1

NEWTON'S LAWS FOR PARTICLES AND RIGID BODIES

In the basic course in dynamics you learned about various coordinate frames and how to express displacement, velocity, and acceleration with respect to these different frames of reference (see, e.g., [1]). You also covered Newton's laws and learned the basics of relating forces to acceleration when the system motion was prescribed and calculations were made for particular instantaneous system configurations.

In your first exposure to engineering dynamics it was appropriate to separate the concepts of kinematics and kinetics, the former dealing with how physical elements fit and move together and the latter dealing with the forces required for the prescribed motion to occur. A further distinction was to treat the motion of single particles as separate from that of rigid bodies. Here we treat kinematics and kinetics as simply all part of the dynamic system analysis, and particle and rigid, body mechanics are treated together. You will see that there really are only a few principles that govern the dynamics of all systems, and that, dynamically, all systems are pretty much the same.

The emphasis in this chapter is on choosing appropriate coordinates for deriving complete equations of motion such that if the resulting equations are solved, the entire motion–time history of the system can be predicted, including any special configurations of interest. The resulting equations of motion for most engineering systems will be nonlinear, and thus solutions will be obtained using simulation. This topic is covered in later sections.

We start by reviewing the most common coordinate frames and learn how to express position, velocity, and acceleration with these coordinates. This is followed by a discussion of free-body diagrams, where internal forces are exposed acting on the inertial components of the system, and how Newton's laws are used to derive the equations of motion. We then show the increased complexity of working with rigid bodies rather than particles. In a later chapter you will see how to solve equations of motion using time-step simulation.

1.1 NEWTON'S SECOND LAW

The fundamental principle upon which dynamics is based is *Newton's second law,* which states that force is equal to mass times acceleration:

$$\mathbf{F} = m\mathbf{a} \tag{1.1}$$

where the boldface notation indicates vector or matrix quantities. Equation (1.1) simply states that the force acting in some direction on a mass particle causes an acceleration of that particle in the same direction as the force. It is implied that the acceleration is measured with respect to an inertial reference frame, and for most engineering problems the inertial reference frame is attached to the Earth. The proportionality constant is the particle mass expressed in kilograms (kg). We use the SI system of units in this book, where force is measured in newtons (N), length is measured in meters (m), and time is measured in seconds (s).

You should remember that for a particle currently at position $\mathbf{r}$ moving at velocity $\mathbf{v}$ and having acceleration $\mathbf{a}$, the position, velocity, and acceleration are related by

$$\begin{aligned} \mathbf{v} &= \frac{d}{dt}\mathbf{r} = \dot{\mathbf{r}} \\ \mathbf{a} &= \frac{d}{dt}\mathbf{v} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} \end{aligned} \tag{1.2}$$

or

$$\begin{aligned} \mathbf{v} &= \int \mathbf{a}\, dt \\ \mathbf{r} &= \int \mathbf{v}\, dt \end{aligned} \tag{1.3}$$

Note in Eq. (1.2) that we introduce the "dot" notation for derivatives. This is used interchangeably with $\frac{d}{dt}$ throughout the book.

Although Eq. (1.1) is often seen emblazed on posters and T-shirts, it is not very applicable in the form shown. To derive equations of motion that lead to predictions of system response, we must be able to express acceleration in terms of position and velocity coordinates. Next we describe the most common coordinates to use for this purpose.

1.2 COORDINATE FRAMES AND VELOCITY AND ACCELERATION DIAGRAMS

In this section the most common coordinate frames for expressing position, velocity, and acceleration are shown.

Rectangular Coordinates

Figure 1.1*a* shows an inertial reference frame and a particle traveling along some path. At the instant shown, the particle has a position vector **r**, velocity vector **v**, and acceleration vector **a**. The velocity vector is aligned tangent to the path of the particle, but the acceleration vector could be in any direction. In Figure 1.1*b* the general vector representation has been removed and the perpendicular components of the vectors are

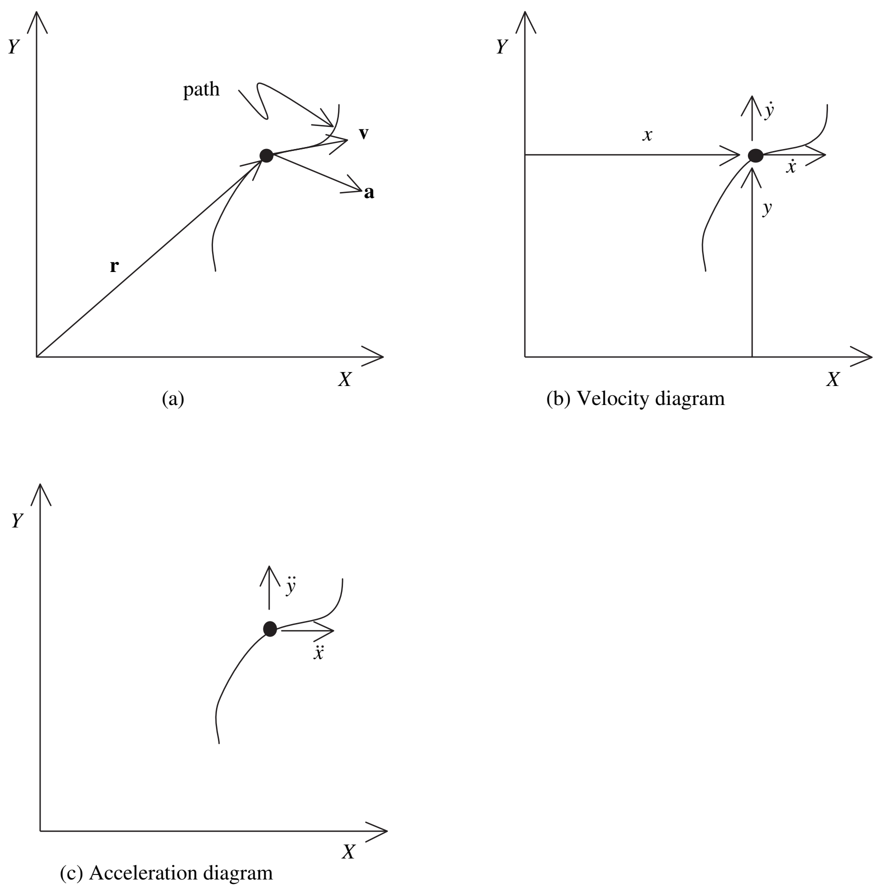


Figure 1.1 Rectangular coordinates.

shown aligned in the inertial X and Y directions. At this point we do not know whether the particle is moving right or left or up or down. We do not know in which direction the acceleration vector is actually pointing. So we choose the directions that we define as positive and point the components in their assumed positive directions. When we ultimately derive the equations of motion for a particular system, the solution will tell us in which direction the system parts are actually moving. In Figure 1.1*b* the velocity components are pointing in the positive X and Y directions. These are the rectangular components of the velocity vector with respect to the inertial X,Y frame and they are displayed on a *velocity diagram*. Figure 1.1*c* shows the components of the acceleration vector displayed on an *acceleration diagram*.

Polar Coordinates

With polar coordinates we start with the position vector shown in Figure 1.1*a* and use the unit vectors $\boldsymbol{\varepsilon}_r$ and $\boldsymbol{\varepsilon}_\theta$ as indicated in Figure 1.2*a*. The unit vector $\boldsymbol{\varepsilon}_r$ is pointing in the positive r-direction and the unit vector $\boldsymbol{\varepsilon}_\theta$ is perpendicular to the r-direction. Polar coordinates are less obvious than rectangular coordinates because when we take the time rate of change of the position vector to obtain the velocity, it is necessary to account for the change in direction of the position vector. In rectangular coordinates this is not necessary. The x-direction component of velocity is always pointing in the x-direction. Its length can change, but its direction never does.

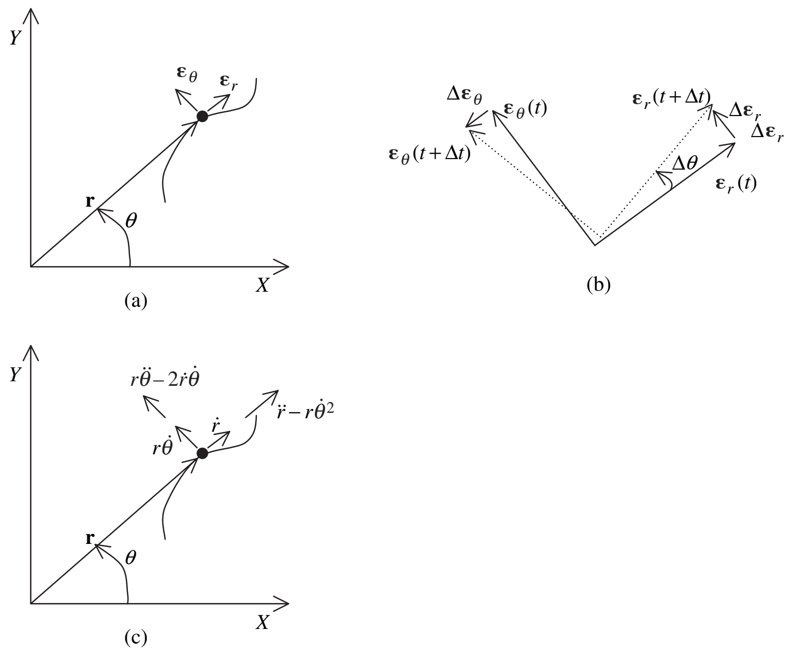


Figure 1.2 Polar coordinates.

Figure 1.2*b* shows an expanded view of the unit vectors at time t and their orientation Δt seconds later. To determine the change of the unit vectors over this time period, we need

$$\begin{aligned} \Delta \boldsymbol{\varepsilon}_r &= \frac{\boldsymbol{\varepsilon}_r(t+\Delta t) - \boldsymbol{\varepsilon}_r(t)}{\Delta t} \\ \Delta \boldsymbol{\varepsilon}_\theta &= \frac{\boldsymbol{\varepsilon}_\theta(t+\Delta t) - \boldsymbol{\varepsilon}_\theta(t)}{\Delta t} \end{aligned} \tag{1.4}$$

and these vectors are shown in Figure 1.2*b*. The reader should remember that the difference between two vectors, say $\mathbf{A} - \mathbf{B}$, is the vector drawn between their tips, from $\mathbf{B}$ pointing toward $\mathbf{A}$. From Figure 1.2*b* it is seen that $\Delta \boldsymbol{\varepsilon}_r$ is pointing in the $\boldsymbol{\varepsilon}_\theta$-direction and that $\Delta \boldsymbol{\varepsilon}_\theta$ is pointing in the negative $\boldsymbol{\varepsilon}_r$-direction. Since $\Delta\theta$ is very small, the magnitudes of $\Delta \boldsymbol{\varepsilon}_r$ and $\Delta \boldsymbol{\varepsilon}_\theta$ are very nearly equal to the magnitude of $\boldsymbol{\varepsilon}_r$ or $\boldsymbol{\varepsilon}_\theta$ (which is unity by definition), times the angle that has been swept through, $\Delta\theta$. Thus,

$$\begin{aligned} \frac{\Delta \boldsymbol{\varepsilon}_r}{\Delta t} &\rightarrow 1 \, \frac{\Delta \theta}{\Delta t} \, \boldsymbol{\varepsilon}_\theta \\ \frac{\Delta \boldsymbol{\varepsilon}_\theta}{\Delta t} &\rightarrow -1 \, \frac{\Delta \theta}{\Delta t} \, \boldsymbol{\varepsilon}_r \end{aligned} \tag{1.5}$$

and the final result is

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}_r &= \dot{\theta} \boldsymbol{\varepsilon}_\theta \\ \dot{\boldsymbol{\varepsilon}}_\theta &= \dot{\theta} \boldsymbol{\varepsilon}_r \end{aligned} \tag{1.6}$$

Later we will see that the change in these unit vectors can be expressed as a cross-product,

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}_r &= \boldsymbol{\omega} \times \boldsymbol{\varepsilon}_r \\ \dot{\boldsymbol{\varepsilon}}_\theta &= \boldsymbol{\omega} \times \boldsymbol{\varepsilon}_\theta \end{aligned} \tag{1.7}$$

where the angular velocity vector $\boldsymbol{\omega}$ has length $\dot{\theta}$ and direction given by the right-hand rule, where the fingers of your right hand are oriented in the $\mathbf{r}$-direction and swept in the direction of defined positive angular motion. The direction of $\boldsymbol{\omega}$ is the direction your thumb is pointing after sweeping your fingers appropriately. For the case of Figure 1.2, $\boldsymbol{\omega}$ is pointing toward the reader, out of the page. We would call this the $\boldsymbol{\varepsilon}_z$-direction.

Starting with the position vector

$$\mathbf{r} = r \boldsymbol{\varepsilon}_r \tag{1.8}$$

the velocity vector is determined from

$$\dot{\mathbf{r}} = \dot{r} \boldsymbol{\varepsilon}_r + r \dot{\boldsymbol{\varepsilon}}_r = \dot{r} \boldsymbol{\varepsilon}_r + r \dot{\theta} \boldsymbol{\varepsilon}_\theta \tag{1.9}$$

and the acceleration vector is determined by differentiating again:

$$\ddot{\mathbf{r}} = \ddot{r}\boldsymbol{\varepsilon}_r + \dot{r}\dot{\boldsymbol{\varepsilon}}_r + \dot{r}\dot{\theta}\boldsymbol{\varepsilon}_\theta + r\ddot{\theta}\boldsymbol{\varepsilon}_\theta + r\dot{\theta}\dot{\boldsymbol{\varepsilon}}_\theta \tag{1.10}$$

which becomes the final result when Eqs. (1.6) are substituted; thus,

$$\ddot{r} = \ddot{r}\boldsymbol{\varepsilon}_r + \dot{r}\dot{\theta}\boldsymbol{\varepsilon}_\theta + \dot{r}\dot{\theta}\boldsymbol{\varepsilon}_\theta + r\ddot{\theta}\boldsymbol{\varepsilon}_\theta - r\dot{\theta}^2\boldsymbol{\varepsilon}_r \quad \text{or} \quad \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{\varepsilon}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\varepsilon}_\theta \tag{1.11}$$

The components of velocity and acceleration in polar coordinates are shown in Figure 1.2*c*.

