Chapter 1

Welcome to the World of Differential Equations

In This Chapter

- Breaking into the basics of differential equations
- Getting the scoop on derivatives
- Checking out direction fields
- Putting differential equations into different categories
- Distinguishing among different orders of differential equations
- Surveying some advanced methods

It's a tense moment in the physics lab. The international team of high-powered physicists has attached a weight to a spring, and the weight is bouncing up and down.

"What's happening?" the physicists cry. "We have to understand this in terms of math! We need a formula to describe the motion of the weight!"

You, the renowned Differential Equations Expert, enter the conversation calmly. "No problem," you say. "I can derive a formula for you that will describe the motion you're seeing. But it's going to cost you."

The physicists look worried. "How much?" they ask, checking their grants and funding sources. You tell them.

"Okay, anything," they cry. "Just give us a formula."

You take out your clipboard and start writing.

"What's that?" one of the physicists asks, pointing at your calculations.

"That," you say, "is a differential equation. Now all I have to do is to solve it, and you'll have your formula." The physicists watch intently as you do your math at lightning speed.

"I've got it," you announce. "Your formula is $y = 10 \sin (5t)$, where y is the weight's vertical position, and t is time, measured in seconds."

"Wow," the physicists cry, "all that just from solving a differential equation?"

"Yep," you say, "now pay up."

Well, you're probably not a renowned differential equations expert — not yet, at least! But with the help of this book, you very well may become one. In this chapter, I give you the basics to get started with differential equations, such as derivatives, direction fields, and equation classifications.

The Essence of Differential Equations

REMEMBER

In essence, differential equations involve *derivatives,* which specify how a quantity changes; by solving the differential equation, you get a formula for the quantity itself that doesn't involve derivatives.

Because derivatives are essential to differential equations, I take the time in the next section to get you up to speed on them. (If you're already an expert on derivatives, feel free to skip the next section.) In this section, however, I take a look at a qualitative example, just to get things started in an easily digestible way.

Say that you're a long-time shopper at your local grocery store, and you've noticed prices have been increasing with time. Here's the table you've been writing down, tracking the price of a jar of peanut butter:

Month	***Price***
1	\$2.40
2	\$2.50
3	\$2.60
4	\$2.70
5	\$2.80
6	\$2.90

Looks like prices have been going up steadily, as you can see in the graph of the prices in Figure 1-1. With that large of a price hike, what's the price of peanut butter going to be a year from now?

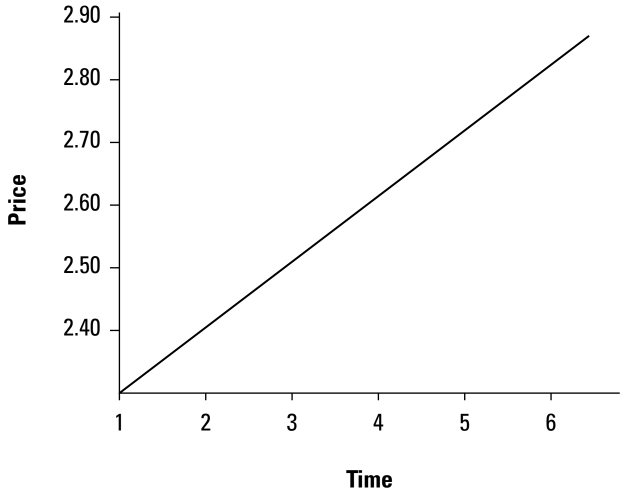

Figure 1-1: The price of peanut butter by month.

You know that the slope of a line is $\Delta y/\Delta x$ (that is, the change in y divided by the change in x). Here, you use the symbols Δp for the change in price and Δt for the change in time. So the slope of the line in Figure 1-1 is $\Delta p/\Delta t$.

Because the price of peanut butter is going up 10 cents every month, you know that the slope of the line in Figure 1-1 is:

$$\frac{\Delta p}{\Delta t} = 10¢/\text{month}$$

The slope of a line is a constant, indicating its rate of change. The derivative of a quantity also gives its rate of change at any one point, so you can think of the derivative as the slope at a particular point. Because the rate of change of a line is constant, you can write:

$$\frac{dp}{dt} = \frac{\Delta p}{\Delta t} = 10¢/\text{month}$$

TIP

In this case, dp/dt is the derivative of the price of peanut butter with respect to time. (When you see the d symbol, you know it's a derivative.)

