1 This Tomb Holds Diophantus

Many centuries ago, in ancient Alexandria, an old man had to bury his son. Heartbroken, the man distracted himself by assembling a large collection of algebra problems and their solutions in a book he called the *Arithmetica*. That is practically all that is known of Diophantus of Alexandria, and most of it comes from a riddle believed to have been written by a close friend soon after his death[1]:

> This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.[2]

The epitaph is a bit ambiguous regarding the death of Diophantus's son. He is said to have died at "half his father's life," but does that mean half the father's age at the time of the son's death, or half the age at which Diophantus himself eventually died? You can work it out either way, but the latter assumption — Diophantus's son lived half the number of years that Diophantus eventually did — is the one with the nice, clean solution in whole numbers without fractional years.

Let's represent the total number of years that Diophantus lived as x. Each part of Diophantus's life is either a fraction of his total life (for example, x divided by 6 for the years he spent as a boy) or a whole number of years (for example, 5 years

[1] Thomas L. Heath, *Diophantus of Alexandria: A Study in the History of Greek Algebra*, second edition (Cambridge University Press, 1910; Dover Publications, 1964), 3.

[2] *Greek Mathematical Works II: Aristarchus to Pappus of Alexandria* (Loeb Classical Library No. 362), translated by Ivor Thomas (Harvard University Press, 1941), 512–3.

from the time he was married to the birth of his son). The sum of all these eras of Diophantus's life is equal to x, so the riddle can be expressed in simple algebra as:

$$\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5+\frac{x}{2}+4=x$$

The least common multiple of the denominators of these fractions is 84, so multiply all the terms on the left and right of the equal sign by 84:

$$14x+7x+12x+420+42x+336=84x$$

Grouping multipliers of x on one side and constants on the other, you get:

$$84x-14x-7x-12x-42x=420+336$$

Or:

$$9x=756$$

And the solution is:

$$x=84$$

So, Diophantus was a boy for 14 years and could finally grow a beard after 7 more years. Twelve years later, at the age of 33, he married, and he had a son 5 years after that. The son died at the age of 42, when Diophantus was 80, and Diophantus died 4 years later.

There's actually a faster method for solving this riddle: If you look deep into the soul of the riddle maker, you'll discover that he doesn't want to distress you with fractional ages. The "twelfth part" and "seventh part" of Diophantus's life must be whole numbers, so the age he died is equally divisible by both 12 and 7 (and, by extension, 6 and 2). Just multiply 12 by 7 to get 84. That seems about right for a ripe old age, so it's probably correct.

Diophantus may have been 84 years old when he died, but the crucial historical question is *when*. At one time, estimates of Diophantus's era ranged from 150 BCE to 280 CE.[3] That's a tantalizing range of dates: It certainly puts Diophantus after early Alexandrian mathematicians such as Euclid (flourished ca. 295 BCE[4]) and Eratosthenes (ca. 276–195 BCE), but might make him contemporaneous with Heron of Alexandria (also known as Hero, flourished 62 CE), who wrote books on mechanics, pneumatics, and automata, and seems to have invented a primitive steam engine. Diophantus might also have known the Alexandrian astronomer Ptolemy (ca. 100–170 CE), remembered mostly for the *Almagest*, which contains the first trigonometry table and established the mathematics for the movement of the heavens that wasn't persuasively refuted until the Copernican revolution of the sixteenth and seventeenth centuries.

[3]Those dates persist in Simon Hornblower and Antony Sprawforth, eds., *Oxford Classical Dictionary*, revised third edition (Oxford University Press, 2003), 483.

[4]All further dates of Alexandrian mathematicians are from Charles Coulston Gillispie, ed., *Dictionary of Scientific Biography* (Scribners, 1970).

Unfortunately, Diophantus probably did not have contact with these other Alexandrian mathematicians and scientists. For the last hundred years or so, the consensus among classical scholars is that Diophantus flourished about 250 CE, and his major extant work, the *Arithmetica*, probably dates from that time. That would put Diophantus's birth at around the time of Ptolemy's death. Paul Tannery, who edited the definitive Greek edition of the *Arithmetica* (published 1893–1895), noted that the work was dedicated to an "esteemed Dionysius." Although a common name, Tannery conjectured that this was the same Dionysius who was head of the Catechist school at Alexandria in 232–247, and then Bishop of Alexandria in 248–265. Thus, Diophantus may have been a Christian.[5] If so, it's a bit ironic that one of the early (but lost) commentaries on the *Arithmetica* was written by Hypatia (ca. 370–415), daughter of Theon and the last of the great Alexandrian mathematicians, who was killed by a Christian mob apparently opposed to her "pagan" philosophies.