The reader may wonder why one would ever use polar coordinates over rectangular coordinates since polar components seem so much more complicated. We demonstrate through examples that there are cases when polar coordinates are far more convenient than rectangular coordinates for setting up and deriving equations of motion. For now it is worthwhile to remember that velocity components come from Eq. (1.9) and acceleration components from Eq. (1.11) when using polar coordinates.

As an example of the use of rectangular and polar coordinates, consider the system in Figure 1.3*a*. It consists of a spring and a mass pendulum in a gravity field. It is desired to derive the equations of motion that, if solved, would predict the motion–time history of this system from any set of initial conditions. We first need to show acceleration and force components so that Newton's laws can be used to derive equations. The position components in both coordinate frames are shown in part *a* of the figure. The velocity components and acceleration components in rectangular coordinates are shown in Figure 1.3*b*, and the velocity and acceleration components in polar coordinates are shown in part (*c*). As introduced above and indicated in Figure 1.3, it is convenient to show velocity components using arrows and symbols on a velocity diagram. Similarly, we show the acceleration components on an acceleration diagram. When these diagrams are coupled with a force diagram we will discover that it is very straightforward to derive the equations of motion directly using Newton's law.

It should be noted that both coordinate representations are describing exactly the same motion. At this instant in time, the mass particle has some absolute velocity vector and absolute acceleration vector measured with respect to the inertial frame. Rectangular coordinates represent these vectors with mutually perpendicular components in the X and Y directions, and polar coordinates use mutually perpendicular components in the r and θ directions. If the respective components from each description were added vectorially, the result would be the instantaneous absolute velocity and acceleration of the particle.

Another example of choosing coordinates is the system of Figure 1.4*a*. In this system the spring and mass pendulum are connected to a cart of mass m_c and the cart is attached to ground through a spring k_c. An external force $F(t)$ is prescribed on the cart. Our ultimate desire is to derive equations of motion which, if solved, would predict the motion–time history of this system. For now we just want to choose

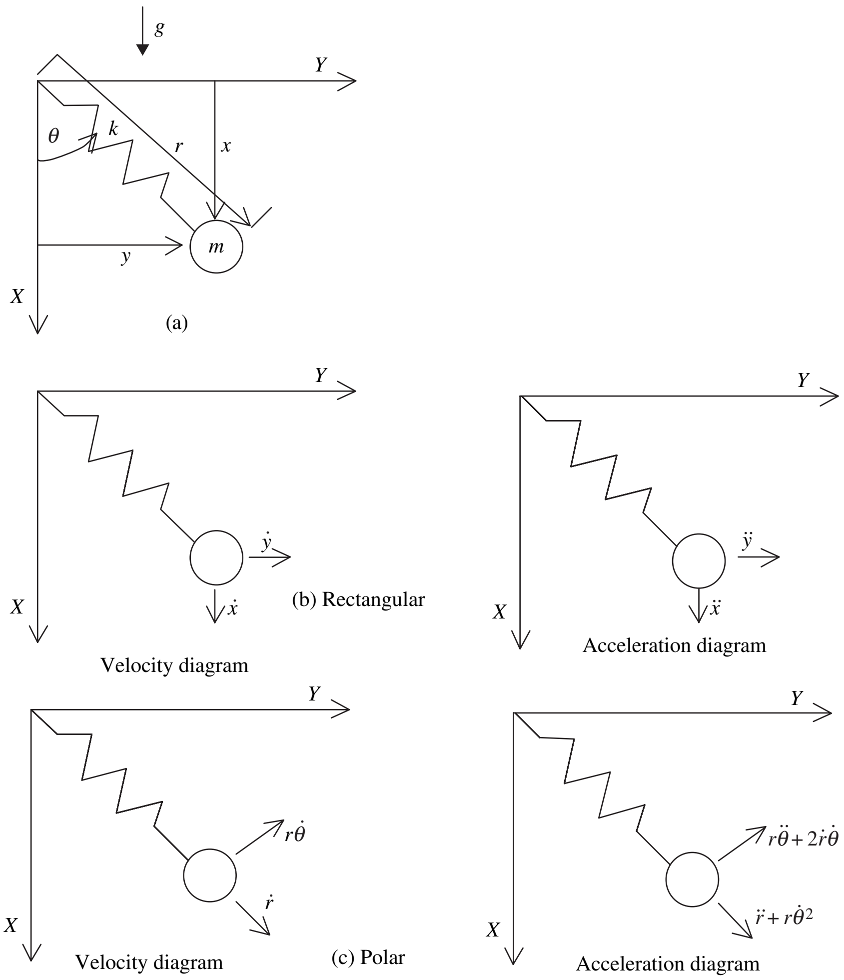


Figure 1.3 Coordinates for a spring and pendulum.

reasonable coordinates with which we can conveniently express the velocity and acceleration of the inertial components.

In Figure 1.4*b* rectangular coordinates are used for both mass elements. We only need one component for the cart, as it is constrained to move only horizontally. We need both the x and y components for the pendulum mass since it can move in the entire plane. In part c of the figure, rectangular coordinates are used for the cart and polar coordinates are used for the pendulum mass. The velocity and acceleration components are indicated. In addition to the polar coordinate components, the velocity and acceleration components of the cart have been transferred to the pendulum mass. This is required because the polar coordinates are defined with respect to a frame

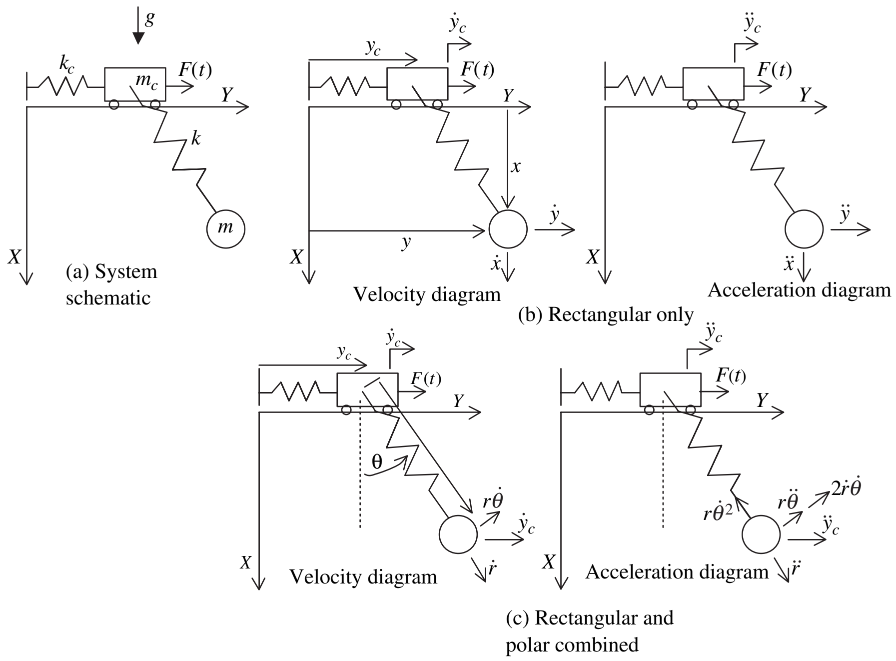


Figure 1.4 Coordinates for spring and pendulum attached to a cart.

that is attached to the cart, and we must account for the fact that the pendulum mass experiences the motion of the cart plus additional motions described by the polar coordinates. In other words, if r and θ were fixed and not permitted to change, it is pretty obvious that the pendulum mass would move identically to the cart. Later we formalize this idea of transferring velocity and acceleration components from one location to another. For now it is simply stated that when the equations of motion are derived for the system of Figure 1.4, the reader will discover that it is much more convenient to use combined coordinates rather than rectangular components only.

Coordinate Choice and Degrees-of-Freedom

A major emphasis of this book is the derivation of equations of motion that allow prediction of the response of a system. These equations will typically be nonlinear differential equations in terms of the coordinates chosen. Choosing the most convenient coordinates for large, complex problems is not an easy task. Sometimes there are multiple choices for appropriate coordinates, and some choices are better than others (meaning easier to work with), but prior to working on the problem, it is difficult to know which choice is better. For smaller systems, as demonstrated primarily in this book, choosing coordinates is not so hard.

We typically need a position coordinate for each direction in which a mass particle can move, which can include an angular position coordinate, and we need a

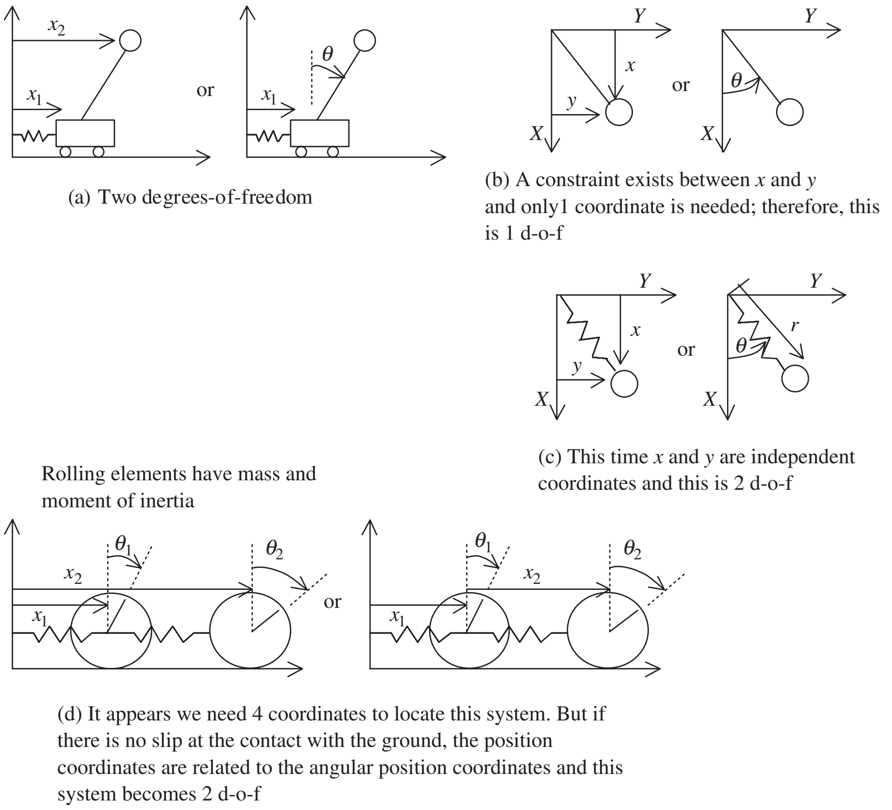


Figure 1.5 Coordinates and degrees of freedom for several systems.

position coordinate for each direction in which the center of gravity of rigid bodies can move. We also need the angular position of each rigid body if that body can rotate. Sometimes there exist constraints among the coordinates, and sometimes we can use the constraint relationships to reduce the number of coordinates needed. With some restrictions, the minimum number of geometric specifications needed to describe any possible position that a system can attain is called its ***degrees-of-freedom***. Thus, a single degrees-of-freedom system requires only one coordinate to describe its motion, a two degree-of-freedom system requires two coordinates, and so on. In Figure 1.5 we show several systems, some possible coordinate choices, and identify the degrees-of-freedom. Equations of motion are derived in terms of these coordinates and their derivatives.

1.3 FREE-BODY DIAGRAMS AND FORCE DIAGRAMS

Newton's second law states that forces in a given direction cause acceleration in that direction. Force is a vector quantity just as position, velocity, and acceleration are vector quantities. There are a few forces that act mysteriously from afar. Electromagnetic and gravity forces are examples of these. All other forces and torques are a direct result of physical contact between system components, and these forces and torques must be identified before equations of motion can be derived. It is convenient to show all forces and torques acting on inertial components of a system using arrows and symbols in a separate diagram called a *force diagram*. With the exception of some prescribed input forces, and gravity forces, we do not know the magnitude or actual direction of any of the system force vectors. So we choose a positive direction and use arrows and symbols to indicate the force components. For plane motion we need to show two mutually perpendicular components for each force. There are some elements that can sustain forces only along their axis. For such elements we need only show one force component.

Consider the system shown in Figure 1.6. It consists of two rigid bodies pinned together at the right end of body 1 and the left end of body 2. Body 1 is pinned to ground at its left end and has an applied torque $\tau_1(t)$. Body 2 is constrained to move only horizontally at its right end, and there is a spring attached at the right end of body 2. In Figure 1.6*b* the bodies are separated and the forces at their connection points are exposed. These force vectors are indicated by their respective components. At the left end of body 1 we have the force components F_{x1} and F_{y1}. At the left end of body 2 we have the force components F_{x2} and F_{y2}. According to Newton's laws, we must show these force components as equal and opposite in direction on the right end of body 1. We do not know if at this instant of time component F_{x2} is acting to the right on body 2 as indicated, but if it is, a force of the same magnitude must be acting to the left on body 1. This is a requirement of Newton's laws which must not be forgotten when making a diagram such as Figure 1.6*b*. At the right end of body 2 the spring force is indicated as F_s and the normal force required to constrain body 2 to move horizontally at its right end is indicated as F_N. This normal force has an equal but opposite effect on the constraining slot. There are equal and opposite forces acting on the ground at the left end of body 1, but these are not shown. The spring force passes

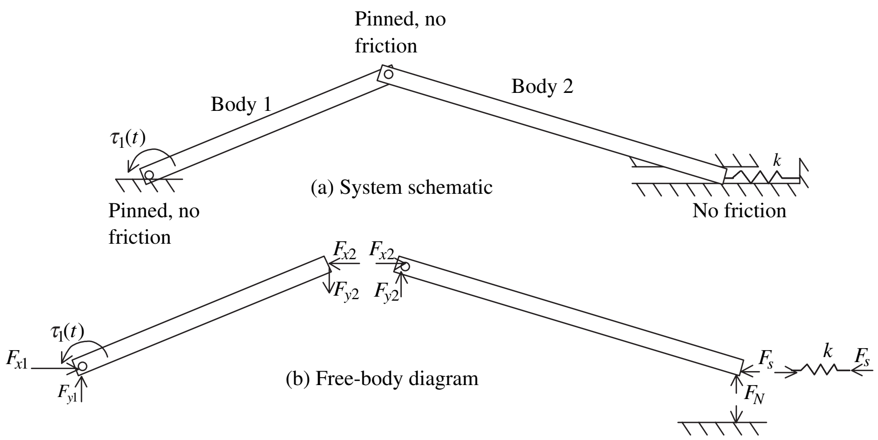


Figure 1.6 Free-body diagram for a two-rigid-body system.

