1

First Principles

This first chapter presents an overview of some basic ideas. Later chapters will expand on these ideas and clarify the subtleties that are frequently encountered. Practical examples will be emphasized. The data to be processed is presented in a sampled-time or sampled-frequency format, using a number of samples that is usually not more than $2^{11} = 2048$. The following "shopping list" of operations is summarized as follows:

1. The user inputs, from a tabulated or calculated sequence, a set of numerical values, or possibly two sets, each with $N = 2^M (M = 3, 4, 5, \ldots, 11)$ values. The sets can be real or complex in the "time" or "frequency" domains, which are related by the Discrete Fourier Transform (DFT) and its companion, the Inverse Discrete Fourier Transform (IDFT). This book will emphasize time and frequency domains as used in electronic engineering, especially communications. The reader will become more comfortable and proficient in both domains and learn to think simultaneously in both.
2. The sequences selected are assumed to span one period of an eternal steady-state repetitive sequence and to be highly separated from

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adjacent sequences. The DFT (discrete Fourier transform), and DFS (discrete Fourier series) are interchangeable in these situations.

3. The following topics are emphasized:
   a. Forward transformation and inverse transformation to convert between "frequency" and "time".
   b. Spectral leakage and aliasing.
   c. Smoothing and windowing operations in time and frequency.
   d. Time and frequency scaling operations.
   e. Power spectrum and cross-spectrum.
   f. Multiplication and convolution using the DFT and IDFT.
   g. Relationship between convolution and multiplication.
   h. Autocorrelation and cross-correlation.
   i. Relations between correlation and power spectrum using the Wiener-Khintchine theorem.
   j. Filtering or other signal-processing operations in the time domain or frequency domain.
   k. Hilbert transform and its applications in communications.
   l. Gaussian (normal) random noise.
   m. The discrete differential (difference) equation.

The sequences to be analyzed can be created by internal algorithms or imported from data files that are generated by the user. A library of such files, and also their computed results, can be named and stored in a special hard disk folder.

The DFT and IDFT, and especially the FFT and IFFT, are not only very fast but also very easy to learn and use. Discrete Signal Processing using the computer, especially the personal computer, is advancing steadily into the mainstream of modern electrical engineering, and that is the main focus of this book.

SEQUENCE STRUCTURE IN THE TIME AND FREQUENCY DOMAINS

A time-domain sequence $x(n)$ of infinite duration $-\infty \leq n \leq +\infty$ that repeats at multiples of N is shown in Fig. 1-1a, where each $x(n)$ is uniquely

