

Part I
Analysis of Functions
of a Single Real Variable

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Chapter 1

The Real Numbers

This investigation of analysis starts with minimal prerequisites. Regarding set theory, the terms “set” and “element” will remain undefined, as is customary in mathematics to avoid paradoxes. The **empty set** \emptyset is the set that has no elements. The statement “ $e \in S$ ” says that e is an element of the set S . The statement “ $A \subseteq B$ ” says that every element of A is an element of B . Sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$. The statement “ $A \subset B$ ” says that $A \subseteq B$ and $A \neq B$. Subsets will be defined as “ $A = \{x \in S : \langle \text{property} \rangle\}$,” that is, with a statement from which set S the elements of A are taken and a property describing them. The **union** of two sets A and B is $A \cup B = \{x : x \in A \text{ or } x \in B\}$, the **intersection** is $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Union and intersection of finitely many sets are denoted $\bigcup_{j=1}^n A_j$ and $\bigcap_{j=1}^n A_j$, respectively, and the **relative complement** of B in A is $A \setminus B = \{x \in A : x \notin B\}$. Further details on set theory are purposely delayed until Section 7.1. Until then, we focus on analytical techniques. Any required notions of set theory will be clarified on the spot.

To define properties, sometimes the **universal quantifier** “ \forall ” (read “for all”) or the **existential quantifier** “ \exists ” (read “there exists”) are used. Formal logic is described in more detail in Appendix A. Finally, the reader needs an intuitive idea what a function, a relation and a binary operation are. Details are relegated to Appendices B.2 and C.2.

The real numbers \mathbb{R} are the “staging ground” for analysis. They can be characterized as the unique (up to isomorphism) mathematical entity that satisfies Axioms 1.1, 1.6, and 1.19. That is, they are the unique linearly ordered, complete field (see Exercise 1-30). In this chapter, we introduce the axioms for the real numbers and some fundamental consequences. These results assure that the real numbers indeed have the properties that we are familiar with from algebra and calculus.

1.1 Field Axioms

The description of the real numbers starts with their algebraic properties.

Axiom 1.1 *The real numbers \mathbb{R} are a field. That is, \mathbb{R} has at least two elements and there are two binary operations, **addition** $+$: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and **multiplication** \cdot : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, so that*

1. Addition is **associative**, that is, for all $x, y, z \in \mathbb{R}$ we have
 $(x + y) + z = x + (y + z)$.
2. Addition is **commutative**, that is, for all $x, y \in \mathbb{R}$ we have
 $x + y = y + x$.
3. There is a **neutral element** 0 for addition, that is, there is an element $0 \in \mathbb{R}$ so that for all $x \in \mathbb{R}$ we have $x + 0 = x$.
4. For every element $x \in \mathbb{R}$ there is an **additive inverse** element $(-x)$ so that
 $x + (-x) = 0$.
5. Multiplication is **associative**, that is, for all $x, y, z \in \mathbb{R}$ we have
 $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
6. Multiplication is **commutative**, that is, for all $x, y \in \mathbb{R}$ we have
 $x \cdot y = y \cdot x$.
7. There is a **neutral element** 1 for multiplication, that is, there is an element $1 \in \mathbb{R}$ so that for all $x \in \mathbb{R}$ we have $1 \cdot x = x$.
8. For every element $x \in \mathbb{R} \setminus \{0\}$ there is a **multiplicative inverse** element x^{-1} so that $x \cdot x^{-1} = 1$.
9. Multiplication is **(left) distributive** over addition, that is, for all $\alpha, x, y \in \mathbb{R}$ we have $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$.

As is customary for multiplication, the dot between factors is usually omitted.

Fields are investigated in detail in abstract algebra. For analysis, it is most effective to remember that the field axioms guarantee the properties needed so that we can perform algebra and arithmetic “as usual.” Some of these properties are exhibited in this section and in the exercises. The exercises also include examples that show that not every field needs to be infinite (see Exercises 1-7–1-9).

Theorem 1.2 *The following are true in \mathbb{R} :*

1. For all $x \in \mathbb{R}$, we have $0x = 0$.
2. $0 \neq 1$.
3. Additive inverses are unique. That is, if $x \in \mathbb{R}$ and x' and \bar{x} both have the property in part 4 of Axiom 1.1, then $x' = \bar{x}$.
4. For all $x \in \mathbb{R}$, we have $(-1)x = -x$.

Proof. Early in the text, proofs will sometimes be interrupted by comments in italics to point out standard formulations and proof techniques.

To prove part 1, let $x \in \mathbb{R}$. Then the axioms allow us to obtain the following equation. $0x \stackrel{\text{Ax.3}}{=} (0+0)x \stackrel{\text{Ax.6}}{=} x(0+0) \stackrel{\text{Ax.9}}{=} x0+x0 \stackrel{\text{Ax.6}}{=} 0x+0x$. This implies

$$0 \stackrel{\text{Ax.4}}{=} 0x + (-0x) \stackrel{\text{above}}{=} (0x+0x) + (-0x) \stackrel{\text{Ax.1}}{=} 0x + (0x + (-0x)) \stackrel{\text{Ax.4}}{=} 0x + 0 \stackrel{\text{Ax.3}}{=} 0x$$

as was claimed. *The proof of part 1 shows how every step in a proof needs to be justified. Usually we will not explicitly justify each step in a computation with an axiom or a previous result. However, the reader should always mentally fill in the justification. The practice of filling in these justifications should be started in the computations in the remainder of this proof.*

To prove part 2, first note that, because \mathbb{R} has at least two elements, there is an $x \in \mathbb{R} \setminus \{0\}$. Now suppose for a contradiction (see *Standard Proof Technique 1.4* below) that $0 = 1$. Then $x = 1 \cdot x = 0 \cdot x = 0$ is a contradiction to $x \in \mathbb{R} \setminus \{0\}$.

For part 3, note that if x' and \bar{x} both have the property in part 4 of Axiom 1.1, then $x' = x' + 0 = x' + (x + \bar{x}) = (x' + x) + \bar{x} = (x + x') + \bar{x} = 0 + \bar{x} = \bar{x} + 0 = \bar{x}$. Note that the statement of part 3 already encodes the typical approach to a uniqueness proof (see *Standard Proof Technique 1.5* below).

Finally, for part 4 note that $x + (-1)x = 1x + (-1)x = (1 + (-1))x = 0x = 0$. Because by part 3 additive inverses are unique, $(-1)x$ must be the additive inverse $-x$ of x . *The last step is a typical application of modus ponens, see Standard Proof Technique 1.3 below.* ■

To familiarize the reader with standard proof techniques, these techniques will be pointed out explicitly in the early part of the text. The techniques presented in Chapter 1 are general proof techniques applicable throughout mathematics. Techniques presented in later chapters are mostly specific to analysis.

Standard Proof Technique 1.3 The simplest mathematical proof technique is a **direct proof** in which a result that says “ A implies B ” is applied after we have proved that A is true. Truth of A and of “ A implies B ” guarantees truth of B . This technique is also called **modus ponens**. An example is in the proof of part 4 of Theorem 1.2. □

Standard Proof Technique 1.4 In a proof by **contradiction**, we suppose the contrary (the negation, also see Appendix A.2) of what is claimed is true and then we derive a contradiction. Typically, we derive a statement and its negation, which is a contradiction, because they cannot both be true. For an example, see the proof of part 2 of Theorem 1.2 above. Given that the reasoning that led to the contradiction is correct, the contradiction must be caused by the assumption that the contrary of the claim is true. Hence, the contrary of the claim must be false, because true statements cannot imply false statements like contradictions (see part 3 of Definition A.2 in Appendix A). But this means the claim must be true.

We will usually indicate proofs by contradiction with a starting statement like “suppose for a contradiction.” □

Standard Proof Technique 1.5 For many mathematical objects it is important to assure that they are the *only* object that has certain properties. That is, we want to assure that the object is unique. In a typical **uniqueness proof**, we assume that there is more than one object with the properties under investigation and we prove that any two of these objects must be equal. Part 3 of Theorem 1.2 shows this approach. \square

Exercises

- 1-1. Prove that $(-1) \cdot (-1) = 1$.
- 1-2. \cdot is right distributive over $+$. Prove that for all $x, y, z \in \mathbb{R}$ we have $(x + y)z = xz + yz$.
- 1-3. **Multiplicative inverses are unique.** Prove that if $x \in \mathbb{R}$ and x' and \bar{x} both have the property in part 8 of Axiom 1.1 then $x' = \bar{x}$.
- 1-4. Prove that 0 does not have a multiplicative inverse.
- 1-5. Prove that if $x, y \neq 0$, then $(xy)^{-1} = y^{-1}x^{-1}$. Conclude in particular that $xy \neq 0$.
- 1-6. Prove each of the **binomial formulas** below. Justify each step with the appropriate axiom.
- (a) $(a + b)^2 = a^2 + 2ab + b^2$ (b) $(a - b)^2 = a^2 - 2ab + b^2$
- (c) $(a + b)(a - b) = a^2 - b^2$
- 1-7. Prove that the set $\{0, 1\}$ with the usual multiplication and the usual addition, except that $1 + 1 := 0$, is a field. That is, prove that the set and addition and multiplication as stated have the properties listed in Axiom 1.1.
- 1-8. Prove that the set $\{0, 1, 2\}$ with the sum and product of two elements being the remainder obtained when dividing the regular sum and product by 3 is a field.
- 1-9. A property and some finite fields.
- (a) Let F be a field and let $x, y \in F$. Prove that $x \cdot y = 0$ if and only if $x = 0$ or $y = 0$.
- (b) Prove that the set $\{0, 1, 2, 3\}$ with the sum and product of two elements being the remainder obtained when dividing the regular sum and product by 4 is *not* a field.
- (c) Prove that the set $\{0, 1, \dots, p - 1\}$ with the sum and product of two elements being the remainder obtained when dividing the regular sum and product by p is a field if and only if p is a prime number.

