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EQUATIONS REPRESENTING PHYSICAL QUANTITIES

Some equations model systems or processes that occur in the real, physical world. Most of the variables that appear in these equations have dimensions, and they carry certain physical units. For example, a variable d describing distance has the dimension of length and carries a specific unit such as meters, microns, or miles. The numerical value of the variable d is given as a multiple of the unit we choose, and the specific unit is usually chosen so that the numerical values are convenient to work with.

Without a unit, the physical meaning of the numerical value associated with a dimensioned variable contains no useful information. For example, to say the distance between points A and B is "$d = 8$" is not useful for scientific and engineering purposes. We also have to specify a unit of length, such as $d = 8\,\text{in}$, $d = 8\,\text{m}$, or $d = 8$ light-years, each of which describes very different quantities in the physical world.

A number that does not carry any physical units, e.g., 1, –2.23, or π, is said to be *dimensionless*. There are some dimensionless quantities that nonetheless can carry units. One well-known example is an angle θ. Angles are dimensionless because they represent the ratio of two lengths, namely the subtended arc length on a circle divided by the radius r of that circle. The natural unit for θ is the radian. The value of the angle for one complete circular revolution is 2π radians, which follows from the fact that the corresponding arc length is the circumference of the circle, or $2\pi r$. Alternatively, the degree is a commonly

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used unit to measure angles. There are 360° in one complete circular revolution, so the conversion factor between radians and degrees is

$$1\,\text{rad} = 360°/2\pi \approx 57.3°. \tag{1.1}$$

We have a choice of which of these units to use.

At first, it might seem like keeping track of the units associated with each variable in an equation is an inconvenience, akin to carrying extra baggage. As explored in this chapter, however, the use of units in fact can help us to better understand equations that contain variables representing physical quantities. Keeping track of dimensions and units can also uncover errors and can simplify work. This theme recurs elsewhere in this book, especially in Chapter 6, where the topics of dimensional analysis and scaling are discussed.

Some units have long and interesting histories, which illustrate their importance in science, engineering, and commerce. In ancient times, balance scales were commonly used to measure weight. The unknown weight of an object was measured by counting the number of unit weights required to counterbalance it. The carob tree is grown in the Mediterranean region, and its fruit is a pod that contains multiple seeds. It was found that the weight of the carob seeds varied little from one to the next. Also, it was relatively easy to get a uniform set of seeds. The heavier or lighter seeds could be eliminated from the collection because their weight correlated well with their size.

So it became convenient to use a group of carob seeds of uniform size to counterbalance the unknown quantity on the other side of the scale. The weight of the carob seeds was also of a convenient magnitude for weighing small objects like gemstones. The relative weight of the unknown object was quite accurately expressed in terms of the equivalent number of carob seeds, and this practice became a standard for commerce. Measured in modern units, a typical carob seed has a mass of approximately 0.20 g. Today, the unit *carat* is used to measure the mass (or the equivalent weight) of gemstones. A carat is defined to be exactly 0.20 g, and its name is derived from the name of the carob tree and its seeds.

1.1 SYSTEMS OF UNITS

Many different systems of units have been devised. For most scientific and engineering work today, the preferred units are in the "SI" system. This designation comes from the French "Système International d'Unités" (International System of Units). SI units are based on quantities with the seven fundamental dimensions listed in Table 1.1. Note that three of these fundamental units, the meter, kilogram, and second, were carried over from the older MKS system of units for the quantities of length, mass, and time when the SI system was developed in 1960.

TABLE 1.1. The Seven Fundamental SI Units.

Dimension	SI Unit
Length	meter or metre (m)
Mass	kilogram (kg)
Time	second (s)
Electric current	ampere (A)
Thermodynamic temperature	kelvin (K)
Amount of substance	mole (mol)
Luminous intensity	candela (cd)

SI units have gained popularity for several reasons. First, they use prefixes (e.g., nano, milli, kilo, mega) based on powers of 10. Prefixes allow the introduction of related units that are appropriate over a wide range of scales. For example, the unit of 1 nm is equal to 10^{-9} m. The powers of 10 also make conversion relatively simple, for example, converting units of area:

$$1\,\mathrm{m}^2 = \left(1\times 10^9\,\mathrm{nm}\right)^2 = 10^{18}\,\mathrm{nm}^2. \tag{1.2}$$