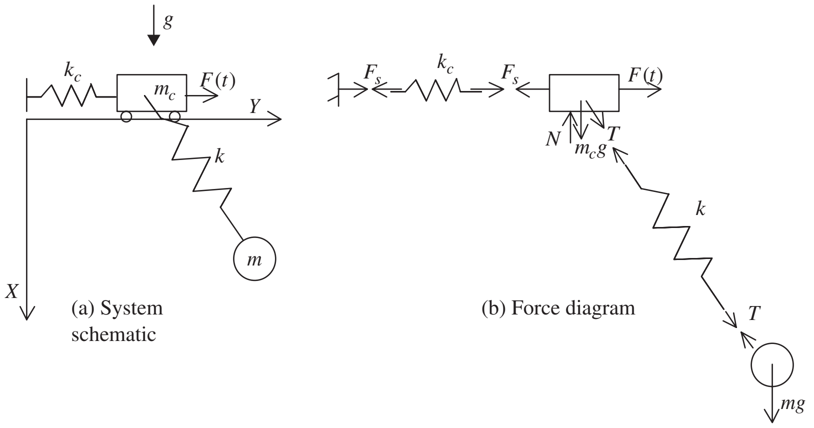


Figure 1.7 Force diagram for the system of Figure 1.4.

through the spring as shown. A diagram in which forces are exposed and arrows and symbols are used to indicate force components is known as a *free-body diagram* or *force diagram*. We will be sketching a lot of these.

Figure 1.7 repeats the system from Figure 1.4. In part *b* the system elements have been separated and the internal forces have been exposed. The forces have been given symbolic names and are pointing in arbitrarily chosen positive directions. It makes absolutely no difference whether or not any of these force components ever actually point in the directions chosen. When a solution is obtained for the dynamic motion of this system and we plot the respective force components, if they are sometimes positive, they are pointing in the directions shown in the figure. If they plot negative, they are pointing in the opposite direction. We ultimately know in what direction the forces are pointing, but we have to wait until we solve the equations of motion. Obviously, we first need to derive the equations of motion.

In Figure 1.7 several different symbols have been used to represent force components. The pendulum and spring can support a force only along the axis; thus, only one force component is shown. It is called T and acts equal and oppositely on the two mass elements connected to the spring. The gravity forces on both masses are shown acting downward. The force between the cart and the ground is called a *normal force* and is given the symbol N in this example. The horizontal spring force is called F_s. Both spring forces have been defined arbitrarily to be positive in tension. With the acceleration diagram of Figure 1.4 and the force diagram of Figure 1.7, the equations of motion are derived straightforwardly, which we do in Chapter 2.

1.4 TRANSFERRING VELOCITY AND ACCELERATION COMPONENTS

The concept of relative motion is introduced in any first course in dynamics [1]. The idea is to be able to express absolute velocity and absolute acceleration when

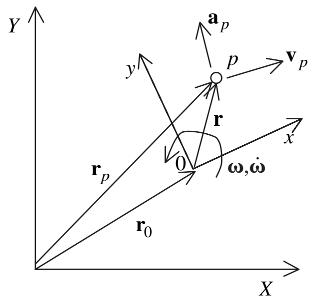


Figure 1.8 Absolute motion of a particle with respect to a moving frame.

coordinate descriptions are stated with respect to an intermediate coordinate frame that accelerates and rotates. Figure 1.8 shows such a situation. The particle p has instantaneous absolute velocity $\mathbf{v}_p$ and absolute acceleration $\mathbf{a}_p$. The particle has absolute position vector $\mathbf{r}_p$, and this vector is the vector sum of the vector that locates the base of the moving frame $\mathbf{r}_0$ and the vector that locates the particle relative to the moving frame $\mathbf{r}$. The frame itself has angular velocity and acceleration $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$, with vector orientation defined by the right hand rule, and the base of the moving frame has absolute velocity $\mathbf{v}_0$ and absolute acceleration $\mathbf{a}_0$. To keep the figure less busy, these vectors are not shown in Figure 1.8. It is straightforward to write

$$
\begin{aligned}
\mathbf{r}_p &= \mathbf{r}_0 + \mathbf{r} \\
\mathbf{v}_p &= \dot{\mathbf{r}}_0 + \dot{\mathbf{r}} = \mathbf{v}_0 + \dot{\mathbf{r}} \\
\mathbf{a}_p &= \ddot{\mathbf{r}}_0 + \ddot{\mathbf{r}} = \mathbf{a}_0 + \ddot{\mathbf{r}}
\end{aligned}
\tag{1.12}
$$

The motion of the base of the frame can be described using either of the coordinate frames already discussed, but the rate of change of the relative vector $\mathbf{r}$ must take into account the rotational motion of the relative frame. It turns out [1] that for any vector $\mathbf{A}$ expressed with respect to a frame that is rotating, its absolute rate of change is given by

$$
\frac{d\mathbf{A}}{dt} = \left.\frac{\partial \mathbf{A}}{\partial t}\right|_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{A} \tag{1.13}
$$

where the first term on the right-hand side symbolizes the change in the vector as observed from the moving frame and accounts for the change in length of the vector. The second term in (1.13) accounts for the change in direction of the vector. We used the second term in (1.13) above in Eq. (1.7) when determining the rate of change of the unit vectors in polar coordinates. Unit vectors do not change their length, so only the second term in Eq. (1.13) is needed.

When the vector in Eq. (1.13) is a position vector like **r**,

$$\dot{\mathbf{r}} = \mathbf{v}_{\mathrm{rel}} + \boldsymbol{\omega} \times \mathbf{r} \tag{1.14}$$

and

$$\ddot{\mathbf{r}} = \dot{\mathbf{v}}_{\mathrm{rel}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}} \quad \text{or} \quad \ddot{\mathbf{r}} = \mathbf{a}_{\mathrm{rel}} + \boldsymbol{\omega} \times \mathbf{v}_{\mathrm{rel}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\mathbf{v}_{\mathrm{rel}} + \boldsymbol{\omega} \times \mathbf{r})$$

where Eq. (1.13) has been used for $\dot{\mathbf{v}}_{\mathrm{rel}}$, to yield

$$\dot{\mathbf{v}}_{\mathrm{rel}} = \mathbf{a}_{\mathrm{rel}} + \boldsymbol{\omega} \times \mathbf{v}_{\mathrm{rel}} \tag{1.15}$$

and Eq. (1.14) has been substituted for $\dot{\mathbf{r}}$. The final result for (1.15) is

$$\ddot{\mathbf{r}} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\mathrm{rel}} + \mathbf{a}_{\mathrm{rel}} \tag{1.16}$$

Combining (1.12), (1.14), and (1.15) yields the final result,

$$\begin{aligned} \mathbf{v}_p &= \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{\mathrm{rel}} \\ \mathbf{a}_p &= \mathbf{a}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\mathrm{rel}} + \mathbf{a}_{\mathrm{rel}} \end{aligned} \tag{1.17}$$

Equation (1.17) expresses symbolically the absolute velocity and acceleration of a point when the position of that point is defined with respect to a frame that is moving and rotating. In (1.16), $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$ is the angular motion of the moving frame and **r** is the vector drawn from the base of the moving frame to point *p*.

It is important to be able to apply Eqs. (1.17) and to transfer velocity and acceleration components from one part of a system to another. Consider the rotating disk and slot of Figure 1.9. The disk is pinned at its center and has angular velocity $\dot{\theta} = \omega$ and angular acceleration $\ddot{\theta} = \dot{\omega}$. The angular motion vectors have the magnitudes $\dot{\theta}$ and $\ddot{\theta}$, respectively, and the direction given by the right-hand rule, in this example pointing out of the page toward the reader. Within the radial slot is a mass particle *m* that can move within the slot. Inertial axes are indicted, as are axes attached to the disk and rotating with the disk. The rotating coordinate frame executes the same motion as the disk.

The velocity and acceleration diagrams are shown in Figure 1.9*b* and *c*. It should be noted that the particle motion could be represented simply using polar coordinates for this system, but this is a good starting example to demonstrate use of the relative motion relationships of Eqs. (1.17). On velocity and acceleration diagrams the components are identified along with the term from Eq. (1.17) from which each comes. For the velocity diagram the $\boldsymbol{\omega} \times \mathbf{r}$ term comes from pointing the fingers of your right hand in the $\boldsymbol{\omega}$-direction (out of the page in this instance) and sweeping your fingers toward the vector **r** (of length *x* in this instance), resulting in your thumb pointing in the component direction indicated in the figure with a magnitude of $x\dot{\theta}$.

The $\mathbf{v}_{\mathrm{rel}}$-component comes from imagining that you are located on the moving frame at the location of the particle and asking if the particle has any additional

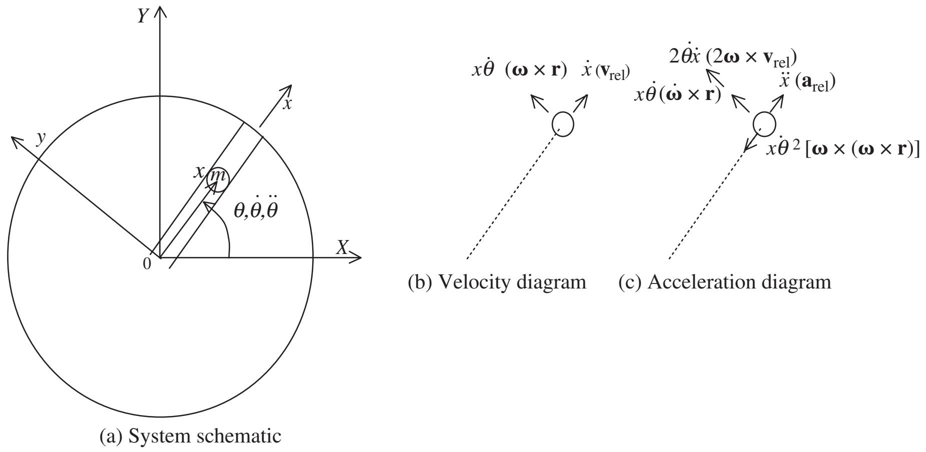


Figure 1.9 Rotating disk with a radial slot.

velocity that you are not experiencing. In this case the particle can move radially within the slot and has an additional velocity component $\dot{x}$ as indicated. Another test for the existence of $\mathbf{v}_{rel}$ is to imagine that the moving frame, as you have defined it, is stopped. Then assess if the particle has additional velocity. In this case the particle could continue moving radially.

The acceleration diagram is shown in Figure 1.9*c*. The cross product for $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ is carried out exactly as for the $\boldsymbol{\omega} \times \mathbf{r}$ term in the velocity diagram. The $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ term is made easier to assess due to the presence of the velocity diagram. Notice that the $\boldsymbol{\omega} \times \mathbf{r}$ term is already exposed on the velocity diagram. To cross $\boldsymbol{\omega}$ into $\boldsymbol{\omega} \times \mathbf{r}$ we take the $\boldsymbol{\omega}$ vector and place it at the base of the $\boldsymbol{\omega} \times \mathbf{r}$ vector and take the cross product using our right hand. Point the fingers of the right hand in the direction of $\boldsymbol{\omega}$ (out of the page for this example) and sweep the fingers toward the $\boldsymbol{\omega} \times \mathbf{r}$ vector. The thumb will be pointing in the resultant direction, in this case toward the center of the disk. Thus, the resulting component has magnitude equal to $|\mathbf{r}| \times |\boldsymbol{\omega}|^2$, or $\times\omega^2$, as shown. Similarly, the cross product for $2\boldsymbol{\omega} \times \mathbf{v}_{rel}$ is obtained by putting the $\boldsymbol{\omega}$ vector at the base of the $\mathbf{v}_{rel}$ vector and carrying out the cross product with the right hand. The resultant vector is oriented as shown in the figure. The relative acceleration $\mathbf{a}_{rel}$ is determined in a manner similar to that described for $\mathbf{v}_{rel}$. If one were either located on the defined moving frame at the location of the particle or if the moving frame were stopped temporarily, we would determine that the particle could still have an acceleration component $\ddot{x}$ directed along the slot. When creating velocity and acceleration diagrams using the relative motion, Eqs. (1.17), it is quite convenient to identify the components as is done in Figure 1.9.

As a second example, consider the rotating pinned disk shown in Figure 1.10. This time a vertical tube is located a distance a from the center and a particle can move within the tube. Inertial and rotating frames are indicated. To use the relative motion, Eqs. (1.17), the $\mathbf{r}$ vector is officially the vector from the base of the moving frame to the particle m. To always take cross products of vectors at right angles, it is convenient to think of the $\mathbf{r}$ vector as the sum of vector components of length a along

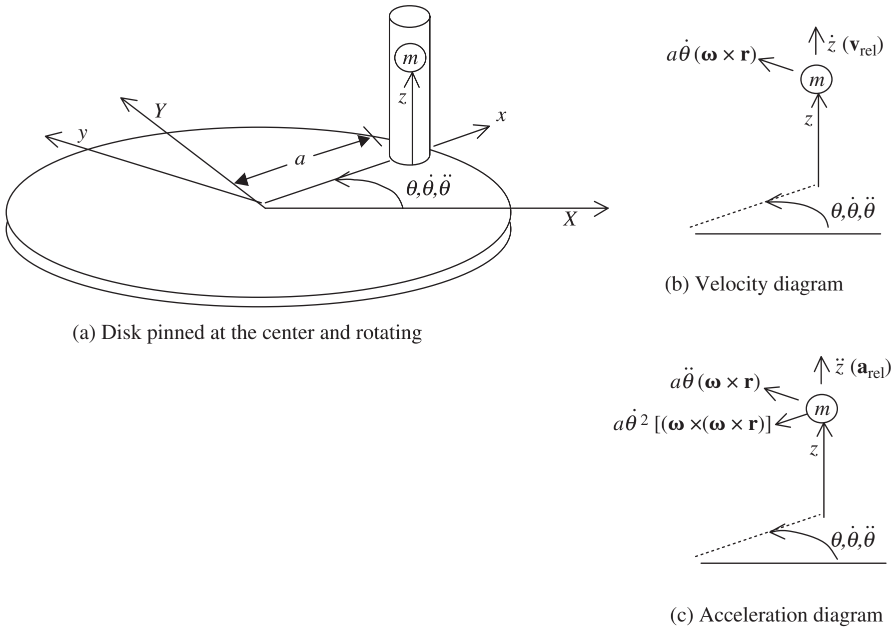


Figure 1.10 Rotating disk with a vertical slot.

the disk plane and of length z vertically upward. When cross products are needed, say $\boldsymbol{\omega} \times \mathbf{r}$, we simply take the cross product of $\boldsymbol{\omega}$ with each component of $\mathbf{r}$. In the velocity diagram, Figure 1.10*b* the $\boldsymbol{\omega} \times \mathbf{r}$ term was generated by first taking the cross product of $\boldsymbol{\omega}$ with the a-component of $\mathbf{r}$, resulting in the component $a\dot{\theta}$ in the direction indicated. The cross product of $\boldsymbol{\omega}$ with the z-component of $\mathbf{r}$ results in zero, since the two vector components are parallel. The relative velocity component $\dot{z}$ comes from the thought experiment described above. If the moving frame were stopped temporarily, the particle could still have a velocity component along the tube.

