CHAPTER 1

MATHEMATICAL MODELS

1.1 APPLYING MATHEMATICS

Recent history has shown us that many problems of our technically oriented society yield to mathematical descriptions and solutions. Problems as complex as sending people into space or maximizing the profit of a giant industrial conglomerate and problems as simple as balancing our own monthly budget or winning at the game of Nim are susceptible to mathematical formulations. This book is concerned with two specific fields of mathematics, linear programming and game theory, that offer insights into certain problems of the real world and techniques for solving some of these problems.

To understand best how one goes about applying a mathematical theory to the solution of some real-world problem, consider the stages that a problem passes through from organization to conclusion. We list four:

- recognition of the problem;
- formulation of a mathematical model;
- solution of the mathematical problem; and
- translation of the results back into the context of the original problem.

These four stages are by no means exclusive or well defined. Other authors have broken down the problem-solving operation in different ways, but the four steps listed indicate the framework in which the applied mathematician works.

The meaning of the first stage, recognition of the problem, is self-explanatory. The meaning of the second stage, formulation of a mathematical model, can be much more mysterious, conjuring visions of a precisely built representation of a small, snow-covered village at a scale of $\frac{1}{87}$. Actually, although the meaning of this step can be made quite clear, it is usually the most critical and difficult step to implement in the entire operation. The development of the mathematical model consists of translating the problem into mathematical terms, that is, into the language and concepts of mathematics. As an example of this process, consider what is called the "word problem" *word problem* of high school algebra. Here the mathematics is trivial and the problems are unrealistic, but many students stumble over the difficulties inherent in translating some concocted word problem into an algebraic equation, that is, in formulating the mathematical model. It was not always easy to determine how

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much 40% antifreeze solution to drain from the 20-qt cooling system to attain a 75% solution by adding a 90% antifreeze mixture.

In the development of a mathematical model of a complex situation, two basic and opposing elements are encountered. On the one hand, one seeks simplifying assumptions and overlooks minor details so that the resulting mathematical problem yields to a successful analysis. On the other hand, the model must adequately reflect reality so that the knowledge gained from the study of the model can be applied to the original problem. The ability to select those elements of a problem that are of major importance and disregard those of minor importance probably comes best from experience. Throughout the text and, in particular, in the next two sections, examples and problems requiring the development of a mathematical model are given. Although in many instances problems from a text may immediately single out the important elements and may seem somewhat artificial, much skill is to be gained by attempting them; practice model building and problem solving whenever possible.

Once the mathematical model has been formulated, one comes to the third stage in the process, the solution of the mathematical problem. It should be emphasized that this can entail much more than just computing the difference of a function at the end points of an interval or finding the solution to a system of equations. Even if the known theory does provide a complete theoretical solution to the problem, the specific answer to the problem at hand must still be calculated. It could very well be that further analysis does not provide any simplification of the problem, and only through involved computations can an estimate of the solution be made. Thus, finding a solution to a problem could mean determining a technique to approximate a solution that is financially feasible to implement within a given computer's capabilities and provides error estimates within given tolerance limits.

The meaning of the fourth step of the operation, the translation of the results back into the context of the original problem, is clear. Of course, more than a simple numerical answer is called for. The simplifying assumptions on which the solution is based must be understood, and the changes in the problem that would invalidate these assumptions should be considered.

We now give two examples of specific and well-known problems and begin the development of the associated mathematical models.

1.2 The Diet Problem

The diet problem is one of the classical illustrations of a problem that leads to a linear programming model. The problem is concerned with providing at minimal cost a diet adequate for a person to sustain himself or herself. Simply stated, what is the least expensive way of combining various amounts of available foods in a diet that meets a person's nutritional requirements?

To develop a mathematical model of this problem, first the various aspects of the problem must be considered. Here the two competing needs for simplification and realism come into play as one attempts to state in precise terms the different components of the problem. For example, just how does one determine the basic nutritional requirements? We must consider the age, sex, size, and activity of our subject. We must determine what nutrients, among the many known nutrients such as calories, proteins, and the multitude of vitamins and minerals, are essential. Can a need for one be met by a combination of others? Is it the case that too much of a certain nutrient is harmful and therefore forces an upper bound on the intake of that quantity? Should we provide for some variety in the diet, hopefully to meet nutritional requirements unknown to us at the present time?

Another component of the problem requiring study is consideration of the foods to be used in the diet. What foods can we assume are available? For example, can we assume that fresh fish, fruits, or vegetables or frozen foods are available? Once the foods that can be used in the problem are established, the nutrient values of these foods must be determined. Here again only approximations can be made, since the nutrient value of a certain type of food, say apples or hamburger, not only varies from sample to sample because of lack of uniformity, but is also contingent on the conditions and duration of storage and the method of preparation for consumption. The cost of a food can also fluctuate due to seasonal and geographical variances.

