in the arbitrary small displacements of any roots of these equations.[58] *The derivatives*

[57] Further, we shall assume that the poles polynomial $P(s)$ is *monic*, i.e. its higher degree coefficient is 1.

[58] See also Remark 1.6 concerning the roots of the closed-loop MIMO systems that do not change as the coefficient k changes.

of multivariable root loci are continuous everywhere except for the denumerable number of the points at which the CTFs $q_i(s)$ become infinity, or their derivatives with respect to s are not determined, or $dq_i(s)/ds = 0$, or, finally, $k = 0$. This can be readily checked, differentiating Equation (1.198) with respect to k, which yields

$$\frac{ds}{dk} = -\frac{q_i(s)}{k(dq_i(s)/ds)}. \tag{1.201}$$

Note that, unlike the SISO case, in which the derivatives of root loci lose their continuity only at the poles of the open-loop system or at the points of the loci intersections, in the multivariable case, we have, in addition, the branch points, at which the branches of root loci pass from one sheet of the Riemann surface to another. At these points, the values of some (two or more) CTFs coincide. In the simplest case of two-dimensional systems, the branch points are those values of s for which the CTFs $q_1(s)$ and $q_2(s)$ are equal. From Equation (1.79), it is evident that these branch points are given by the expression

$$D(s) = tr\{W(s)\}^2 - 4\det W(s) = 0, \tag{1.202}$$

where $D(s)$ [the radicand in Equation (1.79)] is called the *discriminant* of the algebraic function. In the case of MIMO systems of an arbitrary dimension ($N \geq 3$), the branch points are also determined by equating to zero the discriminant; however, in the general case, simple analytical expressions for $D(s)$ do not exist (Bliss 1966; Postlethwaite and MacFarlane 1979).

Rule 3: *Starting and ending points of multivariable root loci*. The root loci commence at the poles of the CTFs $q_i(s)$ (for $k = 0$) and terminate at the zeros of $q_i(s)$ (for $k = \infty$). The condition $k = 0$ physically means breaking the feedback loop of the MIMO system, i.e. the roots of the MIMO system must coincide with poles of the open-loop system and, owing to Equation (1.200), with the set of all poles of the CTFs $q_i(s)$ ($i = 1, 2, \ldots, N$). If k tends to infinity, then, to preserve the equality $kq_i(s) = -1$, the CTFs $q_i(s)$ must tend to zero, i.e. the complex variable s must tend to zeros of $q_i(s)$, or, taking into account Equation (1.200), to zeros of the open-loop transfer matrix $W(s)$. A significant feature of MIMO systems is that in the presence of *branch points*, to the zeros of the ith CTF $q_i(s)$ generally can tend the root loci that begin at the poles of the CTF $q_r(s)(r \neq i)$ belonging to the adjacent sheets of the Riemann surface.

Rule 4: *Number of the branches tending to infinity*. If the ith CTF $q_i(s)$has np_i poles and nz_i zeros and *there are no branch points*, then, as $k \to \infty$, those of the np_i root locus branches of the given CTF which do not tend to nz_i *finite* zeros must tend to infinity, to preserve equality [Equation (1.198)]. This implies that for each $q_i(s)$, we have $e_i = np_i - nz_i$ locus branches tending to infinity and, in all, there are $e = n - m$ such branches, where n and m are the degrees of the poles and zeros polynomials $P(s)$ and $Z(s)$ in [Equation (1.200)].[59] *In the presence of* branch points, on the ith sheet of the Riemann surface can tend to infinity the locus branches that begin at the poles of the CTFs belonging to the adjacent sheets of the Riemann surface. In the latter case, the total number of the root loci approaching infinity, naturally, does not change.

[59] These root trajectories are frequently referred to as tending to *infinite zeros* of the MIMO system.

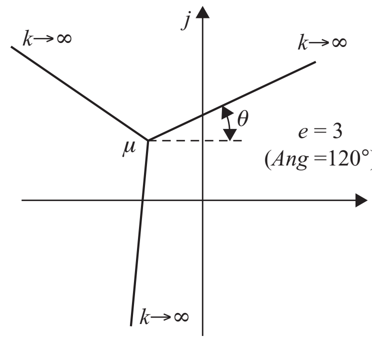

Figure 1.41 A third-order group of asymptotes.

Rule 5: *Angles and pivots of asymptotes of multivariable root loci*. Unfortunately, in contrast to SISO systems, in the case of MIMO systems, there are no *simple* formulae expressing the angles and pivots of the root loci asymptotes through the poles and zeros of the open-loop system. Therefore, we shall give here only some comments concerning the possible asymptotic behaviour of multivariable root loci, explaining, as far as possible, that behaviour in terms of the CTFs method. Since Rule 5 is a logical continuation of the preceding one, the presence of *branch points* is also of great importance here. If the ith CTFs $q_i(s)$ is situated on a *one-sheeted* (isolated) Riemann surface, i.e. it has no branch points, then, in principle, to find the pivot and angles of the asymptotes of that CTF, one can use the expressions analogous to those well known in the classical control theory (Evans 1954; Horowitz 1963).[60] In the general case, i.e. in the presence of branch points, the pivots and angles of asymptotes of the multivariable root loci situated on the ith sheet of the Riemann surface and associated with the ith CTF $q_i(s)$ depend also on the poles, zeros and 'gains' of the adjacent CTFs $q_r(s)(r \neq i)$. It is possible, however, to indicate some general patterns of relationship. In the MIMO system of order N, there may exist N groups of asymptotes constituting the Butterworth configuration, in which each group corresponds to *one* of the CTFs $q_i(s)$ $(i = 1, 2, \ldots, N)$ of the open-loop MIMO system. The pivots of these groups can be situated both on the real axis and at the arbitrary points of the complex plane, where all complex pivots must be complex conjugate in pairs. The latter property ensues from the fact that the multivariable root loci as a whole must be symmetrical with respect to the real axis.

A third-order group of asymptotes with a complex pivot at the point μ is schematically depicted in Figure 1.41, in which the angles between the asymptotes are equal to 120°. The starting angle of the asymptotes θ depends on the specific characteristics of the MIMO system. Another fundamental distinction of the multivariable root loci from the root loci of SISO systems should be pointed out. In the one-dimensional case, if $e = 1$, i.e. if the number of poles exceeds the number of zeros by one, then the corresponding asymptote always belongs to the real axis and the 'redundant' pole[61] p_r tends to infinity remaining on that axis. In the multivariable case, if any ith CTF $q_i(s)$ has the difference $e_i = np_i - nz_i = 1$, then the corresponding pole p_r does not necessarily tend to infinity along the real axis, and its asymptote

[60] We shall discuss such situations in the next two parts of this section.
[61] This pole, naturally, can be only real.

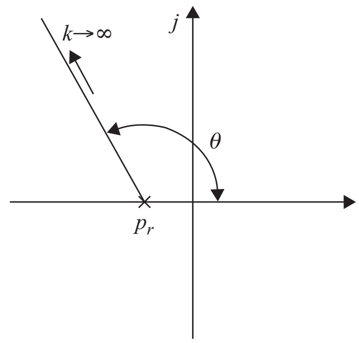

Figure 1.42 A possible asymptote of a real pole of $q_i(s)$ for $e_i = 1$.

may form with that axis a certain constant angle θ (Figure 1.42). In such situations, however, there always exists a CTF $q_r(s)(r \neq i)$ with the identical pole p_r, but having a 'complex conjugate' asymptote situated in the lower half-plane.

Rule 6: *Sections of the real axis, belonging to multivariable root loci.* In the SISO case, the root trajectories coinciding with the finite or semi-infinite intervals of the real axis can be found very easily – those are the sections of the real axis located to the left of the *odd* number of the real-valued singular points[62] of the open-loop transfer function (Evans 1954; Horowitz 1963). To a certain extent, the sections of the multivariable root loci belonging to the real axis can also be found relatively easily, but the main obstacle here is that there is a possibility that at the same point of the real axis, there may exist *more than one branch* of the root loci. In other words, in the multivariable case, we have to determine not only the real sections of the root loci, but also their number in each section. As always, for the high-dimension MIMO systems, this procedure is rather difficult. However, for the case of two-dimensional systems, we can, following Yagle (1981), indicate some simple rules for finding the sought sections. For such systems, based upon Equation (1.198), we have that the real values of complex variable s may belong to the root loci if Equation (1.79), rewritten below,

$$q_{1,2}(s) = \frac{tr\{W(s)\}}{2} \pm \sqrt{\frac{tr\{W(s)\}^2}{4} - \det W(s)} \tag{1.203}$$

produces real and negative values for $q_{1,2}(s)$. As the open-loop transfer matrix $W(s)$ [and, consequently, the trace $tr\{W(s)\}$ and the determinant $\det W(s)$] always becomes a real-valued numerical matrix for real values of s, then, from an inspection of Equation (1.203), it is easy to determine under which conditions the CTFs $q_{1,2}(s)$ are real and negative. For this purpose, we need to consider the possible values of the discriminant $D(s)$ [Equation (1.202)]. Thus, for the negative values $\det W(s) < 0$, only one of the CTFs $q_{1,2}(s)$ [which corresponds to the minus sign before the radical in Equation (1.203)] will be negative and, for that real value of s, we have *one* branch of the root loci at the point s.

[62] Here, and from now on, *singular points* means the poles and zeros.

