CHAPTER 1

INTRODUCTION: INTUITIVE TOPOLOGY

COPYRIGHTED MATERIAL

1.1 INTRODUCTION: INTUITIVE TOPOLOGY

What is topology? One would hope that a book entitled *Topology* would provide a simple answer to this question as a starting point. Unfortunately, it's not terribly easy to give a brief answer without first building up some background. Here's a first attempt: Topology is the study of the qualitative properties of certain objects (called *topological spaces*) that are invariant under certain kinds of transformation (called *continuous maps*), especially those properties which are invariant under a certain kind of equivalence (called a *homeomorphism*). This answer probably raises more questions than it answers, but it makes a good deal of sense when put into an appropriate mathematical context.

Euclidean geometry, for example, is a branch of mathematics that nearly everyone has spent some time studying. If asked for a short definition of Euclidean geometry, most students would have a difficult time distilling a year-long high school course into a single sentence. However, there is a very simple theme that unites nearly all of the varied topics studied in high school geometry – one considers the properties of certain objects (triangles, squares and other planar figures) that are preserved under congruence or similarity. Precisely, we consider certain transformations of the plane, called *Euclidean transformations*: rotations (of the coordinate axes by any angle), reflections (across any line in the plane) and translations (moving the "origin" of the coordinate system to any point in the plane), and transformations obtained as some combination of these. Two triangles are said to be congruent if one can be transformed into the other under some Euclidean transformation. If we add "rescalings" (scalar multiplication of the plane by any positive number) to our list of possible transformations, we can define the concept of similar triangles in much the same way. So our short definition of Euclidean geometry is as follows: geometry is the study of the properties of planar figures that are invariant under Euclidean transformations. We'll look at a similar explanation of topology shortly.

This idea ties in very nicely to another area of mathematics that most students have studied before encountering topology – abstract algebra. The set of Euclidean transformations of the plane forms a *group* under the operation of composition. In fact, Felix Klein defined a "geometry on a space X" to be the study of the properties of subsets of X that are invariant under the action of some transformation group G. [1]

The various areas of study lumped together as abstract algebra provide another important analogy for understanding what topology is all about. In group theory, for example, one studies groups and group homomorphisms. (See Section 2.6 for a primer on groups.) Precisely, a group is a set G with an operation, $\bullet$, satisfying three properties: $\bullet$ is associative ($(g_1 \bullet g_2) \bullet g_3 = g_1 \bullet (g_2 \bullet g_3)$ for all $g_1, g_2, g_3 \in G$); G contains an identity for the operation $\bullet$ (there exists $e \in G$ such that $g \bullet e = g = e \bullet g$ for every $g \in G$) and each element of G has an inverse in G (for every $g \in G$, there exists $g^{-1} \in G$ such that $g \bullet g^{-1} = e = g^{-1} \bullet g$). For example, the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ forms a group under $+$, but not under multiplication. A group homomorphism is a function $f : G \to H$ from a group G to a group H that respects the group structures: $f(g_1 \bullet_G g_2) = f(g_1) \bullet_H f(g_2)$ for every $g_1, g_2 \in G$, where $\bullet_G$ indicates the operation in G and $\bullet_H$ the operation in H. Group theory can be summarized as the study of the properties of groups that are preserved under group homomorphisms, especially those preserved by isomorphisms (equivalences between groups). For example, the property of being *abelian* [2] (or commutative: $g_1 \bullet g_2 = g_2 \bullet g_1$ for every $g_1, g_2 \in G$) is preserved by group homomorphisms (both trivial and nontrivial). So if G is abelian and H is not, then G cannot be isomorphic to H. Thus, the cyclic group of order 6, $\mathbb{Z}_{/6}$, cannot be isomorphic to the dihedral group on three letters, D_3, despite the fact that they both have six elements. This idea shows up in other areas of algebra; including semigroup theory, ring theory and field

---

[1] Felix Klein's (1849–1925) Erlangen program (named after the Universitat Erlange, where he held his chair) defined the concept of geometry entirely in terms of abstract algebra, a notable departure from previous mainstream mathematical thought.

theory. In each case, we have a set plus some other structure, and we study properties preserved by functions that respect this extra structure.