1.2 Order Axioms

Exercises 1-7–1-9c show that the field axioms alone are not enough to describe the real numbers. In fact, fields need not even be infinite. However, aside from executing the familiar algebraic operations, we can also compare real numbers. This section presents the order relation on the real numbers and its properties.

Axiom 1.6 *The real numbers \mathbb{R} contain a subset \mathbb{R}^+ , called the **positive real numbers** such that*

- For all $x, y \in \mathbb{R}^+$, we have $x + y \in \mathbb{R}^+$ and $xy \in \mathbb{R}^+$,
- For all $x \in \mathbb{R}$, exactly one of the following three properties holds.
Either $x \in \mathbb{R}^+$ or $-x \in \mathbb{R}^+$ or $x = 0$.

A real number x is called **negative** if and only if $-x \in \mathbb{R}^+$.

Once positive numbers are defined, we can define an order relation. As usual, instead of writing $y + (-x)$ we write $y - x$ and call it the **difference** of x and y . The binary operation “ $-$ ” is called **subtraction**.

The phrase “if and only if,” which is used in definitions and biconditionals, is normally abbreviated with the artificial word “**iff**.”

Definition 1.7 For $x, y \in \mathbb{R}$, we say x is **less than** y , in symbols $x < y$, iff $y - x \in \mathbb{R}^+$. We say x is **less than or equal to** y , denoted $x \leq y$, iff $x < y$ or $x = y$. Finally, we say x is **greater than** y , denoted $x > y$, iff $y < x$, and we say x is **greater than or equal to** y , denoted $x \geq y$, iff $y \leq x$.

The relation \leq satisfies the properties that define an order relation.

Proposition 1.8 The relation \leq is an **order relation** on \mathbb{R} . That is,

1. \leq is **reflexive**. For all $x \in \mathbb{R}$ we have $x \leq x$,
2. \leq is **antisymmetric**. For all $x, y \in \mathbb{R}$ we have that $x \leq y$ and $y \leq x$ implies $x = y$,
3. \leq is **transitive**. For all $x, y, z \in \mathbb{R}$, we have that $x \leq y$ and $y \leq z$ implies $x \leq z$.

Moreover, the relation \leq is a **total order relation**, that is, for any two $x, y \in \mathbb{R}$ we have that $x \leq y$ or $y \leq x$.

Proof. The relation \leq is reflexive, because it includes equality.

For antisymmetry, let $x \leq y$ and $y \leq x$ and suppose for a contradiction that $x \neq y$. Then $x - y \in \mathbb{R}^+$ and $-(x - y) = y - x \in \mathbb{R}^+$, which cannot be by Axiom 1.6. Thus \leq must be antisymmetric.

For transitivity, let $x \leq y$ and $y \leq z$. There is nothing to prove if one of the inequalities is an equality. Thus we can assume that $x < y$ and $y < z$, which means $y - x \in \mathbb{R}^+$ and $z - y \in \mathbb{R}^+$. But then \mathbb{R}^+ contains $(z - y) + (y - x) = z - x$, and hence $x < z$. We have shown that for all $x, y, z \in \mathbb{R}$ the inequalities $x \leq y$ and $y \leq z$ imply $x \leq z$, which means that \leq is transitive.

For the “moreover” part note that if $x, y \in \mathbb{R}$, then $y - x \in \mathbb{R}$ and we have either $y - x \in \mathbb{R}^+$, which means $x < y$, or $y - x = 0$, which means $y = x$, or $x - y = -(y - x) \in \mathbb{R}^+$, which means $y < x$. Therefore for all $x, y \in \mathbb{R}$ one of $x \leq y$ or $y \leq x$ holds, and hence \leq is a total order. ■

Once an order relation is established, we can define intervals.

Definition 1.9 An **interval** is a set $I \subseteq \mathbb{R}$ so that for all $c, d \in I$ and $x \in \mathbb{R}$ the inequalities $c < x < d$ imply $x \in I$. In particular for $a, b \in \mathbb{R}$ with $a < b$ we define

1. $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$,
2. $(a, b) := \{x \in \mathbb{R} : a < x < b\}$, $(a, \infty) := \{x \in \mathbb{R} : a < x\}$,
 $(-\infty, b) := \{x \in \mathbb{R} : x < b\}$, $(-\infty, \infty) := \mathbb{R}$,

$$3. [a, b) := \{x \in \mathbb{R} : a \leq x < b\}, [a, \infty) := \{x \in \mathbb{R} : a \leq x\},$$

$$4. (a, b] := \{x \in \mathbb{R} : a < x \leq b\}, (-\infty, b] := \{x \in \mathbb{R} : x \leq b\}.$$

The points a and b are also called the **endpoints** of the interval. An interval that does not contain either of its endpoints (where $\pm\infty$ are also considered to be “endpoints”) is called **open**. An interval that contains exactly one of its endpoints is called **half-open** and an interval that contains both its endpoints is called **closed**.

For the first part of this text, the domains of functions will almost exclusively be intervals. Because analysis requires extensive work with inequalities, we need to investigate how the order relation relates to the algebraic operations.

Theorem 1.10 *Properties of the order relation. Let $x, y, z \in \mathbb{R}$.*

1. The number x is positive iff $x > 0$ and x is negative iff $x < 0$.
2. If $x \leq y$, then $x + z \leq y + z$.
3. If $x \leq y$ and $z > 0$, then $xz \leq yz$.
4. If $x \leq y$ and $z < 0$, then $xz \geq yz$.
5. If $0 < x \leq y$, then $y^{-1} \leq x^{-1}$.

Similar results can be proved for other combinations of strict and nonstrict inequalities. We will not state these here, but instead trust that the reader can make the requisite translation from the statements in this theorem.

Proof. Parts 1 and 2 are left to the reader as Exercises 1-10a and 1-10b. Throughout this text, parts of proofs will be delegated to the reader to facilitate a better connection to the material presented.

For part 3, let $x \leq y$ and let $z > 0$. Then, $y - x \in \mathbb{R}^+$ or $y = x$. In case $y = x$, we obtain $yz = xz$ and thus, in particular, $xz \leq yz$. In case $y - x \in \mathbb{R}^+$, note that $z > 0$ means $z \in \mathbb{R}^+$, and hence $yz - xz = (y - x)z \in \mathbb{R}^+$. By definition, this implies $xz < yz$, and in particular $xz \leq yz$. Because we have shown $xz \leq yz$ in each case, the result is established. All proofs in this section are done with the above kind of case distinction (see Standard Proof Technique 1.11).

For part 4, let $x \leq y$ and let $z < 0$. Then, $y - x \in \mathbb{R}^+$ or $y = x$. In case $y = x$, we obtain $yz = xz$, and hence $xz \geq yz$. In case $y - x \in \mathbb{R}^+$, note that $z < 0$ means $-z \in \mathbb{R}^+$, and hence $xz - yz = (x - y)z = (y - x)(-z) \in \mathbb{R}^+$. By definition, this implies $yz < xz$, and hence $yz \leq xz$, which establishes the result.

For part 5, first note that there is nothing to prove if $x = y$. Hence, we can assume that $x < y$. Suppose for a contradiction that $x^{-1} < y^{-1}$. Then by part 3 we have that $1 = x^{-1}x < y^{-1}x$, and hence $x < y \cdot 1 < yy^{-1}x = x$, contradiction. ■

Standard Proof Technique 1.11 When several possibilities must be considered in a proof, the proof usually continues with separate arguments for each possibility. The proof is complete when each separate argument has led to the desired conclusion. This type of proof is also called a proof by **case distinction**. □

We conclude this section by introducing the absolute value function and some of its properties.

Definition 1.12 For $x \in \mathbb{R}$, we set $|x| = \begin{cases} x; & \text{if } x \geq 0, \\ -x; & \text{if } x < 0, \end{cases}$ and we call it the **absolute value** of x .

Theorem 1.13 summarizes the properties of the absolute value. The numbering is adjusted so that properties 1, 2, and 3 correspond to the analogous properties for norms (see Definition 15.38). We will formulate many results in the first part of the text to be analogous or easily generalizable to more abstract settings, but we will usually do so without explicit forward references. In this fashion many abstract situations will be more familiar because of similarities to situations investigated in the first part.

Theorem 1.13 *Properties of the absolute value.*

0. For all $x \in \mathbb{R}$, we have $|x| \geq 0$.
1. For all $x \in \mathbb{R}$, we have $|x| = 0$ iff $x = 0$.
2. For all $x, y \in \mathbb{R}$, we have $|xy| = |x||y|$.
3. **Triangular inequality.** For all $x, y \in \mathbb{R}$, we have $|x + y| \leq |x| + |y|$.
4. **Reverse triangular inequality.** For all $x, y \in \mathbb{R}$, we have $||x| - |y|| \leq |x - y|$.

Proof. For part 0, let $x \in \mathbb{R}$. In case $x \geq 0$, by Definition 1.12 we have $|x| = x \geq 0$. In case $x < 0$, we have $x \notin \mathbb{R}^+$ and by part 2 of Axiom 1.6 we conclude $-x > 0$. Because in this case $|x| = -x > 0$, part 0 follows.

Throughout the text, the two implications of a biconditional “ A iff B ” will be referred to as “ \Rightarrow ,” denoting “if A , then B ” and “ \Leftarrow ,” denoting “if B , then A .”