And so you get this differential equation:

$$\frac{dp}{dt} = 10¢/\text{month}$$

The previous equation is a differential equation because it's an equation that involves a derivative, in this case, *dp*/*dt*. It's a pretty simple differential equation, and you can solve for price as a function of time like this:

$$p = 10t + c$$

In this equation, p is price (measured in cents), t is time (measured in months), and c is an arbitrary constant that you use to match the initial conditions of the problem. (You need a constant, c, because when you take the derivative of $10t + c$, you just get 10, so you can't tell whether there's a constant that should be added to $10t$ — matching the initial conditions will tell you.)

The missing link is the value of c, so just plug in the numbers you have for price and time to solve for it. For example, the cost of peanut butter in month 1 is \$2.40, so you can solve for c by plugging in 1 for t and \$2.40 for p (240 cents), giving you:

$$240 = 10 + c$$

By solving this equation, you calculate that $c = 230$, so the solution to your differential equation is:

$$p = 10t + 230$$

And that's your solution — that's the price of peanut butter by month. You started with a differential equation, which gave the rate of change in the price of peanut butter, and then you solved that differential equation to get the price as a function of time, $p = 10t + 230$.

Want to see the solution to your differential equation in action? Go for it! Find out what the price of peanut butter is going to be in month 12. Now that you have your equation, it's easy enough to figure out:

$$p = 10t + 230$$

$$10(12) + 230 = 350$$

As you can see, in month 12, peanut butter is going to cost a steep \$3.50, which you were able to figure out because you knew the *rate* at which the price was increasing. This is how any typical differential equation may work: You have a differential equation for the rate at which some quantity changes (in this case, price), and then you solve the differential equation to get another equation, which in this case related price to time.

TIP

Note that when you substitute the solution ($p = 10t + 230$) into the differential equation, dp/dt indeed gives you 10 cents per month, as it should.

Derivatives: The Foundation of Differential Equations

As I mention in the previous section, a derivative simply specifies the rate at which a quantity changes. In math terms, the derivative of a function $f(x)$, which is depicted as $df(x)/dx$, or more commonly in this book, as $f'(x)$, indicates how $f(x)$ is changing at any value of x. The function $f(x)$ has to be continuous at a particular point for the derivative to exist at that point.

Take a closer look at this concept. The amount $f(x)$ changes in a small distance along the x axis Δx is:

$$f(x + \Delta x) - f(x)$$

The rate at which $f(x)$ changes over the change Δx is:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

So far so good. Now to get the derivative dy/dx, where $y = f(x)$, you must let Δx get very small, approaching zero. You can do that with a *limiting expression,* which you can evaluate as Δx goes to zero. In this case, the limiting expression is:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

In other words, the derivative of $f(x)$ is the amount $f(x)$ changes in Δx, divided by Δx, as Δx goes to zero.

I take a look at some common derivatives in the following sections; you'll see these derivatives throughout this book.

Derivatives that are constants

The first type of derivative you'll encounter is when $f(x)$ equals a constant, c. If $f(x) = c$, then $f(x + \Delta x) = c$ also, and $f(x + \Delta x) - f(x) = 0$ (because all these amounts are actually the same), so $df(x)/dx = 0$. Therefore:

$$f(x) = c \qquad \frac{df(x)}{dx} = 0$$

How about when $f(x) = cx$, where c is a constant? In this case, $f(x) = cx$, and $f(x + \Delta x) = cx + c\,\Delta x$.

So $f(x + \Delta x) - f(x) = c\,\Delta x$ and $(f(x + \Delta x) - f(x))/\Delta x = c$. Therefore:

$$f(x) = cx \qquad \frac{df(x)}{dx} = c$$

Derivatives that are powers

Another type of derivative that pops up is one that includes raising x to the power n. Derivatives with powers work like this:

$$f(x) = x^n \qquad \frac{df(x)}{dx} = n\ x^{n-1}$$

Raising e to a certain power is always popular when working with differential equations (e is the natural logarithm base, $e = 2.7128\ldots$, and a is a constant):

$$f(x) = e^{ax} \qquad \frac{df(x)}{dx} = a\ e^{ax}$$

And there's also the inverse of e^a, which is the natural log, which works like this:

$$f(x) = \ln(x) \qquad \frac{df(x)}{dx} = \frac{1}{x}$$

Derivatives involving trigonometry

Now for some trigonometry, starting with the derivative of $\sin(x)$:

$$f(x) = \sin(x) \qquad \frac{df(x)}{dx} = \cos(x)$$

And here's the derivative of $\cos(x)$:

$$f(x) = \cos(x) \qquad \frac{df(x)}{dx} = -\sin(x)$$

Derivatives involving multiple functions

The derivative of the sum (or difference) of two functions is equal to the sum (or difference) of the derivatives of the functions (that's easy enough to remember!):

$$f(x) = a(x) \pm b(x) \qquad \frac{df(x)}{dx} = \frac{d\ a(x)}{dx} \pm \frac{d\ b(x)}{dx}$$

The derivative of the product of two functions is equal to the first function times the derivative of the second, plus the second function times the derivative of the first. For example:

$$f(x)=a(x)b(x) \qquad \frac{df(x)}{dx}=a(x)\frac{d\ b(x)}{dx}+b(x)\frac{d\ a(x)}{dx}$$

How about the derivative of the quotient of two functions? That derivative is equal to the function in the denominator times the derivative of the function in the numerator, minus the function in the numerator times the derivative of the function in the denominator, all divided by the square of the function in the denominator:

$$f(x)=\frac{a(x)}{b(x)} \qquad \frac{df(x)}{dx}=\frac{b(x)\frac{d\ a(x)}{dx}-a(x)\frac{d\ b(x)}{dx}}{b(x)^2}$$

Seeing the Big Picture with Direction Fields

It's all too easy to get caught in the math details of a differential equation, thereby losing any idea of the bigger picture. One useful tool for getting an overview of differential equations is a direction field, which I discuss in more detail in Chapter 2. *Direction fields* are great for getting a handle on differential equations of the following form:

$$\frac{dy}{dx}=f\left(x,y\right)$$

REMEMBER

The previous equation gives the slope of the equation $y = f(x)$ at any point x. A direction field can help you visualize such an equation without actually having to solve for the solution. That field is a two-dimensional graph consisting of many, sometimes hundreds, of short line segments, showing the slope — that is, the value of the derivative — at multiple points. In the following sections, I walk you through the process of plotting and understanding direction fields.

Plotting a direction field

Here's an example to give you an idea of what a direction field looks like. A body falling through air experiences this force:

$$F = mg - \gamma v$$

In this equation, F is the net force on the object, m is the object's mass, g is the acceleration due to gravity (g = 9.8 meters/sec^2 near the Earth's surface), γ is the *drag coefficient* (which adds the effect of air friction and is measured in newtons sec/meter), and v is the speed of the object as it plummets through the air.

If you're familiar with physics, consider Newton's second law. It says that $F = ma$, where F is the net force acting on an object, m is its mass, and a is its acceleration. But the object's acceleration is also dv/dt, the derivative of the object's speed with respect to time (that is, the rate of change of the object's speed). Putting all this together gives you:

$$F = ma = m\frac{dv}{dt} = mg - \gamma v$$

Now you're back in differential equation territory, with this differential equation for speed as a function of time:

$$\frac{dv}{dt} = g - \frac{\gamma}{m}v$$

Now you can get specific by plugging in some numbers. The acceleration due to gravity, g, is 9.8 meters/sec^2 near the Earth's surface, and let's say that the drag coefficient is 1.0 newtons sec/meter and the object has a mass of 4.0 kilograms. Here's what you'd get:

$$\frac{dv}{dt} = 9.8 - \frac{v}{4}$$

To get a handle on this equation without attempting to solve it, you can plot it as a direction field. To do so you create a two-dimensional plot and add dozens of short line segments that give the slope at those locations (you can do this by hand or with software). The direction field for this equation appears in Figure 1-2. As you can see in the figure, there are dozens of short lines in the graph, each of which give the slope of the *solution* at that point. The vertical axis is v, and the horizontal axis is t.

Because the slope of the solution function at any one point doesn't depend on t, the slopes along any horizontal line are the same.

Connecting slopes into an integral curve

You can get a visual handle on what's happening with the solutions to a differential equation by looking at its direction field. How? All those slanted line segments give you the solutions of the differential equations — all you have to do is draw lines connecting the slopes. One such solution appears in Figure 1-3. A solution like the one in the figure is called an *integral curve* of the differential equation.

