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LINEAR SPACES AND OPERATORS

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1.1 INTRODUCTION

A major object of study, which runs through all of mathematics, is that of a *function*. The origins of the subject of *functional analysis* can be traced back to a time, near the beginning of the twentieth century,when mathematicians began to deal with a whole collection or space of functions, rather than looking at one function at a time. This idea proved particularly fruitful to those studying differential and integral equations motivated from problems in physics. It gradually became clear that the problems were remarkably similar to familiar ones of solving algebraic systems of equations, but rather than looking for unknown numbers or vectors, they were looking for unknown elements in a space of functions.

In order to carry out the analysis, they needed notions of distance or length in spaces of functions and were able to formulate this by imitating the geometry of finite-dimensional Euclidean spaces using the Lebesgue integral in place of sums. The type of structure that emerged was later termed *Hilbert space*, named after the German mathematician David Hilbert, one of the early pioneers in this endeavor. It soon became clear that there were other applications, such as Weierstrass's theorem on the approximation of continuous functions with polynomials, which used different types of distances and length. In 1920, the Polish school of mathematicians, led by Stefan Banach, formulated axioms for a general abstract structure that applied to various collections of functions and methods of measuring distance. This is known today as a *Banach space*.

Mathematicians did not just conceive of a set of functions as a static object but rather looked at associated dynamic aspects. One often starts with a function and assigns another function to it. Familiar examples abound in elementary calculus through the operations of differentiation or integration. In other cases one takes a

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function and assigns a number, or a constant function, such as taking a limit of a sequence or summing an infinite series. In many applications of mathematics dealing with quantities changing over time, one has a function at one point of time and is interested in the resulting function at a future point of time. Another familiar example, as indicated above, is a system of linear equations. This can be viewed as a transformation on vectors, which are functions on a finite set. In many cases, the correspondence is linear (defined below), and the assignment of one function to another can be expressed through the abstract concept of a linear operator, which is a particular kind of function defined on one space of functions and taking values in another.

Through this abstraction, one then had a powerful approach that combined aspects of analysis, algebra, and geometry. The same techniques could be brought to bear in diverse settings. Subsequently, applications were found in fields outside of mathematics and physics. These include probability theory, statistics, economics and finance, engineering, and many others. As we go along, we will allude to some of these developments.

Although the subject has evolved considerably, we believe that the main goal of a first course is to investigate this original abstract scheme, which can be compactly described as the study of *bounded linear operators on normed linear spaces*. In this brief introductory chapter we will summarize the basic facts about linear spaces and linear operators, somewhat quickly, as we assume that the typical reader has some knowledge of linear algebra. Most examples will be left to the exercises. The concepts of boundedness and norms are introduced in Chapter 2.

1.2 LINEAR SPACES

1.2.1 Basic Definitions

Throughout the book, we will have occasion to refer to the field of real numbers or the field of complex numbers. In most cases, our statements will hold for both, and we will use the symbol $\mathcal{F}$ to denote either field. For the occasions when we definitely need one or the other, we will use $\mathcal{R}$ for the reals and $\mathcal{C}$ for the complex numbers.

We use standard notation for elements of $\mathcal{C}$. Given a complex number $z = x + iy$ with x, y real, we denote x by Re z, y by Im z, and we let $\bar{z} = x - iy$ denote the *complex conjugate* of z. The absolute value $|z|$ is given by $(x^2 + y^2)^{1/2}$. We can write $z = |z|e^{i\Theta}$, where $e^{i\Theta} = \cos(\Theta) + i\ sin(\Theta)$ has absolute value 1. This is known as the *polar* form of z.

A *linear space* over the field $\mathcal{F}$ is a set X equipped with operations of addition and scalar multiplication subject to some natural rules. For completeness, we state these axioms.

Given any x and y in X and α in $\mathcal{F}$, there are elements $x + y$ and αx in X satisfying the following:

Rules of addition:

(i) $x + y = y + x$;
(ii) $(x + y) + z = x + (y + z)$;
(iii) there is an element 0 in X such that $x + 0 = x$ for all $x \in X$;
(iv) for each $x \in X$, there is an element $-x$ such that $x + (-x) = 0$.

Rules of scalar multiplication:

(i) $1x = x$;
(ii) $\alpha(\beta x) = (\alpha\beta)x$.

Rules connecting the two operations:

(i) $(\alpha + \beta)x = \alpha x + \beta x$;
(ii) $\alpha(x + y) = \alpha x + \alpha y$.

We will use the notation $x - y$ to denote $x + (-y)$.