For part 1, note that the direction “ \Leftarrow ” is trivial, because $|0| = 0$. For the direction “ \Rightarrow ,” let $x \in \mathbb{R}$ be so that $|x| = 0$ and suppose for a contradiction that $x \neq 0$. If $x > 0$, then $0 < x = |x| = 0$, a contradiction. (Note that the previous sentence is a short proof by contradiction that is part of a longer proof by contradiction.) Therefore $x < 0$. But then $0 < -x = |x| = 0$, a contradiction. Hence, x must be equal to 0.

For part 2, let $x, y \in \mathbb{R}$. If $x \geq 0$ and $y \geq 0$, then by part 3 of Theorem 1.10 $xy \geq 0$, and hence $|xy| = xy = |x||y|$. If $x \geq 0$ and $y < 0$, then by part 4 of Theorem 1.10 we infer $xy \leq 0$. Hence, $|xy| = -xy = x(-y) = |x||y|$. The case $x < 0$ and $y \geq 0$ is similar and the reader will produce it in Exercise 1-11a. Finally, if $x < 0$ and $y < 0$, then by part 4 of Theorem 1.10 we obtain $xy > 0$. Hence, $|xy| = xy = (-1)(-1)xy = (-x)(-y) = |x||y|$.

To prove the triangular inequality, first note that for all $x \in \mathbb{R}$ we have that $x \leq |x|$. This is clear for $x \geq 0$ and for $x < 0$ we simply note $x < 0 < -x = |x|$. Moreover, (see Exercise 1-11b) for all $x \in \mathbb{R}$ we have $-x \leq |x|$. Now let $x, y \in \mathbb{R}$. If the inequality $x + y \geq 0$ holds, then by part 2 of Theorem 1.10 at least one of x, y is greater than or equal to 0. (Otherwise $x < 0$ and $y < 0$ would imply $x + y < 0$.) Hence, by part 2 of Theorem 1.10 $|x + y| = x + y \leq |x| + y \leq |x| + |y|$. If $x + y < 0$,

then at least one of x and y is less than 0. Hence, by part 2 of Theorem 1.10 we obtain $|x + y| = -(x + y) = -x + (-y) \leq |-x| + (-y) \leq |-x| + |-y| = |x| + |y|$.

Finally, for the reverse triangular inequality, let $x, y \in \mathbb{R}$. Without loss of generality (see *Standard Proof Technique 1.14*) assume that $|x| \geq |y|$. (The proof for the case $|x| < |y|$ is left as Exercise 1-11c.) Then $|x| = |x - y + y| \leq |x - y| + |y|$, which implies $||x| - |y|| = |x| - |y| \leq |x - y|$. ■

Standard Proof Technique 1.14 If the proofs for the cases in a case distinction are very similar, it is customary to assume **without loss of generality** that one of these similar cases is true. This is not a loss of generality, because it is assumed that what is presented enables the reader to fill in the proof(s) for the other case(s). In this text, the omitted part is sometimes included as an explicit exercise for the reader. □

Exercises

1-10. Finishing the proof of Theorem 1.10.

- (a) Prove part 1 of Theorem 1.10.
- (b) Prove part 2 of Theorem 1.10.

1-11. Finishing the proof of Theorem 1.13.

- (a) Let $x, y \in \mathbb{R}$. Prove that if $x \geq 0$ and $y < 0$, then $|xy| = |x||y|$.
- (b) Prove that for all $x \in \mathbb{R}$ we have $-x \leq |x|$.
- (c) Prove that if $|x| < |y|$, then $||x| - |y|| \leq |x - y|$.

1-12. Let $I, J \subseteq \mathbb{R}$ be intervals. Prove that $I \cap J = \{x \in \mathbb{R} : x \in I \text{ and } x \in J\}$ is again an interval.

1-13. Let $a < b$ and let $x, y \in [a, b]$. Prove that $|x - y| \leq b - a$.

1-14. Prove that none of the fields from Exercise 1-9c can satisfy Axiom 1.6 by showing that for these fields part 2 of Axiom 1.6 fails for $x = 1$.

Note. This result shows that Axiom 1.6 distinguishes \mathbb{R} from the finite fields of Exercise 1-9c.

1.3 Lowest Upper and Greatest Lower Bounds

A structure that has the properties outlined in Axioms 1.1 and 1.6 is also called a **linearly ordered field**. The rational numbers satisfy these properties just as well as the real numbers. Thus we are not done with our characterization of \mathbb{R} . The final axiom for the real numbers addresses upper and lower bounds of sets.

Definition 1.15 Let A be a subset of \mathbb{R} .

1. The number $u \in \mathbb{R}$ is called an **upper bound** of A iff $u \geq a$ for all $a \in A$. If A has an upper bound, it is also called **bounded above**.
2. The number $l \in \mathbb{R}$ is called a **lower bound** of A iff $l \leq a$ for all $a \in A$. If A has a lower bound, it is also called **bounded below**.

A subset $A \subseteq \mathbb{R}$ that is bounded above and bounded below is also called **bounded**.

Among all upper bounds of a set, the smallest one (if it exists) plays a special role. Similarly, the greatest lower bound plays a special role if it exists.

Definition 1.16 Let $A \subseteq \mathbb{R}$.

1. The number $s \in \mathbb{R}$ is called **lowest upper bound** of A or **supremum** of A , denoted $\sup(A)$, iff s is an upper bound of A and for all upper bounds u of A we have that $s \leq u$.
2. The number $i \in \mathbb{R}$ is called **greatest lower bound** of A or **infimum** of A , denoted $\inf(A)$, iff i is a lower bound of A and for all lower bounds l of A we have that $l \leq i$.

Formally, it is not guaranteed that suprema and infima are unique, but the next result shows that this is indeed the case. Note that the statement of Proposition 1.17 follows the standard pattern for a uniqueness statement.

Proposition 1.17 Suprema are unique. *That is, if the set $A \subseteq \mathbb{R}$ is bounded above and $s, t \in \mathbb{R}$ both are suprema of A , then $s = t$.*

Proof. Let $A \subseteq \mathbb{R}$ and $s, t \in \mathbb{R}$ be as indicated. Then s is an upper bound of A and, because t is a supremum of A , we infer $s \geq t$. Similarly, t is an upper bound of A and, because s is a supremum of A , we infer $t \geq s$. This implies $s = t$. ■

Standard Proof Technique 1.18 (Also compare with Standard Proof Technique 1.14.) When, as in the proof of Proposition 1.17, two parts of a proof are very similar, it is common to only prove one part and state that the other part is similar. Throughout the text, the reader will become familiar with this idea through exercises that require the construction of proofs that are similar to proofs given in the narrative. □

The proof that infima are unique is similar (see Exercise 1-15). Because suprema and infima are unique if they exist, we speak of *the* supremum and *the* infimum.

The final axiom for the real numbers now states that suprema and infima exist under mild hypotheses.

Axiom 1.19 Completeness Axiom. *Every nonempty subset S of \mathbb{R} that has an upper bound has a lowest upper bound.*

Although the Completeness Axiom formally only guarantees that nonempty subsets of \mathbb{R} that are bounded above have suprema, existence of infima is a consequence.

Proposition 1.20 *Let $S \subseteq \mathbb{R}$ be nonempty and bounded below. Then S has a greatest lower bound.*

Proof. Let $L := \{x \in \mathbb{R} : x \text{ is a lower bound of } S\}$. Then $L \neq \emptyset$. Let $s \in S$. Then for all $l \in L$ we have that $l \leq s$. Because $S \neq \emptyset$ this means that L is bounded above. Because $L \neq \emptyset$, by the Completeness Axiom, L has a supremum $\sup(L)$. Every $s \in S$ is an upper bound of L , which means that $s \geq \sup(L)$ and so $\sup(L)$ is a lower bound of S . By definition of suprema, $\sup(L)$ is greater than or equal to all elements of L ,

that is, it is greater than or equal to all lower bounds of S . By definition of infima, this means that $\sup(L) = \inf(S)$. ■

We will see that suprema and infima are valuable tools in analysis on the real line. The next result shows that in any set with a supremum we can find numbers that are arbitrarily close to the supremum. This fact is important, because analysis ultimately is about objects “getting close to each other.”

Proposition 1.21 *Let $S \subset \mathbb{R}$ be a nonempty subset of \mathbb{R} that is bounded above and let $s := \sup(S)$. Then for every $\varepsilon > 0$ there is an element $x \in S$ so that $s - x < \varepsilon$.*

Proof. Suppose for a contradiction that there is an $\varepsilon > 0$ so that for all $x \in S$ we have that $s - x \geq \varepsilon$. Then for all $x \in S$ we would obtain $s - \varepsilon \geq x$, that is, $s - \varepsilon$ would be an upper bound of S . But $s - \varepsilon < s$ contradicts the fact that s is the lowest upper bound of S . ■

Although the supremum and infimum of a set need not be elements of the set, we have different names for them in case they are in the set.

Definition 1.22 *Let A be a subset of \mathbb{R} .*

1. *If A is bounded above and $\sup(A) \in A$, then the supremum of A is also called the **maximum** of A , denoted $\max(A)$.*
2. *If A is bounded below and $\inf(A) \in A$, then the infimum of A is also called the **minimum** of A , denoted $\min(A)$.*

Although the distinctions between suprema and maxima and between infima and minima are small, the notions are distinct. For example, the open interval $(0, 1)$ has a supremum (1) and an infimum (0) , but it has neither a maximum, nor a minimum.