The acceleration diagram, Figure 1.10 *c*, is generated term by term according to Eq. (1.17). The $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ term is just like the $\boldsymbol{\omega} \times \mathbf{r}$ term, but in terms of angular acceleration instead of angular velocity. The $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ term is generated conveniently, due to the presence of the $\boldsymbol{\omega} \times \mathbf{r}$ term in the velocity diagram. Placing the $\boldsymbol{\omega}$ vector at the base of $\boldsymbol{\omega} \times \mathbf{r}$ and carrying out the cross product with the right hand results in the appropriate component in the acceleration diagram. The $2\boldsymbol{\omega} \times \mathbf{v}_{rel}$ term is zero because the $\boldsymbol{\omega}$ vector is parallel to the $\mathbf{v}_{rel}$ vector. Finally, there is the $\mathbf{a}_{rel}$-component, which comes from stopping the moving frame and assessing any additional acceleration.

As a third example, consider the pinned rotating disk in Figure 1.11, with a slot creating a chord out a distance a from the center. In the slot is a mass particle located with the relative position vector of length y. The inertial frame and a rotating frame are shown in the figure. The position vector is officially the vector drawn from the base of the moving frame out to the mass particle. It is more convenient to consider the position vector as composed of vector components of length a oriented radially and of length y oriented along the slot. The velocity diagram is shown in Figure 1.11*b*.

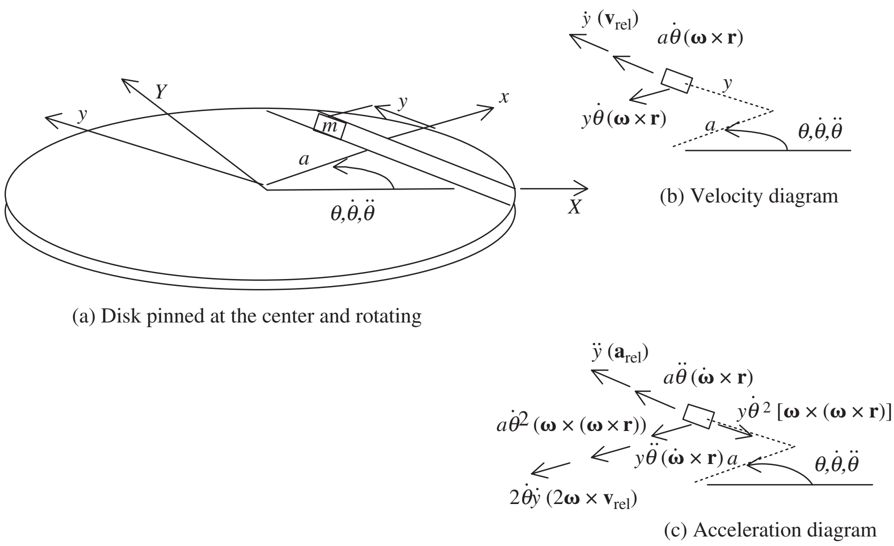


Figure 1.11 Rotating circular disk with a chord slot.

The $\boldsymbol{\omega} \times \mathbf{r}$ term has two components. This comes from first placing the $\boldsymbol{\omega}$ vector at the base of the a component of the position vector and carrying out the cross product with the right hand. This produces $a\dot{\theta}$, as shown. We next place the $\boldsymbol{\omega}$ vector at the base of the y-component of the position vector and carry out the cross product, producing the $y\dot{\theta}$-component. The $\mathbf{v}_{\text{rel}}$ vector comes from the familiar assessment determining any additional velocity of the particle if the moving frame were stopped.

There are quite a few components in the acceleration diagram shown in Figure 1.11c, but each was generated term by term from Eqs. (1.17). The base of the moving frame has no acceleration; thus, $\mathbf{a}_0 = 0$. The $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ term is identical to $\boldsymbol{\omega} \times \mathbf{r}$ but using angular acceleration rather than angular velocity. Then $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ vector benefits from having the $\boldsymbol{\omega} \times \mathbf{r}$ vector exposed on the velocity diagram. The $\boldsymbol{\omega}$ vector is placed at the base of each component of $\boldsymbol{\omega} \times \mathbf{r}$ and the cross product is carried out using the right hand. This produces the two terms shown for components of $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. Having $\mathbf{v}_{\text{rel}}$ exposed on the velocity diagram facilitates generation of $2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}}$. We simply place the base of the $\boldsymbol{\omega}$ vector at the base of the $\mathbf{v}_{\text{rel}}$ vector and carry out the cross product.

1.5 TRANSFERRING MOTION COMPONENTS OF RIGID BODIES AND GENERATING KINEMATIC CONSTRAINTS

Equations (1.17) are very useful when dealing with rigid bodies. It is very typical to need the velocity and acceleration of the center of mass [or center of gravity (c. g.)] of rigid bodies, and it is also very typical to need the velocity and acceleration at attachment points of a rigid body to other rigid bodies or system components. We will find that when rigid bodies share motion at an attachment point, a kinematic constraint is generated. It turns out that using Eqs. (1.17) is actually easier when dealing with rigid bodies than in the particle motion examples done above. The particles that comprise a rigid body by definition do not execute any relative motion. Points on a rigid body located some distance apart remain at that distance regardless of the forces acting. Thus, in Eqs. (1.17) the terms involving relative velocity and relative acceleration are all zero when working with rigid bodies. For rigid bodies,

$$\begin{aligned} \mathbf{v}_p &= \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} \\ \mathbf{a}_p &= \mathbf{a}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \end{aligned} \tag{1.18}$$

We say that the velocity of some point p on a rigid body is equal to the velocity some other point 0 on the rigid body plus $\boldsymbol{\omega}$ of the rigid body crossed into the vector drawn from point 0 to point p. Similarly, we can relate the acceleration of some point on a rigid body to the acceleration of any other point on the rigid body. This is demonstrated through example.

The rigid bodies from Figure 1.6 are shown again in Figure 1.12. The velocity components of the center of mass of each body are shown in inertial directions. For this example, body 1 uses inertial coordinates with the X-axis pointing to the right and positive rotation counterclockwise, and body 2 uses inertial coordinates with the X-axis pointing to the left and positive rotation clockwise. Using the right-hand rule for rotational vectors, the positive direction for the $\boldsymbol{\omega}$ vector of body 1 is pointing out of the page and the positive direction for the $\boldsymbol{\omega}$ vector of body 2 is pointing into the page. The first of Eqs. (1.18) is used to transfer the velocity components to the ends of the two bodies. This is accomplished by treating the center of gravity of the body as point 0 and the end to which velocity is transferred as point p. For example, to transfer

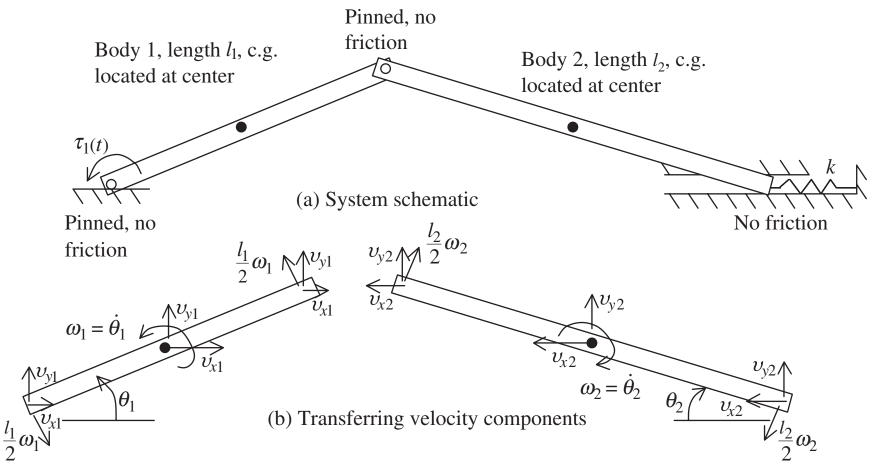


Figure 1.12 Rigid bodies with kinematic constraints.

components from the center of gravity of body 1 to its right end, we first transfer the c.g. velocity components to accomplish the first requirement of Eqs. (1.18) followed by using our right hands to perform the cross-product requirement. For body 1 we point the fingers of the right hand out of the page (positive $\boldsymbol{\omega}$ direction) and sweep the fingers toward the **r** vector (length $l_{1/2}$ and from the center of gravity to the right end of body 1). The thumb is pointing as indicated in Figure 1.12. We similarly transfer velocity components to the left end of body 1, except that this time the **r** vector is directed from the center of gravity toward the left end of body 1. The cross-product term points in the direction shown. Identical steps are carried out for body 2. The only difference is that the positive angular velocity vector points into the page. The cross-product terms are carried out the same way as for body 1 using the right-hand rule.

Kinematic Constraints

The bodies in Figure 1.11 do not move independently, and one of the reasons for transferring velocity components to the attachment points of rigid bodies is to allow specification of kinematic constraints. At the left end of body 1, the body is pinned to inertial ground and the absolute velocity in the X and Y directions must be zero at the left end. From Figure 1.12 we can write down the horizontal and vertical components of velocity as

$$\begin{aligned} v_{H1_L} &= v_{x1} + \frac{l_1}{2}\,\omega_1 \sin\theta_1 = 0 \\ v_{V1_L} &= v_{y1} - \frac{l_1}{2}\,\omega_1 \cos\theta_1 = 0 \end{aligned} \tag{1.19}$$

These components must each equal zero to enforce the kinematic constraint at the left end of body 1. For the right end of body 2 there can be no vertical velocity, as motion is constrained to be horizontal only. Thus, we have the kinematic constraint

$$v_{y2} - \frac{l_2}{2}\omega_2 \cos\theta_2 = 0 \tag{1.20}$$

For a perfect pin joint at the center of the two bodies it is simplest to write down the horizontal and vertical components at the ends of each body and then enforce that the respective components must be equal. Thus, in the horizontal direction,

$$v_{x2} - \frac{l_2}{2}\omega_2 \sin\theta_2 = -\left(v_{x1} - \frac{l_1}{2}\omega_1 \sin\theta_1\right) \tag{1.21}$$

and in the vertical direction,

$$v_{y2} + \frac{l_2}{2}\omega_2 \cos\theta_2 = v_{y1} + \frac{l_1}{2}\omega_1 \cos\theta_1 \tag{1.22}$$

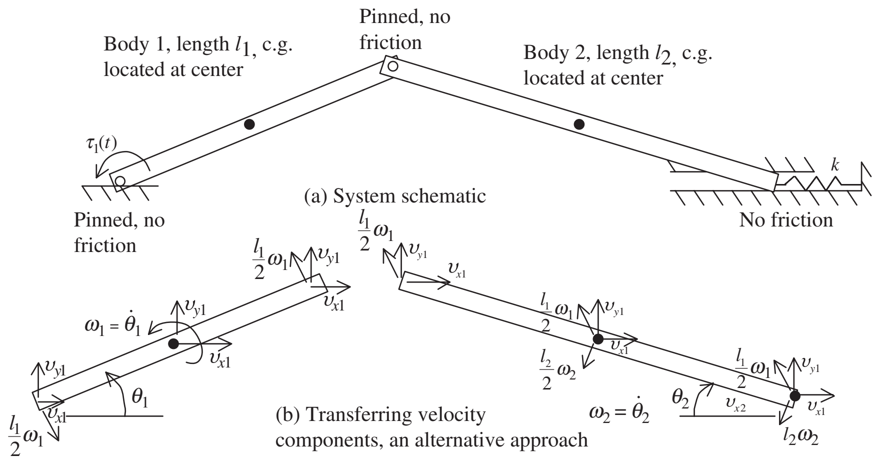


Figure 1.13 Rigid bodies with an alternative approach to kinematic constraints.

These constraint equations can be used to eliminate some variables from the formulation of system equations while retaining others. This is demonstrated in subsequent chapters.