If $\det W(s) > 0$, then, at the point s of the real axis, we may have *either two or no branches*. For $D(s) < 0, q_{1,2}(s)$ [Equation (1.203)] are complex conjugate numbers and do not satisfy the conditions of forming the roots of the closed-loop MIMO system [Equation (1.198)], i.e. the real point s does not belong to the root loci. For $D(s) > 0$, we have to analyse two additional conditions:

(a) if $tr\{W(s)\} > 0$, then *none of the branches* lies on the real axis at the point s, since both values $q_{1,2}(s)$ [Equation (1.203)] will be positive;

(b) if $tr\{W(s)\} < 0$, then we have *two branches* of the root loci, since both values $q_{1,2}(s)$ will be negative.

Finally, the zero value of the discriminant $D(s) = 0$ corresponds, as was stated above, to the *branch point*, at which the CTFs $q_{1,2}(s)$ are equal, i.e. $q_1(s) = q_2(s)$, and we have the transition from one sheet of the Riemann surface to another. At that point, the *direction* of the root trajectories along the real axis may change to the opposite.

In principle, there are similar rules for determining the number of real-valued branches of the multivariable root loci for $N > 2$, under the additional restriction of no repeated real poles and zeros of the open-loop MIMO system (Yagle 1981). We shall not dwell on them, as, for an engineer, it is much simpler to construct directly the root loci of the investigated MIMO system, rather than utilize the mentioned rules for finding the number of the root loci branches at the various points of the real axis. In the next two sections, while discussing the root loci of uniform as well as circulant and anticirculant MIMO systems, we shall give some quite simple rules for determining the real branches of the root loci of the listed classes of MIMO systems, which actually do not yield in simplicity to those in the SISO case.

Rule 7: *Angles of departure and arrival of multivariable root loci.* Here, we once again encounter the situation already familiar to us from some of the preceding rules, namely that the results existing in the literature are not very convenient for practical application (Shaked 1976) and, with the programs for numerical computations of the multivariable root loci available, the formulae for these angles play no role, since the angles of departure and arrival can be found immediately on having constructed and plotted the trajectories. Historically, interest in the angles of departure and arrival was most likely due to the absence of computer aids,[63] when the developer had to calculate the root loci of SISO systems with the help of a calculator or a slide-rule, or by means of a special tool called the *Spirule* (very likely, many contemporary readers are not even familiar with the latter device or the slide-rule) (Evans 1954).

Rule 8: *Breakaway points of multivariable root loci.* In the classical control theory, the term *breakaway point* usually is used for the points at which two or more branches of the root loci intersect (Evans 1954; Horowitz 1963). In the multivariable case, a significant and vital peculiarity exists, which has already been mentioned above. Unlike the common SISO systems in which we always have at the breakaway points a symmetrical picture of the 'approaching' and 'leaving' root trajectories (Figure 1.43), in the case of MIMO systems, two different situations are possible. The first of them corresponds to the intersections of the root loci situated on *one sheet* of the Riemann surface, related to the ith CTF $q_i(s)$, and is completely analogous to that shown in Figure 1.43, i.e. all entering and leaving trajectories form a symmetrical picture. In

[63] It is not excluded, of course, that in some tasks, knowledge of the departure and arrival angles can bring about qualitative estimates about the investigated MIMO system, and therefore can present certain theoretical interest.

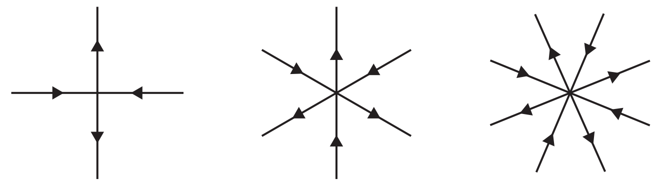

Figure 1.43 The breakaway points of SISO systems. (a) $l = 2$; (b) $l = 3$; (c) $l = 4$ (l is the number of trajectories).

the second situation, the picture is quite different; here, we have an apparent intersection of the root trajectories in which the latter in fact belong to the *different* CTFs and are situated on *different* sheets of the Riemann surface. In this case, the matter concerns not the *approaching* and *leaving* root trajectories, but an apparent *intersection* of two different independent trajectories, with arbitrary relative angles at the intersection point and with another character of the continuation of the trajectories (each of them preserves the previous direction of its motion after the intersection) (Figure 1.44).

The procedure of determining the breakaway points that are situated on the same, say the ith, sheet of the Riemann surface is practically analogous to that known from the classical control theory. Thus, if we have the intersection of two trajectories, then, at the point of intersection $s = s_o$, the ith equation in Equation (1.198) must have two equal roots s_o, i.e. it can be expressed in the form

$$1 + kq_i(s) = (s - s_o)^2 H(s) = 0. \tag{1.204}$$

Therefore, at the point s_o, not only the expression $[1 + kq_i(s)]$ but also its derivative with respect to s must be equal to zero, i.e. at that point, the condition $dq_i(s)/ds = 0$ must also be satisfied. Similarly, if trajectories of three roots of the ith characteristic system pass through the point s_o, then the point $s = s_o$ satisfies the corresponding equation in Equation (1.198) and the first and the second derivatives of $q_i(s)$ with respect to s at the point s_o are equal to zero. All this gives us analytical expressions for determining the breakaway points of the ith characteristic system if, of course, we have an analytical expression of the CTFs $q_i(s)$ itself.

Rule 9: *Intersection of the root loci with the imaginary axis.* Just as in the SISO case (Evans 1954; Horowitz 1963), the intersections of multivariable root loci with the imaginary axis can

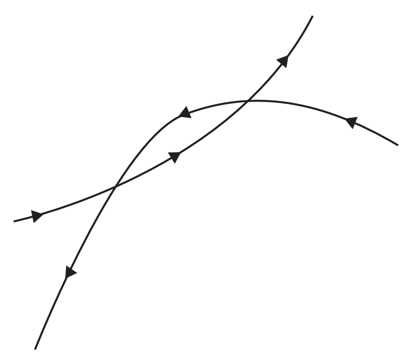

Figure 1.44 Intersections of the root loci of the CTFs situated on different sheets of a Riemann surface.

be easily found with the help of frequency-domain techniques, such as Nyquist, Bode and Nichols plots. Indeed, if we have, for instance, the Nyquist plots of the open-loop CTFs $q_i(j\omega)$ ($i = 1, 2, \ldots, N$), then the frequencies ω at the points of intersections, if any, of these plots with the negative real axis[64] determine the ordinates of the points at which the corresponding branches of root loci intersect the imaginary axis. This follows from the fact that at the points of intersection of $q_i(j\omega)$ with the negative real axis, we have the equality $\arg q_i(j\omega) = \pm 180^\circ$, i.e. the phase condition of forming the roots of the closed-loop MIMO system [the second condition in Equation (1.199)] is satisfied.

Rule 10: *Sum and product of the MIMO system roots.* In accordance with Viete's theorem (Derusso *et al.* 1965), the sum of the roots of a monic algebraic equation of order n is equal to the coefficient of the term of degree $n - 1$ taken with the opposite sign, and the product of the roots is equal to the absolute term multiplied by $(-1)^n$. Based upon these properties, it is possible to obtain some useful relationships between the sum and product of the roots situated on the root loci of a MIMO system and the singular points of the transfer matrix $W(s)$. The characteristic equation of the MIMO system [Equation (1.197)] can be represented in the following expanded form:

$$\det[I + kW(s)] = 1 - ktr\{W(s)\} + \ldots. + (-1)^N k^N \det W(s) = 0, \tag{1.205}$$

where the terms expressed through the sums of the principle minors of different orders [from the second to the $(N - 1)$th] of $W(s)$ are denoted by dots. After reducing to a common denominator, this equation can be rewritten, taking into account Equation (1.200), in the form

$$P(s) - kP_1(s) + \ldots. + (-1)^N k^N Z(s) = 0, \tag{1.206}$$

where $P_1(s) = P(s)tr\{W(s)\}$, i.e. in the form of an equation of degree n with respect to s, where n is the number of MIMO system poles. It is easy to understand that if the difference between the orders of the numerator and denominator polynomials of all diagonal elements of $W(s)$ is greater than two, which is a very typical situation in practice, then the difference in the orders of the poles polynomial $P(s)$ and the polynomial $P_1(s)$ in Equation (1.206) will also be greater than two. Therefore, the coefficient of the term of degree $n - 1$ in Equation (1.206) is equal to the corresponding coefficient of the polynomial $P(s)$ and *does not depend* on the parameter k. This, in turn, implies that as k changes, the sum of the roots of the closed-loop MIMO system must remain constant and equal to the sum of the poles of the open-loop system. As a result, if some of the roots of the characteristic Equation (1.205) move in the complex plane to the left, then some other roots of that equation must move to the right. Besides, the product of the MIMO system poles, which is equal to the absolute term of Equation (1.206), is proportional to k^N and increases infinitely with adequate increasing of k. Consequently, those branches of multivariable root loci that do not terminate at the finite zeros must tend to infinity.