Exercises

- 1-15. Let $A \subseteq \mathbb{R}$ be bounded below and let $s, t \in \mathbb{R}$ both be infima of A . Prove that $s = t$.
- 1-16. Approaching infima. State and prove a version of Proposition 1.21 that applies to infima. Is the proof significantly different from that of Proposition 1.21?
- 1-17. Let $S \subseteq \mathbb{R}$ be bounded above. Prove that $s \in \mathbb{R}$ is the supremum of S iff s is an upper bound of S and for all $\varepsilon > 0$ there is an $x \in S$ so that $|s - x| < \varepsilon$.
- 1-18. Suprema and infima vs. containment of sets.
- (a) Let $A, B \subseteq \mathbb{R}$ be bounded above. Prove that $A \subseteq B$ implies $\sup(A) \leq \sup(B)$.
 - (b) Let $A, B \subseteq \mathbb{R}$ be bounded below. Prove that $A \subseteq B$ implies $\inf(A) \geq \inf(B)$.
- 1-19. Let $A \subseteq \mathbb{R}$ be bounded above. Prove that $\inf\{x \in \mathbb{R} : -x \in A\} = -\sup(A)$.

This section concludes the introduction of the axioms for the real numbers. Exercise 1-30 after the next section shows that the axioms *uniquely* determine the real numbers. We will not explicitly construct a mathematical entity that satisfies these axioms. Readers interested in the construction of \mathbb{R} from \mathbb{Q} can revisit this idea after Theorem 16.89 (see Exercise 16-93). The construction of the rational numbers from the axioms of set theory is sketched in Appendix C.

1.4 Natural Numbers, Integers, and Rational Numbers

Although Axioms 1.1, 1.6 and 1.19 uniquely describe the real numbers, they do not mention familiar subsets, such as natural numbers, integers, and rational numbers. This is because these sets can be constructed from the axioms as subsets of the real numbers. We start with the natural numbers, which are the unique subset with properties as stated in Theorem 1.23. While their existence is easy to establish, the uniqueness of the natural numbers can only be proved in Theorem 1.28 after some more machinery has been developed.

Theorem 1.23 *There is a subset $\mathbb{N} \subseteq \mathbb{R}$, called the **natural numbers**, so that*

1. $1 \in \mathbb{N}$.
2. For each $n \in \mathbb{N}$ the number $n + 1$ is also in \mathbb{N} .
3. **Principle of Induction.** *If $S \subseteq \mathbb{N}$ is such that $1 \in S$ and for each $n \in S$ we also have $n + 1 \in S$, then $S = \mathbb{N}$.*

Proof. Call a subset $A \subseteq \mathbb{R}$ a **successor set** iff $1 \in A$ and for all $a \in A$ we also have $a + 1 \in A$. Successor sets exist, because, for example, \mathbb{R} itself is a successor set. Let \mathbb{N} be the set of all elements of \mathbb{R} that are in all successor sets. Because 1 is an element of every successor set, we infer $1 \in \mathbb{N}$. Moreover, if $n \in \mathbb{N}$, then n is in every successor set, which means $n + 1$ is in every successor set, and hence $n + 1 \in \mathbb{N}$. Finally, any subset $S \subseteq \mathbb{N}$ as given in the Principle of Induction is a successor set. Because the elements of \mathbb{N} are contained in all successor sets, we conclude that $\mathbb{N} \subseteq S$, and hence $\mathbb{N} = S$. ■

Of course, we will denote the natural numbers by their usual names $1, 2, 3, \dots$. As algebraic objects, natural numbers are suited for addition and multiplication (see Proposition 1.24), but they are not so well suited for subtraction (see Proposition 1.25). Although all results until Theorem 1.28 are stated for \mathbb{N} , they hold “for every subset of \mathbb{R} that satisfies the properties in Theorem 1.23.” The reader should keep this in mind and double check, because we will need it in the proof of Theorem 1.28. To avoid awkward formulations, the results up to Theorem 1.28 are formulated for \mathbb{N} , however.

Proposition 1.24 *The natural numbers are closed under addition and multiplication. That is, if $m, n \in \mathbb{N}$, then $m + n$ and mn are in \mathbb{N} also.*

Proof. The key to this result is the Principle of Induction. Let $m \in \mathbb{N}$ be arbitrary and let $S_m := \{n \in \mathbb{N} : m + n \in \mathbb{N}\}$. Then $m \in \mathbb{N}$ implies $m + 1 \in \mathbb{N}$, and hence $1 \in S_m$. Moreover, if $n \in S_m$, then $m + n \in \mathbb{N}$, and hence $m + (n + 1) = (m + n) + 1 \in \mathbb{N}$, which means that $n + 1 \in S_m$. By the Principle of Induction we conclude that $S_m = \mathbb{N}$. Because $m \in \mathbb{N}$ was arbitrary, this means that for any $m, n \in \mathbb{N}$ we have $m + n \in \mathbb{N}$.

The proof for products is similar and left to the reader as Exercise 1-20. ■

Readers familiar with induction recognize the part “ $1 \in S_m$ ” of the preceding proof as the **base step** of an induction and the part “ $n \in S_m \Rightarrow n + 1 \in S_m$ ” as the **induction step**. In this section, we use the “induction on sets” as done in the preceding proof. The more commonly known Principle of Induction is introduced in Theorem 1.39.

Proposition 1.25 *Let $m, n \in \mathbb{N}$ be such that $m > n$. Then $m - n \in \mathbb{N}$.*

Proof. We first show that if $m \in \mathbb{N}$, then $m - 1 \in \mathbb{N}$ or $m - 1 = 0$. To do this, let $A := \{m \in \mathbb{N} : m - 1 \in \mathbb{N} \text{ or } m - 1 = 0\}$. Then $1 \in A$ and if $m \in A$, then $(m + 1) - 1 = m \in A \subseteq \mathbb{N}$, which means $m + 1 \in A$. Hence, $A = \mathbb{N}$ by the Principle of Induction.

Now let $S := \{n \in \mathbb{N} : (\forall m \in \mathbb{N} : m > n \text{ implies } m - n \in \mathbb{N})\}$. If $n = 1$ and $m \in \mathbb{N}$ satisfies $m > 1$, then $m - 1 > 0$ and so by the above $m - 1 \in \mathbb{N}$, which means $1 \in S$. Let $n \in S$. If $m > n + 1$, then $m - 1 > n$, and hence $m - (n + 1) = (m - 1) - n \in \mathbb{N}$, which means $n + 1 \in S$. By the Principle of Induction we conclude that $S = \mathbb{N}$, and hence for all $m, n \in \mathbb{N}$ we have proved that $m > n$ implies $m - n \in \mathbb{N}$. ■

Proposition 1.26 shows that the natural numbers are positive and the smallest difference between any two of them is 1.

Proposition 1.26 *For all $n \in \mathbb{N}$, the inequality $n \geq 1$ holds and there is no $m \in \mathbb{N}$ so that the inequalities $n < m < n + 1$ hold.*

Proof. The proof that all natural numbers are greater than or equal to 1 is left to Exercise 1-21.

Now suppose for a contradiction that there is an $n \in \mathbb{N}$ and an $m \in \mathbb{N}$ so that $n < m < n + 1$. Then $m - n \in \mathbb{N}$ and $m - n < 1$, a contradiction. ■

The Well-ordering Theorem turns out to be equivalent to the Principle of Induction (see Exercise 1-22).

Theorem 1.27 Well-ordering Theorem. *Every nonempty subset of \mathbb{N} has a smallest element.*

Proof. Suppose for a contradiction that $B \subseteq \mathbb{N}$ is not empty and does not have a smallest element. Let $S := \{n \in \mathbb{N} : (\forall m \in \mathbb{N} : m \leq n \text{ implies } m \notin B)\}$. By Proposition 1.26, 1 is less than or equal to all elements of \mathbb{N} , so $1 \notin B$, and hence $1 \in S$. Now let $n \in S$. Then all $m \in \mathbb{N}$ with $m \leq n$ are not in B . But then $n + 1 \in B$ would by Proposition 1.26 imply that $n + 1$ is the smallest element of B . Hence, $n + 1 \notin B$ and we conclude $n + 1 \in S$. By the Principle of Induction, $S = \mathbb{N}$ and consequently $B = \emptyset$, a contradiction. ■

Now we are finally ready to show that the natural numbers are unique.

Theorem 1.28 *The natural numbers \mathbb{N} are the unique subset of \mathbb{R} that satisfies the properties in Theorem 1.23.*

Proof. Examination of the proofs of all results since Theorem 1.23 reveals that any set $S \subseteq \mathbb{R}$ that satisfies the properties in Theorem 1.23 must also have the properties given in these results.

It may feel tedious to go back and verify the above statement. However, mathematical presentations more often than not will ask a reader to use a modification of a known proof to prove a result (also see Standard Proof Technique 1.14). When this occurs, the

reader is expected to verify that the result(s) can indeed be proved with similar methods as were used for earlier results.

Now suppose for a contradiction that there is a set $S \neq \mathbb{N}$ with properties as in Theorem 1.23. Then S is a successor set, so $\mathbb{N} \subseteq S$. Let $B := S \setminus \mathbb{N} = \{s \in S : s \notin \mathbb{N}\}$. Then $B \neq \emptyset$, and hence by the Well-ordering Theorem, which is valid for S , B has a smallest element b . Because $1 \in \mathbb{N}$ we infer $b > 1$, and hence by Proposition 1.25, which is valid for S , we have $b - 1 \in S$. But then $b - 1 \notin \mathbb{N}$, because this would imply $b = (b - 1) + 1 \in \mathbb{N}$. Hence, $b - 1 \in B$, which is a contradiction to the fact that b is the smallest element of B . ■

Once we have constructed the natural numbers, the next number system to consider are the integers.