Addition and scalar multiplication carry over to operations on sets. Given subsets A and B in a linear space and a scalar α, $A + B$ is the set of all elements that can be written as $a + b$ for some $a \in A$ and $b \in B$ and αA is the set of all elements of the form αa for some $\alpha \in \mathcal{F}$ and $a \in A$. In particular, $-1A$ is denoted by just $-A$ and $A + (-B)$ is denoted by $A - B$.

Two trivial examples of linear spaces are $\mathcal{F}$ itself and the set consisting of the single element 0. We discuss more pertinent examples in the next section. Of course, associated with any linear space are several others, namely its linear subspaces defined as follows.

Definition. Let Y be a nonempty subset of the linear space X that is closed under addition and scalar multiplication. That is, for all $x, y \in Y$ and $\alpha \in \mathcal{F}$, the elements $x + y$ and αx are in Y. Then, Y becomes a linear space in its own right under the same operations, and it is called a *linear subspace* of X. (We will normally drop the word *linear* and just use *subspace*.)

Linear spaces are also commonly known as *vector spaces*. Accordingly, we will frequently refer to elements of a linear space as vectors.

We next summarize the concepts of basis and dimension.

Definitions. Given a set B in a linear space, a *finite linear combination of elements of B* is an element x of the form

$$x = \sum_{i=1}^{n} \alpha_i b_i,$$

where for each i we have $\alpha_i \in \mathcal{F}$ and $b_i \in B$. We refer to the set $\{\alpha_i\}$ as the *coefficients*.

We let span(B) denote the set of all finite linear combinations of elements of B. It is the smallest subspace containing the set B, in the sense that if Y is a subspace containing B, then span$(B) \subseteq Y$.

Definition. A set B is said to be a *basis* for the linear space X if every element in X can be written uniquely as a finite linear combination of elements of B with *nonzero* coefficients. (We need nonzero here to ensure uniqueness, for without it we could add an unused element with a zero coefficient.)

Definitions. A linear space is *finite-dimensional* if it has a basis with a finite number of elements. It is a basic theorem of linear algebra that in a finite-dimensional space all bases have the same number of elements, which is known as the *dimension* of the space.

Definition. A set B in a linear space is said to be *linearly independent* if no element in B can be written as a finite linear combination of the others. An equivalent formulation is that if a finite linear combination of elements of $B = 0$, then all coefficients must be zero.

Many authors give the definition of a basis in an alternate form, namely, that it is a linearly independent set B such that span$(B) = X$. That is, the linear independence replaces the uniqueness criterion.We invite the reader to show the equivalence of these two definitions. We prefer to state the definition in the above form, as it is the way in which we most frequently use the concept of basis; additionally, it ties in better with the definition of a Schauder basis, which we present in Chapter 2.

1.2.2 Spaces of Functions

The main reason why linear spaces are important in analysis is that the interesting classes of functions form such an object. To fix the ideas, let S be any set, and let $F(S)$ denote the set of all functions from S to $\mathcal{F}$. By defining $f + g$ and αf in the obvious way as

$$(f + g)(s) = f(s) + g(s), \qquad (\alpha f)(s) = \alpha f(s)$$

for all $s \in S$, it is easy to verify that all of the above rules hold, and $F(S)$ becomes a linear space. The 0 element of this space is just the function that assigns 0 to all points.

When S consists of n elements for some finite n, a function on S can be represented as an ordered n-tuple, and $F(S)$ is nothing more than the familiar n-dimensional space consisting of n-dimensional vectors. Here, $F(S)$ is denoted by $\mathcal{R}^n$ for real scalars or $\mathcal{C}^n$ for complex scalars.

When S is infinite, $F(S)$ is too large to be of much interest, and our focus will be on certain subspaces of $F(S)$. Elementary calculus and analysis courses are filled with examples showing that a certain class of functions forms a subspace of $F(S)$ for various S. When S is an interval of the real line, these include bounded functions, continuous functions, differentiable functions, and polynomial functions, and the list goes on. When S is the set $\mathcal{N}$ of positive integers, our functions are sequences. Interesting subspaces include bounded sequences, convergent sequences, sequences converging to 0, summable sequences, and others. We will investigate many of these subspaces in more detail in the remaining chapters.

Rarely will we again have occasion to verify the rules above in order to demonstrate that we have a linear space. Almost all of the examples we discuss will be subspaces of $F(S)$ for some S, with a couple of variations, which we point out as they arise.

Remark on Notation. To emphasize the functional aspect, we will sometimes write vectors or sequences with arguments in brackets, as opposed to the more traditional use of subscripts. So, for example, when talking about several vectors or sequences, we will often write $x = (x(1), x(2), \ldots, x(n))$ and reserve subscripts to distinguish between different vectors or sequences.