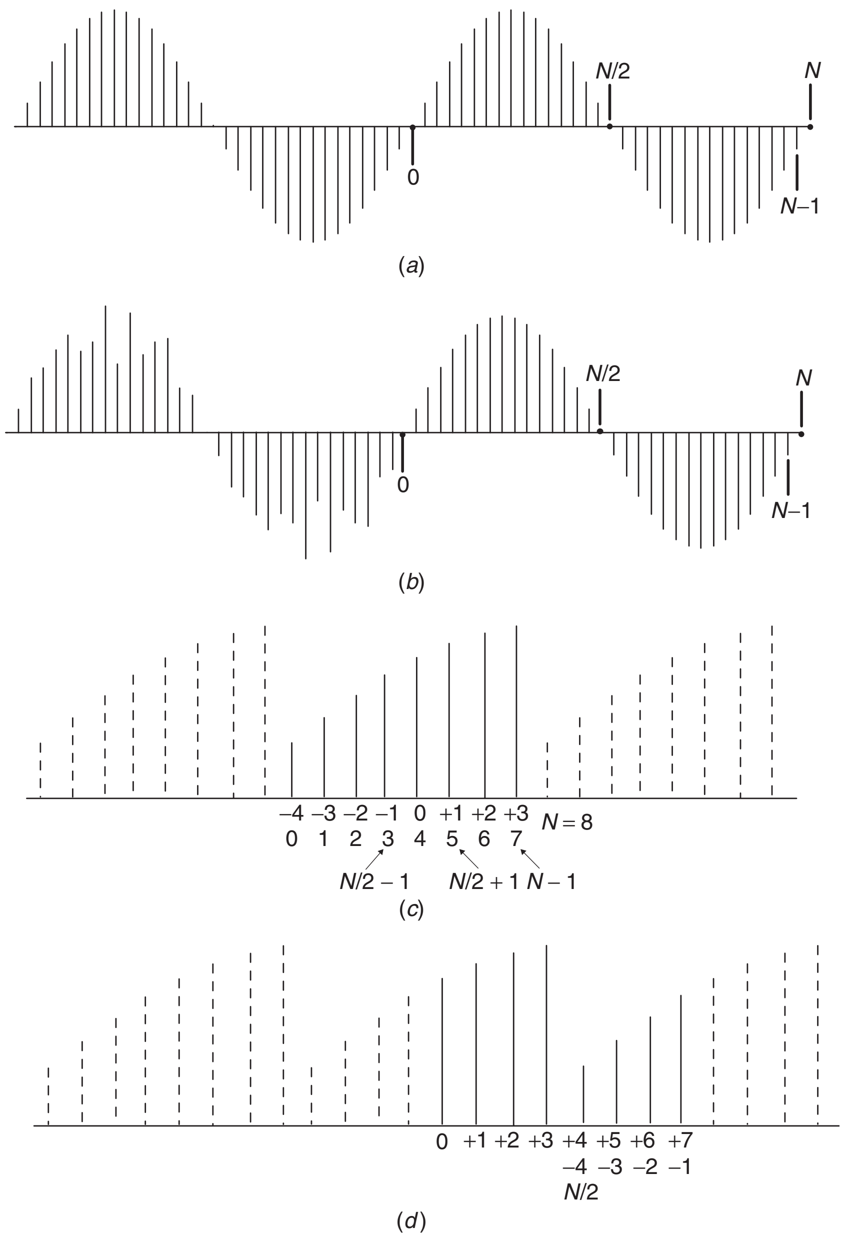

Figure 1-1 Infinite sequence operations for wave analysis. (a) The segment of infinite periodic sequence from 0 to $N - 1$. The next sequence starts at N. (b) The Segment of infinite sequence from 0 to $N - 1$ is not periodic with respect to the rest of the infinite sequence. (c) The two-sided sequence starts at -4 or 0. (d) The sequence starts at 0.

identified in both time and amplitude. If the sequence is nonrepeating (random), or if it is infinite in length, or if it is periodic but the sequence is not chosen to be exactly one period, then this segment is not one period of a truly periodic process, as shown in Fig. 1-1b. However, the wave analysis math assumes that the part of the wave that is selected is actually periodic within an infinite sequence, similar to Fig. 1-1a. The selected sequence can then perhaps be referred to as "pseudo-periodic", and the analysis results are correct for that sequence. For example, the entire sequence of Fig. 1-1b, or any segment of it, can be analyzed exactly as though the selected segment is one period of an infinite periodic wave. The results of the analysis are usually different for each different segment that is chosen. If the 0 to $N - 1$ sequence in Fig. 1-1b is chosen, the analysis results are identical to the results for 0 to $N - 1$ in Fig. 1-1a.

When selecting a segment of the data, for instance experimentally acquired values, it is important to be sure that the selected data contains the amount of information that is needed to get a sufficiently accurate analysis. If amplitude values change significantly between samples, we must use samples that are more closely spaced. There is more about this later in this chapter.

It is important to point out a fact about the time sequences $x(n)$ in Fig. 1-1. Although the samples are shown as thin lines that have very little area, each line does represent a definite amount of energy. The sum of these energies, within a unit time interval, and if there are enough of them so that the waveform is adequately represented (the Nyquist and Shannon requirements) [Stanley, 1984, p. 49], contains very nearly the same energy per unit time interval; in other words very nearly the same average power (theoretically, exactly the same), as the continuous line that is drawn through the tips of the samples [Carlson, 1986, pp. 351 and 624]. Another way to look at it is to consider a single sample at time (n) and the distance from that sample to the next sample, at time $(n + 1)$. The area of that rectangle (or trapezoid) represents a certain value of energy. The value of this energy is proportional to the length (amplitude) of the sample. We can also think of each line as a Dirac "impulse" that has zero width but a definite area and an amplitude $x(n)$ that is a measure of its energy. Its Laplace transform is equal to 1.0 times $x(n)$.

If the signal has some randomness (nearly all real-world signals do), the conclusion of adequate sampling has to be qualified. We will see in later chapters, especially Chapter 6, that one record length (N) of such a signal may not be adequate, and we must do an averaging operation, or other more elaborate operations, on many such records.

Discrete sequences can also represent samples in the frequency domain, and the same rules apply. The power in the adequate set of individual frequencies over some specified bandwidth is almost (or exactly) the same as the power in the continuous spectrum within the same bandwidth, again assuming adequate samples.

In some cases it will be more desirable, from a visual standpoint, to work with the continuous curves, with this background information in mind. Figure 1-6 is an example, and the discrete methods just mentioned are assumed to be still valid.

TWO-SIDED TIME AND FREQUENCY

An important aspect of a periodic time sequence concerns the relative time of occurrence. In Fig. 1-1a and b, the "present" item is located at $n = 0$. This is the reference point for the sequence. Items to the left are "previous" and items to the right are "future". Figure 1-1c shows an 8-point sequence that occurs between -4 and $+3$. The "present" symbol is at $n = 0$, previous symbols are from -4 to -1, and future symbols are from $+1$ to $+3$. In Fig. 1-1d the same sequence is shown labeled from 0 to $+7$. But the $+4$ to $+7$ values are observed to have the same amplitudes as the -4 to -1 values in Fig. 1-1c. Therefore, the $+4$ to $+7$ values of Fig. 1-1d should be thought of as "previous" and they may be relabeled as shown in Fig. 1-1d. We will use this convention consistently throughout the book. Note that one location, $N/2$, is labeled both as $+4$ and -4. This location is special and will be important in later work. In computerized waveform analysis and design, it is a good practice to use $n = 0$ as a starting point for the sequence(s) to be processed, as in Fig. 1-1d, because a possible source of confusion is eliminated.