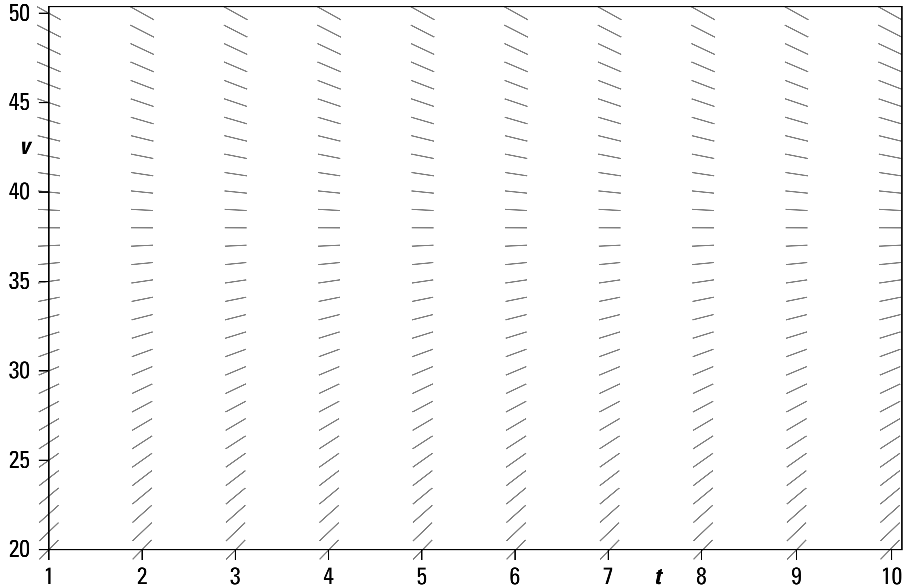

Figure 1-2: A direction field.

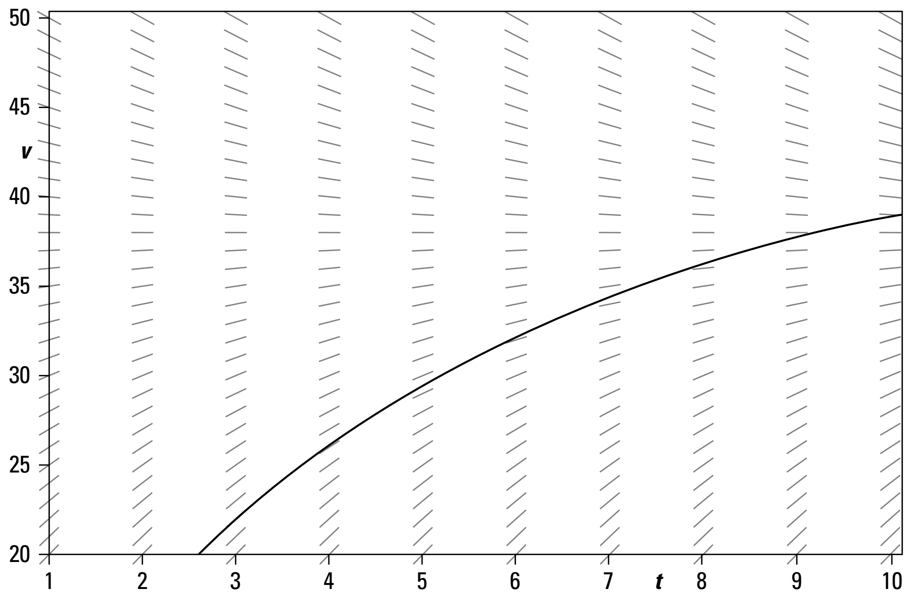

Figure 1-3: A solution in a direction field.

Recognizing the equilibrium value

As you can see from Figure 1-3, there are many solutions to the equation that you're trying to solve. As it happens, the actual solution to that differential equation is:

$$v = 39.2 + ce^{-t/4}$$

In the previous solution, c is an arbitrary constant that can take any value. That means there are an infinite number of solutions to the differential equation.

But you don't have to know that solution to determine what the solutions behave like. You can tell just by looking at the direction field that all solutions tend toward a particular value, called the *equilibrium value.* For instance, you can see from the direction field graph in Figure 1-3 that the equilibrium value is 39.2. You also can see that equilibrium value in Figure 1-4.