Ancient Greek mathematics had traditionally been strongest in the fields of geometry and astronomy. Diophantus was ethnically Greek, but he was unusual in that he assuaged his grief over the death of his son with the "science of numbers," or what we now call *algebra*. He seems to be the source of several innovations in algebra, including the use of symbols and abbreviations in his problems, signifying a transition between word-based problems and modern algebraic notation.

The 6 books of the *Arithmetica* (13 are thought to have originally existed) present increasingly difficult problems, most of which are quite a bit harder than the riddle to determine Diophantus's age. Diophantus's problems frequently have multiple unknowns. Some of his problems are *indeterminate*, which means they have more than one solution. All but one of the problems in *Arithmetica* are abstract in the sense that they are strictly numerical and don't refer to real-world objects.

Another element of abstraction in Diophantus involves powers. Up to that time, mathematicians had been familiar with powers of 2 and 3. Squares were required for calculating areas on the plane, and cubes were needed for the volumes of solids. But Diophantus admitted higher powers to his problems: powers of 4 (which he called a "square-square"), 5 ("square-cube"), and 6 ("cube-cube"). These powers have no physical analogy in the world that Diophantus knew and indicate that Diophantus had little concern for the practicality of his mathematics. This was purely recreational mathematics with no goal but to strengthen the mind.

Here's the first problem from Book IV.[6] Diophantus states it first in general terms:

> To divide a given number into two cubes such that the sum of their sides is a given number.

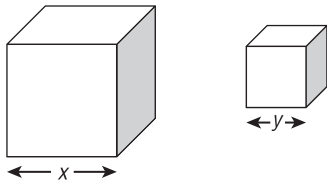
[5]Heath, *Diophantus of Alexandria*, 2, note 2. Heath himself seems to be skeptical.

[6]Heath, *Diophantus of Alexandria*, 168.

Then he provides the two numbers:

Given number 370, given sum of sides 10.

Visualized geometrically, he's dealing with two cubes of different sizes. As a modern algebraist, you or I might label the sides of the two cubes x and y:



The two sides (x and y) add up to 10. The volumes of the two cubes (x^3 and y^3) sum to 370. Now write down two equations:

$$\begin{aligned} x + y &= 10 \\ x^3 + y^3 &= 370 \end{aligned}$$

The first equation indicates that y equals $(10 - x)$, so that could be substituted in the second equation:

$$x^3 + (10 - x)^3 = 370$$

Now multiply $(10 - x)$ by $(10 - x)$ by $(10 - x)$ and pray that the cubes eventually disappear:

$$x^3 + (1000 + 30x^2 - 300x - x^3) = 370$$

Fortunately they do, and after a bit of rearranging you get:

$$30x^2 - 300x + 630 = 0$$

Those three numbers on the left have a common factor, so you'll want to divide everything by 30:

$$x^2 - 10x + 21 = 0$$

Now you're almost done. You have two choices. If you remember the quadratic formula,[7] you can use that. Or, if you've had recent practice solving equations of this sort, you can stare at it and ponder it long enough until it magically decomposes itself like so:

$$(x - 7)\,(x - 3) = 0$$

The lengths of the two sides are thus 7 and 3. Those two sides indeed add up to 10, and their cubes, which are 343 and 27, sum to 370.

Diophantus doesn't solve the problem quite like you or I would. He really can't. Although Diophantus's problems often have multiple unknowns, his notation

[7] For $ax^2 + bx + c = 0$, solve $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

allows him to represent only a single unknown. He compensates for this limitation in ingenious ways. Rather than labeling the sides of the two cubes as x and y, he says that the two sides are $(5 + x)$ and $(5 - x)$. These two sides are both expressed in terms of the single unknown x, and they indeed add up to 10. He can then cube the two sides and set the sum equal to 370:

$$(5 + x)^3 + (5 - x)^3 = 370$$

Now this looks worse than anything we've yet encountered, but if you actually expand those cubes, terms start dropping out like crazy and you're left with:

$$30x^2 + 250 = 370$$

With some rudimentary rearranging and another division by 30, it further simplifies to:

$$x^2 = 4$$

Or x equals 2. Because the two sides are $(5 + x)$ and $(5 - x)$, the sides are really 7 and 3.