A similar but slightly different idea occurs in the frequency-domain sequence, which is usually a two-sided spectrum consisting of positive- and negative-frequency harmonics, to be discussed in detail later. For example, if Fig. 1-1c and d are frequency values $X(k)$, then -4 to -1 in Fig. 1-1c and $+4$ to $+7$ in Fig. 1-1d are negative frequencies. The value at

$k = 0$ is the dc component, $k = \pm 1$ is the $\pm$ *fundamental* frequency, and other $\pm k$ values are $\pm$ *harmonics* of the $k = \pm 1$ value. The frequency $k = \pm N/2$ is special, as discussed later. Because of the assumed steady-state periodicity of the sequences, the Discrete Fourier Transform, often *correctly* referred to in this book as the Discrete Fourier Series, and its inverse transform are used to travel very easily between the time and frequency domains.

An important thing to keep in mind is that in all cases, in this chapter or any other where we perform a summation (Σ) from 0 to $N - 1$, we assume that all of the *significant* signal and noise energy that we are concerned with lies within those boundaries. We are thus relieved of the integrations from $-\infty$ to $+\infty$ that we find in many textbooks, and life becomes simpler in the discrete 0 to $N - 1$ world. It also validates our assumptions about the steady-state repetition of sequences. In Chapters 3 and 4 we look at aliasing, spectral leakage, smoothing, and windowing, and these help to assure our reliance on 0 to $N - 1$. We can also increase N by $2^M (M = 2, 3, 4, \ldots)$ as needed to encompass more time or more spectrum.

DISCRETE FOURIER TRANSFORM (SERIES)

A typical example of discrete-time $x(n)$ values is shown in Fig. 1-2a. It consists of 64 equally spaced real-valued samples $0 \le n \le 63$ of a sine wave, peak amplitude $A = 1.0$ V, to which a dc bias of Vdc $= +1.0$ V has been added. Point $n = N = 64$ is the beginning of the next sine wave plus dc bias. The sequence $x(n)$, including the dc component, is

$$x(n) = A \sin\left(2\pi \frac{n}{N} K_x\right) + \text{Vdc} \qquad \text{volts} \tag{1-1}$$

where K_x is the number of cycles per sequence length: in this example, 1.0. To find the frequency spectrum $X(k)$ for this $x(n)$ sequence (Fig. 1-2b), we use the DFT of Eq. (1-2) [Oppenheim et al., 1983, p. 321]:

$$X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi \frac{n}{N} \cdot k} \qquad \text{volts}, \qquad k = 0 \text{ to } N - 1 \tag{1-2}$$

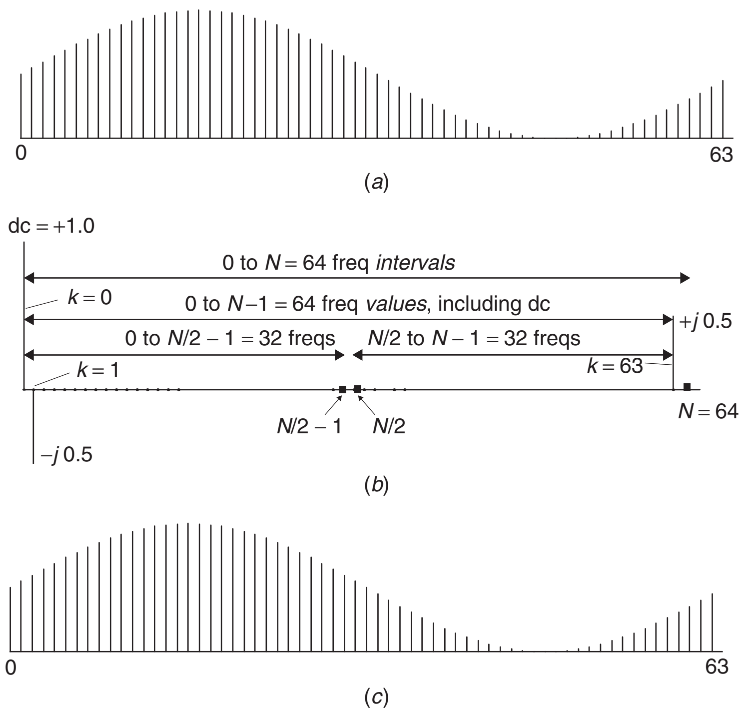

Figure 1-2 Sequence (a) is converted to a spectrum (b) and reconverted to a sequence (c). (a) 64-point sequence, sine wave plus dc bias. (b) Two-sided spectrum of w to count freq part (a) showing ho values and frequency intervals. (c) The spectrum of part (b) is reconverted to the time sequence of part (a).