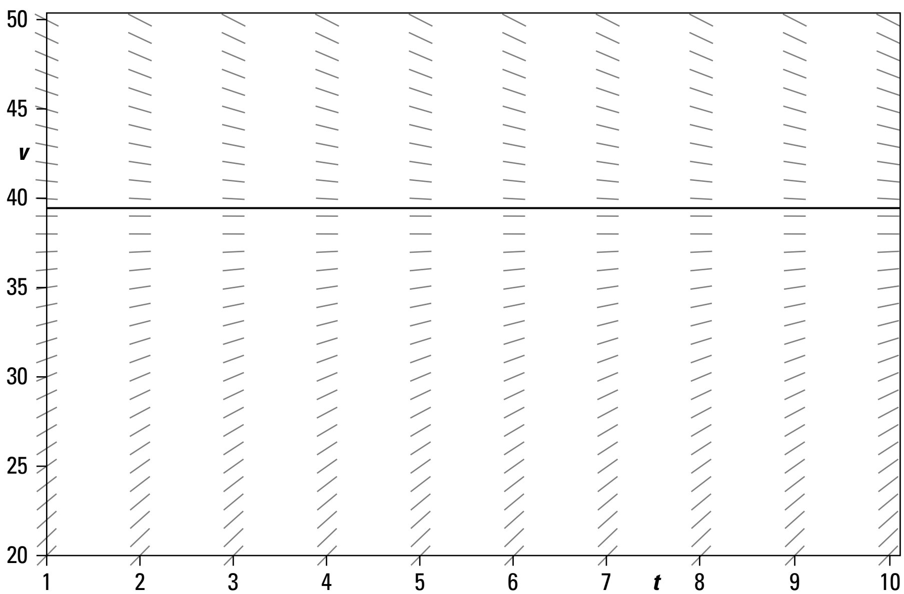

Figure 1-4: An equilibrium value in a direction field.

Classifying Differential Equations

Tons of differential equations exist in Math and Science Land, and the way you tackle them differs by type. As a result, there are several *classifications* that you can put differential equations into. I explain them in the following sections.

Classifying equations by order

The most common classification of differential equations is based on *order.* The order of a differential equation simply is the order of its highest derivative. For example, check out the following, which is a first order differential equation:

$$\frac{dy}{dx} = 5x$$

Here's an example of a second order differential equation:

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} = 19x + 4$$

And so on, up to order n:

$$9\frac{d^n y}{dx^n} - 16\frac{d^{n-1} y}{dx^{n-1}} + \ldots + 14\frac{d^2 y}{dx^2} + 12\frac{dy}{dx} - 19x + 4 = 0$$

As you might imagine, first order differential equations are usually the most easily managed, followed by second order equations, and so on. I discuss first order, second order, and higher order differential equations in a bit more detail later in this chapter.

Classifying ordinary versus partial equations

You can also classify differential equations as *ordinary* or *partial.* This classification depends on whether you have only ordinary derivatives involved or only partial derivatives.

An *ordinary (non-partial) derivative* is a full derivative, such as dQ/dt, where you take the derivative of all terms in Q with respect to t. Here's an example of an ordinary differential equation, relating the charge $Q(t)$ in a circuit to the electromotive force $E(t)$ (that is, the voltage source connected to the circuit):

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Here, Q is the charge, L is the inductance of the circuit, C is the capacitance of the circuit, and $E(t)$ is the electromotive force (voltage) applied to the circuit. This is an ordinary differential equation because only ordinary derivatives appear.

On the other hand, partial derivatives are taken with respect to only one variable, although the function depends on two or more. Here's an example of a partial differential equation (note the squiggly d's):

$$\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

In this heat conduction equation, α is a physical constant of the system that you're trying to track the heat flow of, and $u(x, t)$ is the actual heat.

Note that $u(x, t)$ depends on both x and t and that both derivatives are partial derivatives — that is, the derivatives are taken with respect to one or the other of x or t, but not both.

In this book, I focus on ordinary differential equations, because partial differential equations are usually the subject of more advanced texts. Never fear though: I promise to get you your fair share of partial differential equations.