A straightforward extension of these statements to the case of individual characteristic systems, the combination of the root loci of which constitutes the overall root loci of the MIMO systems, is not so evident. If no branch points exist and the CTFs $q_i(s)$ can be represented as a quotient of the two algebraic polynomials $M_i(s)$ and $D_i(s)$ (we know that this is possible for uniform, circulant and anticirculant systems), then, for those $q_i(s)$ for which the difference

[64] The gain margins of the characteristic systems are determined at these points (see Section 1.1.4).

between the numbers of poles and zeros exceeds two, i.e. $e_i = np_i - nz_i > 2$, the sum of the roots along the root loci of the given CTF is constant and does not depend on k. The reason is that in the absence of branch points, the root loci of the corresponding CTFs lie on the *isolated one-sheeted surfaces*. Then, this rule is a direct consequence of the same Viete's theorem. If $e_i > 2$, then, instead of Equation (1.198), we can write

$$D_i(s) + kM_i(s) = s^{np_i} + d_1 s^{np_i - 1} + (d_2 + km_0)s^{np_i - 2} + \ldots\ldots + (d_{np_i} + km_{nz_i}) = 0, \tag{1.207}$$

where the d_1 coefficient, taken with the opposite sign, is equal to the sum of all roots of that equation, and is equal to the sum of all poles of the ith open-loop CTF. Since the coefficient of the term of degree $np_i - 1$ in Equation (1.207) does not depend on k, then, if some roots on the ith sheet of the Riemann surface move to the right, the other roots must move to the left, and their sum should be constant and equal to the sum of the poles of $q_i(s)$. Besides, the product of the roots of Equation (1.207) increases with the increase in k, and, for $d_{np_i} = 0$, which is equivalent to the presence of zero poles in $q_i(s)$, this product is directly *proportional* to the value of k. These properties are complete analogues of the corresponding properties of the roots of common SISO systems (Evans 1954; Horowitz 1963).

In the presence of branch points, the picture becomes far more complicated. The chief problem is that in this case, some trajectories pass from one sheet of the Riemann surface to another. Therefore, as the coefficient k changes, there may be a change in *the number of roots* on a certain sheet of the Riemann surface, with a corresponding change in the number of roots on the adjacent sheet. This circumstance excludes generally the application to the individual CTFs of the above-stated simple rules concerning behaviour of the root loci of the MIMO systems taken as a whole.

In the classical control theory, one can find some other rules (Horowitz 1963), but we shall restrict our discussion to the listed ones, since they are quite sufficient for understanding the general features of multivariable root loci and their relationship with the root loci of common SISO systems.

Remark 1.13 In Sections 1.2.2 and 1.2.3, the definition of the *absolute poles* and *zeros*, as poles and zeros that are *common* to all CTFs $q_i(s)$ of the open-loop MIMO system was given. From the above exposition, it ensues that from any absolute pole originate, taking into account its multiplicity, exactly N branches of the root loci, and at any absolute zero terminate, taking into account its multiplicity, exactly N branches of the root loci of the MIMO system; such (generally, non-coincident) branches has each characteristic system. This can serve as another equivalent definition of the absolute poles and zeros of MIMO systems.

Remark 1.14 In Remark 1.6, we indicated that in the presence of zero entries in the open-loop transfer matrix $W(s)$, the situations in which some transfer functions of the cross-connections do not appear in the determinant $\det W(s)$ and their poles do not change on introducing the feedback and become the poles of the closed-loop MIMO system are possible. These poles represent the so-called *degenerate* branches of the multivariable root loci (Postlethwaite and MacFarlane 1979), consisting of a set of isolated points that preserve their positions as the coefficient k changes from zero to infinity.

Below, we give some specific examples which illustrate, on the one hand, the above-stated general rules of behaviour of multivariable root loci and which show, on the other hand, how

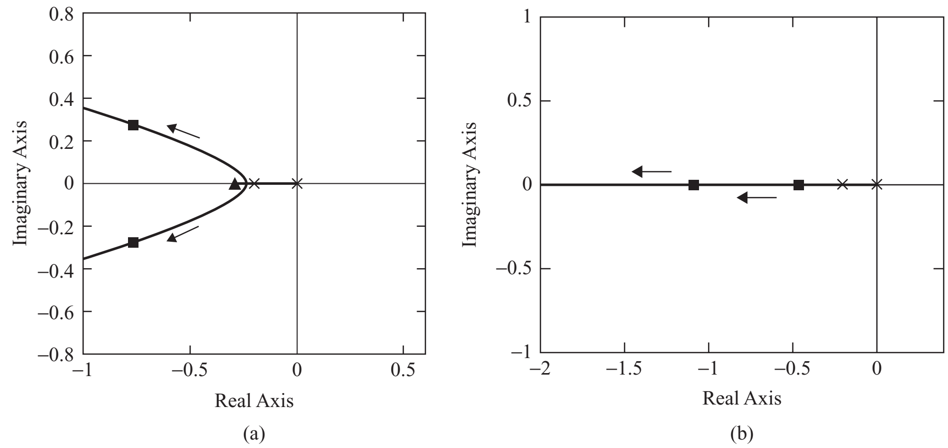

Figure 1.45 Root loci of the two-axis guidance system with transfer functions [Equation (1.65)]. (a) $\alpha_1 = 30^\circ, \alpha_2 = 20^\circ$; (b) $\alpha_1 = 30^\circ, \alpha_2 = -20^\circ$.

difficult may be in some cases to predict, based upon the given open-loop transfer matrix $W(s)$, even the general character of the roots' trajectories.

Example 1.12 Consider the two-axis general guidance system of Example 1.2. The root loci of that system in the case of the transfer functions [Equation (1.65)] and for the same two combinations of angles α_1 and α_2, as in Example 1.2: $\alpha_1 = 30^\circ, \alpha_2 = 20^\circ$ and $\alpha_1 = 30^\circ$, $\alpha_2 = -20^\circ$, are shown in Figure 1.45. The crosses in Figure 1.45 denote the open-loop system poles ($p_1 = 0$, $p_2 = -0.2$), the squares the closed-loop system roots (corresponding to $k = 1$) and the triangles the branch points. The arrows show the directions of movements of the roots as the coefficient k increases. As can be seen from these figures, we have two completely different pictures of the root loci.

For $\alpha_1 = 30^\circ, \alpha_2 = 20^\circ$, when the eigenvalues of the cross-connections matrix R [Equation (1.60)] are complex conjugate numbers, there is one branch point on the real axis at -0.2575. For $k = 0$, the root trajectories begin at the indicated above poles of the open-loop system, which are situated on the *different* sheets of the Riemann surface. As k increases, both poles first shift along the real axis to the left. Then, when the left root reaches the branch point, it passes to the adjacent sheet and begins moving in the opposite direction, towards the second root. For $k = 0.2039$, both roots meet (notice, on the *second* sheet of the Riemann surface!) at the point -0.2362, then depart from the real axis in opposite directions, and begin moving to infinity along two asymptotes making angles $\approx \pm 17.85^\circ$ with the negative real axis. For $k = 1$, the closed-loop system has two complex conjugate poles: $-0.768 \pm j0.2755$. Here, we have encountered the specific feature of multivariable root loci, when some branches, beginning their motion on one sheet of the Riemann surface, tend to infinity on the other sheet. This confirms once more the fact that the CTFs constitute a unique mathematical entity and should generally be considered a single whole.

For $\alpha_1 = 30^\circ$, $\alpha_2 = -20^\circ$, when the eigenvalues of R are real, the pattern of behaviour of the closed-loop system roots drastically differs from that considered above. Here, *there are no* branch points and the roots of each CTF move along the real axis, remaining on their own sheet of the Riemann surface. The configuration of root loci for each CTF in this case is quite

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analogous to that of the root loci of the SISO systems, having one zero pole or one real negative pole. For $k = 1$, this system has two real negative roots at the points -1.0764 and -0.4595.

Hence, depending on the specific forms of cross-connections and transfer functions of MIMO systems, the root trajectories may have quite different characters, and it is very difficult to predict the general behaviour of multivariable root loci. At the same time, if the root loci have already been constructed, then it can be verified that they indeed obey the above-listed rules.

Let us proceed now to a system with more complicated transfer functions $W_1(s)$ and $W_2(s)$ [Equation (1.66)]. The root loci of the systems for $\alpha_1 = 0°$, $\alpha_2 = 0°$, i.e. the root loci of the isolated separate channels of the system, are given in Figure 1.46(a) and (b), which show that as the coefficient k indefinitely increases, six branches of the root loci tend to infinity (two branches in the first channel and four in the second).

The root loci of the cross-connected system for $\alpha_1 = 30°$, $\alpha_2 = 20°$, that is for complex conjugate eigenvalues of the matrix R, are shown in Figure 1.46(c). The analysis of these root loci gives the following picture. Since $N = 2$, we have *two* sheets of the Riemann surface.