In this equation, for each discrete value of (k) from 0 to $N-1$, the function $x(n)$ is multiplied by the complex exponential, whose magnitude $=$ 1.0. Also, at each (n) a constant negative (clockwise) phase lag increment $(-2\pi nk/N)$ radians is added to the exponential. Figure 1-2b shows that the spectrum has just two lines of amplitude $\pm j\ 0.5$ at $k=1$ and 63, which is correct for a sine wave of frequency 1.0, plus the dc at $k=0$.

These two lines combine coherently to produce a real sine wave of amplitude $A=1.0$. The peak power in a 1.0 ohm resistor is not the sum of the peak powers of the two components, which is $(0.5^2+0.5^2)=0.5$ W; instead, the peak power is the square of the sum of the two components, which is $(0.5+0.5)^2=1.0$ W. If the spectrum component $X(k)$ has a real part and an imaginary part, the real parts add coherently and the imaginary parts add coherently, and the power is complex (real watts and imaginary vars). There is much more about this later.

If $K_x = 1.2$ in Eq. (1-1), then 1.2 cycles would be visible, the spectrum would contain many frequencies, and the final phase would change to $(0.2 \cdot 2\pi)$ radians. The value of the phase angle in degrees for each complex $X(k)$ is

$$\phi(k) = \arctan\left(\frac{\mathrm{Im}\left(X(k)\right)}{\mathrm{Re}\left(X(k)\right)}\right) \cdot \frac{180}{\pi} \qquad \text{degrees} \tag{1-3}$$

For an example of this type of sequence, look ahead to Fig. 1-6. A later section of this chapter gives more details on complex frequency-domain sequences.

At this point, notice that the complex term $\exp(j\omega t)$ is calculated by Mathcad using its powerful and efficient algorithms, eliminating the need for an elaborate complex Taylor series expansion by the user at each value of (n) or (ω). This is good common sense and does not derail us from our discrete time/frequency objectives.

At each (k) stop, the sum is performed at 0 to $N - 1$ values of time (n), for a total of N values. It may be possible to evaluate accurately enough the sum at each (k) value with a smaller number of time steps, say $N/2$ or $N/4$. For simplicity and best accuracy, N will be used for both (k) and (n). Using Mathcad to find the spectrum without assigning discrete (k) values from 0 to $N - 1$, a very large number of frequency values are evaluated and a continuous graph plot is created. We will do this from time to time, and the summation (Σ) becomes more like an integral $\left(\int\right)$, but this is not always a good idea, for reasons to be seen later.

Note also that in Eq. (1-2) the factor $1/N$ ahead of the sum and the minus sign in the exponent are used but are not used in Eq. (1-8) (look ahead). This notation is common in engineering applications as described by [Ronald Bracewell, 1986] and is also an option in Mathcad (functions FFT and IFFT). See also [Oppenheim and Willsky et al., 1983, p. 321]. This agrees with the practical engineering emphasis of this book. It also agrees with our assumption that each record, 0 to $N - 1$, is one replication of an infinite steady-state signal. These two equations, used together and consistently, produce correct results.

Each (k) is a harmonic number for the frequency sequence $X(k)$. To repeat a few previous statements for emphasis, $k = 1$ is the fundamental frequency, $k = 2$ is second harmonic, etc. A two-sided (positive and negative) phasor spectrum is produced by this equation (we will learn to appreciate the two-sided spectrum concept). N, an integral power of 2, is chosen large enough to provide adequate resolution of the spectrum (sufficient harmonics of $k = 1$). The dc component is at $k = 0$ [where the exp(0) term = 1.0] and

$$X(0) = \frac{1}{N}\sum_{n=0}^{N-1} x(n) = \langle x(n) \rangle \qquad \text{volts} \tag{1-4}$$

which is the *time average* over the entire sequence, 1.0, in Fig. 1-2.

Equation (1-2) can be used directly to get the spectrum, but as a matter of considerable interest later it can be separated into two regions having an equal number of data points, from 0 to $N/2 - 1$ and from $N/2$ to $N - 1$ as shown in Eq. (1-5). If $N = 8$, then k (positive frequencies) = 1, 2, 3 and k (negative frequencies) = 7, 6, 5. Point N is the beginning of the next periodic continuation. Dc is at $k = 0$, and $N/2$ is not used, for reasons to be explained later in this chapter.