Definition 1.29 The set $\mathbb{Z} := \{m \in \mathbb{R} : m \in \mathbb{N} \text{ or } m = 0 \text{ or } -m \in \mathbb{N}\}$ is called the set of integers.

We leave several proofs of natural properties of the integers to the reader.

Proposition 1.30 The integers are closed under addition, subtraction and multiplication. Moreover, for any two integers k, l with $k > l$ we have that $k - l \geq 1$, every nonempty set $A \subseteq \mathbb{Z}$ that is bounded below has a minimum, and every nonempty set $A \subseteq \mathbb{Z}$ that is bounded above has a maximum.

Proof. To prove that \mathbb{Z} is closed under addition, let $m, n \in \mathbb{Z}$. In case both are natural numbers or in case one of them is zero, there is nothing to prove. Moreover, in case $-m, -n \in \mathbb{N}$ we have $m + n = -((-m) + (-n))$, which is in \mathbb{Z} , because $(-m) + (-n) \in \mathbb{N}$. Now consider the case $m \in \mathbb{N}$ and $-n \in \mathbb{N}$. If $m = -n$, we obtain $m + n = 0 \in \mathbb{Z}$. If $m > -n$, then by Proposition 1.25 we conclude that $m + n = m - (-n) \in \mathbb{N} \subseteq \mathbb{Z}$. Finally, if $m < -n$ again by Proposition 1.25 we conclude that $-(m + n) = (-n) - m \in \mathbb{N}$, which means by definition of \mathbb{Z} that $m + n \in \mathbb{Z}$. The case $-m \in \mathbb{N}$ and $n \in \mathbb{N}$ is treated similarly (see Exercise 1-23a).

Closedness under subtraction and multiplication as well as the claim about differences are left to Exercises 1-23b–1-23d.

Now let $A \subseteq \mathbb{Z}$ be nonempty and bounded below. Then, because $A \subseteq \mathbb{R}$, it has an infimum a . By the version of Proposition 1.21 for infima, there is an integer $m \in A$ with $m - a < 1$. Because the absolute value of the difference between any two distinct integers is at least 1, m is the only integer in $[a, a + 1)$. Hence, m is below all elements of A that are not in $[a, a + 1)$. Because m is the only element of A in $[a, a + 1)$, m must be the minimum of A .

The proof of the corresponding result for nonempty subsets $A \subseteq \mathbb{Z}$ that are bounded above is left to Exercise 1-23e. ■

A key property of the natural numbers is that any real number is exceeded by a natural number. To prove this, we need the usual fractions, which are easily introduced.

Definition 1.31 For all $a \in \mathbb{R} \setminus \{0\}$ we set $\frac{1}{a} := a^{-1}$ and call it the reciprocal of a . For $b \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$ we set $\frac{b}{a} := b \cdot \frac{1}{a} = ba^{-1}$ and call it a fraction.

Because $\frac{1}{2} + \frac{1}{2} = 2^{-1} + 2^{-1} = (1 + 1) \cdot 2^{-1} = 2 \cdot 2^{-1} = 1$ we can now prove the following.

Theorem 1.32 *For every $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ so that $n \geq x$.*

Proof. For a contradiction, suppose that x is such that for all $n \in \mathbb{N}$ we have that $n < x$. Then $B := \{y \in \mathbb{R} : (\forall n \in \mathbb{N} : n < y)\}$ is not empty. Moreover, B is bounded below by all $n \in \mathbb{N}$. By the Completeness Axiom, B has an infimum, call it b . Then $b - \frac{1}{2} \notin B$, which means there is an $n \in \mathbb{N}$ with $n \geq b - \frac{1}{2}$. But then $n + 1 \geq b + \frac{1}{2}$ is a lower bound of B , a contradiction to $b = \inf(B)$. ■

Because $\mathbb{N} \subseteq \mathbb{Z}$ and because subsets of \mathbb{Z} that are bounded below have a minimum, we infer that for every real number x there is a unique smallest integer that is greater than or equal to x . Similarly there is a unique largest integer that is less than or equal to x . These numbers are useful when we need integers instead of real numbers, so we define the following.

Definition 1.33 *For every $x \in \mathbb{R}$, let $\lceil x \rceil$ be the smallest integer greater than or equal to x . Moreover, let $\lfloor x \rfloor$ be the largest integer less than or equal to x . As functions from \mathbb{R} to \mathbb{Z} , $\lceil \cdot \rceil$ is called the **ceiling function** and $\lfloor \cdot \rfloor$ is called the **floor function**.*

The last subset of \mathbb{R} that we introduce is the set of rational numbers. Rational numbers are naturally defined as fractions.

Definition 1.34 *The set $\mathbb{Q} := \left\{ \frac{n}{d} : n \in \mathbb{Z}, d \in \mathbb{N} \right\}$ is called the set of **rational numbers**. The set $\mathbb{R} \setminus \mathbb{Q} := \{x \in \mathbb{R} : x \notin \mathbb{Q}\}$ is called the set of **irrational numbers**.*

Proposition 1.35 *The rational numbers are closed under addition, subtraction and multiplication. Moreover, if $q, r \in \mathbb{Q}$ and $r \neq 0$, then $\frac{q}{r} \in \mathbb{Q}$.*

Proof. Let $m, n \in \mathbb{Z}$, let $c, d \in \mathbb{N}$ and consider the rational numbers $\frac{m}{c}$ and $\frac{n}{d}$. Then \mathbb{Q} is closed under addition because

$$\begin{aligned} \frac{m}{c} + \frac{n}{d} &= mc^{-1} + nd^{-1} = mdd^{-1}c^{-1} + ncc^{-1}d^{-1} \\ &= (md + nc)c^{-1}d^{-1} = \frac{md + nc}{cd}. \end{aligned}$$

For multiplication, note that $\frac{m}{c} \frac{n}{d} = mc^{-1}nd^{-1} = mnc^{-1}d^{-1} = \frac{mn}{cd}$. The remainder is left to Exercise 1-24. ■

Rational numbers can be found between any two real numbers and Exercise 1-45 will establish a similar result for irrational numbers.

Theorem 1.36 *Let $a, b \in \mathbb{R}$ with $a < b$. Then there is a rational number $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. By Theorem 1.32, there is an $n \in \mathbb{N}$ so that $0 < \frac{1}{b-a} < n$. By part 5 of Theorem 1.10, we obtain $\frac{1}{n} < b-a$. Now let $u := \min \left\{ m \in \mathbb{Z} : \frac{m}{n} \geq b \right\}$ and similarly let $l := \max \left\{ m \in \mathbb{Z} : \frac{m}{n} \leq a \right\}$. Then $\frac{u}{n} - \frac{l}{n} \geq b-a > \frac{1}{n}$, which means $\frac{l+1}{n} < \frac{u}{n}$. Hence, by definition of l and u we infer $a < \frac{l+1}{n} < b$. ■

We conclude with a simple looking result that is actually at the heart of a standard proof technique (see Standard Proof Technique 2.7). Exercise 1-25 extends Theorem 1.37 to inequalities.

Theorem 1.37 Let $x \in \mathbb{R}$. If $x \geq 0$ and for all $\varepsilon > 0$ we have $x \leq \varepsilon$, then $x = 0$.

Proof. Let x be as indicated and suppose for a contradiction that $x > 0$. Then $\varepsilon := \frac{x}{2}$ is positive and $x \leq \varepsilon = \frac{x}{2}$ implies $1 \leq \frac{1}{2}$, a contradiction. ■

Exercises

1-20. Prove that if $m, n \in \mathbb{N}$, then $mn \in \mathbb{N}$.

Hint. Same idea as the first part of the proof of Proposition 1.24 with sets $S_m := \{n \in \mathbb{N} : mn \in \mathbb{N}\}$.

1-21. Prove that if $n \in \mathbb{N}$, then $n \geq 1$.

Hint. Use $S := \{n \in \mathbb{N} : n \geq 1\}$.

1-22. Use the Well-ordering Theorem to prove the Principle of Induction.

1-23. Finish the proof of Proposition 1.30 by proving the following.

- (a) Finish the proof that \mathbb{Z} is closed under addition. That is, prove that if $-m \in \mathbb{N}$ and $n \in \mathbb{N}$, then $m+n \in \mathbb{Z}$.
- (b) Prove that \mathbb{Z} is closed under subtraction. That is, prove that $m-n \in \mathbb{Z}$ for all $m, n \in \mathbb{Z}$.
- (c) Prove that \mathbb{Z} is closed under multiplication. That is, prove that $mn \in \mathbb{Z}$ for all $m, n \in \mathbb{Z}$.
- (d) Prove that for any two integers m, n with $m > n$ we have $m-n \geq 1$.
Hint. Find a contradiction to Proposition 1.26.
- (e) Prove that every nonempty set $A \subseteq \mathbb{Z}$ that is bounded above has a maximum.

1-24. Finish the proof of Proposition 1.35. That is,

- (a) Prove that \mathbb{Q} is closed under subtraction.
- (b) Prove that if $q, r \in \mathbb{Q}$ and $r \neq 0$, then $\frac{q}{r} \in \mathbb{Q}$.

Hint. First show that for $n \in \mathbb{Z} \setminus \{0\}$ and $d \in \mathbb{N}$ we have that $\left(\frac{n}{d}\right)^{-1} = \frac{d}{n}$.