Classifying linear versus nonlinear equations

Another way that you can classify differential equations is as *linear* or *nonlinear*. You call a differential equation *linear* if it exclusively involves linear terms (that is, terms to the power 1) of y, y', y'', and beyond to $y^{(n)}$. For example, this equation is a linear differential equation:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

TIP

Note that this kind of differential equation usually will be written this way throughout this book. And this form makes the linear nature of this equation clear:

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

On the other hand, nonlinear differential equations involve nonlinear terms in any of y, y', y'', up to $y^{(n)}$. The following equation, which describes the angle of a pendulum, is a nonlinear differential equation that involves the term *sin* θ (not just θ):

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

TIP

Handling nonlinear differential equations is generally more difficult than handling linear equations. After all, it's often tough enough to solve linear differential equations without messing things up by adding higher powers and other nonlinear terms. For that reason, you'll often see scientists cheat when it comes to nonlinear equations. Usually they make an approximation that reduces the nonlinear equation to a linear one.

For example, when it comes to pendulums, you can say that for small angles, $\sin\theta \approx \theta$. This means that the following equation is the standard form of the pendulum equation that you'll find in physics textbooks:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

As you can see, this equation is a linear differential equation, and as such, it's much more manageable. Yes, it's a cheat to use only small angles so that $\sin\theta \approx \theta$, but unless you cheat like that, you'll sometimes be reduced to using numerical calculations on a computer to solve nonlinear differential equations; obviously these calculations work, but it's much less satisfying than cracking the equation yourself (if you're a math geek like me).

Solving First Order Differential Equations

Chapters 2, 3, and 4 take a look at differential equations of the form $f'(x) = f(x, y)$; these equations are known as first order differential equations because the derivative involved is of first order (for more on these types of equations, see the earlier section "Classifying equations by order."

First order differential equations are great because they're usually the most solvable. I show you all kinds of ways to handle first order differential equations in Chapters 2, 3, and 4. The following are some examples of what you can look forward to:

- As you know, first order differential equations look like this: $f'(x) = f(x, y)$. In the upcoming chapters, I show you how to deal with the case where $f(x, y)$ is linear in x — for example, $f'(x) = 5x$ — and then nonlinear in x, as in $f'(x) = 5x^2$.
- You find out how to work with separable equations, where you can factor out all the terms having to do with y on one side of the equation and all the terms having to do with x on the other.
- I also help you solve first order differential equations in cool ways, such as by finding integrating factors to make more difficult problems simple.

REMEMBER

Direction fields, which I discuss earlier in this chapter, work only for equations of the type $f'(x) = f(x, y)$ — that is, where only the first derivative is involved — because the first derivative of $f(x)$ gives you the slope of $f(x)$ at any point (and, of course, connecting the slope line segments is what direction fields are all about).

Tackling Second Order and Higher Order Differential Equations

As noted in the earlier section "Classifying equations by order," second order differential equations involve only the second derivative, d^2y/dx^2, also known as y''. In many physics situations, second order differential equations are where the action is.

For example, you can handle physics situations such as masses on springs or the electrical oscillations of inductor-capacitor circuits with a differential equation like this:

$$y'' - ay = 0$$

In Part II, I show you how to tackle second order differential equations with a large arsenal of tools, such as the *Wronskian matrix determinant,* which will tell you if there are solutions to a second (or higher) order differential equation. Other tools I introduce you to include the method of *undetermined coefficients* and the method of *variation of parameters.*

After first and second order differential equations, it's natural to want to keep the fun going, and that means you'll be dealing with higher order differential equations, which I also cover in Part II. With these high-end equations, you find terms like d^ny/dx^n, where $n > 2$.

TIP

The derivative d^ny/dx^n is also written as $y^{(n)}$. Using the standard syntax, derivatives are written as y', y'', y''', y^{iv}, y^{v}, and so on. In general, the *n*th derivative of y is written as $y^{(n)}$.

Higher order differential equations can be tough; many of them don't have solutions at all. But don't worry, because to help you solve them I bring to bear the wisdom of more than 300 years of mathematicians.

Having Fun with Advanced Techniques

You discover dozens of tools in Part III of this book; all of these tools have been developed and proved powerful over the years. Laplace Transforms, Euler's method, integrating factors, numerical methods — they're all in this book.

These tools are what this book is all about — applying the knowledge of hundreds of years of solving differential equations. As you may know, differential equations can be broken down by type, and there's always a set of tools developed that allows you to work with whatever type of equation you come up with. In this book, you'll find a great many powerful tools that are just waiting to solve all of your differential equations — from the simplest to the seemingly impossible!