In contrast, the imperial (sometimes called "British") system of units contains conversion factors that are usually not integer powers of 10. For example, to express 1 yd^2 in terms of square inches, we have to calculate 36×36:

$$1\,\mathrm{yd}^2 = (36\,\mathrm{in})^2 = 1296\,\mathrm{in}^2. \tag{1.3}$$

Another advantage of the SI system is that it contains many named, derived units such as the watt to measure power. The addition of these derived units is one of the major changes between the MKS and SI systems. The derived units in the SI system are *coherent*, that is, each one can be expressed in terms of a product of the fundamental units (or other derived units) and a numerical multiplier that is equal to 1. For example, the SI unit of electrical charge is the coulomb, and

$$\begin{aligned} &1 \text{ coulomb} = 1 \text{ ampere} \times 1 \text{ second} = 1\ \mathrm{A\cdot s} \\ &1 \text{ coulomb} = 1 \text{ farad} \times 1 \text{ volt} = 1\ \mathrm{F\cdot V} \\ &1 \text{ coulomb} = 1 \text{ volt} \times 1 \text{ second} \div 1 \text{ ohm} = 1\ \mathrm{V\cdot s\cdot \Omega^{-1}}. \end{aligned} \tag{1.4}$$

The derived SI unit for power is the watt, which is equal to 1 joule per second. On the other hand, a unit of power in the imperial system, 1 horsepower, equals to 550 ft·lb/s. The simple conversion factors in the SI system also make it easy to decompose all of the derived units back into integer powers of the fundamental units. For example,

$$\begin{aligned} 1\text{ newton} &= 1\,\text{m}\cdot\text{kg}\cdot\text{s}^{-2}\,(\text{force}),\\ 1\text{ watt} &= 1\,\text{m}^2\cdot\text{kg}\cdot\text{s}^{-3}\,(\text{power}),\\ 1\text{ ohm} &= 1\,\text{m}^2\cdot\text{kg}\cdot\text{s}^{-3}\cdot\text{A}^{-2}\,(\text{electrical resistance}). \end{aligned} \tag{1.5}$$

As discussed in Chapter 6, the decomposition illustrated in Equation 1.5 is particularly useful for dimensional analysis.

1.2 CONVERSION OF UNITS

Scientists and engineers are trained to work with SI units. Inevitably, however, we encounter units from other dimensional systems that require conversion back and forth to the SI system. For example, the speed of a car is commonly expressed in miles per hour or kilometers per hour, but rarely in the SI unit of meter per second. Similarly, household electrical energy usage is billed in kilowatt-hours rather than the SI unit of joules. Sometimes, the scale of the SI unit is not very convenient. For example, a kilogram is a very large unit in which to express the mass of an individual molecule, and a meter is a very short unit for interstellar distances. Rather than relying solely on the power-of-10 prefixes mentioned previously, more convenient, non-SI units like the atomic mass unit (amu or u) or light-year are sometimes used. Fortunately, the conversion of units is straightforward, as illustrated in the following example.

EXAMPLE 1.1

Convert 1 mi/h to the SI unit for speed, meter per second. There are 5280 ft per mile, 12 in per foot, and 2.54 cm per inch.

ANSWER

Unit conversion is readily accomplished with multiplication by a string of conversions factors, each of which is dimensionless and equals to 1, such as 1 = (2.54 cm)/(1.00 in) = 2.54 cm/in. We multiply together powers (or inverse powers) of the conversion factors so that all of the units "cancel," except for the desired result:

$$1\frac{\text{mi}}{\text{h}} = 1\frac{\text{mi}}{\text{h}} \times \frac{5280\,\text{ft}}{\text{mi}} \times \frac{12\,\text{in}}{\text{ft}} \times \frac{2.54\,\text{cm}}{\text{in}} \times \frac{1\,\text{m}}{100\,\text{cm}} \times \frac{1\,\text{h}}{60\,\text{min}} \times \frac{1\,\text{min}}{60\,\text{s}}. \tag{1.6}$$

Gathering the numerical terms,

$$1\,\text{mi/h} = \frac{12 \times 5280 \times 2.54}{100 \times 60 \times 60}\,\text{m/s} = 0.44704\,\text{m/s}. \tag{1.7}$$

As elementary as Example 1.1 seems, errors in unit conversion are not uncommon and can have disastrous consequences. A well-known example occurred on September 23, 1999, when an unmanned orbiting satellite approached Mars at too low an altitude and crashed into the red planet. A subsequent investigation by NASA revealed that engineers failed to properly convert the imperial system unit of force used to measure rocket thrust (the pound-force) into the SI unit force, the newton.