1-25. Prove that if $a, b \in \mathbb{R}$ are such that for all $\varepsilon > 0$ we have $a \leq b + \varepsilon$, then $a \leq b$.

1-26. Prove that for every real number x there is an integer n so that $n \leq x$.

1-27. Prove that for any real numbers $x, \varepsilon > 0$ there is an $n \in \mathbb{N}$ so that $\frac{x}{n} < \varepsilon$.

Hint. Theorem 1.32.

1-28. Prove that $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$.

1-29. A rational number r is called a **dyadic rational number** iff there are $p \in \mathbb{Z}$ and $n \in \mathbb{N}$ so that $r = \frac{p}{2^n}$. Dyadic rational numbers are useful in analysis because they can provide a sequence of “grids” such that each new grid contains the old one (see part 1-29a below), the whole set is the union of the “grids” (see part 1-29b) and between any two real numbers there is a dyadic rational number (see part 1-29c).

Let D be the set of dyadic rational numbers and for each $n \in \mathbb{N}$ let $D_n := \left\{ \frac{p}{2^n} : p \in \mathbb{Z} \right\}$.

(a) Prove that for all $n \in \mathbb{N}$ we have $D_n \subset D_{n+1}$.

(b) Let $\bigcup_{n=1}^{\infty} D_n := \left\{ x \in \mathbb{R} : (\exists n \in \mathbb{N} : x \in D_n) \right\}$ and prove that $D = \bigcup_{n=1}^{\infty} D_n$.

(c) Prove that for any $x, y \in \mathbb{R}$ with $x < y$ there is a dyadic rational number d so that $x < d < y$.

1-30. In this exercise, we will prove that the **real numbers are the (up to isomorphism) unique linearly ordered complete field**. That is, we will prove that every mathematical object that satisfies Axioms 1.1, 1.6, and 1.19 is in a certain sense (defined below) “the same as \mathbb{R} .”

First notice that, similar to the proof of Theorem 1.28, *all* results proved so far hold for *any* object that satisfies Axioms 1.1, 1.6, and 1.19 (because the results are derived from these axioms). That is, every set $\tilde{\mathbb{R}}$ that satisfies Axioms 1.1, 1.6, and 1.19 contains subsets $\tilde{\mathbb{N}}$, $\tilde{\mathbb{Z}}$, and $\tilde{\mathbb{Q}}$ that have the properties that we have proved up to now for the natural numbers, the integers and the rational numbers.

(a) Prove that for all $x \in \mathbb{R}$ we have that $x = \sup\{r \in \mathbb{Q} : r \leq x\}$.

(b) Now let $\tilde{\mathbb{R}}$ be a set that satisfies Axioms 1.1, 1.6, and 1.19 and let $\tilde{\mathbb{N}}$, $\tilde{\mathbb{Z}}$, and $\tilde{\mathbb{Q}}$ be subsets of $\tilde{\mathbb{R}}$ that have the properties that we have proved up to now for the natural numbers, the integers and the rational numbers, including Exercise 1-30a.

i. Define a function $f : \mathbb{Q} \rightarrow \tilde{\mathbb{R}}$ as follows. For $n \in \mathbb{N}$, let $f(1) := \bar{1}$ and once $f(n)$ is defined let $f(n+1) := f(n) \dot{+} \bar{1}$. Also let $f(-n) := \tilde{-}n$. For $n \in \mathbb{Z}$ and $d \in \mathbb{N}$ let $f\left(\frac{n}{d}\right) := \frac{\tilde{n}}{d}$. Prove that for all $x \in \mathbb{Q}$ the above definition is not self-contradictory by

proving that it assigns exactly one value to each $x \in \mathbb{Q}$. Then prove that $f(x) \in \tilde{\mathbb{Q}}$ for each $x \in \mathbb{Q}$ and that f preserves the order, that is, if $x < z$, then $f(x) < f(z)$.

ii. For $x \in \mathbb{R}$ let $f(x) := \sup \{ f(r) : r \in \mathbb{Q} \text{ and } r \leq x \}$. Prove that for all $x \in \mathbb{R}$ the above definition is not self-contradictory by proving it assigns exactly one value to each $x \in \mathbb{R}$.

(Formally this says that f is **well-defined**.)

iii. Prove that the above function does not map any two points to the same image by proving that for all $x, y \in \mathbb{R}$ the inequality $x \neq y$ implies that $f(x) \neq f(y)$.

(Formally, this says that the function f is **one-to-one** or **injective**.)

iv. Prove that the above function “reaches” every element of $\tilde{\mathbb{R}}$ by proving that for all $\tilde{x} \in \tilde{\mathbb{R}}$ there is an $x \in \mathbb{R}$ so that $f(x) = \tilde{x}$.

(Formally, this says that the function f is **onto** or **surjective**.)

v. Prove that the above function is consistent with the algebraic operations by proving that for all $x, y \in \mathbb{R}$ we have that $f(x+y) = f(x) \dot{+} f(y)$ and $f(x \cdot y) = f(x) \dot{\cdot} f(y)$.

(Formally, this says that f is a **field isomorphism**.)

vi. Prove that the above function is consistent with the order relation by proving that for all $x, y \in \mathbb{R}$ we have that $x \leq y$ implies that $f(x) \dot{\leq} f(y)$.

(Formally, this says that f is an **order isomorphism**.)

The above steps show that the points and operations in \mathbb{R} and in $\tilde{\mathbb{R}}$ can be identified with each other in such a way that it does not matter if we are working in \mathbb{R} or in $\tilde{\mathbb{R}}$. Thus for all intents and purposes, \mathbb{R} and $\tilde{\mathbb{R}}$ are “the same.” This is the essence of saying that the real numbers are up to isomorphism the unique linearly ordered, complete field.

1.5 Recursion, Induction, Summations, and Products

A recursive definition defines an entity X_n that depends on a natural number n first for $n = 1$ and then it defines X_{n+1} in terms of X_n . By the Principle of Induction the set $S = \{n \in \mathbb{N} : X_n \text{ is defined}\}$ is equal to \mathbb{N} , which means that a recursive definition defines the entity X_n for all natural numbers n . In this fashion, the sum of finitely many numbers can be defined.

Definition 1.38 For each $j \in \mathbb{N}$ let $a_j \in \mathbb{R}$. Define the **sum** $\sum_{j=1}^1 a_j := a_1$ and for $n \in \mathbb{N}$ define the **sum** $\sum_{j=1}^{n+1} a_j := a_{n+1} + \sum_{j=1}^n a_j$. For $m \in \mathbb{N} \cup \{0\}$, set $\sum_{j=1}^{-m} a_j := 0$. The parameter j is also called the **summation index**.

In particular, note that a sum whose index starts at 1 and ends at a number smaller than 1 is always zero. It is also called an **empty sum**. Summations that start at numbers other than 1 are defined similarly (Exercise 1-31). By their nature, recursive definitions are closely linked to induction. Unlike what is stated in Theorem 1.23, induction normally is used to prove statements about natural numbers. This is possible, because a proof that a statement is true for all natural numbers is the same as a proof that a certain set is equal to \mathbb{N} .

Theorem 1.39 Principle of Induction. Let $P(n)$ be a statement about the natural number n . If $P(1)$ is true and if for all $n \in \mathbb{N}$ truth of $P(n)$ implies truth of $P(n+1)$, then $P(n)$ holds for all natural numbers.

Proof. Let P be as indicated and consider the set $S := \{n \in \mathbb{N} : P(n) \text{ is true}\}$. Then $1 \in S$. For every $n \in S$ the statement $P(n)$ is true, hence $P(n+1)$ is true, which means $n+1 \in S$. By Theorem 1.23 we conclude $S = \mathbb{N}$ and thus $P(n)$ is true for all $n \in \mathbb{N}$. ■

Standard Proof Technique 1.40 In the form of Theorem 1.39, **induction** is a standard proof technique. It involves a two-step process. In the first step, called the **base step**, $P(1)$ is proved. Then, in the **induction step**, $P(n)$ is used to prove $P(n+1)$. In this context, $P(n)$ is also called the **induction hypothesis**. All proofs in this section rely on induction. Moreover, Exercise 1-32 exhibits another way to carry out an induction (sometimes called strong induction). □

Example 1.41 For all $n \in \mathbb{N}$, the **summation formula** $\sum_{j=1}^n j = \frac{1}{2}n(n+1)$ holds.

Proof. The statement is $P(n) = \left[\sum_{j=1}^n j = \frac{1}{2}n(n+1) \right]$.

Base step. We prove $P(n)$ for $n = 1$. $\sum_{j=1}^1 j = 1 = \frac{1}{2}1(1+1)$, so $P(1)$ holds.

Induction step. Under the induction hypothesis $\sum_{j=1}^n j = \frac{1}{2}n(n+1)$ we must prove

$\sum_{j=1}^{n+1} j = \frac{1}{2}(n+1)((n+1)+1)$. A standard step in induction for recursively defined quantities is to split off the last term. This is done in the first step here.

$$\begin{aligned} \sum_{j=1}^{n+1} j &= (n+1) + \sum_{j=1}^n j && \text{Now we can apply the induction hypothesis to the sum and the rest is algebra.} \\ &= (n+1) + \frac{1}{2}n(n+1) = \frac{1}{2}2(n+1) + \frac{1}{2}n(n+1) \\ &= \frac{1}{2}(n+2)(n+1). \end{aligned}$$

Further examples of similar inductions can be found in Exercise 1-33. □

Similar to sums we can define products. Although products occur less frequently than sums, they are useful to define powers.