Given $s \in S$, we will let $\delta_s \in F(S)$ denote the function that takes the value 1 at the point s and 0 elsewhere. For a general subset $A \subseteq S$, we will use 1_A to denote the function that takes the value of 1 on A and 0 on the complement of A. (This is often referred to as *the characteristic function* of A. Note that this term is used in probability theory with a completely different meaning.)

1.3 LINEAR OPERATORS

1.3.1 Basic Definitions

If X and Z are two linear spaces, a *linear operator* from X to Z is a rule T that associates to each $x \in X$ a unique element $Tx \in Z$ such that, for all $x, y \in X$ and $\alpha \in \mathcal{F}$, we have

$$T(x + y) = Tx + Ty, \qquad T(\alpha x) = \alpha Tx.$$

Linear operators are also referred to as *linear transformations* or *linear mappings*.

We will use the familiar notation $T : X \to Z$ to denote a mapping from X to Z, and unless other specified, this notation will imply that the mapping is linear.

For most readers, one of the most familiar examples of a linear operator will likely be the operation on $\mathcal{R}^n$ of multiplying a vector by a matrix. Indeed, all linear operators on $\mathcal{R}^n$ can be so expressed, as we show in Section 1.3.4. Various examples of operators on sequence spaces appear in Exercises 1.1, 1.3, and 1.4.

1.3.2 Null Spaces and Ranges

To every $T : X \to Z$ we associate two important subspaces. The *null space* (also known as the *kernel*), denoted by $N(T)$, consists of all $x \in X$ such that $Tx = 0$.

The *range* of T (also known as *the image*), denoted by $R(T)$, consists of all $z \in Z$ such that $Tx = z$ for some $x \in X$. These are easily seen to be subspaces of X and Z, respectively.

By definition, a linear operator T is *surjective* (onto) if and only if $R(T) = Z$. We next verify that a linear operator T is *injective* (one-to-one) if and only if $N(T) = 0$. It is easily seen that we always have $T(0) = 0$, so if x is a nonzero vector in $N(T)$, then $Tx = T(0)$ and the map is not injective. Conversely, if $N(T) = 0$, then by invoking linearity, we have $Tx = Ty$ if and only if $T(x - y) = 0$ if and only if $x - y = 0$ if and only if $x = y$.

1.3.3 Equations and Inverses

Many problems that arise in mathematics are of the following type. We are given $T : X \to Z$, an element $z \in Z$, and we want to find an element $x \in X$ satisfying the equation

$$Tx = z.$$

Systems of linear equations can be put into this form. Indeed, the beginning of functional analysis, to which we referred to in the introduction, involved problems of this type with T an integral operator on continuous functions or a differential operator on differentiable functions.

The first questions one usually asks about such an equation is whether a solution exists and, if so, whether it is unique. Answers can be easily formulated in terms of the subspaces of the previous section. A solution exists for a given z if and only if $z \in R(T)$; therefore, one exists for all z if and only if $R(T) = Z$. Similarly, if one exists, it is unique if and only if $N(T) = 0$ since $Tx = Ty$ if and only if $x - y \in N(T)$.

If $N(T) = 0$, we can define the *inverse* transformation $T^{-1} : R(T) \to X$ by setting $T^{-1}z$ equal to the unique x such that $Tx = z$. It is straightforward to show that T^{-1} is linear. In this case, the solution to the above equation is given for all $z \in R(T)$ by $x = T^{-1}(z)$. This shows the importance of inverses, and we will deal much more with this concept in the chapters to follow.

1.3.4 Space of Linear Operators

We now observe one of the variations to which we alluded in Section 1.2 when we said that nearly all of the linear spaces which we would encounter would be linear subspaces of $F(S)$. In defining the linear space of functions on S we can allow the functions to take values in any linear space Z, rather than be restricted to $\mathcal{F}$. The definitions of addition and scalar multiplication of functions are the same as given, using the corresponding operations of Z. Additionally, suppose that our domain S is itself a linear space X. Then, the subset of the space of functions from X to Z consisting of the linear operators is easily seen to be a subspace of all the functions on X with values in Z.

To summarize, given any two linear spaces X and Z, we get a third linear space, namely, the linear operators from X to Z, with the usual pointwise addition and scalar multiplication inherited from Z. We will denote this space by $\mathcal{L}(X,Z)$. When $X = Z$, we shorten this to $\mathcal{L}(X)$.