Another example of confusion caused by the improper conversion of units occurred over 300 years earlier in connection with Sir Isaac Newton's work on the theory of gravitation. In 1679, Newton consulted a sailor's manual to obtain numerical values that he used to check the predictions of his theory for the speed of the Moon as it orbits the Earth. That speed was known in Newton's time from the Moon's observed orbital period and its estimated distance from the Earth, which was deduced from the observation of eclipses. Newton, however, did not know that the term "mile" in the sailor's manual referred to a nautical mile, which is approximately 15% longer than the statute mile (5280 ft) with which he was familiar. This confusion led to a 15% discrepancy between his prediction for the speed of the Moon and the accepted value. Discouraged by this, Newton abandoned his correct approach and searched for an alternative theory. This detour delayed Newton's work on gravity by approximately 5 years. Eventually, of course, Newton discovered the error concerning units, and his theory of gravitation has become the basis for much of modern space flight.

1.3 DIMENSIONAL CHECKS AND THE USE OF SYMBOLIC PARAMETERS

Anytime we equate one term to another, they both must have the same dimensions for the expression to make physical sense. We cannot equate a term with the dimension of length to a term with the dimension of mass. Using the basic rules of algebra, we can extend this principle to say that whenever we add or subtract terms, they must also have the same dimensions. We say that such an expression is dimensionally consistent or dimensionally *homogeneous*. We can always add zero to or subtract zero from any equation. Whenever we do so, we will assume that the zero carries the appropriate dimensions.

If an equation is dimensionally inconsistent, we can recognize immediately that it must be flawed. The inconsistency might have arisen because the equation's construction was based on faulty principles, or because its derivation contained an algebraic error. The converse is not necessarily true. If an equation is dimensionally consistent, it does not mean that it is necessarily correct, only that it *could be* correct.

Consider the following alternative expressions both intended to describe the height y of a ball thrown in the air, as a function of time t,

$$y(t) = 3 + 2t - 4.9t^2 \tag{1.8}$$

and

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2, \quad y_0 = 3\,\mathrm{m},\, v_{0y} = 2\ \mathrm{m/s}, \text{and } g = 9.8\,\mathrm{m/s}^2, \tag{1.9}$$

where t is measured in seconds. These two expressions might appear equivalent at first glance, and Equation 1.8 is more compact. Retaining symbols as in Equation 1.9, however, has several important advantages. First, a quick, visual check confirms that we are adding and subtracting terms that all have the same units, in this case meters. Equation 1.8 is not dimensionally homogeneous unless we assume that the units are implied, i.e., in the first term, "3" really means "3 m," and similarly for the other numerical coefficients. It is easy to apply this assumption inconsistently, leading to errors. On the other hand, retaining symbols and checking units can bring our attention to a typographic or careless error. For example, the incorrect exponent can be spotted easily

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^3 \quad [\text{incorrect}], \tag{1.10}$$

because the last term has the incorrect units of m·sec instead of meters. Common errors like these can be quite difficult to uncover when the numerical format of Equation 1.8 is used, where the dimensions of the numerical factors are implied, rather than given explicitly. The use of the symbolic format as in Equation 1.9 avoids many of these problems.

The symbolic format of Equation 1.9 also allows the acceleration and initial height and initial velocity for the trajectory of the ball to be easily extracted. The initial height is $y(t)|_{t=0} = y_0$, and differentiating $y(t)$ with respect to time yields the initial velocity and the acceleration:

$$\begin{aligned} \left.\frac{dy}{dt}\right|_{t=0} &= v_{0y} \\ \frac{d^2y}{dt^2} &= -g \end{aligned}. \tag{1.11}$$

Equation 1.8 can also be differentiated with respect to time. With a proper choice of notation, however, symbols such as v_{0y} generally do a better job evoking the physical meaning of the parameters than the numerical values appearing in Equation 1.8. Usually, the more complicated the analytic expression, the greater the advantage of retaining the symbolic notation becomes.

If we assign numerical values to quantities appearing in equations, we also have to be careful about their units. For example, the expression (1 m + 1 cm) combines two quantities that have the dimension of length but are measured in different units. To reduce the expression correctly to 1.01 m or 101 cm (instead of "2") naturally requires proper unit conversion.

Finally, the basic expression for $y(t)$ in Equation 1.9 remains valid regardless of which system of units is chosen, which is not true for Equation 1.8. To convert the expression for $y(t)$ in Equation 1.9 into units of feet, we only need to convert the given parameters to a new set with the desired units. This is easy to do using the method illustrated in Example 1.1. Using Equation 1.9, the values of the coefficients become $y_0 = 9.84\,\text{ft}$, $v_{0y} = 6.56\,\text{ft/s}$, and $g = 32.2\,\text{ft/s}^2$.

To summarize, the symbolic format of Equation 1.9 is preferred over the numerical format of Equation 1.8 because it facilitates dimensional checks, makes no assumptions about the dimensions of the parameters, and is easier to translate between systems of units. For computational work, the symbolic format of Equation 1.9 is also preferable to the "hard-coded" format of Equation 1.8, because it provides an easier and less error-prone way to pass the values of the coefficients between program modules such as subroutines.