An alternative way to enforce kinematic constraints for the system of Figure 1.12 is shown in Figure 1.13. The velocity components of the center of gravity of body 1 are transferred to the left and right ends exactly is in Figure 1.12. The velocity components at the right end of body 1 are shared by the left end of body 2. Then the velocity components are transferred to the center of gravity of body 2 and to the right end of body 2 using the first of Eqs. (1.18). We never introduce independent specification of the velocity of body 2. The kinematic constraint at the pin joint of bodies 1 and 2 is enforced automatically, and the constraint of zero vertical velocity at the right end of body 2 becomes

$$v_{y1} + \frac{l_1}{2}\omega_1 \cos\theta_1 - l_2\omega_2 \cos\theta_2 = 0 \tag{1.23}$$

As a final example of transferring velocity components and enforcing kinematic constraints, consider the slider–crank mechanism of Figure 1.14. This mechanism comprises the fundamental geometry of virtually every internal combustion engine. The astute reader will notice that the slider–crank device is really identical to the system of Figure 1.12. It is shown here to introduce the concept of body-fixed coordinates. In Figure 1.14 the absolute velocity vector of the center of gravity of the connecting rod is shown as two mutually perpendicular components, one aligned along the rod and the other oriented perpendicular to the rod. The axes shown in Figure 1.14*a* are called *body fixed* in that they are attached to the body and execute all the motions of the body. The velocity components referred to this coordinate frame, called *body-fixed velocity components*, differ from inertial direction components in that they change direction

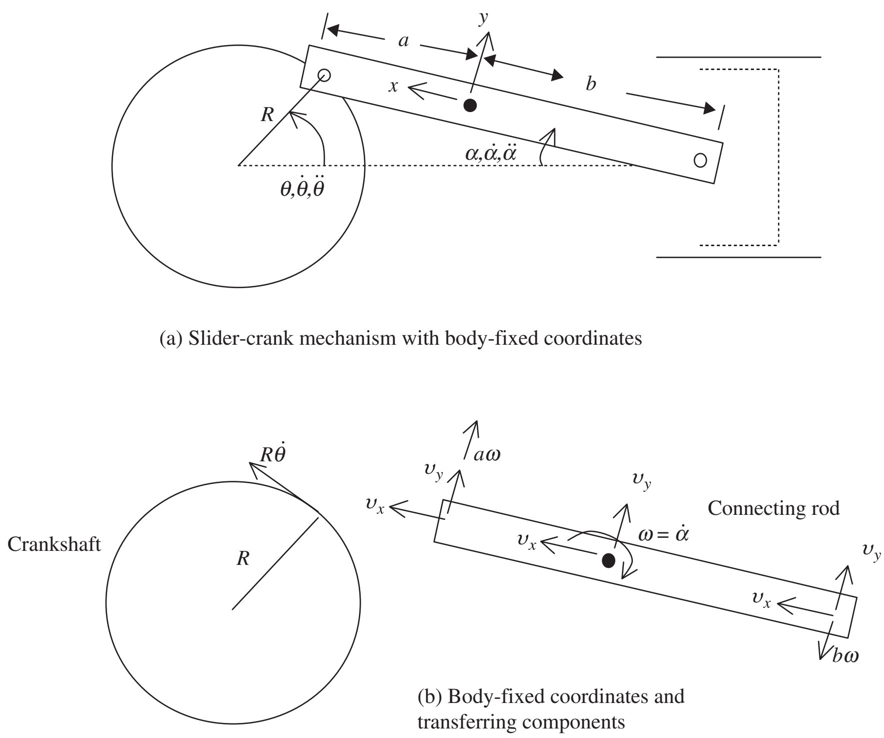


Figure 1.14 Slider–crank mechanism.

as the body moves. More will be said of body-fixed coordinates in later chapters. For the problem here we simply want to transfer the components to the ends of the rod in order to generate kinematic constraints. The angular motion of the connecting rod is indicated by the angle α and its derivatives. The angular velocity vector is pointing into the page by the right-hand rule.

Applying Eqs. (1.18) allows us to transfer the body-fixed velocity components at the center of gravity to the body-fixed components at the left and right ends of the rod. The crankshaft rotational motion is indicated by the angle θ and its derivatives. For the crankshaft we use polar coordinates and indicate the velocity at the connection with the rod as shown. At the right end of the rod only horizontal motion is permitted, so the constraint that the vertical velocity at the right end equals zero becomes

$$v_x \sin\alpha + (v_y - b\omega)\cos\alpha = 0 \tag{1.24}$$

At the left end of the rod, one way to enforce the kinematic constraint is to equate the horizontal and vertical velocity components from the crankshaft and connecting rod; thus,

$$R\dot{\theta}\ \sin\theta = v_x \cos\alpha - (v_y + a\omega)\sin\alpha$$

for the horizontal direction and

$$R\dot{\theta}\cos\theta = v_x \sin\alpha + (v_y + a\omega)\cos\alpha \tag{1.25}$$

for the vertical direction. Another way to enforce the constraint at the left end of the rod is to equate components in the direction tangent to the crankshaft and set to zero the component perpendicular to the crankshaft. Thus,

$$\begin{aligned} v_x \sin(\theta+\alpha) + (v_y + a\omega)\cos(\theta+\alpha) &= R\dot{\theta} \\ v_x \cos(\theta+\alpha) - (v_y + a\omega)\sin(\theta+\alpha) &= 0 \end{aligned} \tag{1.26}$$

It is left as an exercise to show that Eqs. (1.25) and (1.26) are the same.

1.6 REVIEW OF CENTER OF MASS, LINEAR MOMENTUM, AND ANGULAR MOMENTUM FOR RIGID BODIES

In elementary courses in dynamics using texts such as ref. [1] or very similar texts, the concepts of center of mass, linear momentum, and angular momentum are presented. For plane motion the concept is as shown in Figure 1.15. The rigid body is composed of many small particles glued together such that no single particle executes relative motion with respect to any other; each particle obeys Newton's laws. One such particle is shown in Figure 1.15. Axes are attached at an arbitrary point 0, and a special point called the *center of mass* or center of gravity of the rigid body is located by the position vector $\mathbf{r}_{c/0}$. The *i*th particle of mass dm_i, which is part of the rigid body, is located by the position vector $\mathbf{r}_i$ and has the absolute velocity $\mathbf{v}_i$. Acting on the particle is the internal force vector $\mathbf{f}_i$, which is a result of contact with other particles surrounding this one, and the external force $\mathbf{F}_i$, which is due to contact at that point with some external element not shown.

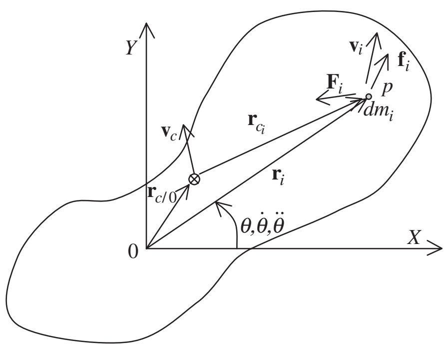


Figure 1.15 Rigid body that demonstrates angular momentum.

The linear momentum of a particle is a vector quantity equal to the product of mass and velocity. Thus, for the mass particle that is part of the rigid body of Figure 1.15, the momentum contribution to the to the total momentum of the body is

$$d\mathbf{p}_i = dm_i\, \mathbf{v}_i \tag{1.27}$$

The total momentum of the body comes from summing all the momentum contributions from all the particles:

$$\mathbf{p} = \sum \mathbf{v}_i\, dm_i = \sum (\mathbf{v}_c + \boldsymbol{\omega} \times \mathbf{r}_{c_i})\, dm_i \tag{1.28}$$

where the transfer relationship has been used to express the velocity of the *i*th particle in terms of the velocity of the center of mass plus the cross-product term, which accounts for the rotation of the position vector $\mathbf{r}_{c_i}$. As the number of particles approaches infinity, Eq. (1.28) can be expressed as

$$\mathbf{p} = \int (\mathbf{v}_c + \boldsymbol{\omega} \times \mathbf{r}_c)\, dm = m\mathbf{v}_c + \boldsymbol{\omega} \times \int \mathbf{r}_c\, dm \tag{1.29}$$

where integration takes place over the continuous mass distribution. Since $\mathbf{v}_c$ is independent of any *dm* particle, it can be taken outside the integral to produce the first term on the right-hand side of (1.29). Also, $\boldsymbol{\omega}$ is the angular velocity of the frame and is independent of any *dm* particle or its location; thus, $\boldsymbol{\omega}$ can be taken outside the integral, resulting in the second term of (1.29).

The final integral on the right side of (1.29) is zero, due to the definition of the center of mass. Consider the axes through zero in Fig. 1.15. The center of mass is defined as

$$m\mathbf{r}_{c/0} = \int \mathbf{r}\, dm \tag{1.30}$$

To actually find the center of mass of a rigid body or collection of mass particles, we would cast Eq. (1.30) into component form and carry out the operations. In this way we could locate the *x*, *y*, *z* location of the center of mass relative to the original axes. A nice physical way to think about the center of gravity is to imagine pinning the body to a wall with a pin through the center of gravity. You could rotate the body to any orientation, release it, and it would remain in that location and not move under the influence of gravity.

For our purposes here we need to recognize that if we are working with axes attached at the center of gravity as in Eq. (1.29), $\mathbf{r}_{c/0}$ is zero and the integral on the right side of (1.29) is zero when measuring from the center of gravity [1]. Thus, the linear momentum of a rigid body is equal to the mass of the body times the velocity of the center of mass,

$$\mathbf{p}_c = m\mathbf{v}_c \tag{1.31}$$

where the subscript c has been appended to the momentum to enforce that it is the momentum of the center of mass of the rigid body that characterizes the total momentum.

Angular momentum is defined as the moment of the momentum vector with respect to some axes. The moment of a force is probably familiar to the reader. In Figure 1.15, if we desired the moment of the force $\mathbf{F}_i$ about the axes through zero, we would need to resolve the force into components perpendicular to $\mathbf{r}_i$ and parallel to $\mathbf{r}_i$. The moment of the component parallel to $\mathbf{r}_i$ would be zero, the moment of the component perpendicular to $\mathbf{r}_i$ would be the magnitude of the perpendicular component times the length of $\mathbf{r}_i$, and the direction of the moment vector would be given by the right-hand rule. The vector notation that accounts for the total moment $\mathbf{M}_i$ of the force vector $\mathbf{F}_i$ is given by

$$\mathbf{M}_i = \mathbf{r}_i \times \mathbf{F}_i \tag{1.32}$$

This relationship is easy to demonstrate from Figure 1.16. The force vector $\mathbf{F}$ is resolved into two components and the position vector $\mathbf{r}$ is resolved into two components. If we carry out the operation $\mathbf{r} \times \mathbf{F}$, we need to cross each component of $\mathbf{r}$ into each component of $\mathbf{F}$. We first cross the x-component of $\mathbf{r}$ into the F_x-component and then into the F_y-component of $\mathbf{F}$. Since x and F_x are parallel, their cross product is zero. As shown in Figure 1.16, by placing the x-component at the base of the F_y component and using our right hands to take the cross product, we align out fingers along the x-component and sweep them toward the F_y-component, with the result indicated in Figure 1.16*b*. We do a similar operation to cross the y-component of position into the F_x-component of force, with the result indicated in the figure. The final result is

$$\mathbf{r} \times \mathbf{F} = xF_y - yF_x \quad \text{with a vector direction out of the page} \tag{1.33}$$

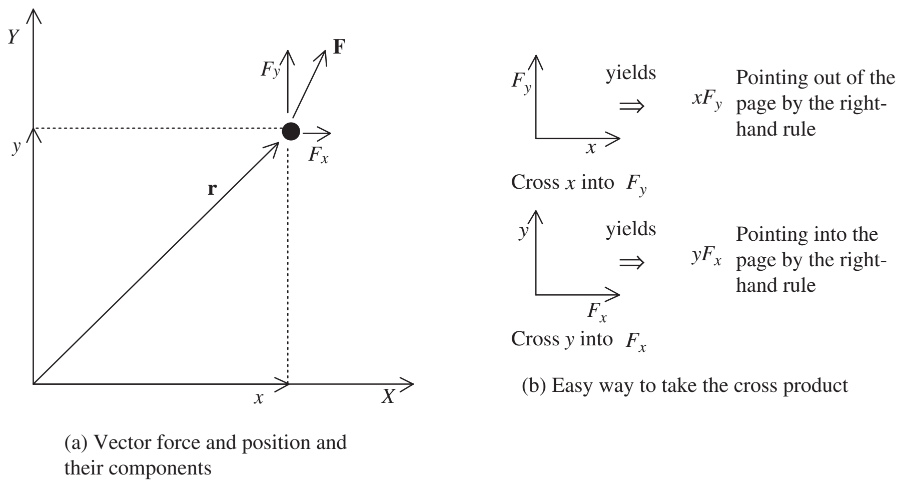


Figure 1.16 Demonstration of the vector moment equation.

We see that carrying out the operations indicated by Eq. (1.32) results in properly taking the moment of a force with respect to some axes.

Let's return to angular momentum, which is the moment of the momentum vector. With respect to the axes through zero in Figure 1.15, the angular momentum contribution of the ith particle is

$$d\mathbf{h}_i = \mathbf{r}_i \times \mathbf{v}_i \, dm_i = \mathbf{r}_i \times (\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_i) \, dm_i \tag{1.34}$$

using the transfer of velocity relationship. The total angular momentum of all the particles comes from summing the contribution from each:

$$\mathbf{H}_0 = \sum \mathbf{r}_i \, dm_i \times \mathbf{v}_0 + \sum \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \, dm_i \tag{1.35}$$

As the number of particles becomes infinite, the summations become integrations over the continuous mass distribution, and the angular momentum of a rigid body with respect to axes through the general point zero is

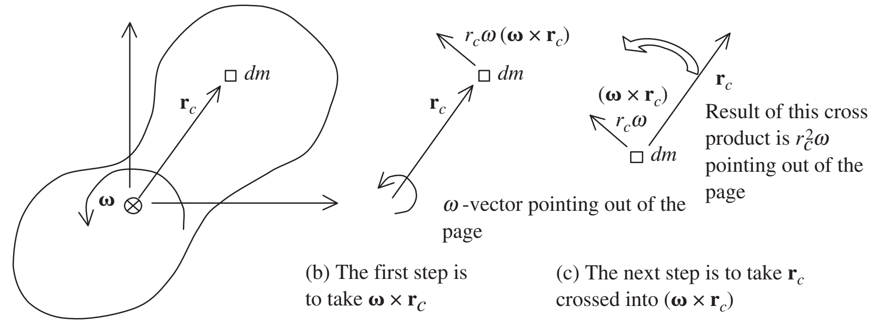
$$\mathbf{H}_0 = \int \mathbf{r} \, dm \times \mathbf{v}_0 + \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \, dm \tag{1.36}$$

We now make a simplification that is used throughout the book. The first term on the right side of (1.36) adds unnecessary complexity. If the rigid body happens to have a fixed point (i.e., a point that is literally pinned to inertial ground) and the base of the reference frame is placed there, then for a fixed point $\mathbf{v}_0 = 0$ and the first term of (1.36) is zero. If the base of the reference frame happens to be on the center of gravity, then $\int \mathbf{r} \, dm = 0$. This comes from the definition of the center of mass as described in Eq. (1.30). Thus, if the base of the reference frame is placed on a fixed point or the center of gravity, the angular momentum of the body is given by

$$\begin{aligned} \mathbf{H}_0 &= \int \mathbf{r}_0 \times (\boldsymbol{\omega} \times \mathbf{r}_0) \, dm \\ \mathbf{H}_c &= \int \mathbf{r}_c \times (\boldsymbol{\omega} \times \mathbf{r}_c) \, dm \end{aligned} \tag{1.37}$$

where sub 0 and sub c are used to indicate measurements from a fixed point or the center of gravity, respectively. For all the applications discussed in this book we will always work with axes attached at a fixed point if one exists or at the center of gravity. If you want to remember only one thing, it would be always to attach axes to rigid bodies at the center of gravity. Then the angular momentum is always the second of Eqs. (1.37). We next have to interpret the operational meaning of Eqs. (1.37).