On one of them, which we shall call the first, there are situated *five* poles of the CTF $q_1(s)$: *two* zero poles $p^1_{1,2} = 0$ and three real poles: $p^1_3 = -2$, $p^1_4 = -4$, $p^1_5 = -8$. On the second

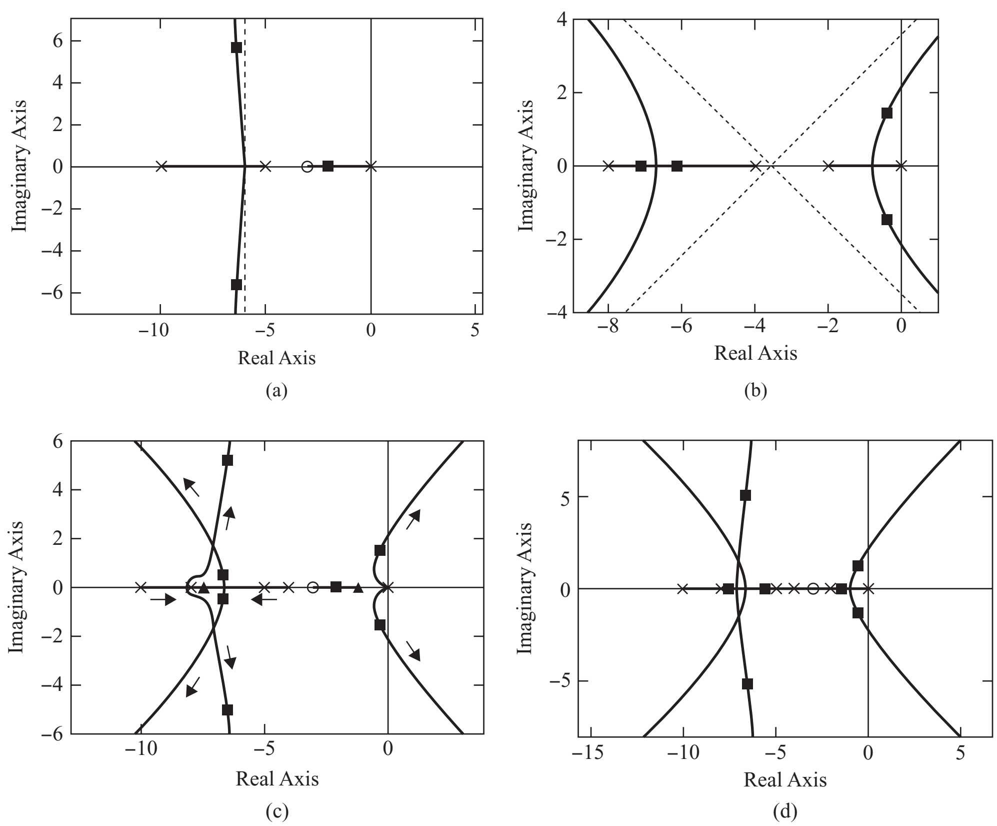

Figure 1.46 The root loci of the two-dimensional system [Equation (1.66)] for different combinations of angles α_1 and α_2. (a) $\alpha_1 = 0°$, $\alpha_2 = 0°$ (the first channel); (b) $\alpha_1 = 0°$, $\alpha_2 = 0°$ (the second channel); (c) $\alpha_1 = 30°$, $\alpha_2 = 20°$; (d) $\alpha_1 = 30°$, $\alpha_2 = -20°$.

sheet, which corresponds to the CTF $q_2(s)$, there are *two* poles $p_1^2 = -5$, $p_2^2 = -10$ and *one* zero $z_1^2 = -3$. *Three* branch points exist in the complex plane, at $s = -1.1545$, -7.66 and -7.954, denoted in Figure 1.46(c) by the small triangles. As k increases from the zero value, the zero poles of the CTF $q_1(s)$ form two complex conjugate branches of the root loci and move to the left; the root emerging from the pole $p_3^1 = -2$ moves along the real axis to the right, towards the roots that emerged from the origin.

The poles at points -4 and -5 first move to the left, and the poles at -8 and -10 move to the right; thus, they are situated in pairs on the different sheets of the Riemann surface. For $k = 0.1151$, the root on the first sheet, moving to the right from the pole $p_5^1 = -8$, reaches the branch point at -7.954, and, passing to the second sheet of the Riemann surface, begins moving in the opposite direction, towards the root from the pole $p_2^2 = -10$. For $k = 0.1975$, both these real roots meet at the point -8.1622 and then go into the complex plane, making angles $\pm 90^\circ$ with the real axis; thus, they form two branches of the root trajectories on the second sheet of the Riemann surface which tend to infinity in parallel with the imaginary axis (the second-order Butterworth pattern). The root of the first CTF, which emerged from the pole $p_3^1 = -2$, reaches, for $k = 0.307$, the branch point at -1.1545 and, passing to the second sheet of the Riemann surface, begins moving in the opposite direction, towards the zero $z_1^2 = -3$.[65] The two roots that emerged, on the first sheet, from the origin also change at that moment the moving direction to the opposite, towards the right half-plane. For $k = 0.3271$, the root of the second characteristic system, which emerged from the pole $p_1^2 = -5$, reaches the branch point at -7.66 and, after passing to the first sheet, begins moving to the right, towards the root from -4. These two roots meet, for $k = 0.9188$, at the point -6.634 and, breaking away from the real axis (and making with the latter the angles $\pm 90^\circ$), move in the first sheet to the left, thereby forming with the two branches from the origin a fourth-order Butterworth pattern.

It is interesting to note that in this example, we have encountered both variants of root loci intersections; in Figure 1.46(c), we can see the root loci intersections on the same sheet (the intersections on the real axis), as well as intersections of the branches situated on the *different* sheets of the Riemann surface (the intersection of the two complex-valued branches). Note also that the closed-loop system roots corresponding to $k = 1$ and marked in Figure 1.46(c) by the squares indicate the system stability. This agrees with the results obtained in Example 1.2 by means of the generalized Nyquist criterion.

The root loci of the system for $\alpha_1 = 30^\circ$, $\alpha_2 = -20^\circ$, when the eigenvalues of the cross-connection matrix R are real numbers, have quite another nature. These root loci are shown in Figure 1.46(d) and, roughly speaking, can be viewed as the superposition of the root loci of the isolated separate channels in Figure 1.46(a) and (b). In this case, there are no branch points and therefore all trajectories remain within their own sheets of the Riemann surface. However, the distribution of the poles of the CTFs $q_1(s)$ and $q_2(s)$ does not coincide with that of the poles of transfer functions $W_1(s)$ and $W_2(s)$ [Equation (1.66)]. The inspection of the root trajectories in Figure 1.46(d) shows that $q_1(s)$ and $q_2(s)$ can be represented, up to the values of the real 'gains' K_1 and K_2, as

$$q_1(s) = \frac{K_1(s+2)}{s(s+5)(s+8)}, \quad q_2(s) = \frac{K_2}{s(s+2)(s+4)(s+10)}. \tag{1.208}$$

[65] Note that in the case of rigid (static) cross-connections and a diagonal transfer matrix of the separate channels, the zeros and poles of the open-loop MIMO system coincide with the zeros and poles of the separate channels.

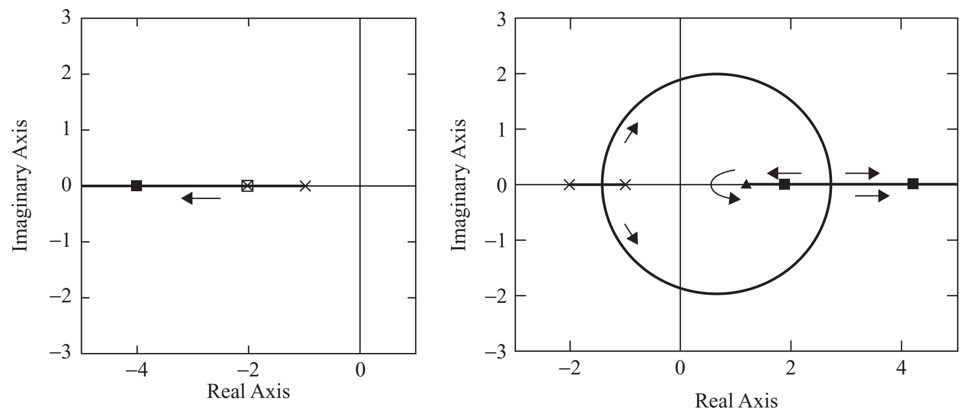

Figure 1.47 Root loci of the two-dimensional system of Example 1.3. (a) initial system; (b) varied system.

Comparison of Equation (1.208) with Equation (1.66) shows that the distribution of poles and zeros of $q_1(s)$ and $q_2(s)$ differs from that of $W_1(s)$ and $W_2(s)$ only by the fact that the poles -10 and -8 have interchanged. Correspondingly, the pairs of the poles that form the complex conjugate branches of the root loci in Figure 1.46(d) have changed.

Example 1.13 In Example 1.3, we discussed a peculiar two-dimensional system [Equation (1.67)], which, having, at first sight, an infinite gain margin, becomes unstable under the *simultaneous* additive increase of the first channel gain by 0.13 and decrease of the second channel gain by 0.12. A question arises of how these changes affect the root loci of the system. The root trajectories of the initial system are shown in Figure 1.47(a) and those of the varied system with the diagonal regulator [Equation (1.70)] in Figure 1.47(b). The comparison of these graphs confirms once more that the simultaneous change in the gains of separate channels, but with opposite signs, results in drastic changes in the internal structure of the system. The root loci of the initial system represent the combination of the root loci of two usual first-order systems with the poles at -1 and -2, i.e. here, we have two real branches along which the closed-loop system roots tend to $-\infty$, remaining on their sheets of the Riemann surface.[66] However, under the indicated variations of the gains, the picture completely changes. For $k = 0$, both poles of the MIMO system belong to *one* CTF and are situated on one sheet of the Riemann surface, at the points -1 and -2. As k increases, the closed-loop system roots, emerging from these poles, move towards each other and (for $k = 0.0144$) meet at the breakaway point s $= -1.4346$, where they depart from the real axis. Then, the complex conjugate poles move along a circle to the right half-plane and meet (for $k = 0.9284$) on the real axis at the point s $= 2.7288$. It should be accentuated that all these trajectories belong to one sheet of the Riemann surface. After meeting on the real axes, the roots begin moving along that axis in opposite directions, where the left root moves towards the branch point at s $= 1.2916$. For $k = 1.7191$, the left root reaches the branch point, then passes to the *second* sheet of the Riemann surface and, changing to the opposite moving direction, begins moving (like the root

[66] In Figure 1.47(a), the root of the *closed-loop* first characteristic system coincides at point -2 with the pole of the *open-loop* second characteristic system.

remaining on the first sheet) in the positive direction of the real axis. All this is shown in Figure 1.47(b) by arrows. Note that the moving direction of both poles along the real axis to $+\infty$ happens because the product of the roots must increase together with the increase in k (Rule 10).