1.4 ARGUMENTS OF TRANSCENDENTAL FUNCTIONS

Because added or subtracted terms must have the same dimensions, the following section will show that we can infer that the arguments of many transcendental functions are dimensionless. The trigonometric functions like sine, cosine, tangent, and secant are transcendental, as are the exponential functions, which also include hyperbolic sine and hyperbolic cosine. They are distinguished from algebraic functions, which include polynomials, square roots, and other simpler functions.

These functions are often expressed in terms of an infinite series expansion. Consider the well-known series expansion for the sine function:

$$\sin(u) = u - \frac{u^3}{6} + \frac{u^5}{120} - \frac{u^7}{5040} + \ldots. \tag{1.12}$$

Because the coefficients of 1/6, 1/120, etc., are dimensionless numbers, u cannot carry any physical units either. Suppose, for example, that u had the dimensions of power, measured in units of "watts." Because we cannot subtract a term with units of watts^3 from a term with units of watts, Equation 1.12 would not make physical sense. Therefore, the argument of the transcendental function $\sin(u)$ must always be dimensionless.

Typically, the arguments of trigonometric functions are angles. As mentioned earlier, angles are dimensionless, and their SI unit is the radian. It is important to remember that many common mathematical formulas involving trigonometric functions such as the series expansion

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} + \ldots \tag{1.13}$$

or the derivative

$$\frac{d}{d\theta}\sin\theta = \cos\theta \tag{1.14}$$

are not valid unless the angle θ is measured in radians. For example, Equation 1.12 implies that the sine of a small angle ($\ll 1\,\text{rad}$) is approximately equal to the angle itself, $\sin\theta \approx \theta$. With the use of a scientific calculator, the reader can easily verify that, to five significant figures, sin(0.01 rad) = 0.01000. On the other hand, sin(0.01 °) = 0.00017453, which reflects the conversion factor ($2\pi/360 \approx 0.01745$) stated in Equation 1.1.

Next, consider an exponential function and its series expansion:

$$f(t) = e^{3t} = 1 + 3t + 4.5t^2 \ldots \tag{1.15}$$

Equation 1.15 is dimensionally consistent provided that t is a dimensionless variable. If, however, the variable t represents time measured in seconds, then Equation 1.15 does not make sense unless we assume that the units are implied, i.e., "3" really means "$3\,\text{s}^{-1}$" and "4.5" really means "$4.5\,\text{s}^{-2}$." The discussion following Equation 1.8 showed how this type of assumption can lead to problems. Instead, it is better to write

$$f(t) = e^{\lambda t} = 1 + \lambda t + \frac{(\lambda t)^2}{2} + \ldots \tag{1.16}$$

with $\lambda = 3\text{s}^{-1}$. The argument of the transcendental function in Equation 1.16 is now the product λt, which is dimensionless, as is the third term in the expansion, $\frac{1}{2}(\lambda t)^2$.

Among the transcendental functions, the logarithm provides an interesting special case. Consider $\log(x/a)$, where the ratio (x/a) is dimensionless. For example, both x and a might have the dimension of length, measured in units of meters. Suppose that $x = 3\,\text{m}$ and $a = 2\,\text{m}$. Logarithms reduce the operation of division to subtraction, i.e.,

$$\log\left(\frac{x}{a}\right) = \log\left(\frac{3\,\text{m}}{2\,\text{m}}\right) = \log\left(\frac{3}{2}\right) = \log(3) - \log(2). \tag{1.17}$$

All of the operations in Equation 1.17 are valid, and note that Equation 1.17 holds regardless of the logarithm's base, e.g., 10, 2, or e.

We might encounter a symbolic expression containing a term $\log(x)$, where x is not dimensionless. We cannot immediately conclude that the *entire* expression is incorrect. The expression might also include another term of the form $-\log(a)$, or equivalently $+\log(1/a)$, where a has the same units as x. Then, even though the variable x appearing in $\log(x)$ is dimensioned, the entire expression can be correct.