Consider the following manipulations of Eq. (1-2):

$$X(k) = \frac{1}{N}\left[\sum_{n=0}^{N/2-1} x(n)e^{-jk2\pi\left(\frac{n}{N}\right)} + \sum_{n=N/2}^{N-1} x(n)e^{-jk2\pi\left(\frac{n}{N}\right)}\right] \tag{1-5}$$

The last exponential can be modified as follows without changing its value:

$$e^{-jk2\pi\frac{n}{N}} = \underbrace{e^{j(2\pi n)}}_{360^\circ} e^{-jk2\pi\frac{n}{N}} = e^{j2\pi n\left(1-\frac{k}{N}\right)} = e^{j2\pi(\mathrm{N}-\mathrm{k})\frac{\mathrm{n}}{\mathrm{N}}} \tag{1-6}$$

and Eq. (1-2) becomes

$$X(k) = \frac{1}{N}\left[\sum_{n=0}^{N/2-1} x(n)e^{-jk2\pi\frac{n}{N}} + \sum_{n=N/2}^{N-1} x(n)e^{j2\pi(N-k)\frac{n}{N}}\right] \tag{1-7}$$

The second exponential is the phase conjugate ($e^{-j\theta} \rightarrow e^{+j\theta}$) of the first and is positioned as shown in Fig. 1-2b for $k = N/2$ to $N - 1$. At $k = 0$ we see the dc. The two imaginary components $-j0.5$ and $+j0.5$, are at $k = 1$ and $k = 63$ (same as $k = -1$), typical for a sine wave of length 64. We use this method quite often to convert two-sided sequences into one-sided (positive-time or positive-frequency) sequences (see Chapter 2 for more details).

INVERSE DISCRETE FOURIER TRANSFORM

The inverse transformation (IDFT) in Eq. (1-8) [Oppenheim et al., 1983, p. 321] takes the two-sided spectrum $X(k)$ in Fig. 1-2b and exactly recreates the original two-sided time sequence $x(n)$ shown in Fig. 1-2c:

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{jk2\pi\left(\frac{n}{N}\right)} \tag{1-8}$$

At each value of (n) the spectrum values $X(k)$ are summed from $k = 0$ to $k = N - 1$. In Eq. (1-8) the phase increments are in the counter-clockwise (positive) direction. This reverses the negative phase increments that were introduced into the DFT [Eq. (1-2)]. This step helps to return each *complex* $X(k)$ in the frequency domain to a *real* $x(n)$ in the time domain. See further discussion later in the chapter.

It is interesting to focus our attention on Eqs. (1-2) and (1-8) and to observe that in both cases we are simultaneously in the time and frequency domains. We must have data from both domains to travel back and forth. This confirms that we are learning to be comfortable in both domains at once, which is exactly what we need to do.

So far, Eqs. (1-2) and (1-8) have been used directly, without any need for a faster method, the FFT (the Fast Fourier Transform), described later. Modern personal computers are usually fast enough for simple problems using just these two equations. Also, Eqs. (1-2) and (1-8) are quite accurate and very easy to use in computerized analysis (however, Mathcad also has very excellent tools for numerical and symbolic integration that we will use frequently). We do not have to worry about those two discrete equations in our applications because they have been thoroughly tested. It is a good idea to use Eqs. (1-2) and (1-8) together as a pair. To narrow the time or frequency resolution, multiply the value of N by $2^M(m = 1,$ 2, 3, ...), as shown in the next section.

FREQUENCY AND TIME SCALING

Suppose a signal spectrum extends from 0 Hz to 30 MHz (Fig. 1-3) and we want to display it as a 32-point ($=2^5$) two-sided spectrum. The positive side of the spectrum has 15 $X(k)$ values from 1 to $N/2 - 1$ (not counting 0 and $N/2$), and the negative side of the spectrum also has 15 $X(k)$ values from $N/2 + 1$ to $N - 1$ (not counting $N/2$ and N). The frequency range 0 to 30 MHz consists of a fundamental frequency k_1 and $2^4 - 1 = 15$ harmonics of k_1. The fundamental frequency $k1$ is determined by

$$k_1 \cdot 15 = 3 \cdot 10^7 \quad \therefore \quad k_1 = \frac{3 \cdot 10^7}{15} = 2\,\text{MHz} \tag{1-9}$$

and this is the best resolution of frequency that can be achieved with 15 points (positive or negative frequencies) of a 30-MHz signal using a 32-point two-sided spectrum. If we use 2048 data points, we can get 29.31551-kHz resolution using Eq. (1-9).

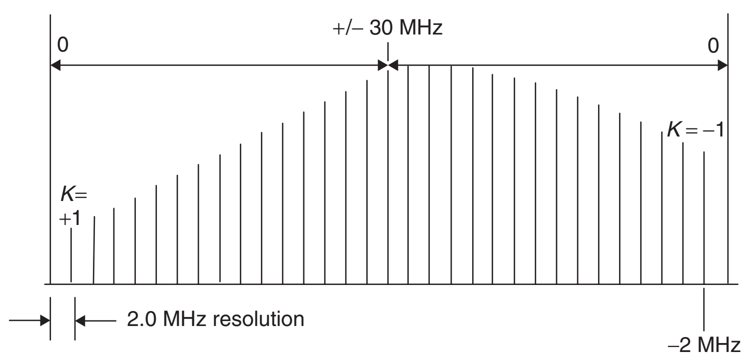

Figure 1-3 A 30-MHz two-sided spectrum with 32 frequency samples, including 0.