Once suitable approximations for the nutritional requirements of our subject and the nutrient values and cost of the available foods have been determined, a mathematical problem involving finding the minimum of a linear function can be formulated. To demonstrate this, we will consider a much simplified version of the diet problem.

Suppose we wish to minimize the cost of meeting our daily requirements of proteins, vitamin C, and iron with a diet restricted to apples, bananas, carrots, dates, and eggs. The nutrient values and cost of a unit of each of these five foods, along with the meaning of a unit of each, are given in the following table.

Food	*Measure of a Unit*	*Protein (g/unit)*	*Vitamin C (mg/unit)*	*Iron (mg/unit)*	*Cost (cents/unit)*
Apples	1 med.	0.4	6	0.4	8
Bananas	1 med.	1.2	10	0.6	10
Carrots	1 med.	0.6	3	0.4	3
Dates	$\frac{1}{2}$ cup	0.6	1	0.2	20
Eggs	2 med.	12.2	0	2.6	15

Our daily diet requires at least 70 g of protein, 50 mg of vitamin C, and 12 mg of iron. Since we are assuming that our supply of these foods is unlimited, it is obvious that we can find a diet that meets our needs; for example, a diet consisting of 6 units of eggs and 5 units of bananas would be more than adequate, as the reader can easily verify.

Our problem then is to determine the least expensive way of combining various amounts of the five foods to meet our three daily requirements. Hence the decision to be made involves the number of units of each of the five foods to consume daily. To translate this question into a mathematical problem, introduce five variables A, B, C, D, and E, where A is defined as the number of units of apples to be used in the daily diet, B the number of units of bananas, C the number of units of carrots, D the number of units of dates, and E the number of units of eggs. The cost in cents of such a diet is given by the function $f(A,B,C,D,E) = 8A + 10B + 3C + 20D + 15E$, found by using the cost column in the above table. It is this function that we wish to minimize.

However, there are clearly restrictions imposed by the problem on the possible values of the variables A, B, C, D, and E, that is, restrictions on the domain of the function f. First, all the variables must be nonnegative. And to guarantee that the daily nutritional requirements are fulfilled, the following three inequalities must be satisfied:

$$\begin{aligned} 0.4A + 1.2B + 0.6C + 0.6D + 12.2E &\geq 70 \\ 6A + 10B + 3C + 1D &\geq 50 \\ 0.4A + 0.6B + 0.4C + 0.2D + 2.6E &\geq 12 \end{aligned}$$

These inequalities are determined by considering the total input of the three required nutrients in a diet consisting of A units of apples, B units of bananas, and so on. For example, since 1 unit of apples contains 0.4 g of protein, A units contain $0.4A$ g. Similarly, B units of bananas contain $1.2B$ g of protein, C units of carrots $0.6C$ units, D units of dates $0.6D$ units, and E units of eggs $12.2E$ units. Adding these five terms gives the total intake of protein. Since our daily requirement of 70 g of protein is a minimal requirement and more is allowable, we have the first inequality. Similarly, the other two inequalities follow.

In sum, the resulting mathematical problem is to determine the minimum value of the function

$$f(A,B,C,D,E) = 8A + 10B + 3C + 20D + 15E$$

with the possible values of A, B, C, D, and E restricted by the inequalities

$$\begin{aligned} 0.4A + 1.2B + 0.6C + 0.6D + 12.2E &\geq 70 \\ 6A + 10B + 3C + 1D &\geq 50 \\ 0.4A + 0.6B + 0.4C + 0.2D + 2.6E &\geq 12 \\ A,B,C,D,E &\geq 0 \end{aligned}$$

In 1945 George Stigler [1] considered the general diet problem. Stigler discussed the questions we raised and others, and he justified modifications and simplifications. For human nutritional requirements, Stigler decided on nine common nutrients (calories, protein, calcium, iron, vitamins A, B_1, B_2, C, and niacin) and estimated their needs from data supplied by the National Research Council. Stigler initially considered 77 types of foods and determined their average nutrient values and costs. From this he was able to construct a diet that satisfied all the basic nutritional requirements and cost only \$39.93 a year (less than 11 cents/day) for the year 1939. The diet consisted solely of wheat flour, cabbage, and dried navy beans.