Remark 1.15 In Example 1.13, we encountered an interesting situation in which, in the presence of branch points, some CTFs may have no poles or zeros on their sheets of the Riemann surface, but those sheets contain branches of multivariable root loci, starting from a certain value of k.

In the classical control theory, there is one more rarely used rule called the s*hift of the root loci starting points* (Horowitz 1963). According to that rule, if we have found the roots of a closed-loop SISO system which correspond to some value $k = k_0$, then these roots may serve as the *poles* of a certain new *open-loop* system, whose zeros coincide with the zeros of the initial system. If the root loci of the new system are constructed as the *new* gain $k' = k - k_0$ changes from zero to infinity, then these loci will be the continuation of the root loci of the initial system, as k changes from 0 to k_0. Or, similarly, the new root loci are part of the root loci of the initial system as k changes from zero to infinity. In other words, the poles of the initial system can always be replaced by a set of other poles that are situated at the corresponding points of the root loci.

As applied to our situation, we can regard the root of the MIMO system which has appeared, for some $k = k_0$, through a branch point on the ith sheet of the Riemann surface as a *fictitious pole* of a new CTF. Then, we can consider this sheet independently of all other sheets, assuming that the new gain $k' = k - k_0$ changes from zero to infinity. Such an approach may be useful for understanding, to a certain extent, the properties of multivariable root loci within each sheet of the Riemann surface.

Remark 1.16 The examination of the above examples allows us to draw another heuristic conclusion about the branch points of multivariable root loci. With respect to the branches of root loci, which begin on the given sheet of the Riemann surface, the branch points may be regarded as some *fictitious zeros* to which the roots of the closed-loop characteristic systems approach, obeying the general properties and features of the root loci. However, unlike the *actual* finite or infinite zeros belonging to the same sheet, the roots of the closed-loop system are ‘compensated’ (or ‘merged’) by the branch points for *finite* values of the gain k. The reader can ascertain this conclusion by inspection of the root loci in Figures 1.46 and 1.47. In particular, the behaviour of the root trajectories of the varied two-dimensional system in Figure 1.47(b) immediately becomes apparent.

Hence, combining the last two remarks, we can say that from the ‘entrance’ side, each branch point may be viewed as a *fictitious zero* to which one of the root trajectories of a certain characteristic system approaches, obeying the customary rules. From the ‘exit’ side, each branch point may be viewed as a *fictitious pole* from which one of the root trajectories of some other characteristic system originates. It is of importance here that the ‘compensation’ of a pole on one sheet of the Riemann surface is accompanied by the ‘origination’ of a new pole on another sheet, and all this occurs for some *finite* value of the parameter k.

1.5.2 Root loci of uniform systems

In Section 1.3, we pointed out that the structural features of uniform systems, i.e. MIMO systems with identical channels and rigid cross-connections, enable us to bring very closely the methods of their investigation to the corresponding methods for usual SISO systems. Largely, this concerns also the problem of constructing root loci of uniform systems. In fact, that problem is reduced to the task of constructing N *separate* root loci of SISO characteristic systems, and the rules of constructing these loci appear as a slight modification of the standard rules known from the classical control theory (Evans 1954; Horowitz 1963).

As we know,[67] the CTFs $q_i(s) = \lambda_i w(s)$ of uniform systems coincide, up to the constant 'gains' λ_i, with the transfer function of the separate channels $w(s) = M(s)/D(s)$, where λ_i are the eigenvalues of the cross-connections matrix R, and these eigenvalues can be either real or in complex conjugate pairs. We also know that all poles and zeros of $w(s)$ are *absolute* zeros and poles of the open-loop uniform system and, as a result, are *common* to all $q_i(s)$. Besides, the CTFs $q_i(s)$ do not have branch points and are situated on N isolated *one-sheeted* Riemann surfaces. The latter means that the root loci of each characteristic system can be constructed independently of the root loci of other characteristic systems, i.e. the task of finding the root loci of uniform systems indeed reduces, as was mentioned above, to N independent one-dimensional tasks (Gasparyan and Vardanyan 2007).

Let us rewrite Equation (1.173), taking into account the above comments, in the form

$$1 + k\lambda_i w(s) = D(s) + k\lambda_i M(s) = 0 \quad \text{or} \quad k\lambda_i w(s) = k\lambda_i \frac{M(s)}{D(s)} = -1, \quad i = 1, 2, \ldots, N. \tag{1.209}$$

The last of these equations may also be written down in the equivalent form

$$kw(s) = -\frac{1}{\lambda_i} = -\frac{1}{|\lambda_i|}\exp\{-j\arg\lambda_i\}, \tag{1.210}$$

from which we have somewhat different conditions of the formation of root trajectories in comparison with Equation (1.174):

$$|kw(s)| = \frac{1}{|\lambda_i|}, \quad \arg w(s) = \pm(2r+1)180^\circ - \arg\lambda_i, \quad r = 0, 1, 2, \ldots. \tag{1.211}$$

From Equations (1.209)–(1.211), the following *additional* rules describing the properties of the root loci of uniform systems immediately ensue.[68]

Rule 1: *Number of branches of the root loci*. The number of root trajectories of an N-dimensional uniform system is equal to Nnp_s, where np_s is the number of the absolute poles, i.e. the number of the poles of $w(s)$.

Rule 2: *Starting and ending points of the root loci*. For $k = 0$, the root trajectories begin at the roots of the equation $D(s) = 0$, i.e. at the absolute poles, and terminate, for $k = \infty$, at the roots of the equation $M(s) = 0$, i.e. at the absolute zeros of the uniform system. From each absolute pole originate exactly N trajectories and, at any absolute zero, terminate exactly N trajectories. In other words, the root loci of each of N characteristic systems begin at the poles of the transfer function $w(s)$ and terminate at its zeros.

[67] See Equation (1.94).
[68] For simplicity, we shall list these rules starting from number 1. The reader should remember that they are *additional* to the general rules established in the previous section.

Rule 3: *Number of the root trajectories tending to infinity*. The number of root trajectories of each characteristic system tending to infinity as k increases indefinitely is equal to $e_s = np_s - nz_s$, where nz_s is the number of zeros of $w(s)$ (the number of absolute zeros of the uniform system). Respectively, the number of root trajectories of the uniform system tending to infinity is equal to Ne_s.

Rule 4: *Angles of the asymptotes of the root loci*.[69] It can be shown that the set of e_s approaching infinity root trajectories of the ith characteristic system tend indefinitely to the Butterworth pattern of order e_s, whose evenly spaced lines (the root trajectory asymptotes) make with the positive direction of the real axis angles

$$\gamma_r = \frac{(2r+1)180^\circ + \arg\lambda_i}{e_s}, \quad r = 0, 1, \ldots, e_s - 1. \tag{1.212}$$

As is evident from Equation (1.212), the angles of root trajectory asymptotes of the characteristic systems with real-valued λ_i, i.e. with $\arg\lambda_i = 0$, are equal to the angles of asymptotes of a common SISO system with an open-loop transfer function $w(s)$. Further, each Butterworth pattern corresponding to a complex-valued λ_i is nonsymmetrical with respect to the real axis. However, taking into account that all complex eigenvalues λ_i must occur in complex conjugate pairs, for each such pattern, there exists a 'complex conjugate' pattern, which is obtained from the first one by mirror mapping around the real axis.

Rule 5: *Pivots of the asymptotes*. It can also be shown that the pivots (centres) of the asymptotes of all characteristic systems are situated at the same point on the real axis and coincide with the common centre of asymptotes A_c of isolated separate channels of the uniform system:

$$A_c = \frac{\sum_{r=1}^{np_s} p_r - \sum_{r=1}^{nz_s} z_r}{np_s - nz_s}, \tag{1.213}$$

where p_r and z_r denote the poles and zeros of the transfer function $w(s)$.

Rule 6: *Coincident symmetrical root trajectories*. The root trajectories of the characteristic systems corresponding to *all* real eigenvalues λ_i of R are symmetrical with respect to the real axis, coincide with each other and coincide with the root trajectories of isolated separate channels of the uniform system. This property becomes apparent if we consider the phase (the second) condition in Equation (1.211), which, for real λ_i, i.e. for $\arg\lambda_i = 0$, just transforms into the analogous condition for common SISO systems with an open-loop transfer function $w(s)$. The location of the roots of the corresponding characteristic systems on these trajectories is different and depends on the magnitudes $|\lambda_i|$.

Rule 7: *Complex conjugate and coincident nonsymmetrical root loci*. The root loci of the characteristic systems with complex conjugate eigenvalues λ_i are nonsymmetrical with respect to the real axis. All of these root loci can be obtained from each other by mirror

[69] The proof of Rules 4 and 5 is left to the reader as an exercise.

mapping around the real axis. All the root trajectories of a uniform system corresponding to λ_i with the same argument $\arg \lambda_i$ but, possibly, with different magnitudes $|\lambda_i|$ coincide. This rule is a logical continuation of the preceding ones and follows from Equation (1.212).