1.5 DIMENSIONAL CHECKS TO GENERALIZE EQUATIONS

The use of dimensional checks allows us to generalize equations and even generate new ones. Suppose we use integration by parts or a table of integrals and find

$$F(x) = \int xe^{x}dx = e^{x}(x-1)+C, \tag{1.18}$$

where C is a constant. We know that the variable x in Equation 1.18 must be dimensionless, because it is the argument of the exponential function. Using dimensional checks, we can generalize Equation 1.18 to evaluate

$$G(x,a) = \int xe^{ax}dx \tag{1.19}$$

without the need for additional integration. The product (ax) must be dimensionless because it appears as the argument of the exponential Equation 1.19. We are free to assume that the variable x has dimensions of length, measured in the unit of meters, and we will do so. In that case, a must have units of m^{-1}. So the task is to use dimensional checks to place the appropriate power of a into each term of the right-hand side of Equation 1.18. Recognizing that each of the terms of $G(x,a)$ must have units of meters squared (because the differential dx carries the same units as x, see exercise 1.12), we can infer that

$$G(x,a) = \int xe^{ax}dx = \frac{e^{ax}}{a}\left(x-\frac{1}{a}\right)+C'. \tag{1.20}$$

Along with the dimensional check, we also used the fact that $G(x, a)$ reduces to $F(x)$ when $a = 1\,\mathrm{m}^{-1}$. Naturally, for a further check, we can differentiate the right-hand side of Equation 1.20. Although we do not know the values of the integration constants C and C', we do know their units. C is dimensionless, and C' carries the same units as x/a, namely meters squared.

Equation 1.20, which was generated with the aid of a dimensional check, can be extended to evaluate related integrals, up to the additive constant (see exercises 1.2 and 1.13). The partial derivative of $G(x, a)$ with respect to a yields

$$\begin{aligned}\frac{\partial G(x,a)}{\partial a} &= \int x\left(\frac{\partial e^{ax}}{\partial a}\right)dx = \int x^2 e^{ax}\,dx \\ &= \frac{\partial}{\partial a}\left[\frac{e^{ax}}{a}\left(x-\frac{1}{a}\right)+C'\right] = \frac{e^{ax}}{a}\left(x^2-\frac{2x}{a}+\frac{2}{a^2}\right)\end{aligned} \tag{1.21}$$

Equation 1.21 implies that

$$\int x^2 e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^2-\frac{2x}{a}+\frac{2}{a^2}\right)+C''. \tag{1.22}$$

Starting with Equation 1.18, the use of dimensional checks followed by partial differentiation yielded a new expression given in Equation 1.22. This sequence of operations can be quite useful for evaluating a variety of indefinite and definite integrals, but we always have to be very careful to follow the rules of calculus. In the example of Equation 1.21, it would *not* be valid to differentiate with respect to x, because x is the integration variable.

1.6 OTHER TYPES OF UNITS

Up to this point, Chapter 1 has focused on variables that carry physical units that describe length, mass, electrical charge, etc. This allowed us to draw useful inferences about equations composed of these variables. There are many other types of units, however, that are not associated with the physical sciences. One example is a monetary unit, like a dollar or a euro. Another example is a unit like the number of soldiers per battalion, which might be used in a logistical calculation to find the required amount of rations for a month.

Although units like \$ or battalion^{-1} are not part of the SI system, they are often very convenient (for example, see exercise 1.14). Equations containing variables that carry these types of units still must be dimensionally homogeneous, provided that the system of units is applied in a consistent manner. Thus, the dimensional checks and unit conversion methods introduced previously in this chapter apply.

Some equations seem to model the physical world quite well but appear to be dimensionally inconsistent. For example, we might find that the time t it takes to finish a task in the office is well described by the equation "t = 20 min + three times the number of phone call interruptions received." This equation seems to be dimensionally inconsistent, because an apparently dimensionless quantity (three times the number of phone calls) is added to a dimensioned quantity (20 min). Whenever this type of expression accurately models the physical world, however, there is an implied conversion factor. In this case, the implied conversion factor is 3 min/phone call.

1.7 SIMPLIFYING INTERMEDIATE CALCULATIONS

Calculations often require many intermediate steps to obtain the desired result. Sometimes when performing calculations with symbolic variables, it is convenient to *temporarily* choose a dimensional system (i.e., a set of units) so that the numerical values of some of the physical constants are equal to 1. This trick can simplify algebraic manipulation, whether it is performed with paper and a pencil or with symbolic manipulation software. Symbolic coefficients, such as those appearing in Equation 1.9, can also temporarily be set equal to 1. After the calculation is completed, the symbols are replaced to make the result dimensionally consistent, as was done to derive Equation 1.20. The entire procedure is illustrated in Example 1.2.

Setting quantities equal to 1, even temporarily, seems like it could lead to incorrect or confusing results. Trouble can be avoided, however, by following these two rules:

Rule 1: Never set a dimensionless quantity equal to 1.

Violating this rule clearly can lead to logical inconsistencies. For example, if we set the dimensionless number 2 equal to 1, we can immediately write an incorrect equation such as "$1 + 1 = 4$." To perform rough *estimates* (as opposed to exact calculations), we sometimes neglect factors of 2, 4, π, and so on. Estimation is discussed in Chapter 5 and is not our focus here.