The field of topology works similarly. The objects we study are topological spaces, where a *topological space*is a set X with an extra structure, called a *topology* on X, which has to do with how "close" points in X are to each other. A function between spaces that respects these structures is called a *continuous function* or *continuous map*, with an equivalence referred to as a *homeomorphism*. Examples of topological spaces that are commonly studied are curves in $\mathbb{R}^2$ or $\mathbb{R}^3$, surfaces in $\mathbb{R}^3$ or $\mathbb{R}^4$, and spaces of functions between surfaces.

It's rather complicated to give a description of what a topology on a set X entails, but it's much easier to give an intuitive idea of what is meant by a continuous function between spaces. For subsets of $\mathbb{R}^n$, a Euclidean transformation is any composition of rotations (of the coordinate axes by any angle), reflections [across any (codimension 1) hyperplane (or subvector space) in $\mathbb{R}^n$] and translations (moving the "origin" of the coordinate system to any point in $\mathbb{R}^n$). Subsets of $\mathbb{R}^n$ are congruent if one can be transformed into the other by a Euclidean transformation. We add "rescalings" (scalar multiplication of $\mathbb{R}^n$ by any positive number) to the list of Euclidean transformations to define when two sets are similar. These concepts are generalized further in defining topological transformations. We add to the list of Euclidean transformations some new ones; we allow any transformations that involve bending or stretching, but not tearing. For example, an ellipse with eccentricity $e < 1$ is not geometrically equivalent to a circle, because such an ellipse has nonconstant curvature, while the circle's curvature is constant. The two planar curves are topologically equivalent (homeomorphic), however, because a composition of Euclidean transformations and bending/stretching can change one into the other. So, while curvature is a nice geometric property (preserved by Euclidean transfomations), it is not a topological property.

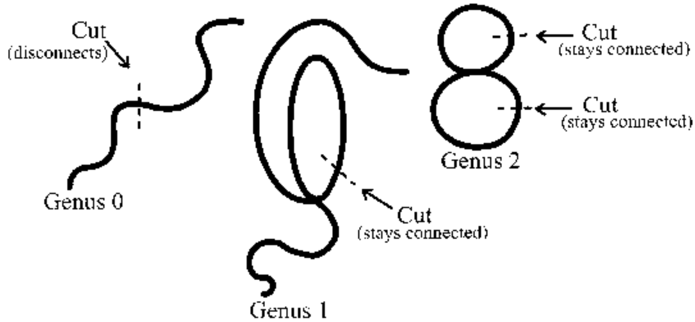
Topological properties are more "qualitative" than curvature. Examples of topological properties can be explained only intuitively at this point, but some of the central ideas can be made clear. *Connectedness* is perhaps the easiest topological property to look at. A topological space X is said to be connected if X is in a single "continuous" piece. For example, the two planar figures illustrated here (an "8" and an "81")

---

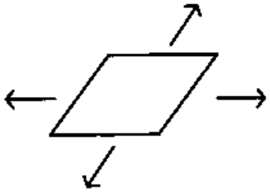
[2] The use of the term *abelian* for commutative groups is in honor of Nils H. Abel (1802-1828), a Danish mathematician whose work on the unsolvability of the quintic polynomial was part of a movement in mathematics that helped start the field of group theory. Abel's work is closely tied into that of Evariste Galois, who also died quite young, but in more romantic circumstances (reputedly in a duel over a barmaid, rather than by tuberculosis). These two exemplify the maxim that a mathematician's greatest work is most often accomplished before age 30. We'll see some striking counterexamples to this statement later.

8 81

cannot be homeomorphic because one is connected, the other disconnected. A second topological property, closely related to connectivity, is that of *genus*, which is quite easy to explain in the setting of planar curves. For a given connected planar curve X, a "cutpoint" of X is a point $x \in X$ such that X becomes disconnected if the point x is removed. A planar curve X is said to have genus 0 if every point of X (except the endpoints, if X has any) is a cut point. More generally, a connected planar curve X has genus n if some set of n points removed from X leaves it connected, but every set of $n+1$ "cuts" disconnects X (again, avoiding endpoints). Here are some examples of planar curves and their genera (plural of genus), with some cuts shown:

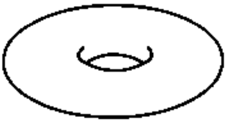


Finally, a very interesting example of a topological property is that of *orientability* in the context of surfaces. A *surface* is a topological space X that is "locally homeomorphic" to the open disk in $\mathbb{R}^2$ (meaning that each point of X has a "neighborhood" that is topologically equivalent to $B^2 = \{(x, y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2$). Some simple examples of surfaces are:



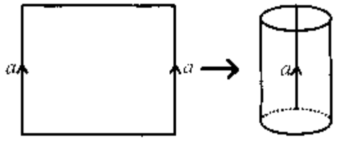
$\mathbb{R}^2$ (plane)

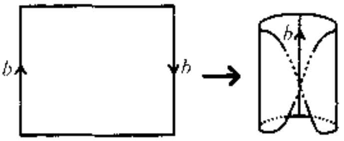
S^2 (sphere)



T^2 (torus)

These are all *oriented* surfaces, in that there is a consistently defined "outward" direction on each. Precisely, at each point $x \in X$, we have a neighborhood equivalent to the planar disk, so we have two sides, which we'll call "inward" and "outward." If these directions can be defined *consistently* for each $x \in X$, we say that X is oriented. The key word here is "consistently"; if one takes any simple closed curve on the surface (one forming a simple loop), the "outward" direction must be preserved throughout the curve. It may seem counterintuitive, but it's quite easy to construct nonorientable surfaces. For example, a Moebius band is built by identifying edges of a rectangle as indicated in the following diagrams, which construct a cylinder and a Moebius band:

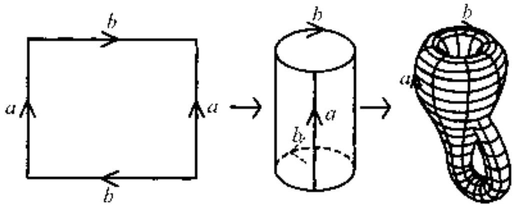




These are not surfaces (since each point on the edge has no neighborhood equivalent to the planar disk) but they are examples of "surfaces with boundary." Note that the cylinder has a consistent orientation, as one can check by looking at the curve that gives the "equator." However, the Moebius band is a "one- sided surface," without a consistently definable outward direction, as one can see by tracing a path along the equator of the band. An example of a true surface that is nonorientable is the *Klein bottle*[3] :

---

[3] Yes, the same Felix Klein as before. The term "Klein bottle" is used almost universally for this beast, but seems to have been an unintended name. The original German term apparently should have been translated



So the Klein bottle cannot be homeomorphic to any oriented surface, despite its similarities to the torus, for example. This is a very good example of how one uses topological properties to tell spaces apart. We'll explore the Klein bottle in more detail in Section 10.2.

In some sense, the goal of topology should be to come up with an exhaustive set of topological properties that would allow one to "classify" all topological spaces. Such a classification program has been completed, for example, in the setting of finite simple groups, where we recall that a simple group is a group with no nontrivial normal subgroups. The Classification Theorem for Finite Simple Groups was completed around 1980, culminating decades of work by scores of mathematicians. Given a finite simple group, one can identify it, up to isomorphism, by knowing its order (the number of elements in the group) and its character table (a complicated invariant, defined by representation theory). Such a classification theorem is nowhere in sight for topological spaces in general, but we do have such a result for connected, compact surfaces (where the term *compact* will be explained in excruciating detail in Chapter 7). The statement and proof of this theorem will occupy us in Chapter 10.

Exercises

Consider the letters of the English alphabet drawn as follows:

A B C D E F G H I J K L M N O P Q R

S T U V W X Y Z

1. Classify the letters by homeomorphism; that is, group the letters together so that all the letters in each set are homeomorphic to each other. You will need a total of nine sets.

as "Klein surface." An inaccurate translation rendered this as "Klein bottle," which has stuck since, for obvious reasons. The surface is often described as a bottle which can hold no water. For more concrete examples, see the website of the Acme Klein Bottle Corporation, `www.kleinbottle.com`.

2. Classify the letters by genus; that is, group the letters together so that all the letters in a set have the same genus. You will need a total of three sets.