An excellent way to improve this example is to frequency-convert the signal band to a much lower frequency, for example 3 MHz, using a very stable local oscillator, which would give us a 2931.55-Hz resolution for this example. Increasing the samples to 2^{14} at 3 MHz provides a resolution of 366.26 Hz, and so forth for higher sample numbers. This is basically what spectrum analyzers do.

The good news for this problem is that a hardware frequency translator may not be necessary. If the signal is narrowband, such as speech or low-speed data or some other bandlimited process, the original 30-MHz problem might be restated at 3 MHz, or maybe even at 0.3 MHz, with the same signal bandwidth and with no loss of correct results, but with greatly improved resolution. With programs for personal computer analysis, very large numbers of samples are not desirable; therefore, we do not try to push the limits too much. The waveform analysis routines usually tell us what we want to know, using more reasonable numbers of samples. Designing the frequency and time scales is very helpful.

Consider a time scaling example, a sequence (record length) that is 10 μsec long from start of one sequence to the start of the next sequence, as shown in Fig. 1-4. For $N = 4$ there are 4 time *values* (0, 1, 2, 3) and 4 time *intervals* (1, 2, 3, 4) to the beginning of the next sequence, which is $10^{-5}/4 = 2.5$ μsec per interval. In the first half there are 2 intervals for a total of 5.0 μsec. For the second half there are also 2 intervals, for a total of 5.0 μsec. Each interval is a "band" of possibly smaller time increments. The total time is 10.0 μsec.

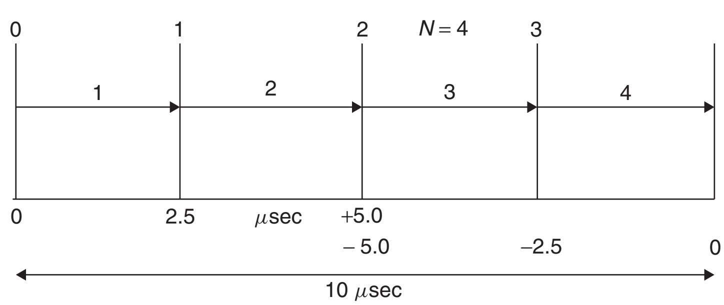

Figure 1-4 A 10-μsec time sequence with positive and negative time values.

For $N = 2^M$ points there are N *values*, including 0, and N *intervals* to the beginning of the next sequence. For a two-sided time sequence the special midpoint term $N/2$ can be labeled as $+5.0$ μsec and also -5.0 μsec, as shown in Fig. 1-4. It is important to do this time scaling correctly.

Figure 1-2b shows an identical way to label frequency values and frequency intervals. Each value is a specific frequency and each interval is a frequency "band". This approach helps us to keep the spectrum more clearly in mind. If amplitude values change too much within an interval, we will use a higher value of N to improve frequency resolution, as discussed previously. The same idea applies in the time domain. The term *picket fence effect* describes the situation where the selected number of integer values of frequency or time does not give enough detail. It's like watching a ball game through a picket fence.

NUMBER OF SAMPLES

The sampling theorem [Carlson, 1986, p. 351] says that a single sine wave needs more than two, preferably at least three, samples per cycle. A frequency of 10,000 Hz requires $1/(10{,}000 \cdot 3) = 3.33 \cdot 10^{-5}$ seconds for each sample. A signal at 100 Hz needs $1/(100 \cdot 3) = 3.33 \cdot 10^{-3}$ seconds for each sample. If both components are present in the same composite signal, the minimum required total number of samples is $(3.33 \cdot 10^{-3})/(3.33 \cdot 10^{-5}) = 10^2 = 100$. In other words, 100 cycles of the 10,000-Hz component occupy the same time as 1 cycle of the 100-Hz component. Because the time sequence is two-sided, positive time and negative time, 200 samples would be a better choice. The nearest preferred value of N is $2^8 = 256$, and the sequence is from $0 \le n \le N - 1$. The plot of the DFT phasor spectrum $X(k)$ is also two-sided with 256 positions. $N = 256$ is a good choice for both time and frequency for this example.

If a particular waveform has a well-defined time limit but insufficient nonzero data values, we can improve the time resolution and therefore the frequency resolution by adding *augmenting zeros* to the time-domain data. Zeros can be added before and after the limited-duration time signal. The total number of points should be $2^M (M = 2, 3, 4, \ldots)$, as mentioned before. Using Eq. (1-8) and recalling that a time record N produces $N/2$

positive-frequency phasors and $N/2$ negative-frequency phasors, the frequency resolution improves by the factor (total points)/(initial points). The spectrum can sometimes be distorted by this procedure, and *windowing* methods (see Chapter 4) can often reduce the distortion.

COMPLEX FREQUENCY DOMAIN SEQUENCES

We discuss further the complex frequency domain $X(k)$ and the phasor concept. This material is very important throughout this book.