Rule 2: When setting a collection of dimensioned quantities equal to 1, never choose this group to be sufficiently large so that a dimensionless product can be formed from them. To do so would result in a dimensionless quantity, i.e., that product, being set equal to 1 in violation of rule 1.

The meaning of rule 2 is illustrated by the following case. Suppose we temporarily set each of three dimensioned quantities q_1, q_2, and q_3 equal to 1 and then we find any exponents a, b, and c such that the product $q_1^a \times q_2^b \times q_3^c$ is dimensionless. Then, rule 2 says that we have gone too far. We need to restore at least one of the q's back to its original value. Rule 2 also implies that we should never simultaneously set two different quantities of the same dimension equal to 1, for example, the height and width of a rectangle. Rule 2 ensures that there is only a single way to return the dimensioned variables back into the final expression when making it (explicitly) dimensionally consistent. This uniqueness property is further explored in exercise 1.9.

EXAMPLE 1.2

To illustrate why it is convenient to temporarily set constants equal to 1, consider the example of Compton scattering, named in honor of the American physicist Arthur H. Compton who published a paper on this effect in 1923. Compton scattering describes the scattering of an X-ray photon from a free electron, which is assumed to initially be at rest. The X-ray photon loses some of its energy to the electron, which results in a reduction of the frequency of the scattered X-ray. The amount of energy the X-ray photon loses (and therefore the frequency of the scattered radiation) depends on the angle through which it is scattered. If it is scattered straight back in the direction from which it came (i.e., $\phi = 180°$), then it loses the maximum possible amount of energy. Compton scattering is important for many applications, including medical imaging methods that use X-rays. One such method is computed tomography. The concepts from modern physics that are applied to set up the equations are described in detail in many physics books, and we only give a brief outline here.

In this example, we will solve the equations for Compton scattering with two procedures: (a) using standard units and (b) using a dimensional system where Planck's constant and the speed of light each equal to 1. Part (b) of this example illustrates the algebraic simplification that can result.

ANSWER

(a) When the X-ray photon scatters from the electron, conservation of energy yields the relation

$$hf_0 + Mc^2 = hf + E, \tag{1.23}$$

where h is Planck's constant, f_0 is the frequency of the incident X-ray, M is the rest mass of the electron, c is the speed of light, f is the frequency of the scattered X-ray, and E is the final, total energy of the electron. There is also an identity from the special theory of relativity that relates the final energy, mass, and momentum P of the electron:

$$E^2 = \left(Mc^2\right)^2 + (Pc)^2. \tag{1.24}$$

Equations 1.23 and 1.24 are both dimensionally consistent. The magnitudes of the initial and final values of the momentum of the X-ray are given by (hf_0/c) and (hf/c), respectively. The X-ray is scattered through an angle ϕ, so that the conservation of momentum and the law of cosines imply

$$P^2 = \left(\frac{hf_0}{c}\right)^2 + \left(\frac{hf}{c}\right)^2 - 2\left(\frac{h^2 f_0 f}{c^2}\right)\cos\phi. \tag{1.25}$$

The final energy E and momentum P of the electron can be eliminated from Equations 1.23–1.25. After some algebra (see exercise 1.10), the desired result for the final frequency of the X-ray is obtained:

$$f = \frac{f_0}{1 + \dfrac{hf_0}{Mc^2}(1 - \cos\phi)}. \tag{1.26}$$

(b) The appearance of Equations 1.23–1.25 is simplified if we temporarily set Planck's constant h and the speed of light c both equal to 1. (This is clearly not true in SI units, in which $h = 6.626068 \times 10^{-34}$ J·s and $c = 2.9979 \times 10^8$ m/s.) The simplified versions of Equations 1.23–1.25 become

$$\begin{aligned} f_0 + M &= f + E, \\ E^2 &= M^2 + P^2, \\ P^2 &= f_0^2 + f^2 - 2f_0 f \cos\phi. \end{aligned} \tag{1.27}$$

The algebraic manipulation is now considerably simpler, because we do not have to carry the factors of h and c through each step. After the simplified algebra, we find that

$$"f" = \frac{f_0}{1 + \dfrac{f_0}{M}(1 - \cos\phi)} \tag{1.28}$$

We know that Equation 1.28 cannot be the final, correct result for f, because it is not dimensionally consistent; the dimensioned term $(f_0/M) \times (1 - \cos\phi)$ is added to the dimensionless quantity 1 in the denominator. We must properly reinsert the dimensioned quantities into the final expression to get the correct answer.

