PART I

ADVANCED CALCULUS IN ONE VARIABLE

CHAPTER 1

REAL NUMBERS AND LIMITS OF SEQUENCES

1.1 THE REAL NUMBER SYSTEM

During the 19th century, as applications of the differential and integral calculus in the physical sciences grew in importance and complexity, it became apparent that intuitive use of the concept of limit was inadequate. Intuitive arguments could lead to seemingly correct or incorrect conclusions in important examples. Much effort and creativity went into placing the calculus on a rigorous foundation so that such problems could be resolved. In order to see how this process unfolded, it is helpful to look far back into the history of mathematics.

Approximately 2000 years ago, Greek mathematicians placed Euclidean geometry on the foundations of deductive logic. Axioms were chosen as assumptions, and the major theorems of geometry were proven, using fairly rigorous logic, in an orderly progression. These ancient mathematicians also had concepts of numbers. They used *natural numbers*, known also as *counting numbers*, the set of which is denoted by

$$\mathbb{N} = \{1, 2, 3, \ldots, n, n+1, \ldots\}.$$

This is the endless sequence of numbers beginning with 1 and proceeding without end by adding 1 at each step. Also used were *positive rational numbers*, which we

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denote as

$$\mathbb{Q}^+ = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{N} \right\}.$$

These numbers were regarded as representing proportions of positive whole numbers.

Members of the Pythagorean school of geometry discovered that there was no ratio of positive whole numbers that could serve as a square root for 2. (See Exercise 1.11.) This was disturbing to them because it meant that the side and the diagonal of a square must be *incommensurable*. That is, the side and the diagonal of a square cannot both be measured as a whole number multiple of some other line segment, or *unit*. So great was these geometers' consternation over the failure of the set of rational numbers to provide the proportion between the side and the diagonal of a square that confidence in the logical capacity of algebra was diminished. Mathematical reasoning was phrased, to the extent possible, in terms of geometry.

For example, today we would express the area of a circle algebraically as $A = \pi r^2$. We could express this common formula alternatively as $A = \frac{\pi}{4}d^2$, where d is the diameter of the circle. But the ancient Greeks put it this way: The areas of two circles are in the same proportion as the areas of the *squares on their diameters*. The squares were constructed, each with a side coinciding with the diameter of the corresponding circle, and the areas of the squares were in the same proportion as the areas of the circles. Much later, in the 17th century, Isaac Newton continued to be influenced by this perspective. In his celebrated work on the calculus, *Principia Mathematica*, we can see repeatedly that where we would use an algebraic calculation, he used a geometrical argument, even if greater effort is required. The reader interested in the history of mathematics may enjoy the book *The Exact Sciences in Antiquity* by Otto Neugebauer [15] and the one by Carl Boyer [3], *The History of the Calculus*.

It took until the 19th century for mathematicians to liberate themselves from their misgivings regarding algebra. It came to be understood that the *real numbers*, the numbers that correspond to the points on an endless geometrical line, could be placed on a systematic logical foundation just as had been done for geometry nearly two thousand years earlier. Most of the axioms that were needed to prove the properties of the real number system were already quite familiar from the arithmetic of the rational numbers. There was one crucial new axiom needed: the *Completeness Axiom of the Real Number System*. Once this axiom had been added, the theorems of the calculus could be proven rigorously, and future development of the subject of *Mathematical Analysis* in the 20th century was facilitated.

Although we will not attempt the laborious task of rigorously proving every familiar property of the real number system, we will sketch the axioms that summarize familiar properties, and we will explain carefully the completeness axiom. With the latter axiom in hand, we will develop the theory of the calculus with great care. Students interested in studying the full and formal development of the real number system are referred to J. M. H. Olmsted's book [16], or to a stylistically distinctive classic by E. Landau [12].

In addition to the set $\mathbb{N}$ of natural numbers, we will consider the set $\mathbb{Z}$ of *integers*, or whole numbers. Thus

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} = \{\pm n \mid n \in \mathbb{N}\} \cup \{0\}.$$

We need also the full set of rational numbers:

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

We list in Table 1.1 the axioms for a general *Archimedean Ordered Field* $\mathbb{F}$. You will observe that the set $\mathbb{Q}$ is an Archimedean ordered field. However, the set $\mathbb{R}$ of *real numbers*, which we will define in Section 1.3, will obey all the axioms for an Archimedean ordered field together with one more axiom, called the *Completeness Axiom*, which is *not* satisfied by $\mathbb{Q}$.