Rule 8: *Sections of the real axis belonging to root loci.* A uniform system can have root trajectories coinciding with finite or semi-infinite sections of the real axis only in the presence of real eigenvalues λ_i. In this case, such sections are those located to the left of the *odd* singular points[70] of $w(s)$. This rule also ensues from the previous rules. Concerning the characteristic systems with complex λ_i, such systems *cannot* have the root trajectories on the real axis. This is evident from the conditions in Equation (1.210), which cannot be satisfied for complex λ_i and real s, when the transfer function $w(s)$ becomes real-valued.

Rule 9: *Value of k.* The value of the parameter k corresponding to any point s_0 on the root trajectories of the ith characteristic system can be found from Equation (1.211) and is equal to $k = 1/|\lambda_i w(s_0)|$.[71]

Hence, we have formulated a number of additional rules describing the properties of root loci of uniform systems, which enable the engineer or researcher to construct those loci without any difficulty. The reader might have realized that these rules are quite similar to the standard rules of constructing root loci for usual SISO systems. Unlike general MIMO systems, in the case of uniform systems, the root loci represent the combination of N root loci corresponding to N isolated, i.e. having no branch points, characteristic systems. This statement, as well as the fact that the CTFs $q_i(s) = \lambda_i w(s)$ of the discussed systems can be expressed analytically for any number of channels N,[72] implies that the problem of construction of root loci for uniform MIMO systems is practically solved.

Example 1.14 Consider the two-axis uniform guidance system of Example 1.4 with the gain of the transfer function decreased by a factor of 10 [Equation (1.103)]. This decrease in the gain is not essential at all, but it provides system stability for both combinations of angles α_1 and α_2. The root loci of the system are shown in Figure 1.48, where the dashed lines represent the asymptotes. The overall view of the root loci for $\alpha_1 = 40^\circ$ and $\alpha_2 = 35^\circ$, which corresponds to complex conjugate eigenvalues λ_1 and λ_2 of the cross-connections matrix R (see Example 1.4), is given in Figure 1.48(a). The root trajectories of both characteristic systems, for the above values of α_1 and α_2, are shown in Figure 1.48(b) and (c). As is evident from Figure 1.48(a)–(c), for complex conjugate eigenvalues of R, the root loci of the guidance system do not have sections situated on the real axis, and the root loci and asymptotes of the characteristic systems are nonsymmetrical with respect to the real axis and can be obtained from each other by complex conjugation.

For $\alpha_1 = 40^\circ$ and $\alpha_2 = -35^\circ$, when the eigenvalues λ_1 and λ_2 are real, the root loci of the two-dimensional guidance system coincide with the root loci of the isolated separate channels, but we have here two identical, superimposed patterns of the root trajectories belonging to two different characteristic systems [Figure 1.48(d)].

[70] That is, if the total number of poles and zeros of $w(s)$ to the left of the section is *odd*.
[71] This rule may be rather useful for the development of interactive program packages intended for computer-aided investigation of linear uniform systems.
[72] Recall that modern application packages enable the user to calculate the eigenvalues of numerical matrices of, in fact, any order.

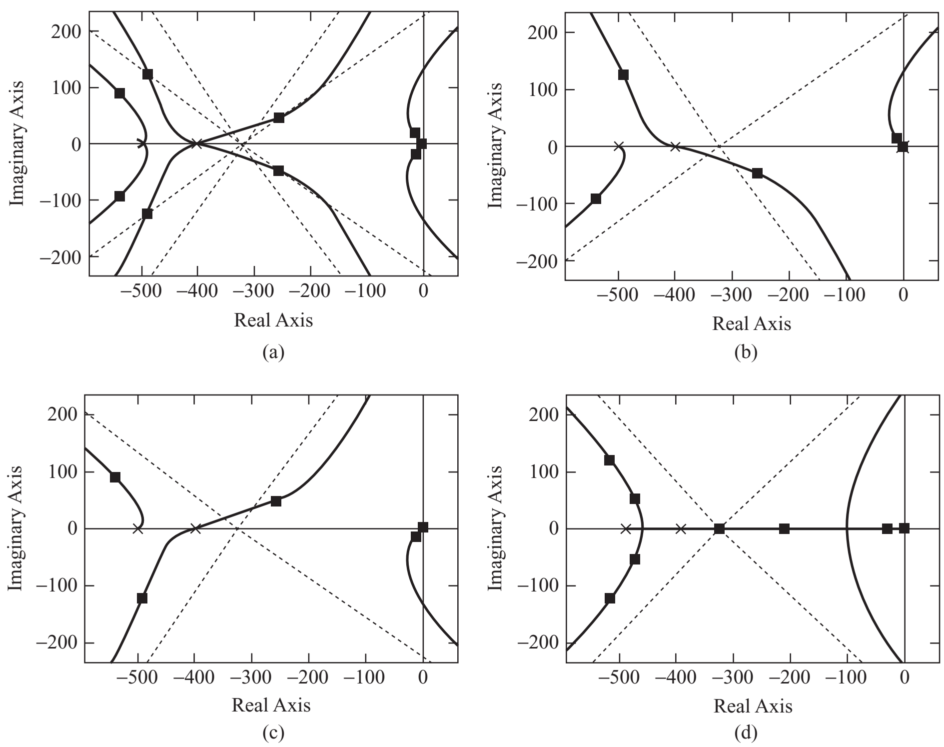

Figure 1.48 The root loci of the two-axis uniform guidance system of Example 1.4. (a) $\alpha_1 = 40^\circ$, $\alpha_2 = 35^\circ$ (overall view); (b) $\alpha_1 = 40^\circ$, $\alpha_2 = 35^\circ$ (the first characteristic system); (c) $\alpha_1 = 40^\circ$, $\alpha_2 = 35^\circ$ (the second characteristic system); (d) $\alpha_1 = 40^\circ$, $\alpha_2 = -35^\circ$ (overall view).

1.5.3 Root loci of circulant and anticirculant systems

Internal structural characteristics and properties of circulant and anticirculant systems are so much alike that we shall discuss only the root loci of circulant systems below, noting, at the same time, some specific features inherent in anticirculant systems. The salient feature of circulant systems is that root loci of their characteristic systems are situated, similarly to uniform systems, on *isolated one-sheeted* Riemann surfaces, i.e. here, also, the problem of constructing root loci splits, without any assumptions or simplifications, into N independent one-dimensional tasks. Further, we know that the CTFs $q_i(s)$ of circulant systems can be represented analytically for any number N of separate channels as proper rational functions in complex variable s. The poles of all CTFs are *absolute* and equal to all poles of the first (or any other) row elements of the open-loop transfer matrix $W(s)$. As a result, their distribution is symmetrical with respect to the real axis.[73] Concerning the zeros of the CTFs, and these

[73] We do not consider here the situations in which zeros of some other circulant transfer matrix can cancel some absolute poles of a circulant system (see also Example 1.10, in which zeros of the CTFs cancel poles of the common scalar multiplier, and the poles of the resulting CTFs are complex conjugate to each other). For simplicity, we shall also assume that the circulant transfer matrix $W(s)$ does not have any scalar multipliers (i.e. multipliers in the form of scalar transfer functions), which originate *absolute poles and zeros* of circulant systems, *common* to all $q_i(s)$.

zeros give the zeros of a circulant system, in the general case, they are roots of the polynomials with complex-valued coefficients; therefore, their distribution for each characteristic system is nonsymmetrical. At the same time, to any such nonsymmetrical location of zeros corresponds a 'complex conjugate' location of zeros of some other characteristic system, resulting in the symmetrical picture for *all* zeros of the circulant system. Finally, it is worth noting that the *number* of zeros of all CTFs of the circulant system is the *same*, which ensues from the form of the CTFs $q_i(s)$ [Equation (1.133)]. All this suggests that the properties of circulant systems' root loci must be quite close to the corresponding properties of uniform systems. This is indeed the case, but, here, there are some essential distinctions due to the presence of *dynamical* cross-connections, or, in other words, due to the presence of N generally different transfer functions in any row of the open-loop matrix $W(s)$, whereas, in the case of uniform systems, we have *only one* transfer function $w(s)$.

If we denote by $D(s)$ the *identical* denominator polynomials of all CTFs $q_i(s)$ of circulant systems and by $M_i(s)$ the numerator polynomials, i.e. if we represent the CTFs in the form $q_i(s) = M_i(s)/D(s)$, then Equations (1.198) and (1.199) can be rewritten as

$$1 + kq_i(s) = D(s) + kM_i(s) = 0 \quad \text{or} \quad kq_i(s) = k\frac{M_i(s)}{D(s)} = -1. \quad i = 1, 2, \ldots, N \tag{1.214}$$

These conditions enable us to formulate some rules for constructing the root loci in a form that allows for the specific structural features of circulant systems. We shall list these rules, beginning, as in the previous section, from number 1.

Rule 1: *Number of branches of the root loci*. The number of root trajectories of an N-dimensional circulant system is equal to Nnp_0, where np_0 is the number of absolute poles, i.e. the number of all poles of the elements of the first row of $W(s)$.

Rule 2: *Starting and ending points of the root loci*. For $k = 0$, the root trajectories of each (say, the ith) characteristic system begin at the same roots of the equation $D(s) = 0$, i.e. at the absolute poles, and terminate, for $k = \infty$, at the roots of the equation $M_i(s) = 0$. If we denote by nz_0 the *same* order of all polynomials $M_i(s)(i = 1, 2, \ldots, N)$, then this implies that nz_0 root trajectories of each characteristic system approach, as $k \to \infty$, nz_0 finite zeros of the corresponding CTF $q_i(s)$.