These expressions for angular momentum are applicable in three dimensions, and such use is described in later chapters. It is much easier to get a physical understanding of Eqs. (1.37) by first considering plane motion. Angular momentum is a vector quantity and the vector direction is given by the cross-product terms in the integrand of (1.37) using the right-hand rule. Figure 1.17 shows a planar rigid body with axes



(a) Rigid body with c.g. attached axes

Figure 1.17 Interpretation of angular momentum.

attached at the center of gravity. The angular velocity vector is defined positive counterclockwise and is pointing out of the page by the right-hand rule. In part *b* of the figure, the cross-product $\boldsymbol{\omega} \times \mathbf{r}_c$ has been done using the right hand. It is pointing orthogonally to the position vector and has length $r_c\omega$. If the vector $\mathbf{r}_c$ is placed at the base of $\boldsymbol{\omega} \times \mathbf{r}_c$ and the required cross product is taken, the result will be a vector pointing out of the page with magnitude $r_c^2\omega$. Thus, for plane motion, Eq. (1.37) becomes

$$\mathbf{H}_c = \omega \int r_c^2\, dm = I_c\omega \quad \text{pointing out of the page} \tag{1.38}$$

The quantity $I_c = \int r_c^2\, dm$, called the *moment of inertia*, is a property of the body. Because we are working with axes attached at the center of gravity, I_c is called the *centroidal moment of inertia*. For simple geometric shapes such as disks and rods, the moment of inertia can be calculated directly from its integral definition. For complex shapes such as engine case castings, the moment of inertia might be determined by testing or by finite-element analyses. The thing that changes in Eq. (1.37) if a fixed point is used rather than the center of gravity is the moment of inertia, since measurements are made from a different location for the two axes choices. The reader may recall the parallel axis theorem, which allows transfer of inertial properties from a set of axes through the center of gravity to another set of parallel axes at some other location on a rigid body. We discuss this further in another section.

1.7 NEWTON'S LAW APPLIED TO RIGID BODIES

Returning to Figure 1.15, Newton's second law applied to the particle shown has the result

$$\mathbf{F}_i + \mathbf{f}_i = dm_i \frac{d\mathbf{v}_i}{dt} = \frac{d}{dt}\mathbf{p}_i \tag{1.39}$$

where $\mathbf{p}_i$ is the momentum of the ith particle. Summing over all the particles yields

$$\sum \mathbf{F}_i + \sum \mathbf{f}_i = \frac{d}{dt} \sum \mathbf{v}_i \, dm_i = \frac{d}{dt} \sum \mathbf{p}_i \tag{1.39a}$$

The first summation on the left side of (1.39a) is the sum of all the external forces acting on the collection of particles that make up the rigid body. The second summation on the left is zero since the internal forces are a result of physical contact among particles, and each internal force will have an equal but opposite counterpart. As the summation is carried out, each positive internal force will be canceled by its negative counterpart, resulting in zero net force. The summation on the right side of (1.39a) is the momentum of the rigid body as developed above, resulting in Eq. (1.31). Thus, the final result becomes

$$\mathbf{F} = m \frac{d\mathbf{v}_c}{dt} = \frac{d\mathbf{p}_c}{dt} \tag{1.39b}$$

where $\mathbf{F}$ is the vector sum of all the external forces acting on the body, and these forces cause an acceleration of the center of mass or center of gravity of the rigid body.

Taking the moment of the forces about the center of gravity from Figure 1.15 results in

$$\mathbf{r}_{c_i} \times \mathbf{F}_i + \mathbf{r}_{c_i} \times \mathbf{f}_i = \mathbf{r}_{c_i} \times \frac{d\mathbf{v}_i}{dt} \, dm_i \tag{1.40}$$

Using the velocity transfer relationship, we get

$$\mathbf{r}_{c_i} \times \mathbf{F}_i + \mathbf{r}_{c_i} \times \mathbf{f}_i = \mathbf{r}_{c_i} \times \frac{d(\mathbf{v}_c + \boldsymbol{\omega} \times \mathbf{r}_{c_i})}{dt} \, dm_i \tag{1.41}$$

We next sum over all the particles, yielding

$$\sum \mathbf{r}_{c_i} \times \mathbf{F}_i + \sum \mathbf{r}_{c_i} \times \mathbf{f}_i = \sum \mathbf{r}_{c_i} \, dm_i \times \frac{d\mathbf{v}_c}{dt} + \frac{d}{dt} \sum \mathbf{r}_{c_i} \times (\boldsymbol{\omega} \times \mathbf{r}_{c_i}) \, dm_i \tag{1.42}$$

The first term on the left side is the moment of all the external forces about the center of gravity. The second term on the left is zero because these are the moments of the internal forces. This means that for every positive contribution there will be an equal and opposite contribution. In other words, if particle a pushes on particle b with an internal force, and this force has a moment about the center of gravity, then particle b is pushing on particle a with equal but opposite internal force and the moment contribution is canceled. The first term on the right side of (1.42) would become an integral over the mass distribution as the number of particles goes toward infinity. The first summation or integral will be zero from the definition of the center of mass since we are measuring from the center of gravity. The second term on the right side of (1.42) as it becomes an integral is the time rate of change of the angular momentum as it was defined in Eq. (1.37). Thus, we have the rotational equation of motion,

$$\mathbf{M}_c = \frac{d}{dt}\mathbf{H}_c \tag{1.43}$$

where $\mathbf{H}_c$ comes from Eq. (1.38).

In Chapter 2 we make use of velocity, acceleration, and force diagrams to derive equations of motion which, if solved, would predict the motion–time history of the physical system represented.

REFERENCE

[1] F. P. Beer, E. R. Johnston, Jr., and W. E. Clausen, *Vector Mechanics for Engineers: Dynamics*, 8th ed., McGraw-Hill, New York, 2004.

PROBLEMS

1.1 For the systems shown in Figure P1.1, state the number of degrees-of-freedom and choose and name coordinates that could be used to describe the motion of the systems.

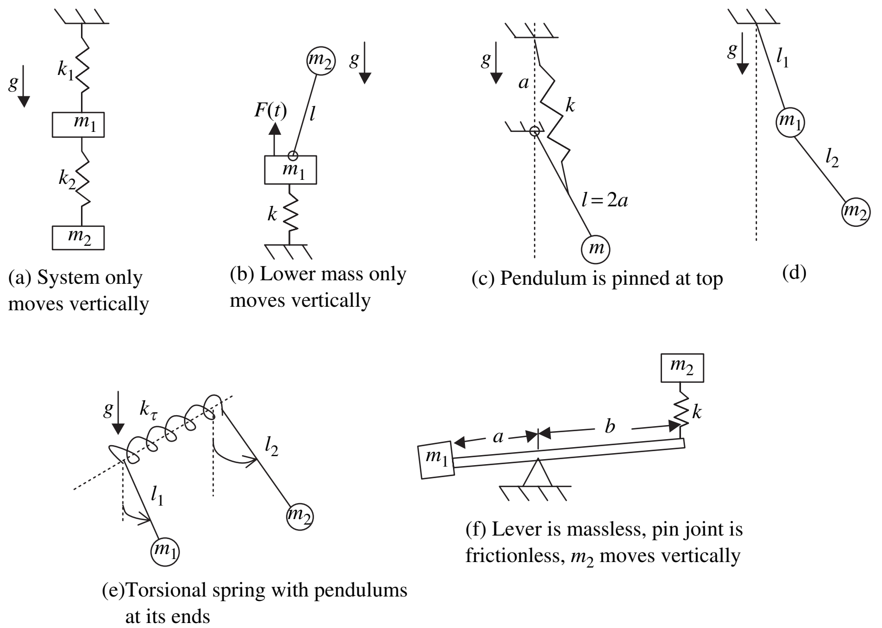


Figure P1.1

1.2 The system shown in Figure P1.2 is sometimes called the *half-car model*. It can be thought of as the side view of a car, with the right side being the front and the left side being the rear. It can also be thought of as the front view of a car where the right side of the figure is the left/front of the car and the left side of the figure is the right/front of the car. The rigid body has mass and c.g. moment of inertia. The suspension elements have a spring and a damper at each end. The unsprung mass is represented by the point masses m_r and m_l, and the tire stiffnesses are represented by the springs k_t. The ground inputs are indicated at the bottom of the right and left sides of the figure. Identify the number of degrees of freedom and show several different choices for coordinates that would fully describe the motion of the system. Make it clear from which reference point each coordinate is measured.

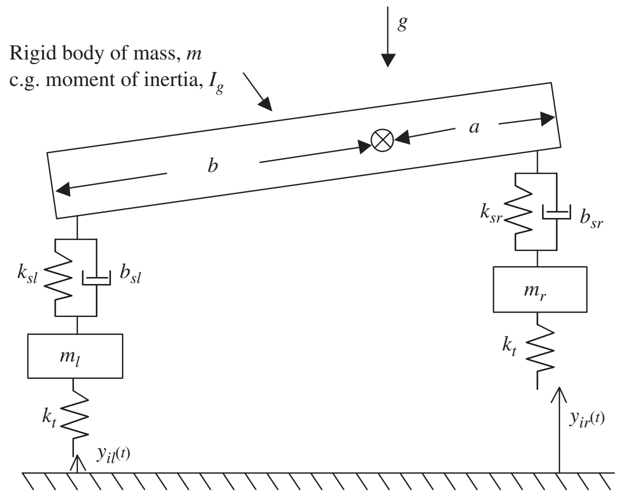


Figure P1.2

1.3 For the systems of Problem 1.1, use your chosen coordinates and show velocity, acceleration, and force diagrams. An example is shown in Figure P1.3 from Problem 1.1 (b). The coordinate y is measured from equilibrium in a gravity field such that the spring force, positive in tension, is $F_s = k(y - y_{\text{eq}})$ where $y_{\text{eq}} = [(m_1 + m_2)g]/k$. Note that you must make use of the velocity transfer formula from Eq. (1.17).

1.4 For the half-car model of Problem 1.2, using the coordinates indicated in the Figure P1.4, show velocity, acceleration, and force diagrams. You can define spring free lengths if you find it necessary. The coordinates are defined such that when all are zero, the system is at equilibrium under the influence of gravity.

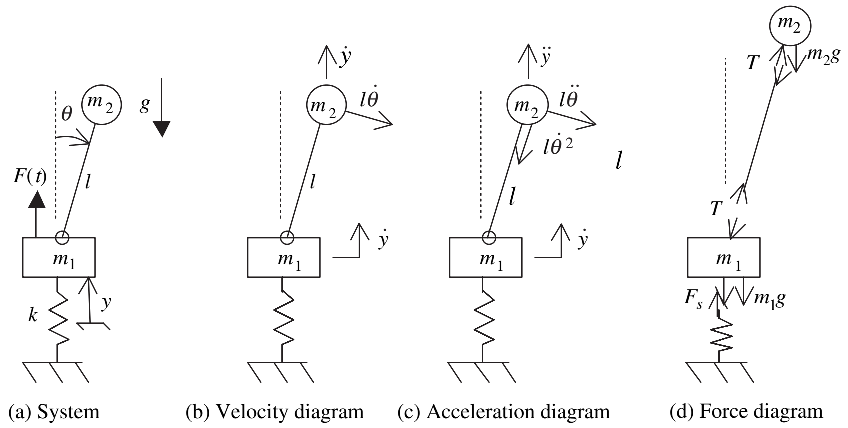


Figure P1.3

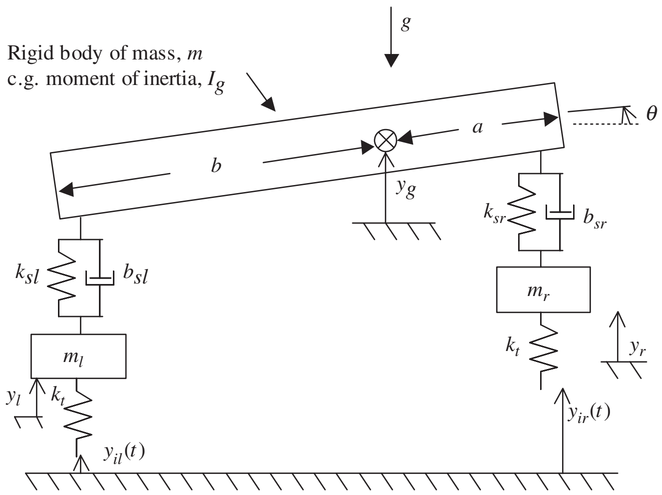


Figure P1.4

1.5 Figure P1.5 shows the front view of what might be an engine block in a heavy truck. The engine is represented as a rigid body and is suspended by soft mounts at the right and left at the bottom. What would probably be a single self-contained isolator in application is shown here as separate horizontal and vertical stiffness elements. The horizontal springs are constrained to remain horizontal and the vertical springs are constrained to remain vertical. The coordinates that describe the possible motions of the engine block are the inertial location of the center of gravity x_g, y_g, and the angular rotation θ.