Definition 1.42 For each $j \in \mathbb{N}$, let $a_j \in \mathbb{R}$. Define the **product** $\prod_{i=1}^1 a_j := a_1$ and

for all $n \in \mathbb{N}$ define the **product** $\prod_{j=1}^{n+1} a_j := a_{n+1} \cdot \prod_{j=1}^n a_j$. For all $m \in \mathbb{N} \cup \{0\}$, set

$\prod_{j=1}^{-m} a_j := 1$. The parameter j is also called the **product index**.

Products that start at numbers other than 1 are defined similarly (Exercise 1-31). Products that end at an index that is smaller than the starting index are set to 1 and are also called **empty products**.

Definition 1.43 For all $a \in \mathbb{R}$, and all $n \in \mathbb{N} \cup \{0\}$, we define the n^{th} **power** $a^n := \prod_{j=1}^n a$.

Aside from integer powers of numbers, we want to work with rational powers. To define rational powers, we need n^{th} roots of nonnegative real numbers. To formally prove their existence, we need the Binomial Theorem. As a start we need binomial coefficients and one of their key properties.

Definition 1.44 For all $n \in \mathbb{N} \cup \{0\}$, we define $n! := \prod_{j=1}^n j$ and call it the **factorial**

of n . For all $n, k \in \mathbb{N} \cup \{0\}$ with $k \leq n$, we define the **binomial coefficient** as

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

Theorem 1.45 The equation $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ holds for all $n, k \in \mathbb{N}$ with $k \leq n$.

Proof. This result can be proved by direct computation.

$$\begin{aligned} \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!k}{k!(n+1-k)!} + \frac{n!(n+1-k)}{k!(n+1-k)!} = \frac{n!(k+n+1-k)}{k!(n+1-k)!} \\ &= \frac{(n+1)!}{k!(n+1-k)!}. \end{aligned}$$

■

Now we are ready to prove the Binomial Theorem.

Theorem 1.46 The Binomial Theorem. For all real numbers $a, b \in \mathbb{R}$, and all $n \in \mathbb{N}$, we have $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$.

Proof. Throughout the proof we will freely use the properties of sums proved in Exercise 1-34. The proof is by induction on n , with $P(n)$ being the statement about $(a+b)^n$.

Base step. For $n = 1$, note that

$$(a+b)^1 = a+b = \binom{1}{0} a^0 b^{1-0} + \binom{1}{1} a^1 b^{1-1} = \sum_{k=0}^1 \binom{1}{k} a^k b^{1-k},$$

which proves the base step.

Induction step. Assuming that the result holds for n , we must prove it for $n+1$. First note that it follows easily from Definition 1.43 that for all $x \in \mathbb{R}$ and all $m \in \mathbb{N}$ we have $x \cdot x^m = x^{m+1}$.

$$(a+b)^{n+1}$$

The first step is to split off the last term of the power.

$$= (a+b)(a+b)^n$$

Now we can apply the induction hypothesis to $(a+b)^n$.

$$= (a+b) \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

$$= \sum_{k=0}^n \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k}$$

After multiplying out parentheses, we want to combine the sums. In order to do this we shift the indices to obtain similar terms in both sums. In the first sum we set $j := k + 1$ and in the second sum we set $j := k$.

$$= \sum_{j=1}^{n+1} \binom{n}{j-1} a^j b^{n+1-j} + \sum_{j=0}^n \binom{n}{j} a^j b^{n+1-j}$$

To combine the sums, the indices must start and end at the same numbers. Thus we split off the last term of the first sum and the first term of the second sum. Then we can combine the sums.

$$= \binom{n}{n} a^{n+1} b^{n+1-(n+1)} + \binom{n}{0} a^0 b^{n+1-0} + \sum_{j=1}^n \left[\binom{n}{j-1} + \binom{n}{j} \right] a^j b^{n+1-j}$$

Now we can apply Theorem 1.45. Moreover, by rewriting the terms outside the sum, we see that they fit the requisite pattern and can be absorbed into the sum.

$$= \binom{n+1}{n+1} a^{n+1} b^{n+1-(n+1)} + \binom{n+1}{0} a^0 b^{n+1-0} + \sum_{j=1}^n \binom{n+1}{j} a^j b^{n+1-j}$$

$$= \sum_{j=0}^{n+1} \binom{n+1}{j} a^j b^{n+1-j}.$$

■

With the Binomial Theorem, we can prove that n^{th} roots exist. The proof of Theorem 1.47 is the first proof in this text in which we have to choose a number to make another number smaller than a given bound. That is, this is our first proof with a distinct analytical flavor.

Theorem 1.47 *Let $n \in \mathbb{N}$. For every nonnegative real number a , there exists a unique nonnegative real number r such that $r^n = a$.*

Proof. We first prove the existence of r . Let $R := \{x \in \mathbb{R} : x \geq 0 \text{ and } x^n \leq a\}$. Then $0 \in R$ and R is bounded above by $\max\{1, a\}$. Let $r := \sup(R)$. To show that $r^n = a$, we will show that $r^n \neq a$ and $r^n \neq a$. First, suppose for a contradiction that $r^n < a$. Then there is a $\delta > 0$ so that $r^n + \delta < a$. By Theorem 1.32 (or Exercise 1-27), for each $k \in \{1, \dots, n\}$ we can find an $m_k \in \mathbb{N}$ so that $\binom{n}{k} r^{n-k} \frac{1}{m_k^k} < \frac{\delta}{n}$. Let $m := \max\{m_1, \dots, m_n\}$. Then by the Binomial Theorem we conclude

$$\left(r + \frac{1}{m}\right)^n = \sum_{k=0}^n \binom{n}{k} r^{n-k} \left(\frac{1}{m}\right)^k$$

$$= r^n + \sum_{k=1}^n \binom{n}{k} r^{n-k} \frac{1}{m^k}$$

Split off the zeroth term.

Now use the definition of m .

$$< r^n + \sum_{k=1}^n \frac{\delta}{n} = r^n + \delta < a.$$

The above shows that $r + \frac{1}{n} \in R$, contradicting the fact that $r = \sup(R)$. Hence, $r^n \neq a$. The proof that $r^n \neq a$ is similar and left to the reader as Exercise 1-36.

For uniqueness, suppose for a contradiction that there is another $b \geq 0$ with $b^n = a$. Then $b < r$ or $b > r$. But if $b > r$, then with $\delta := b - r$ we obtain $a = b^n = (r + \delta)^n = r^n + \sum_{k=1}^n \binom{n}{k} r^{n-k} \delta^k > a$, a contradiction. Hence, $b < r$. But

then with $\delta := r - b$ we have $a = r^n = (b + \delta)^n = b^n + \sum_{k=1}^n \binom{n}{k} b^{n-k} \delta^k > a$, a contradiction. Therefore r is unique. ■

We conclude by defining rational powers of nonnegative numbers and by proving some of their properties.

Definition 1.48 Let $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be nonnegative. The unique nonnegative real number r such that $r^n = a$ is called the n^{th} **root of a** , denoted $\sqrt[n]{a}$. For $n = 2$ the root is called the **square root**, denoted \sqrt{a} .

Existence of n^{th} roots is another property that distinguishes \mathbb{R} from \mathbb{Q} . Although Theorem 1.36 indicates that there are “many” rational numbers, the rational number system has some shortcomings when it comes to powers.

Proposition 1.49 There is no rational number r such that $r^2 = 2$.

Proof. We first prove by induction as stated in Exercise 1-32 (strong induction) that if $n^2 = 2z$ for some $z \in \mathbb{N}$, then $n = 2z'$ for some $z' \in \mathbb{N}$. The base step for $n = 1$ is **vacuously true**. That is, because the hypothesis $1^2 = 2z$ leads to the contradiction $1 = 1^2 = 2z = z + z > 1$, the hypothesis is never true, which means that the implication is automatically true (see Definition A.2 in Appendix A).

For the induction step, first note that the result is trivial for $n = 2$, because $2 = 2 \cdot 1$. Now assume that $n > 2$ and the statement has been proved for all natural numbers less than n . Then $2z = n^2 = (n - 2 + 2)^2 = (n - 2)^2 + 4(n - 2) + 4$ implies that $(n - 2)^2 = 2\bar{z}$ for some $\bar{z} \in \mathbb{N}$. By induction hypothesis, we conclude that $n - 2 = 2\bar{z}'$ for some $\bar{z}' \in \mathbb{N}$, and hence $n = 2\bar{z}' + 2 = 2z'$ for some $z' \in \mathbb{N}$. This proves that if $n^2 = 2z$ for some $z \in \mathbb{N}$, then $n = 2z'$ for $z' = \bar{z}' + 1 \in \mathbb{N}$.

Now suppose for a contradiction that there are $n \in \mathbb{Z}$ and $d \in \mathbb{N}$ so that $\left(\frac{n}{d}\right)^2 = 2$ and such that there is no $k \in \mathbb{N} \setminus \{1\}$ such that $n = n_k \cdot k$ and $d = d_k \cdot k$. But by the above $n^2 = 2d^2$ implies $n = n_2 \cdot 2$. Consequently, $2d^2 = (n_2 \cdot 2)^2$, that is, $d^2 = n_2^2 \cdot 2$, which implies $d = d_2 \cdot 2$, a contradiction. ■

We conclude from Theorem 1.47 and Proposition 1.49 that $\sqrt{2}$ is irrational.

For odd natural numbers (that is, natural numbers of the form $n = 2k + 1$), it is possible to define the n^{th} root of a negative number $a < 0$ as $\sqrt[n]{a} := -\sqrt[n]{|a|}$. For the most part, powers are considered for nonnegative numbers, though.