Rule 3: *Number of root trajectories tending to infinity*. The number of root trajectories of each characteristic system tending to infinity (as k increases indefinitely) is the same and equal to $e_0 = np_0 - nz_0$. Respectively, the total number of root trajectories of a circulant system tending to infinity is equal to Ne_0.

Rule 4: *Angles of root loci asymptotes*. Let us represent $q_i(s)$ in the factored form:

$$q_i(s) = \frac{M_i(s)}{D(s)} = \frac{K_c^i \prod_{j=1}^{nz_0} \left(s - z_j^i\right)}{\prod_{j=1}^{np_0} (s - p_j)}, \quad i = 1, 2, \ldots, N, \tag{1.215}$$

where p_j are the poles (they are the same for all i), z_j^i the zeros and K_c^i the 'gains' (generally complex-valued) of the characteristic systems. It can be shown[74] that the set of e_0 approaching infinity root trajectories of the ith characteristic system tend indefinitely to the Butterworth pattern of order e_0, whose evenly spaced lines (the root trajectory asymptotes) make, with the positive direction of the real axis, angles equal to

$$\gamma_r = \frac{(2r+1)180^\circ + \arg K_c^i}{e_0}, \quad r = 0, 1, \ldots, e_0 - 1. \tag{1.216}$$

Comparison of Equation (1.216) with Equation (1.212) shows that the coefficients K_c^i play here the same role as the eigenvalues λ_i in the case of uniform systems, with all the geometrical features indicated in Rule 4 of the previous section.

Rule 5: *Pivots of the asymptotes*. The pivot of the Butterworth pattern for the ith characteristic system is located on the complex plane at the point

$$A_{ci} = \frac{\sum_{r=1}^{np_0} p_r - \sum_{r=1}^{nz_0} z_r^i}{np_0 - nz_0}, \quad i = 1, 2, \ldots, N \tag{1.217}$$

where the designations are the same as in Equation (1.215).

Here, we have encountered an essential difference between the pivots of circulant and uniform systems. In the case of uniform systems, the pivots of asymptotes for all characteristic systems are at the same point on the real axis. For circulant systems, these pivots may be, in the general case, *complex-valued*. This is because the complex zeros z_j^i in Equation (1.217), being generally the roots of the polynomials with complex coefficients, are not complex conjugate. However, as the reader has most likely guessed, a complex conjugate pivot of some other characteristic system with the complex conjugate numerator polynomial of the CTF always corresponds to each complex pivot A_{ci}. For the circulant systems of *odd* order N, the number of such complex-valued pivots is equal to $N-1$ and, for *even* order N, their number is $N-2$.

It should be pointed out that all pivots of anticirculant systems of *even* order are complex. This can be easily verified by examination of Figure 1.33, in which the eigenvalues of the anticirculant permutation matrix U_- [Equation (1.156)] for different values of N are shown. For anticirculant systems of *odd* order N, one pivot on the real axis generated by the eigenvalue $\beta_{(N+1)/2} = -1$ always exists. Finally, all pivots of *simple symmetrical MIMO systems* lie on the real axis, since all the coefficients of their CTFs [Equations (1.138) and (1.139)] are real numbers.

Rule 6: S*ymmetrical root trajectories.* Let us consider the eigenvalue distribution of the permutation matrix U [Equation (1.128)] (Figure 1.29). We know that, independently of order N, the first eigenvalue β_1 of U is always equal to unity. Further, for even N, the $(1+N/2)$-th eigenvalue $\beta_{1+(N/2)}$ is always equal to minus one. Therefore, all CTFs $q_i(s)$ generated by these eigenvalues will have, owing to Equation (1.133), polynomials with real coefficients, not only in the denominator, but also in the numerator. Correspondingly, the root loci of these characteristic systems possess all the well known properties of the SISO systems' root loci.

[74] We address the proof of Rules 4 and 5 to the reader as an exercise.

This concerns the symmetry of root trajectories with respect to the real axis, sections of the root loci belonging to that axis, breakaway points, etc.

It is easy to understand that *simple symmetrical MIMO systems*, having for any N only two distinct characteristic systems with real coefficients [Equations (1.138) and (1.139)], also have the root loci completely analogous (considering each characteristic system separately) to root loci of usual SISO systems. It is also easy to understand that anticirculant systems of *even* order have only complex-valued CTFs, and therefore cannot have sections of the root loci on the real axis. As for *odd* N (the case of no actual interest in practice), only *one* characteristic system with real parameters exists.

The above simple rules reflect the specific character of the internal structure of circulant and anticirculant systems. The new and very significant point here is that the pivots of the root loci asymptotes are situated not only on the real axis, but can also be complex conjugate. At the same time, taking into account that for the pivots and angles of asymptotes, we have simple expressions [Equations (1.216) and (1.217)], as well as the fact that the CTFs $q_i(s)$ can be expressed in analytical form for any number of channels N, the construction of root loci for circulant and anticirculant systems can be carried out without any difficulty.

Example 1.15 As an example of MIMO systems having complex pivots of the root loci asymptotes, consider a two-dimensional antisymmetrical system described by a transfer matrix

$$W(s) = \frac{50}{(s+2)(s+3)(s+4)} \begin{pmatrix} 1 & \dfrac{4}{(s+8)} \\ -\dfrac{4}{(s+8)} & 1 \end{pmatrix}. \tag{1.218}$$

The CTFs $q_{1,2}(s)$ of that system are

$$q_{1,2}(s) = \frac{50(s+8 \pm j4)}{(s+2)(s+3)(s+4)(s+8)} \tag{1.219}$$

and have two complex-conjugate zeros at $z_1 = -8 - j4$ and $z_2 = -8 + j4$. The root loci of the system are shown in Figure 1.49(a), from which it can be seen that the pivots are located at the complex conjugate points $A_{c1} = -3 + j1.333$ and $A_{c2} = -3 - j1.333$ (these pivots are marked in Figure 1.49 by small black circles). The nonsymmetrical root loci of each characteristic system are presented in Figure 1.49(b) and (c).

Example 1.16 In the last example of this section, we return to a nonrobust antisymmetrical system taken from Doyle (1984) and Packard and Doyle (1993) and discussed in Example 1.10. This system is of interest to us, not only as an *anticirculant* system, but also as a system which becomes, under slight perturbations of the parameters, a *general* MIMO system with rather remarkable features. In the case of an ideal unit regulator $K = diag\{K_i\} = I$, the characteristic equations of one-dimensional systems [Equation (1.180)] have, taking into account the 'root loci coefficient' k, the form

$$\begin{aligned} s - ja + k(1 + ja) &= 0, \\ s + ja + k(1 - ja) &= 0. \end{aligned} \tag{1.220}$$

It is easy to see that as k changes from zero to infinity, the trajectories of the roots of Equation (1.220) are straight lines, beginning at the poles $p_{1,2} = \pm ja$ of the open-loop system and

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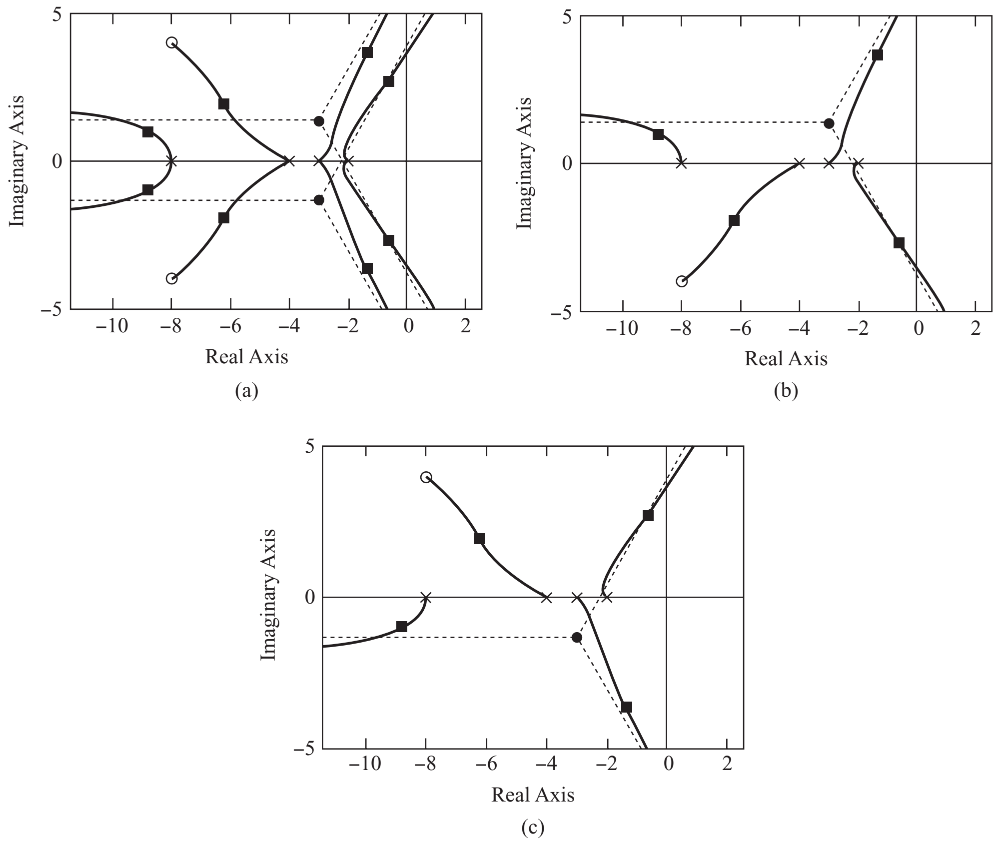