1.3 The Prisoner's Dilemma

In the context of game theory, the word game in general refers to a situation or contest involving two or more players with conflicting interests, with each player having partial but not total control over the outcome of the conflict. The following is an example of such a situation. However, at this stage we are not yet able to translate the conflicting interests represented in the example into a precise mathematical problem, in contrast to the example developed in the previous section. Indeed, one of the major contributions of game theory is the resulting study of the question of what it means to solve a game.

The situation we consider is as follows. A certain democratic republic has a unicameral legislature with a membership drawn primarily from two major political parties. Before the assembly is a bill sponsored by a citizens' group designed to restrict the power and influence of the senior members of each political party. On this issue the legislators can be divided into three approximately equal groups – two groups whose members will follow the directives of their respective party leaders and a third group of responsible representatives who consider passage of the bill more important than the maintenance of party loyalties and will support the bill regardless of circumstances.

Consider now this situation from the viewpoint of the leaders of the two parties. Due to the nature of things they would like to see the bill defeated, but their constituents overwhelmingly support the bill. However, an impending general election complicates matters. Because they are fairly adaptable people, the leaders know that they could, in fact, work moderately well within the limits set by the bill, so each group believes that the most beneficial outcome of the vote on the bill would be for their party to profess support for the bill while the opposition party opposes the bill. Of course, this would mean that the bill would pass, but the wave of public support generated for the one party voting for the bill would be a prevailing factor in the impending election. Thus the problem is, how should each group of leaders direct their respective faithful party members to vote on the bill?

To answer this question, the leaders of one of the parties gather to consider the various possible outcomes of the vote on the bill. The most favorable outcome, as far as they are concerned, is for their party to support the measure and the opposition to oppose it. They denote this outcome by the ordered pair (Y,N) (they vote "yea" and the opposition votes "nay"). The least favorable outcome is the reverse of this situation, with their party members opposing but the opposition favoring passage of the bill (the (N,Y) outcome). The two remaining possible outcomes are for both parties to support the bill (outcome (Y,Y)) and for both parties to oppose the bill (outcome (N,N)). Neither of these outcomes would be a factor in the election, since the public reaction, either good or bad, would be balanced evenly between the two parties. However, outcome (N,N) is preferred over outcome (Y,Y), on the grounds that if both parties oppose the bill, it would be defeated and so the power of the party leaders would remain unaffected. Thus the leaders of the party linearly order the four possible outcomes, from most to least favorable, as follows:

$$(Y,N) > (N,N) > (Y,Y) > (N,Y)$$

Wishing to make this analysis even more precise and, hopefully, instructive, some of the leaders propose to assign numerical weights to each of these outcomes. They claim that such an assignment not only could reflect the above linear ordering, but also could measure how much more one outcome is preferred over another. They point out, for example, that a consideration in some contest of the three outcomes win \$3, win \$2, and win \$1 would not be identical to a consideration of the three outcomes win \$100, win \$2, and win \$1. Seeing the merits of this proposition, the leaders continue their deliberations on the four possible outcomes of the vote on the bill. Since outcomes (Y,N), (Y,Y), and (N,Y) all result in passage of the bill, their relative merits can be measured only by their effects in the impending election. Moreover, because of the equivalent strengths across the country of the two parties, the leaders believe that the advantage of (Y,N) over (Y,Y) is equal to the advantage of (Y,Y) over (N,Y). In fact, they argue that public reaction to support of the bill by only one party could be the determining factor in the election contests in up to 12 representative districts. Accepting this as a general unit and arbitrarily assigning the value 0 to outcome (Y,Y), they set (Y,N) to be worth 12 units and (N,Y) to be worth -12 units. There remains to be considered outcome (N,N), which lies between (Y,N) and (Y,Y) in the linear ordering. The assigning of a weight to this outcome is not immediate but, after a subcommittee review, prolonged debate, and various trade-offs in other matters, the political leaders accept the value of 6 units for this outcome.

Suppose that the leaders of the other party conduct similar deliberations and, since the positions of the two parties are comparable, reach the same conclusions. Then, to each possible outcome is attached two numerical weights, the value of that outcome to each party. Let us denote this pair of weights by an ordered pair of numbers, with the first component being the value of that particular outcome to one fixed party, called Party D, and the second component being the value to the other party, Party R. Then this situation can be represented by the following tableau:

		Party R	
		Vote "yea"	*Vote "nay"*
Party D	*Vote "yea"*	$(0,0)$	$(12,-12)$
	Vote "nay"	$(-12,12)$	$(6,6)$

Thus, for example, the outcome of a "nay" vote by Party D and a "yea" vote by Party R is $(-12,12)$; that is, that outcome is worth -12 units for Party D and 12 units for Party R.