Most readers will have previously encountered a version of $\mathcal{L}(X,Z)$ in the finite-dimensional case as a space of matrices. Given spaces X with a basis $\{b_1, b_2, \ldots, b_n\}$ and Z with a basis $\{c_1, c_2, \ldots, c_m\}$, any $T \in \mathcal{L}(X,Z)$ is determined by its value on the b_j's. This follows since, for $x = \sum_{j=1}^{n} \alpha_j b_j$, the linearity of T (which extends from two to any finite number of terms by induction) shows that

$$Tx = \sum_{j=1}^{n} \alpha_j \, Tb_j.$$

Now, for each j, we can write

$$Tb_j = \sum_{i=1}^{m} a_{i,j} \, c_i$$

for some unique scalars $a_{i,j}$. All the information we need about the operator is encoded by the matrix A, whose entry in the ith row and jth column is $a_{i,j}$. In fact, multiplying the column vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ on the left by the matrix A gives us, as a column vector, the coefficients of Tx with respect to the given basis of Z. Moreover, the matrix associated with the sum of two operators is the usual pointwise sum of two matrices, and the matrix associated with αT is that of T multiplied by the scalar α. So, after choosing the two bases, we can identify $\mathcal{L}(X,Y)$ as the linear space of all $m \times n$ matrices.

1.4 PASSAGE FROM FINITE- TO INFINITE-DIMENSIONAL SPACES

An example of an n-dimensional function space is $F(S)$ in which $S = \{1, 2, \ldots, n\}$. We can take $\{\delta_i : i = 1, 2, \ldots, n\}$ as a basis since for any $f \in F(S)$ we can write f uniquely as

$$f = \sum_{i=1}^{n} f(i)\delta_i.$$

This is a special case, and nearly all of the interesting function spaces are infinite-dimensional in the sense that they do not possess a finite basis. When one passes to the infinite-dimensional case, several of the conclusions and properties of finite-dimensional spaces no longer hold. However, many of these results can be recovered if one deals with *closed* linear subspaces and *continuous* linear operators, in place of arbitrary linear subspaces and operators, provided that we take suitable definitions for these terms. Such notions are often not needed in problems involving finite-dimensional spaces, since the linear structure dominates in that case. In finite dimensions, linear subspaces are automatically closed and linear operators are automatically continuous for any reasonable definitions of these concepts, as we will show later. Therefore, in studying infinite-dimensional spaces, we must go beyond linear algebra and incorporate topological notions that allow us to speak of such things as convergence, continuity, and closure. We will begin to do this in the next chapter.

EXERCISES

In Exercises 1.1 through 1.4, let s denote the linear space consisting of all sequences.

1.1 Let c_{00} denote the set of all sequences that have only finitely many nonzero entries.
   (a) Show that c_{00} is an infinite-dimensional subspace of s. Find a basis.
   (b) Show that the mapping $T : c_{00} \to \mathcal{F}$, given by $Tx = \sum_{i=1}^{\infty} x(i)$, is linear.

1.2 For any subset A of positive integers, let s_A denote the set of all sequences x such that $x(n) = 0$ for all $n \in A$. Show that s_A is a subspace of s.

1.3 Given any $y \in s$, define a function $M_y : s \to s$ by $M_y(x)(n) = y(n)x(n)$.
   (a) Show that M_y is a linear operator.
   (b) Describe the spaces $N(M_y)$ and $R(M_y)$.

1.4 Define functions S and T from s to itself by

$$S(x(1), x(2), x(3), \ldots) = (0, x(1), x(2), \ldots),$$
$$T(x(1), x(2), x(3), \ldots) = (x(2), x(3), \ldots).$$

   (a) Show that S and T are linear operators.
   (b) For both S and T, find the null space and range.
   (c) Compute the compositions ST and TS.

1.5 Verify the assertion made in Section 1.4 that $\mathcal{L}(X, Z)$ is a subspace of the space of all functions from X to Z.

1.6 Consider the space $\mathcal{L}(X, Z)$, where X and Z are infinite-dimensional. Which of the following, if any, are subspaces?
   (a) $\{T : N(T) \text{ is finite-dimensional }\}$;
   (b) $\{T : R(T) \text{ is finite-dimensional }\}$.

1.7 Suppose that for each λ in some index set Λ we are given a subspace Y_λ of a linear space X such that for all $\lambda, \mu \in \Lambda$ either $Y_\lambda \subset Y_\mu$ or $Y_\mu \subset Y_\lambda$. Show that $\cup_{\lambda \in \Lambda} Y_\lambda$ is a subspace of X.

1.8 Let S be an infinite set, and let X be a linear space of functions that includes δ_s for all $s \in S$. Show that X is infinite-dimensional.

1.9 Show that the two definitions of a basis given in this chapter are equivalent.