Diophantus's skill in solving this problem with less sweat than the modern student results from his uncanny ability to express the two sides in terms of a single variable in precisely the right way. Will this technique work for the next problem? Maybe. Maybe not. Developing general methods for solving algebraic equations is really *not* what Diophantus is all about. As one mathematician observed, "Every question requires a quite special method, which often will not serve even for the most closely allied problems. It is on that account difficult for a modern mathematician even after studying 100 Diophantine solutions to solve the 101st problem."[8]

Of course, it's obvious that when Diophantus presents the problem of cubes adding to 370 and sides adding to 10, he's not pulling numbers out of thin air. He knows that these assumptions lead to a solution in whole numbers. Indeed, the term *Diophantine equation* has come to mean an algebraic equation in which only whole number solutions are allowed. Diophantine equations can have multiple unknowns, and these unknowns can be raised to powers of whole numbers, but the solutions (if any) are always whole numbers. Although Diophantus often uses subtraction in formulating his problems, his solutions never involve negative numbers. "Of a negative quantity *per se*, *i.e.*, without some positive quantity to subtract it from, Diophantus had apparently no conception."[9] Nor does any problem have zero for a solution. Zero was not considered a number by the ancient Greeks.

[8]Hermann Hankel (1874) as quoted in Heath, *Diophantus of Alexandria*, 54–55. Other mathematicians find more explicit patterns in Diophantus's methods. See Isabella Grigoryevna Bashmakova, *Diophantus and Diophantine Equations* (Mathematical Association of America, 1997), ch. 4.

[9]Heath, *Diophantus of Alexandria*, 52–53.

Modern readers of Diophantus — particularly those who are already acquainted with the requirement that Diophantine equations have solutions in whole numbers — can be a bit startled when they encounter *rational numbers* in Diophantus. Rational numbers are so named not because they are logical or reasonable in some way, but because they can be expressed as the *ratio* of two whole numbers. For example,

$$\frac{3}{5}$$

is a rational number.

Rational numbers show up in the only problem in the *Arithmetica* that involves actual real-world objects, in particular those perennial favorites: drink and drachmas. It doesn't seem so in the formulation of the problem, but rational numbers are required in the solution:

> A man buys a certain number of measures of wine, some at 8 drachmas, some at 5 drachmas each. He pays for them a *square* number of drachmas; and if we add 60 to this number, the result is a square, the side of which is equal to the whole number of measures. Find how many he bought at each price.[10]

By a "square number," Diophantus means a result of multiplying some number by itself. For example, 25 is a square number because it equals 5 times 5.

After a page of calculations,[11] it is revealed that the number of 5-drachma measures is the rational number:

$$\frac{79}{12}$$

and the number of 8-drachma measures is the rational number:

$$\frac{59}{12}$$

Let's check these results. (Verifying the solution is much easier than deriving it.) If you multiply 5 drachmas by 79/12, and add to it 8 drachmas times 59/12, you'll discover that the man paid a total of $72\frac{1}{4}$ drachmas. Diophantus says the man pays "a *square* number of drachmas." The amount paid has to be a square of something. Curiously enough, Diophantus considers $72\frac{1}{4}$ to be a square number because it can be expressed as the ratio

$$\frac{289}{4}$$

and both the numerator and denominator of this ratio are squares: of 17 and 2, respectively. So, $72\frac{1}{4}$ is the square of 17/2 or $8\frac{1}{2}$. Diophantus further says that "if

[10]Heath, *Diophantus of Alexandria*, 224.

[11]Heath, *Diophantus of Alexandria*, 225.

we add 60 to this number, the result is a square, the side of which is equal to the whole number of measures." That phrase "whole number of measures" is *not* referring to whole numbers. What Diophantus (or rather, his English translator, Sir Thomas Heath) means is the *total* number of measures. Adding 60 to $72\frac{1}{4}$ is $132\frac{1}{4}$, which is the rational number:

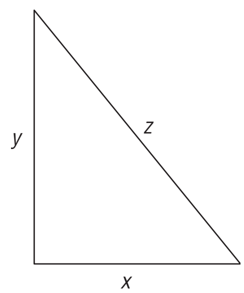
$$\frac{529}{4}$$

Again, Diophantus considers that to be a square because both the numerator and denominator are squares: of 23 and 2, respectively. Thus, the total number of measures purchased is 23/2 or $11\frac{1}{2}$, which can also be calculated by adding 79/12 and 59/12.