This completes our analysis of this situation for the time being. It will be resumed in Chapter 10. We have formulated a two-person, non-zero-sum game in which each player has two possible moves, but we do not yet have a precisely stated mathematical problem to be solved. A primary component of game theory is the analysis accompanying an attempt to define exactly what one would mean by a solution to the game or a resolution of the conflict. Such an analysis for a certain type of game is made in Chapter 9, where a complete mathematical model is formulated for finite, two-person, zero-sum games and the resulting mathematical problems are resolved (terms such as *zero-sum* are defined there).

The assigning of meaningful weights to the various possible outcomes is not properly a part of game theory but is the function of utility theory (see Section 10.1). In the example of this section the use of game theory actually begins with the above tableau. Moreover, it is assumed in the theory that the information contained in that tableau is known to both parties. However, the theory does distinguish various interpretations of the conflict situation, such as whether or not the players can communicate with each other before the event, whether or not they can cooperate with each other, and whether or not agreements made are actually binding.

A word of explanation as to the meaning of the title of this section is in order. The game that has been developed in the section is an example of a certain type of two-person game. The archetype of games in this category, and the game that lends its name to the category, is the following example of a *prisoner's dilemma*.

Two men are arrested on suspicion of armed robbery. The district attorney is convinced of their guilt but lacks sufficient evidence for conviction at a trial. He points out to each prisoner separately that he can either confess or not confess. If one prisoner confesses and the other does not, the district attorney promises immunity for the confessor and a 2-year jail sentence for the convicted partner. If both confess, he promises leniency and the probable result of a 1-year jail sentence for each prisoner. If neither confesses, he promises to throw the book at each of them on a concealed weapons charge, with a 6-month jail sentence resulting for each.

The possible actions and the corresponding outcomes for the two prisoners are given by the following tableau. The outcomes are stated in terms of ordered pairs, with the first component representing the length of a prison term in months for Prisoner 1 and the second component the length for Prisoner 2.

		Prisoner 2	
		Confess	*Not Confess*
Prisoner 1	*Confess*	$(-12,-12)$	$(0,-24)$
	Not Confess	$(-24,0)$	$(-6,-6)$

The negative signs indicate the undesirable nature of the outcomes (certainly a 12-month sentence is more favorable than a 24-month sentence, that is, $-12 > -24$). The similarity between this tableau and the previous one should be apparent, since the positions of the numbers in the linear ordering of the preferences and in the tableaux correspond. In fact, in this particular case, all the corresponding entries in the two tableaux differ by a fixed amount, 12.

1.4 The Roles of Linear Programming and Game Theory

Using as a base the four-step description of the operation of applying mathematics given in Section 1.1, an outline of how the fields of linear programming and game theory fit into this general scheme can be given.

In Section 1.2 an example of a linear programming problem was given. Many problems that occur in business, industry, warfare, economics, and so on can be reduced to problems of this type, problems of finding the optimal value of some given linear function while the domain of the function is restricted by a system of linear equations or inequalities. The major concern here is not to determine whether or not an optimal value exists, but to develop a technique to determine quickly and easily the optimal value and where it occurs. Thus, from a mathematical point of view, we wish to develop for linear programming problems a method to use in the third stage of the process, finding the solution of the mathematical problem; and in particular, because realistic problems arising from a complex situation may have many variables and many constraints, we need a computationally efficient method of solution. Moreover, since the users of an algorithm need to know if the algorithm will always work, the question of completeness of the solution technique must be addressed.

In Section 1.3 an example of a game theory problem was given. Our first concern with games will be with two-person, zero-sum games. Although the extent of our assumptions may seem to limit the applicability of the theory, this theory still serves as the foundation for the study of more complex games. Moreover, two-person, zero-sum games provide the opportunity to consider at a theoretical level the second stage in the process of applying mathematics, the formulation of the mathematical model. What one means by the solution to a game is not at all apparent, and axioms must be established that define this concept precisely and adequately reflect the economic or social situations to which game theory might be applied. This is in contrast to linear programming problems, where the desire to maximize profits or minimize costs translates immediately into a problem of optimizing a particular function.

From our discussion so far, the problems of game theory and linear programming may seem to be totally unrelated, but this is not the case. Once our mathematical model for two-person, zero-sum games is developed, the problems of existence and calculation of a solution to a game will be related to the theory of linear programming. Here the unifying concept will be the notion of duality. Duality will be introduced in Chapter 4, and the main theorem of that chapter, the Duality Theorem, will provide the answer to the principal question of our study of games, that is, the question of existence of a solution.