The complex plane in Fig. 1-5 shows the locus of imaginary values on the vertical axis and the locus of real values on the horizontal axis. The directed line segment Ae^{je}, also known as a *phasor*, especially in electronics, has a horizontal (real) component $A\cos\theta$ and a vertical (imaginary) component $jA\sin\theta$. The phasor rotates counter-clockwise at a positive angular rate (radians per second) $= 2\pi f$. At the *frozen* instant of time in the diagram the phase *lead* of phasor 1 relative to phasor 2 becomes $\theta = \omega\Delta t = 2\pi f\,\Delta t$. That is, phasor 1 will reach its maximum amplitude (in the vertical direction) *sooner* than phasor 2 therefore, *phasor* 1 *leads phasor* 2 in phase and also in time. A time-domain sine-wave diagram of phasor 1 and 2 verifies this logic. We will see this again in Chapter 5.

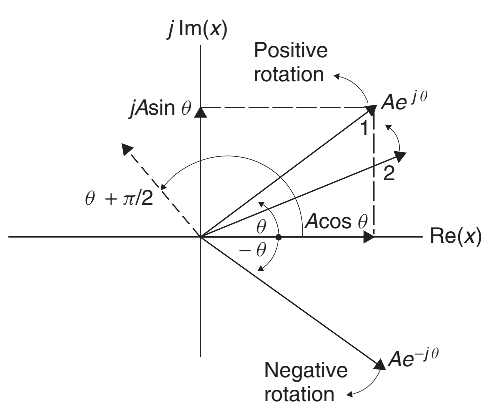

Figure 1-5 Complex plane and phasor example.

The letter j has dual meanings: (1) it is a mathematical *operator*,

$$e^{j\pi/2} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = 0 + j1 = j \qquad (1\text{-}10)$$

that performs a 90° (*quadrature*) counter-clockwise *leading* phase shift on any phasor in the complex plane, for example from 45° to 135°, and (2) it is used as a *label* to tell us that the quantity following it is on the imaginary axis: for example, $R + jX$, where R and X are both real numbers. The *conjugate* of the phase-leading phasor at angle (θ) is the phase-lagging clockwise-rotating phasor at angle ($-\theta$). The *quadrature* angle is $\theta \pm 90^\circ$.

TIME *x(n)* VERSUS FREQUENCY *X(k)*

It is very important to keep in mind the concepts of two-sided time and two-sided frequency and also the idea of complex-valued *sequences* $x(n)$ in the time domain and complex-valued *samples* $X(k)$ in the frequency domain, as we now explain.

There is a distinction between a sample in time and a sample in frequency. An individual time sample *x(n)*, where we define x to be a real number, has two attributes, an amplitude value x and a time value (n). There is no "phase" or "frequency" associated with this $x(n)$, *if viewed by itself*. A special clarification to this idea follows in the next paragraph. Think of the $x(n)$ *sequence* as an oscilloscope screen display. This sequence of time samples may have some combination of frequencies and phases that are defined by the variations in the amplitude and phase of the sequence. The DFT in Eq. (1-2) is explicitly designed to give us that information by examining the time sequence. For example, a phase change of the entire sequence slides the entire sequence left or right. A sine wave sequence in phase with a 0° reference phase is called an (I) wave and a sine wave sequence that is at 90° with respect to the *(I)* wave sequence is called a (Q or jQ) quadrature wave. Also, an individual time sample *x(n)* can have a "phase identifier" by virtue of its position in the time sequence. So we may speak in this manner of the phase and frequency of an $x(n)$ time sequence, but we must avoid confusion on this issue. In this book, each $x(n)$ in the time domain is assumed to be a "real" signal, but the "wave" may be complex in the sense that we have described.

A special circumstance can clarify the conclusions in the previous paragraph. Suppose that instead of $x(n)$ we look at $x(n)\exp(j\theta)$, where θ is a *constant* angle as suggested in Fig. 1-5. Then (see also p. 46)

$$x(n)\exp(j\theta) = x(n)\cos(\theta) + jx(n)\sin\theta = I(n) + jQ(n) \qquad (1\text{-}11)$$

and we now have two *sequences* that are in phase quadrature, and each sequence has real values of *x(n)*. Finally, suppose that the constant θ is replaced by the time-varying θ*(n)* from $n = 0$ to $N - 1$. Equation (1-11) becomes $x(n)\exp[j\theta(n)]$, which is a *phase modulation* of $x(n)$. If we plug this into the DFT in Eq. (1-2) we get the spectrum

$$\begin{aligned} X(k) &= \frac{1}{N}\sum_{n=0}^{N-1}\left[x(n)\exp\big[j\theta(n)\big]\right]\exp\left(-j2\pi\frac{n}{N}k\right) \\ &= \frac{1}{N}\sum_{n=0}^{N-1}x(n)\exp\left\{-j\left[2\pi\frac{n}{N}k - \theta(n)\right]\right\} \end{aligned} \qquad (1\text{-}12)$$

where k can be any value from 0 to $N - 1$ and the time variations in $\theta(n)$ become part of the spectrum of a phase-modulated signal, along with the part of the spectrum that is due to the *peak* amplitude variations (if any) of $x(n)$. Equation (1-12) can be used in some interesting experiments. Note the ease with which Eq. (1-12) can be calculated in the discrete-time/frequency domains. In this book, in the interest of simplicity, we will assume that the *x(n)* values are real, as stated at the outset, and we will complete the discussion.