Definition 1.50 For all real numbers $a \geq 0$, all $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and all $q \in \mathbb{Q}$ with $q > 0$ we define

1. $a^{\frac{1}{n}} := \sqrt[n]{a}$. That is, the $\left(\frac{1}{n}\right)^{\text{th}}$ power of a is the n^{th} root of a .
2. $a^{\frac{m}{n}} := (a^m)^{\frac{1}{n}}$.
3. $a^{-q} := (a^q)^{-1} = \frac{1}{a^q}$ for $a \neq 0$.

Theorem 1.51 For all positive numbers a and b and all rational numbers x and y , the following power laws hold:

$$\begin{aligned} a^x a^y &= a^{x+y} & (ab)^x &= a^x b^x \\ \frac{a^x}{a^y} &= a^{x-y} = \frac{1}{a^{y-x}} & \left(\frac{a}{b}\right)^x &= \frac{a^x}{b^x} \\ (a^x)^y &= a^{xy} \end{aligned}$$

Proof. We first prove $(ab)^x = a^x b^x$. For exponents $n \in \mathbb{N}$, this is an easy induction. The base step $n = 1$ is trivial and the induction step from n to $n + 1$ is $(ab)^{n+1} = ab(ab)^n = aba^n b^n = aa^n bb^n = a^{n+1} b^{n+1}$.

For rational exponents $\frac{n}{d}$ with $n, d \in \mathbb{N}$, we have that $\left((ab)^{\frac{n}{d}}\right)^d = (ab)^n$ and $\left(a^{\frac{n}{d}} b^{\frac{n}{d}}\right)^d = \left(a^{\frac{n}{d}}\right)^d \left(b^{\frac{n}{d}}\right)^d = a^n b^n = (ab)^n$. Note that in both equalities we used the definition of fractional powers, not the power law that we are currently proving. Because all numbers involved are positive and d^{th} roots are unique, we conclude that $(ab)^{\frac{n}{d}} = a^{\frac{n}{d}} b^{\frac{n}{d}}$.

For $x = 0$, the equality $(ab)^x = a^x b^x$ is trivial. Finally, for all positive $x \in \mathbb{Q}$ we note $(ab)^{-x} a^x b^x = (ab)^{-x} (ab)^x = 1$. Therefore $(ab)^{-x}$ is the multiplicative inverse of $a^x b^x$, that is, $(ab)^{-x} = a^{-x} b^{-x}$. Thus $(ab)^x = a^x b^x$ for all $a, b > 0$ and all $x \in \mathbb{Q}$.

To prove that $a^{x+y} = a^x a^y$ we proceed similarly. For exponents $m, n \in \mathbb{N}$, the proof for arbitrary m is an induction on n . The base step $a^m a^1 = a^m a = a^{m+1}$ follows straight from the definition of powers with natural exponents. For the induction step from n to $n + 1$, note that $a^m a^{n+1} = a^m a^n a = a^{m+n} a = a^{m+(n+1)}$, which proves the result for exponents $m, n \in \mathbb{N}$.

For positive rational exponents x and y , note that there are $m, n, d \in \mathbb{N}$ so that $x = \frac{m}{d}$ and $y = \frac{n}{d}$. Then, using the equality we already proved, we obtain

$$a^x a^y = a^{\frac{m}{d}} a^{\frac{n}{d}} = (a^m)^{\frac{1}{d}} (a^n)^{\frac{1}{d}} = (a^m a^n)^{\frac{1}{d}} = (a^{m+n})^{\frac{1}{d}} = a^{\frac{m+n}{d}} = a^{x+y}.$$

The equality is trivial if one of x and y is zero.

In case both exponents are negative, note that for all positive $x, y \in \mathbb{Q}$ we have $a^{x+y} a^{-x} a^{-y} = a^x a^y a^{-x} a^{-y} = 1$, which means $a^{-x} a^{-y} = a^{-x+(-y)}$ as was to be

proved. This leaves the case in which one exponent is positive and the other is negative. Let $x, y \in \mathbb{Q}$ be positive and consider a^{x-y} . If the inequality $|x| > |y|$ holds we have $a^{x-y}a^ya^{-x} = a^xa^{-x} = 1$, which means that $a^{x-y} = a^xa^{-y}$. If $|x| < |y|$ we have $a^{y-x}a^xa^{-y} = a^ya^{-y} = 1$, which means that $a^{y-x} = a^ya^{-x}$. If $|x| = |y|$ the claim is trivial. Thus $a^{x+y} = a^xa^y$ for all $a > 0$ and all $x, y \in \mathbb{Q}$.

We leave the remaining three equalities as Exercise 1-37. ■

Power laws for $a \leq 0$ and $b \leq 0$ (as applicable) can be proved similarly. To conclude, note that the results presented in this chapter guarantee that the real numbers have the properties we expect them to have. We will therefore use the usual notation (fractions, etc.) and laws of algebra throughout this text without further qualms about the need to justify that we are indeed allowed to do so.

Exercises

1-31. Let $k, m \in \mathbb{Z}$ and for each $j \in \mathbb{Z}$ let $a_j \in \mathbb{K}$. Define the sum $\sum_{j=k}^m a_j$ and the product $\prod_{j=k}^m a_j$.

1-32. Let $P(n)$ be a statement about the natural number n . Prove that if $P(1)$ is true and if for all $n \in \mathbb{N} \setminus \{1\}$ truth of $P(1), \dots, P(n-1)$ implies truth of $P(n)$, then $P(n)$ holds for all natural numbers. This type of induction is sometimes called **strong induction**.

Hint. Consider $S := \{n \in \mathbb{N} : (\forall k < n : P(k) \text{ holds})\}$.

1-33. Prove each of the following by induction.

(a)
$$\sum_{j=1}^n j^2 = \frac{n}{6}(n+1)(2n+1)$$

(b)
$$\sum_{j=1}^n j^3 = \frac{1}{4}n^2(n+1)^2$$

(c)
$$\sum_{j=1}^n j^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

(d) **Bernoulli's inequality.** Prove that for all real numbers $x > -1$, $x \neq 0$ and $n \geq 2$ we have that $(1+x)^n > 1+nx$.

1-34. **Properties of sums and products.** Let $c \in \mathbb{R}$ and for all $j \in \mathbb{N}$ let a_j and b_j be real numbers.

(a) Prove that for all $n \in \mathbb{N}$ we have
$$\sum_{j=1}^n (a_j + b_j) = \sum_{j=1}^n a_j + \sum_{j=1}^n b_j.$$

(b) Prove that for all $n \in \mathbb{N}$ we have
$$\sum_{j=1}^n (ca_j) = c \sum_{j=1}^n a_j.$$

(c) Prove that for all $n \in \mathbb{N}$ we have
$$\sum_{j=1}^n 1 = n.$$

(d) Prove that for all $n \in \mathbb{N}$ we have
$$\prod_{j=1}^n (a_j \cdot b_j) = \left(\prod_{j=1}^n a_j \right) \cdot \left(\prod_{j=1}^n b_j \right).$$

1-35. **Reindexing sums.** Let $s \in \mathbb{Z}$, $n \in \mathbb{N}$ and for $j \in \mathbb{Z}$ let $a_j \in \mathbb{K}$. Prove that
$$\sum_{j=s}^{s+n} a_j = \sum_{k=1}^{n+1} a_{k+s-1}.$$

1-36. Finish the proof of Theorem 1.47 by showing that $r^n \neq a$.

Hint. Suppose $r^n > a$ and prove that then for some $\varepsilon > 0$ and all $\delta \in (0, \varepsilon)$ we have $r^n - \delta \notin R$.

1-37. Finish the proof of Theorem 1.51. That is, let a and b be positive real numbers, let $x, y \in \mathbb{Q}$ and prove each of the following.

$$(a) \frac{a^x}{a^y} = a^{x-y} = \frac{1}{a^{y-x}} \quad (b) \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \quad (c) (a^x)^y = a^{xy}$$

1-38. Let $0 \leq a < b$ and let $q > 0$ be rational. Prove that $a^q < b^q$.

1-39. Let $a, x \in (0, \infty)$ and let x be a rational number.

(a) Prove that if $a > 1$ and $x > 1$, then $a^x > a$.

Hint. Let $p, q \in \mathbb{N}$ be so that $x = \frac{p}{q}$ and compare a^p and a^q .

(b) Prove that if $a < 1$ and $x < 1$, then $a^x > a$.

(c) Prove that if $a > 1$ and $x < 1$, then $a^x < a$.

(d) Prove that if $a < 1$ and $x > 1$, then $a^x < a$.

1-40. Let $n \in \mathbb{N}$. Prove that $\binom{n}{0} = 1$ and that $\binom{n}{n} = 1$.

1-41. Prove that there is no rational number r such that $r^2 = 3$.

1-42. Prove that for any n real numbers x_1, \dots, x_n the inequality $\left| \sum_{i=1}^n x_i \right| \leq \sum_{i=1}^n |x_i|$ holds.

1-43. Prove that for all $a, b \geq 0$ the inequality $\sqrt{ab} \leq \frac{a+b}{2}$ holds.

1-44. (a) Prove that for all $a, b \in \mathbb{R}$ we have $\sqrt{a^2 + b^2} \leq |a| + |b|$.

(b) Prove that for any $a_1, \dots, a_n \in \mathbb{R}$ we have $\sqrt{\sum_{j=1}^n a_j^2} \leq \sum_{j=1}^n |a_j|$.

1-45. Let $a, b \in \mathbb{R}$ with $a < b$. Prove that there is an irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.

Hint. Use that $\sqrt{2}$ is irrational and Exercise 1-27 and mimic the proof of Theorem 1.36.