(a) Show a velocity diagram for the center of gravity and transfer these components to the bottom right and left corners so that the spring displacements

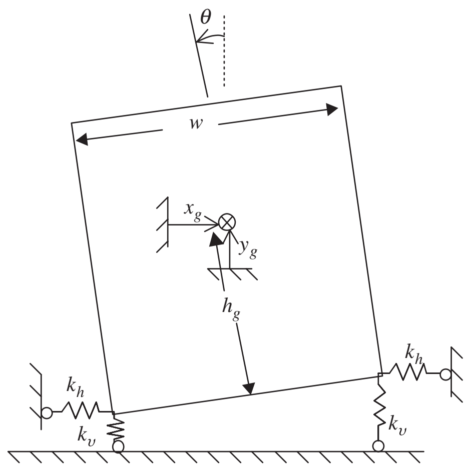


Figure P1.5

could ultimately be determined. You will be making use of the velocity transfer Eq. (1.18), $\mathbf{v}_p = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}$.

(b) Show acceleration and force diagrams. You will need to assign appropriate names for the spring forces.

(c) Try to express the spring forces in terms of the coordinates.

1.6 The system shown in the Figure P1.6 is a cylinder that rolls without slip on a horizontal ground surface and has an offset center of mass. If this device were given some initial velocity to the left, it would roll in a not-very-smooth motion. We ultimately desire the equations of motion for this system, but first we need the velocity, acceleration, and force diagrams. The constraint of rolling without slip makes this a single degree-of-freedom system, and either the location of

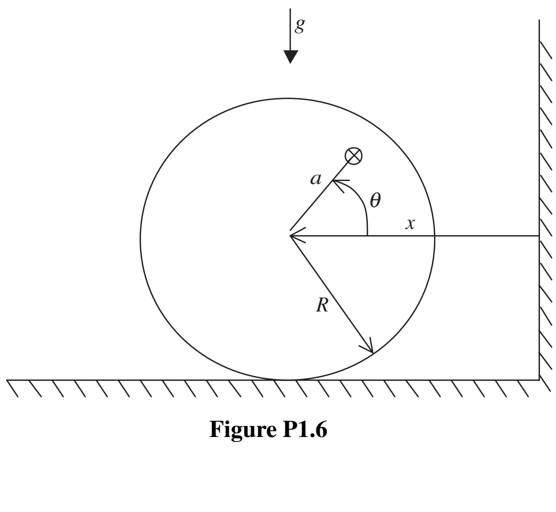


Figure P1.6

the center of the cylinder x or the angular position of the center of gravity, θ can be used as the coordinate for the problem. Assuming that $\theta = 0$ when $x = 0$, the constraint is $x = R\theta$.

The mass of the cylinder is m and the c.g. moment of inertia is I_g. Construct velocity, acceleration, and force diagrams for this rigid body in terms of coordinates x and in terms of the coordinate θ.

1.7 A cylinder of mass m_c and c.g. moment of inertia I_c rolls without slip on a horizontal surface shown in Figure P1.7 . On the inner surface of the cylinder, a mass particle m can slide without friction. This is a two degree-of-freedom system and two coordinates are needed to describe the system motion. Since the cylinder rolls without slip, either X or θ can be used to locate the cylinder. The angle ϕ is used to locate the mass particle. Using arrows and symbols, show velocity, acceleration, and force diagrams for the cylinder and the mass particle. Imagine that a set of axes is attached to the cylinder at the center of gravity and is rotating with the cylinder. Use the transfer of velocity and acceleration formulas, Eqs. (1.17), to identify all the motion components of the mass particle. Be methodical—there are a lot of individual components.

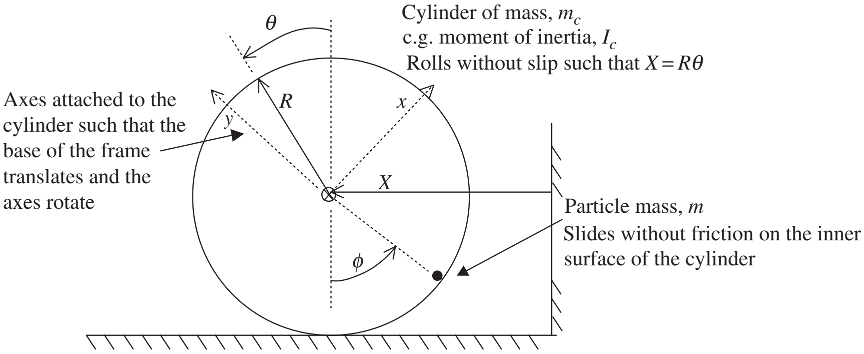


Figure P1.7

1.8 This is basically the same problem as Problem 1.7 except that this time a small cylinder rolls without slip on the inner surface of a big cylinder (Figure P1.8). This remains a two degree-of-freedom system, and either X or θ can be used to locate the big cylinder, with ϕ locating the small cylinder. Using arrows and symbols for the coordinates, construct velocity, acceleration, and force diagrams for this system. When you show the forces, be sure to include a rolling friction force between the bottom of the small cylinder and the inner surface of the big cylinder.

1.9 A cylinder of radius R rolls without slip on a horizontal surface shown in Figure P1.9. The construction is such that there is an inner radius r wrapped with string that extends out horizontally as shown. This is somewhat like a spool of thread. A force F is applied to the end of the string and the cylinder will execute some

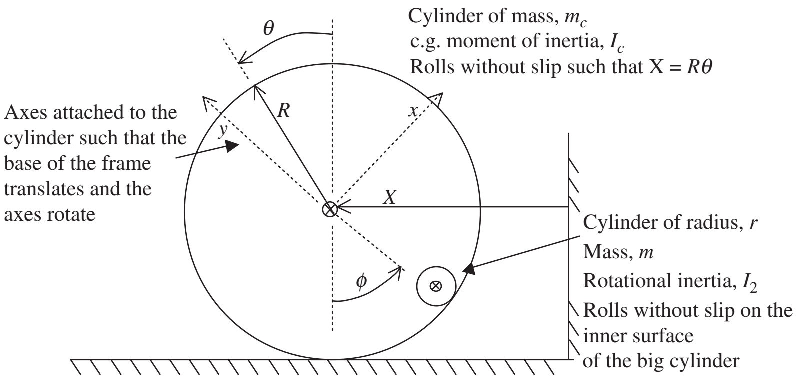


Figure P1.8

motion. Either X or θ can be used as the coordinate for this single degree-of-freedom system. Using arrows and symbols, construct velocity, acceleration, and force diagrams for this system.

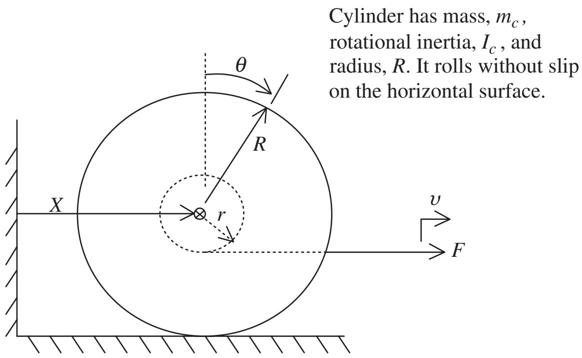


Figure P1.9

1.10 This problem is similar to Problem 1.6, but this time a spring is attached at a height R on the left side wall and extends to the edge of the cylinder diametrally across from the offset center of gravity (Figure P1.10). The cylinder still rolls without slip on the horizontal surface such that either x or θ can be used as the coordinate for the single degree-of-freedom system. The system is set up such that when $\theta = 0$, $x = 0$ and the spring is horizontal and relaxed with no force. In this position the distance from the left wall to the center of the cylinder is D. The mass of the cylinder is m and the c.g. moment of inertia is I_g. Starting from the velocity diagram for the center of gravity of the cylinder, transfer the velocity components to the attachment point of the spring on the cylinder and determine the velocity component in the instantaneous longitudinal spring direction. You might want to use the angle α to make evaluation easier, but as a last step, try to express α as a function of θ.

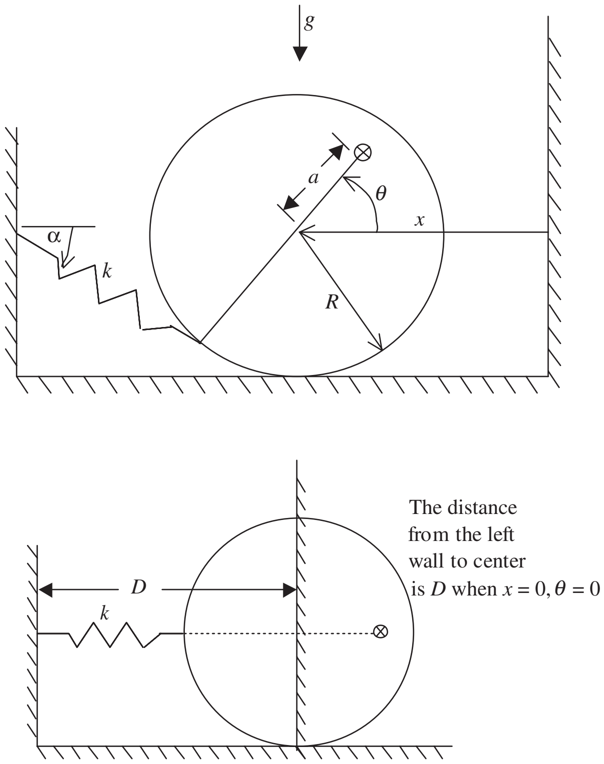


Figure P1.10

1.11 Figure P1.11 shows the top view of a vehicle that has mass m and c.g. moment of inertia about the axis out of the page, I_g. The center of gravity is located a distance a from the front axle and a distance b from the rear axle. The half-width of the vehicle is $w/2$. The front wheels can be steered, indicated by the steer angle δ. A body-fixed coordinate frame is attached to the vehicle at its center of gravity and aligned as shown. The body-fixed velocity components of the center of gravity and the yaw angular velocity are indicated.

(a) Using arrows and symbols, transfer the c.g. velocity to body-fixed directions at the four wheels.

(b) If each wheel is constrained to have no velocity perpendicular to the plane of the wheel, state the kinematic constraints for each wheel.

1.12 This system is identical to that of Problem 1.11, but this time inertial coordinates are used to locate the center of gravity of the vehicle, and the angle θ indicates the angular orientation (Figure P1.12).

(a) Using arrows and symbols, transfer the c.g. velocity to body-fixed directions at the four wheels.

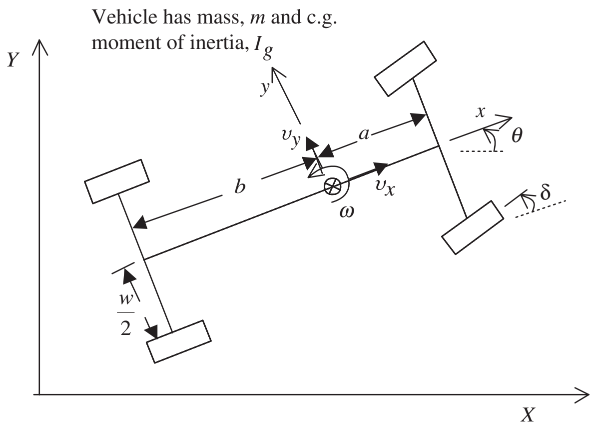


Figure P1.11

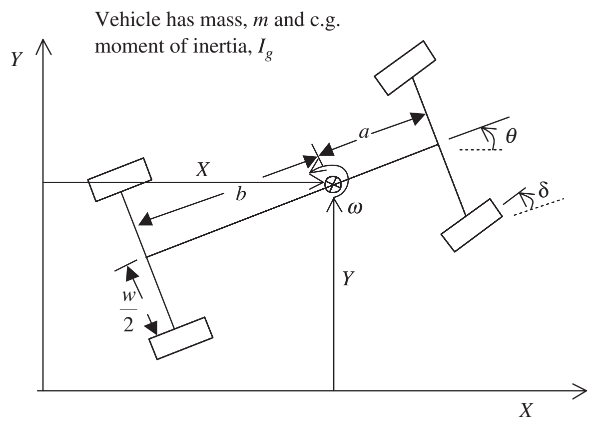


Figure P1.12

(b) If each wheel is constrained to have no velocity perpendicular to the plane of the wheel, state the kinematic constraints for each wheel.

1.13 This is the same problem as Problem 1.5, with an engine block of mass m and c.g. moment of inertia I suspended at the base by some horizontal and vertical springs (Figure P1.13). In this problem the inertial coordinates have been replaced with body-fixed coordinates and the body-fixed velocity components of the center of gravity are indicated.

(a) Using arrows and symbols, show a velocity diagram where the body-fixed c.g. components are transferred to body-fixed directions at the attachment points of the springs.

(b) Resolve the body-fixed components at the attachment points into inertial components and write an expression for the velocity at the end of each spring. For each spring state whether the spring velocities are compressing or extending the spring.

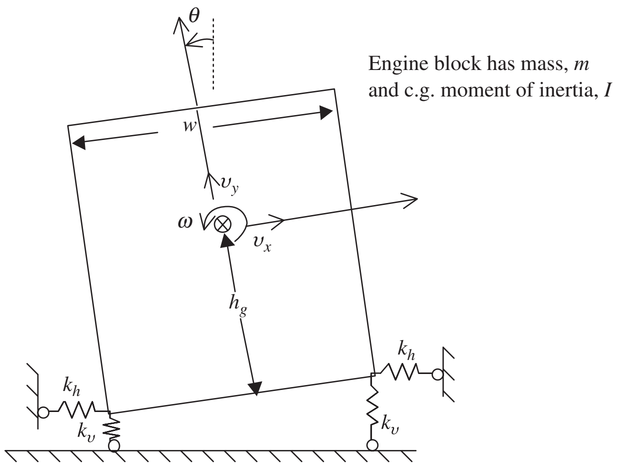


Figure P1.13

1.14 This is the same vehicle as Problem 1.12, but this time the vehicle is towing a trailer of mass m_t and c.g. moment of inertia I_t. The trailer is pinned to the towing vehicle as shown in Figure P1.14 and the center of gravity of the trailer is located c distance behind the hitch point. The rear axle of the trailer is located d from the trailer center of gravity. Body-fixed coordinates are used for the vehicle, and the body-fixed c.g. velocity components v_x, and v_y are shown.