To replace the correct factors of h and c to Equation 1.28, it is convenient to decompose each of the dimensions of each of the variables into fundamental SI units. It is conventional to denote the units of a variable by placing square brackets around it. For the variables in this example,

$$\begin{aligned} [f_0] &= [f] = \mathrm{s}^{-1}, \\ [M] &= \mathrm{kg}, \\ [\phi] &= \mathrm{rad}, \\ [c] &= \mathrm{m \cdot s^{-1}}, \\ [h] &= \mathrm{J \cdot s} = \mathrm{m^2 \cdot kg \cdot s^{-1}}. \end{aligned} \tag{1.29}$$

Naturally, $[f_0/M] = [f_0]/[M]$, so that the ratio (f_0/M) has units $\mathrm{kg^{-1}\cdot s^{-1}}$. We now reintroduce the powers of h and c needed for Equation 1.28 to be dimensionally consistent. The result is Equation 1.26. This can be seen by inspection, but we can also reach the same conclusion more methodically, as described next.

Because f_0 and f both have the same dimensions, all that is required to make Equation 1.28 dimensionally consistent is to replace the powers of h and c so that the term (f_0/M) in the denominator becomes dimensionless. We seek values of the exponents a and b such that

$$\left[h^a c^b f_0 M^{-1}\right] = 1 = 1\,\mathrm{m}^0 \times 1\,\mathrm{s}^0 \times 1\,\mathrm{kg}^0, \tag{1.30}$$

where we omitted the other four fundamental units (like amperes and candelas) because they do not enter into this particular equation. From Equations 1.29 and 1.30, we obtain

$$\left[h^a c^b f_0 M^{-1}\right] = \left(\mathrm{m^2\cdot kg\cdot s^{-1}}\right)^a \left(\mathrm{m\cdot s^{-1}}\right)^b \left(\mathrm{s^{-1}}\right)\left(\mathrm{kg^{-1}}\right) = \mathrm{m}^{2a+b}\mathrm{s}^{-a-b-1}\mathrm{kg}^{a-1}. \tag{1.31}$$

Equating the exponents of each factor in Equations 1.30 and 1.31 yields a set of linear equations:

$$\begin{aligned} 2a + b &= 0, \\ -a - b - 1 &= 0, \\ a - 1 &= 0. \end{aligned} \tag{1.32}$$

These can be readily solved to give $a = 1$ and $b = -2$, which agree with Equation 1.26.

In Example 1.2, temporarily choosing a dimensional system so that $c = 1$ and $h = 1$ has allowed us to get the correct answer while simplifying the algebra. Note that from Equation 1.29, simultaneously setting both $c = 1$ and $h = 1$ did not violate rule 2, because no power of c can cancel the basic SI unit kilogram appearing in h to form a dimensionless quantity.

Although the procedure illustrated in Example 1.2b can simplify algebraic calculation, it is not to everyone's liking. Even if the two stated rules are followed carefully, the ability to perform dimensional checks on each intermediate step of the calculation is lost. Instead, many prefer to solve the problem using standard units as in Example 1.2a but with variable substitutions such as

$$\begin{aligned} u &= Mc^2, \\ v &= hf_0, \\ w &= hf. \end{aligned} \tag{1.33}$$

This method allows for algebraic simplification and still permits dimensional checks at each step. One drawback of the substitutions of Equation

1.33 is that the constants can reappear in the expression when we calculate derivatives or integrals. For example, the constant c reappears if we need to differentiate some function y with respect to M:

$$\frac{dy}{dM} = c^2 \frac{dy}{du}. \tag{1.34}$$

The choice of which, if any, of these simplification methods to use depends on the complexity of the specific problem and is mainly a matter of personal preference.

EXERCISES

(1.1) Convert 1 light-year into meters. Use the speed of light $c = 2.9979 \times 10^8$ m/s.

(1.2) Suppose that

$$\int x^3 \sin x dx = \left(3x^2 - 6\right)\sin x + \left(6x - x^3\right)\cos x + C.$$

(a) Without further integration, use dimensional checks to evaluate $\int x^3 \sin(ax)dx$. Assume that x has the dimensions of length.
(b) Use the resulting expression to evaluate $\int x^4 \cos(ax)dx$ by partial differentiation.

(1.3) A dimensionless variable u is raised to the power b. Show that the exponent b must also be dimensionless. Hint: use $u^b = \exp(\ln u^b) = \exp(b \ln u)$.

(1.4) Suppose that $g = 9.8\,\text{m/s}^2$, $v = 50\,\text{m/s}$, $h = 12\,\text{m}$, $\omega = 30\,\text{rad/s}$, $\theta = 2.5\,\text{rad}$, and t is time measured in seconds. Identify which of the following equations cannot be valid based on a dimensional check:

(a) $v = ht$;
(b) $v = \sqrt{2gh}$;
(c) $\theta = \sin \omega t + \dfrac{h}{vt}$;
(d) $h = vt + \exp(-gt)$.