A frequency *sample* $X(k)$, which we often call a *phasor*, is also a voltage or current value X, but it also has *phase* $\theta(k)$ relative to some reference θ_R, and *frequency* k as shown on an $X(k)$ graph such as Fig. 1-2b, $k = +1$ and $k = +63$ (same as -1). The phase angle $\theta(k)$ of each phasor can

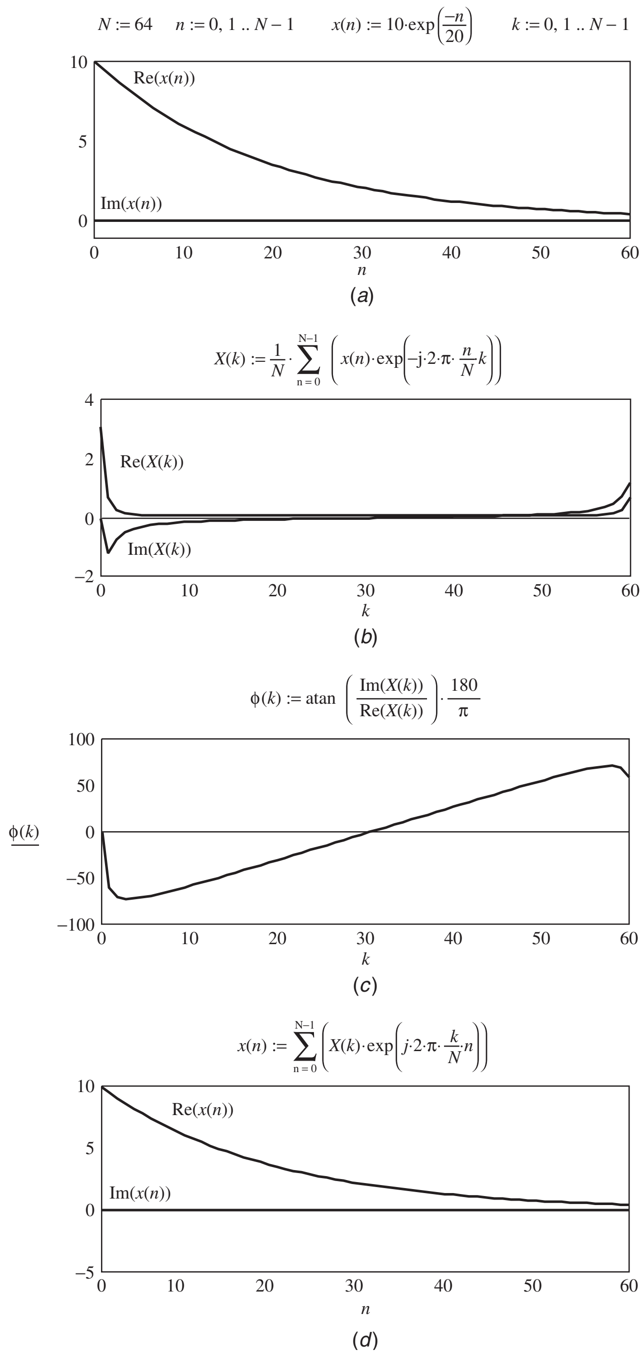

Figure 1-6 Example of time to frequency and phase and return to time.

be shown, if we like, on a separate phase-angle graph (Fig. 1-6). Finally, to reconstruct the time plot in Fig. 1-2c, the two rotating $X(k)$ phasors in Fig. 1-2b re-create the sinusoidal time sequence $x(n)$, using the IDFT of Eq. (1-8). Figure 1-6 should be studied as an example of converting $\exp(-n/20)$ from time to frequency and phase and back to time. Note that parts (a) and (d) show only the positive-time part of the $x(n)$ waveform. The negative-time part is a mirror image and is occasionally not shown, but it is never ignored.

There is one other thing about sequences. Because in this book they are steady-state signals in which all transients have disappeared, it does not matter where they came from. They can be solutions to differential equations, or signal generator output at the end of a long nonlinear transmission line, etc., etc. The DFT and IDFT do not identify the source of the sequences, only tell the relationship between the steady-state time domain and the steady-state frequency domain. We should avoid trying to make anything more than that out of them. Other methods do a much better job of tracing the *origins* of sequences in time and frequency. The Appendix shows a simple example of this interesting and very important activity.

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