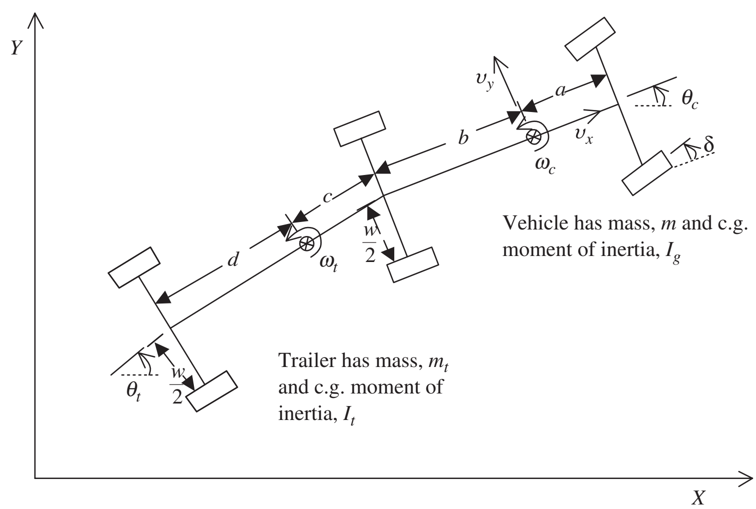


Figure P1.14

(a) Using arrows and symbols, transfer the velocity components of the vehicle to body-fixed directions at the wheels and at the trailer hitch point.

(b) Transfer the velocity components at the hitch point to the center of gravity of the trailer and to its wheels. Show this on a velocity diagram.

(c) If the wheels of the vehicle and the trailer are constrained to have no sideways velocity, state the kinematic constraints that enforce this for all the wheels of the system.

1.15 On fixed tabletop sliding without friction is a mass particle attached to a string (Figure P1.15). The string runs through a hole in the center of the table and another mass particle is attached hanging vertically in a gravity field. The string has length L and this is a two degree-of-freedom system in that position r and angular position θ fully locate the entire system. The mass particle and radial string on the table have been given some initial angular velocity $\dot{\theta}_{\text{ini}}$ and some initial radius r_{ini} and released. Ultimately, we want the equations of motion that would predict the motion–time history of this system, but right now we want the velocity, acceleration, and force diagrams in terms of the two coordinates and their derivatives.

(a) Using arrows and symbols, construct velocity, acceleration, and force diagrams for this system. The string can only support a tensile force, and you will have to give this force a symbolic name.

(b) You may have noticed that there is no force perpendicular to the r-direction. This means that the acceleration in that direction must be zero. Using your acceleration diagram, write an expression which states that the acceleration perpendicular to the r-direction is zero.

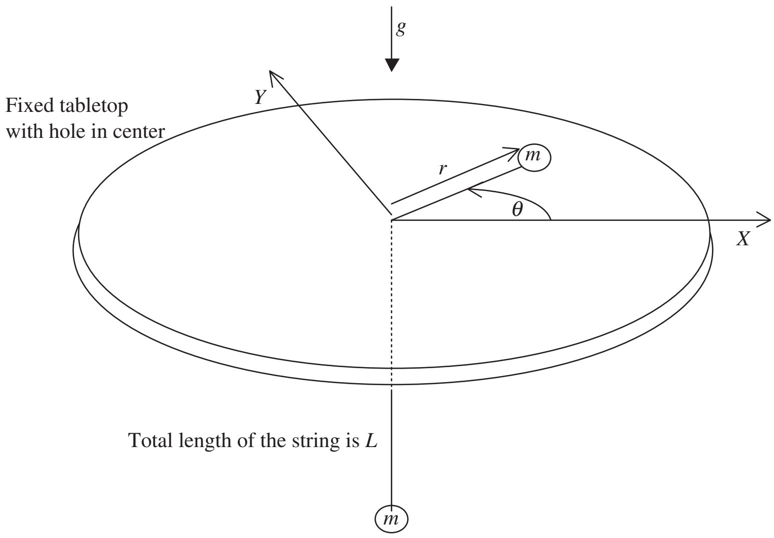


Figure P1.15

(c) The angular momentum of the mass on the table about the axes through the center is the moment of the momentum vector. From your velocity diagram, confirm that the angular momentum is $h = mr^2\dot{\theta}$. Since the tensile force in the string has no moment about these axes, the angular momentum must be constant; thus, $mr^2\dot{\theta} = \text{const}$. Differentiate this expression with respect to time and see if it states the result from part (b).

1.16 A rigid body of mass m, c.g. moment of inertia I_g, and length L rotates without slip on a fixed rigid cylinder (Figure P1.16). Because the rigid body rotates without slip, this is really only a single degree-of-freedom system and only the angle θ is needed to locate the rotated body. Several coordinate systems are shown. The angle θ tracks the contact point on the cylinder, the inertial coordinates X, and Y locate the center of gravity, and body-fixed velocity components are shown in the body-fixed x, and y directions. Springs attached to the rigid body at its ends are constrained to remain vertical regardless of the rotational motion of the rigid body.

(a) For each of the coordinate systems indicated, show velocity diagrams for the attachment points of the springs. Derive the velocity components in the vertical spring directions.

(b) For the θ and X,Y coordinates, show acceleration diagrams for the center of gravity of the rigid body.

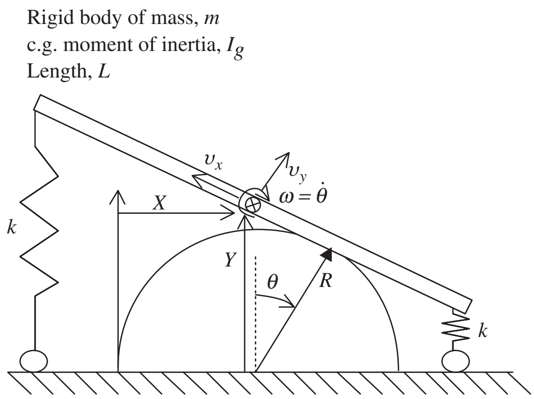


Figure P1.16

1.17 Angular momentum is the moment of the momentum vector about some specified axes. In this chapter, expressions for the angular momentum were derived for rigid bodies with axes attached at a fixed point and for axes attached to the center of gravity. In this problem a rigid body is executing plane motion and has c.g. velocity $\mathbf{v}_g$ and angular velocity ω with respect to inertial X,Y axes. The rigid body is constructed from an infinite number of particles of mass dm, and the ith particle is shown in Figure P1.17. From the inertial X,Y axes, the center of gravity is located by the position vector $\mathbf{r}_g$, the ith particle is located by the position vector $\mathbf{r}_0$, and the ith particle is located with respect to the center of gravity by the position vector $\mathbf{r}$. Using procedures discussed in the chapter text, derive an expression for the angular momentum with respect to the inertial axes in terms of the c.g. velocity and the angular velocity. You will need to recognize that $\mathbf{r}_0 = \mathbf{r}_g + \mathbf{r}$, and you will need to make use of the definition of the center of gravity, where $\int \mathbf{r}\, dm = 0$. The result of your work should be $\mathbf{H}_0 = \mathbf{r}_g \times m\mathbf{v}_g + \int (\mathbf{r} \times \boldsymbol{\omega} \times \mathbf{r})\, dm$, and the interpretation of this result is that the angular momentum of a rigid body with respect to inertial axes is the angular momentum of the center of gravity plus angular momentum due to rotation about the center of gravity. Keep in mind that angular momentum is a vector quantity with direction given by the right-hand rule.

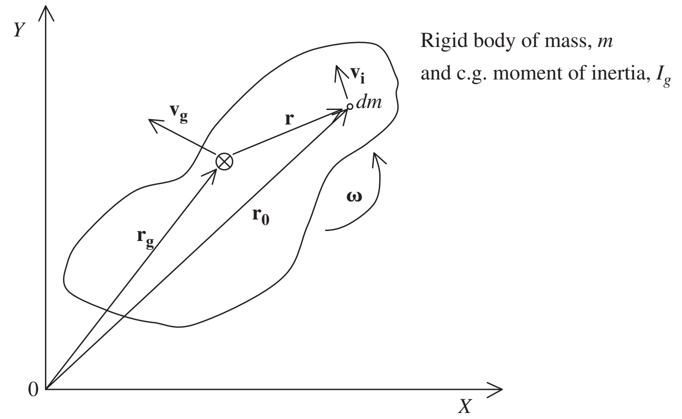


Figure P1.17

1.18 A rigid body with the inertial properties indicated in the Figure P1.18 is traveling at constant initial velocity v_{g_i} toward the left. It is about to encounter an impediment that stops the lower left corner instantaneously. When this happens the body attains a rotational speed ω and the center of gravity attains a velocity v_{g_f}.

(a) Before impact with the impediment, what is the angular momentum of the body with respect to axes attached to the impediment?

(b) Using the results from Problem 1.17, derive an expression for the angular momentum just after impact in terms of v_{g_f} and ω.

(c) Relate v_{g_f} to ω and derive an expression for the angular momentum after impact in terms of ω only. Using your knowledge of the angular momentum of a rigid body about a fixed point and your knowledge of the parallel axis theorem, does your result support what must be the angular momentum of the body just after impact?

(d) It turns out that angular momentum is conserved during the short duration of the impact. Set the initial and final angular momentum equal to each other and derive an expression for the angular velocity just after impact.

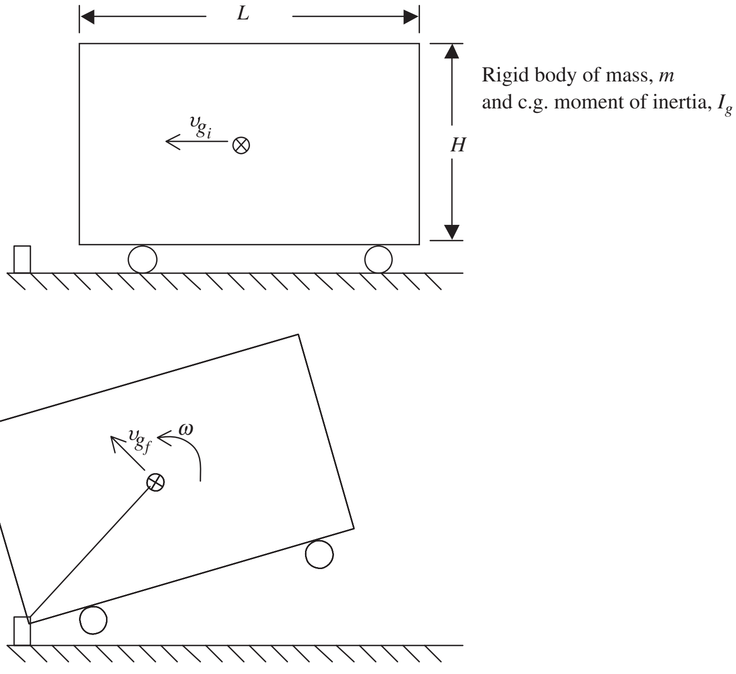


Figure P1.18

1.19 This is a pretty interesting system (Figure P1.19). A pinned cylinder is rotating at an initial angular velocity ω_i. A string is wrapped around the cylinder and a mass particle *m* is attached to the end of the string and is somehow attached to the surface of the cylinder. At some instant the mass is released and begins to swing out from the cylinder until it attains the position shown in Figure P1.19 *b*. It is released at this point and the mass flies away at some final velocity vf

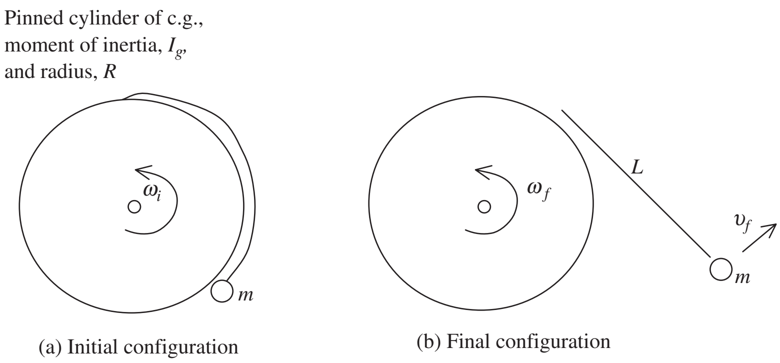


Figure P1.19

and the cylinder has some final angular velocity ω_f. What makes this system interesting is that it is possible to choose the proper mass and proper string length such that the final angular velocity is absolutely zero.

(a) Derive expressions for the initial angular momentum and final angular momentum of the system.

We have not yet covered energy of rotating and translating bodies, but from a first course in dynamics the kinetic energy of the cylinder in configuration a is $T_{c_i} = \frac{1}{2} I_g \omega_i^2$ and the kinetic energy of the mass particle is $T_{m_i} = \frac{1}{2} m (R\omega_i)^2$. The final kinetic energy in configuration b is $T_{c_f} = \frac{1}{2} I_g \omega_f^2$ for the cylinder and $T_{m_f} = \frac{1}{2} m v_f^2$ for the mass particle.

(b) It turns out that both angular momentum and energy are conserved during the motion from configuration a to b. Equate initial and final angular momentum and initial and final total system energy, postulate that $\omega_f = 0$, and determine a length L when the mass is released that will make the postulate true.

1.20 In this system a rigid body can move without friction on a horizontal surface, and it can rotate with positive direction clockwise (Figure P1.20). This is a two degree-of-freedom system and it is decided to use coordinate x to locate the bottom of the body and coordinate θ to locate the angular orientation.

(a) Using arrows and symbols, construct velocity, acceleration, and force diagrams for this system. You will have to assign names to the forces.

(b) Write an expression for the angular momentum of the body with respect to the X,Y axes.

(c) Write an expression for the linear momentum of the body in the inertial directions.

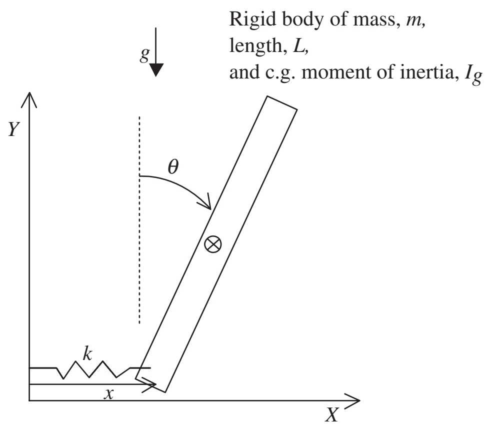


Figure P1.20