Perhaps the most famous problem in the *Arithmetica* is Problem 8 of Book II: "To divide a given square number into two squares," that is, to find x, y, and z such that:

$$x^2 + y^2 = z^2$$

This problem has a geometrical interpretation in the relationship of the sides of a right triangle as described by the Pythagorean Theorem:



The problem has many solutions in whole numbers, such as x, y, and z equaling 3, 4, and 5, respectively. (The squares 9 and 16 sum to 25.) Such a simple solution apparently doesn't appeal to Diophantus, who sets the "given square number" (that is, z^2) to 16, which makes the other two sides the rational numbers 144/25 and 256/25. To Diophantus, these are both squares, of course. The first is the square of 12/5 and the second the square of 16/5, and the sum is the square of 4:

$$\left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 = 4^2$$

It doesn't really matter that Diophantus allows a solution in rational numbers because the solution is equivalent to one in whole numbers. Simply multiply both sides of the equation by 5^2 or 25:

$$12^2 + 16^2 = 20^2$$

Or, 144 plus 256 equals 400. It's really the same solution because it's only a different way of measuring the sides. In Diophantus's statement of the problem, the hypotenuse is 4. That could be 4 inches, for example. Now use a different ruler that measures in units of a fifth of an inch. With that ruler, the hypotenuse is 20 and the sides are 12 and 16.

Whole numbers came about when people started counting things. Rational numbers probably came about when people started measuring things. If one carrot is as long as the width of three fingers, and another carrot is as long as the width of four fingers, then the first carrot is $\frac{3}{4}$ the length of the second.

Rational numbers are sometimes called *commensurable* numbers because two objects with lengths expressed as rational numbers can always be remeasured with whole numbers. You just need to make your unit of measurement small enough.

Diophantus wrote the *Arithmetica* in Greek. At least parts of the work were translated into Arabic. It was first translated into Latin in 1575 and then into a better edition in 1621, when it began to have an influence on European mathematicians. Pierre de Fermat (1601–1665) owned a copy of the 1621 Latin translation and covered its margins with extensive notes. In 1670, Fermat's son published those notes together with the Latin *Arithmetica*. One such note accompanied the problem just described. Fermat wrote:

> On the other hand it is impossible to separate a cube into two cubes, or a biquadrate [power of 4] into two biquadrates, or generally *any power except a square into two squares with the same exponent*. I have discovered a truly marvelous proof of this, which however the margin is not large enough to contain.[12]

Fermat is asserting, for example, that

$$x^3 + y^3 = z^3$$

has no solutions in whole numbers, and neither does any similar equation with powers of 4, or 5, or 6, and so forth. This is not obvious at all. The equation:

$$x^3 + y^3 + 1 = z^3$$

is very, very close to

$$x^3 + y^3 = z^3$$

and it has many solutions in whole numbers, such as x, y, and z equaling 6, 8, and 9, respectively. The equation

$$x^3 + y^3 - 1 = z^3$$

[12]Heath, *Diophantus of Alexandria*, 144, note 3.

is also quite similar, and it too has many solutions in whole numbers, for example, 9, 10, and 12. Why do these two equations have solutions in whole numbers but

$$x^3 + y^3 = z^3$$

does not?

All the problems that Diophantus presented in *Arithmetica* have solutions, but many Diophantine Equations, such as the ones Fermat described, seemingly have no solutions. It soon became more interesting for mathematicians not necessarily to *solve* Diophantine Equations, but to determine whether a particular Diophantine Equation has a solution in whole numbers at all.

Fermat's nonexistent proof became known as Fermat's Last Theorem (sometimes known as Fermat's Great Theorem), and over the years it was generally acknowledged that whatever proof Fermat *thought* he had, it was probably wrong. Only in 1995 was Fermat's Theorem proved by English mathematician Andrew Wiles (b. 1953), who had been interested in the problem since he was ten years old. (For many special cases, such as when the exponents are 3, it had been determined much earlier that no solutions exist.)

Obviously, proving that some Diophantine Equation has *no* possible solution is more challenging than finding a solution if you know that one exists. If you know that a solution exists to a particular Diophantine Equation, you could simply test all the possibilities. The only allowable solutions are whole numbers, so first you try 1, then 2, and 3, and so forth. If you'd rather not do the grunt work yourself, just write a computer program that tests all the possibilities for you. Sooner or later, your program will find the solution.

But if you don't know that a solution exists, then the brute-force computer approach is not quite adequate. You could start it going, but how do you know when to give up? How can you be sure that the very next series of numbers you test won't be the very ones for which you're searching?

That's the trouble with numbers: They're just too damn *infinite*.