Figure 1.49 The root loci of antisymmetrical system [Equation (1.218)]. (a) the overall view of the root loci; (b) the first characteristic system; (c) the second characteristic system.

intersecting, for $k = 1$, at the point $(-1, j0)$, at which both poles of the closed-loop system are located [Figure 1.50(a)]. Note also that each trajectory in Figure 1.50(a) is situated on an isolated one-sheeted Riemann surface. Assume, now, that the coefficients of the static regulator K have been changed to new values $K_1 = 1.1$ and $K_2 = 0.9$. As was indicated in Example 1.10, with these values of K_1 and K_2, the two-dimensional antisymmetrical system becomes an unstable *general* MIMO system. This fact is completely confirmed by the root loci of the varied system presented in Figure 1.50(b). As is evident from that figure, the root trajectories have drastically changed. Now, because the system is not circulant, its root loci are situated on a *two-sheeted* Riemann surface and have *two* branch points. For $k = 0$, both poles of the open-loop system are situated on *one* (say, the first) sheet, i.e. they belong to the first characteristic system. As k increases, the roots of the first characteristic system move in that sheet to the real axis and, for $k = 0.9087$, meet at the breakaway point –0.9087 on that axis. As k increases further, both these roots depart from the breakaway point in opposite directions and move along the real axis towards the branch points (towards the *fictitious zeros* of the first characteristic system) at $s_1 = -2.005$ and $s_2 = +0.005$ (notice that the second branch point is located in the right

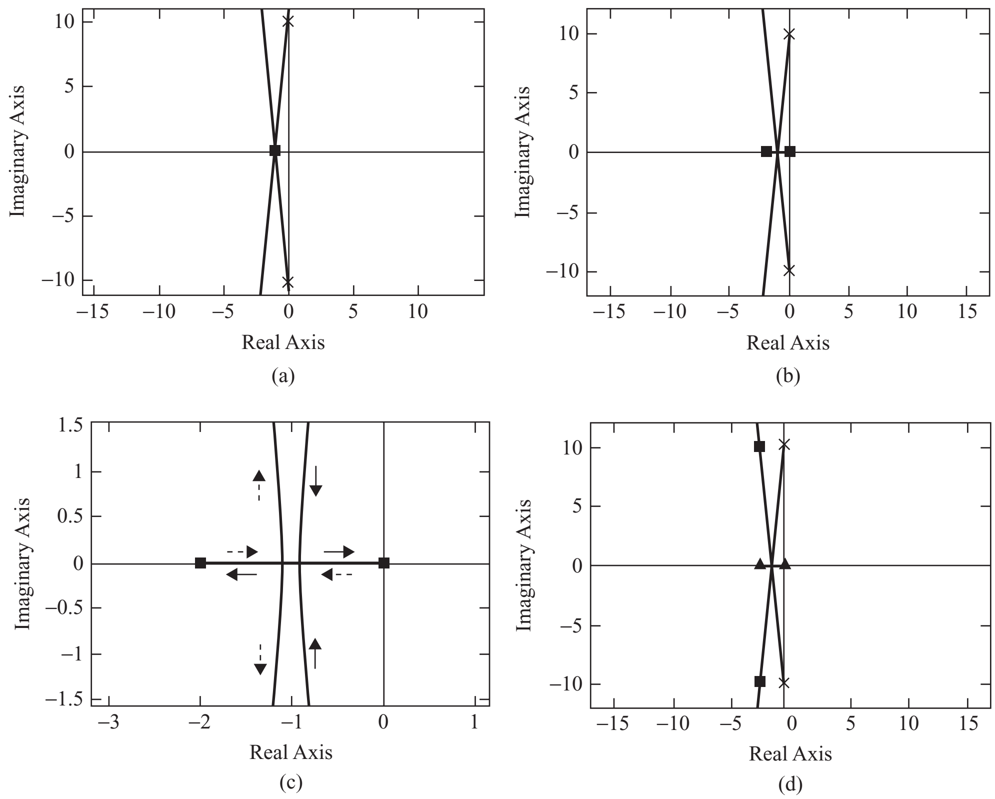

Figure 1.50 The root loci of the non-robust antisymmetrical system from Doyle (1984). (a) the root trajectories of the ideal system (for $K = I$); (b) the root trajectories of the varied system; (c) the root trajectories of the varied system (the enlarged view); (d) the location of the roots of the varied system (for $k = 2$).

half-plane). For $k = 1$, both roots simultaneously (!) reach the branch points[75] and, as k increases further, they pass to the *second* sheet of the Riemann surface. Thus, we have obtained that for the nominal values $K_1 = 1.1$ and $K_2 = 0.9$ of the varied regulator (for $k = 1$), the roots of the closed-loop system are equal to $s_1 = -2.005$ and $s_2 = +0.005$, i.e. one of the roots is in the right half-plane and the system is *unstable*. However, as k increases further, the roots of the system begin moving on the second sheet from the branch points (from the *fictitious* poles) along the real axis towards each other, i.e. the ‘unstable’ right root begins moving in the opposite direction towards the left half-plane, reaching the imaginary axis for $k = 1.0099$. Further, for $k = 1.1117$, both roots meet on the second sheet of the Riemann surface at the breakaway point -1.1117[76] and depart from the real axis into the complex plane along the asymptotes that make, with the positive direction of the real axis, angles approximately equal to $\pm 95.7^\circ$. All of this is illustrated in Figure 1.50(c) by arrows, where the solid arrows correspond to the motion in the first sheet and the dashed arrows to that in the second sheet. For more clarity, the

[75] The right root reaches the imaginary axes for $k = 0.995$.

[76] The equality of the values of k at both breakaway points to the absolute values of these points abscissas –0.9087 and –1.1117 is not a misprint – it is obtained by calculations.

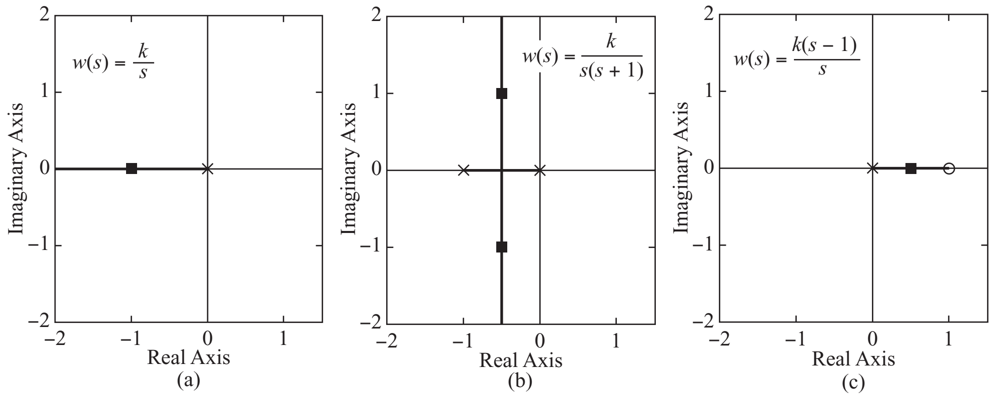

Figure 1.51 The root loci of the SISO systems of Remark 1.17.

roots of the varied system for $k = 2$, which are equal to $-2 \pm j9.7959$, are shown in Figure 1.50(d). Hence, under the varied parameters of the regulator, the discussed system is unstable over a very small range of changes of k, from 0.995 to 1.0099, and this conclusion is obtained by an inspection of the root loci in Figure 1.50.

Remark 1.17 In the scientific and technical literature, much attention is devoted to the limitations on the stability and performance of MIMO systems imposed by the *zeros situated in the right half-plane* (Skogestad and Postlethwaite 2005; Havre and Skogestad 1998). Having the multivariable root loci plotted, these limitations can be understood and assessed adequately, which allows (if needed) one to undertake the corresponding *justified* steps for diminishing the influence of the right half-plane zeros (frequently called the RHP-zeros). In principle, that problem is also inherent in the SISO case. As elementary examples, the root loci of two simple SISO systems having one pole at the origin in the open-loop state are shown in Figure 1.51(a) and (b). As is evident from these root loci, the closed-loop systems are stable for any arbitrary large values of the gain k. However, introducing a RHP-zero into these systems drastically changes the situation, and the systems become *unstable under any arbitrary small* values of k [in Figure 1.51(c), this is illustrated for the system in Figure 1.51(a)]. Similar examples can readily be given for the multivariable case. That can be done especially easily for uniform systems with real eigenvalues of the cross-connections matrix, for which the root loci of characteristic systems will qualitatively have, the transfer functions of the separate channels being chosen correspondingly, just the same form as in Figure 1.51. Rather, the problem here is that for general MIMO systems, as well as for circulant and anticirculant systems, it is very difficult, based on the open-loop transfer matrix $W(s)$, to judge the possible location (and even just the presence) of the zeros[77] in the right half-plane. As the location of these RHP-zeros may be quite unexpected, it may take a great deal of effort for the researcher or engineer to eliminate or diminish the inevitably negative influence of such zeros. To a certain extent, this issue is somewhat simpler for circulant and anticirculant systems, in which the system zeros coincide with the zeros of the CTFs, which can always be expressed analytically. Finally, it should be noted that for general MIMO systems, the *branch points located in the right half-plane* (see Figure 1.47) also impose similar limitations on the stability.

[77] For uniform systems, these zeros always coincide with the zeros of the identical separate channels.