(1.5) Each of the following mathematical expressions has one term that is dimensionally inconsistent with the other two. Find the inconsistent term, and correct it by inserting the appropriate power of a. Assume x is a length:

(a) $a^2 e^{ax} + \dfrac{9\sin(ax)}{x^2} - 3(ax)^5$;

(b) $\dfrac{\sqrt{ax}}{2+(ax)^2}+\dfrac{4\sin ax}{x}+\exp\left[-\dfrac{1}{2}(ax)^2\right]$;

(c) $x\arccos\left(a^{0.25}\sqrt{x}\right)-x\sin\left(ax^2\right)+3$.

(1.6) Look up the definitions of each of these derived SI units: (a) henry, (b) gray, and (c) lux. For each unit, describe what physical quantity it measures and then express the unit as a monomial containing powers of the seven fundamental units (i.e., see Equation 1.5). (A "monomial" is a polynomial with only one term.)

(1.7) Consider the quadratic equation $ax^2 + bx + c = 0$. Assume that x has the dimensions of length and that a is dimensionless. What are the dimensions of b and c? Verify that the solution

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

is dimensionally consistent. If instead b is dimensionless, then what are the dimensions of a and c? In that case, is the solution for x still dimensionally consistent?

(1.8) Suppose a particle of mass m and kinetic energy E collides with another particle of mass m, which is initially at rest. We analyze the problem using the special theory of relativity, and to simplify the resulting equations, we temporarily set $m = 1$ and $c = 1$, where c is the speed of light. After some algebra, we find that the kinetic energy in the center-of-mass reference frame is given by

$$E_{cm}=2\left(\sqrt{1+\frac{E}{2}}-1\right).$$

Reintroduce factors of m and c so that the expression is (explicitly) dimensionally consistent.

(1.9) To reduce the algebraic complexity of a problem, we set three dimensioned physical constants q_1, q_2, and q_3 each equal to 1, similar to Example 1.2. After completing the algebra, we then go on to replace powers of q_1, q_2, and q_3 so that the final expression is dimensionally consistent. We find that for one of the terms in our expression, we can replace the physical constants in two distinct ways: $\sqrt{q_1}\times q_2\times q_3^2$ and $q_1^{3/2}\times\sqrt{q_2}\times q_3$ both make the expression dimensionally consistent. What went wrong? Hint: show that rule 2 was violated. (By this same method, show that, in general, whenever the replacement of the physical constants is not unique, rule 2 must be violated. This implies that if rule 2 is satisfied, then the replacement is unique.)

(1.10) Derive Equation 1.26 from Equations 1.23–1.25. Start by isolating E in Equation 1.23 and then squaring the result. Then derive Equation

1.28 from Equation 1.27. Do you find that the simplification justifies the potential pitfalls of setting the physical constants equal to 1?

(1.11) Interpret each of the following equations from a dimensional perspective. For example, the Pythagorean theorem $c^2 = a^2 + b^2$ is dimensionally consistent only if a, b, and c all have the same units, such as length:

(a) formula for an ellipse: $\left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{b}\right)^2 = 1$;

(b) derivative of a function raised to a power: $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$.

(1.12) Show that the differential dx must have the same units as the variable x. Hint: consider the integral $\int dx$.

(1.13) Suppose we know that $\int_{-\infty}^{\infty} \exp(-x^2)\,dx = \sqrt{\pi}$.

(a) Use a dimensional check to evaluate $\int_{-\infty}^{\infty} \exp(-ax^2)\,dx$.

(b) Use partial differentiation to evaluate $\int_{-\infty}^{\infty} x^4 \exp(-x^2)\,dx$.

(1.14) Suppose that four painters can paint two houses in 2 days. How many days does it take for three painters to paint five houses? Hint: form an expression for the house-painting rate r_p, measured in non-SI units:

$$r_p = \frac{2 \text{ houses}}{4 \text{ painters} \times 2 \text{ days}} = 0.25 \text{ house} \cdot \text{painter}^{-1}\text{day}^{-1}.$$

FURTHER READING

http://physics.nist.gov/cuu/Units/index.html

The web pages located in this URL from the National Institute of Standards and Technology contain many helpful links and much useful information about systems of units.

Szirtes T. 1997. *Applied Dimensional Analysis and Scaling*. New York: McGraw-Hill.

This book gives a very thorough discussion about dimensional consistency and systems of units. Among many other topics, Szirtes provides an interesting discussion regarding the dimensions of "dimensionless" quantities, which the reader might find useful for interpreting Equation 1.30.