Table 1.1 Archimedean Ordered Field

An **Archimedean Ordered Field** $\mathbb{F}$ is a set with two operations, called addition and multiplication. There is also an *order relation*, denoted by $a < b$. These satisfy the following properties:

1. *Closure:* If a and b are elements of $\mathbb{F}$, then $a + b \in \mathbb{F}$ and $ab \in \mathbb{F}$.
2. *Commutativity:* If a and b are elements of $\mathbb{F}$, then $a + b = b + a$ and $ab = ba$.
3. *Associativity:* If a, b, and c are elements of $\mathbb{F}$, then $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$.
4. *Distributivity:* If a, b, and c are elements of $\mathbb{F}$, then $a(b + c) = ab + ac$.
5. *Identity:* There exist elements 0 and 1 in $\mathbb{F}$ such $0 + a = a$ and $1a = a$, for all $a \in \mathbb{F}$. Moreover, $0 \neq 1$.
6. *Inverses:* If $a \in \mathbb{F}$, then there exists $-a \in \mathbb{F}$ such that $-a + a = 0$. Also, for all $a \neq 0$, then there exists $a^{-1} = \frac{1}{a} \in \mathbb{F}$ such that $a\frac{1}{a} = 1$.
7. *Transitivity:* If $a < b$ and $b < c$, then $a < c$.
8. *Preservation of Order:* if $a < b$ and if $c \in \mathbb{F}$, then $a + c < b + c$. Moreover, if $c > 0$, then $ac < bc$.
9. *Trichotomy:* For all a and b in $\mathbb{F}$, exactly one of the following three statements will be true: $a < b$, or $a = b$, or $a > b$ (which means $b < a$).
10. *Archimedean Property:* If $\epsilon > 0$ and if $M > 0$, then there exists $n \in \mathbb{N}$ such that $n\epsilon > M$. (In this general context, $\mathbb{N}$ is defined as the smallest subset of $\mathbb{F}$ that contains 1 and is closed under addition.)

There is an old adage that loosely paraphrases the Archimedean Property found in the table: If you save a penny a day, eventually you will become a millionaire (or a billionaire, etc.).

From the axioms for an Archimedean ordered field, many familiar properties of the real numbers can be deduced. In particular, the behavior of all the operations used in solving equations and inequalities follows directly, with the exception that we have not established yet that roots of positive numbers, such as square roots, exist. Here we will concentrate on those properties that received less emphasis in elementary mathematics courses.

The order axioms are particularly useful for analysis. In this connection, it is important to make the following definition.

Definition 1.1.1 *We define*

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

We think of $|a|$ as representing the *distance* of a from 0 on the number line. Note that $|a|$ is always nonnegative. The absolute value satisfies a vital inequality known as the *Triangle Inequality.*

Theorem 1.1.1 *For all a and b in $\mathbb{R}$, $|a + b| \leq |a| + |b|$.*

Proof: Observe that

$$-|a| \leq a \leq |a|,$$

and

$$-|b| \leq b \leq |b|,$$

so that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|. \tag{1.1}$$

Thus, if $a + b \geq 0$,

$$|a + b| = a + b \leq |a| + |b|.$$

But if $a + b < 0$, then from the first inequality in Equation (1.1), we obtain

$$|a + b| = -(a + b) \leq |a| + |b|.$$

We see that whether $a + b$ is negative or nonnegative, we have in either case that $|a + b| \leq |a| + |b|$. ■

Remark 1.1.1 If the student has not yet *read the Introduction*, including the discussion of *Learning to Write Proofs* on page xxiii, this should be done now. It was explained that in order to learn to write proofs, the student must learn first how to study the theorems and proofs that are presented in this book. Let us note how the remarks made there apply to the short proof of the first theorem in this book.

First we read carefully the statement of Theorem 1.1.1. We note that this is a theorem about absolute values, so we reread Definition 1.1.1 to insure that we know the meaning of this concept. Since the absolute value of a number a depends upon the sign of a, we should test the claimed inequality in the theorem with several pairs of numbers: two positive numbers, two negative numbers, and two numbers of opposite sign. The reader should *do this*, with examples of his or her choice of numbers, noting that the triangle inequality in real application gives either *equality*, if the two numbers have the same sign, or else strict inequality, if the two numbers have opposite sign. This gives us an intuitive appreciation that the triangle inequality ought to be true. Now how do we prove it? Testing more examples will not suffice, because infinitely many pairs are possible. Many correct proofs can be given, but we will discuss the one chosen by the author.

The next step in writing a proof requires some playfulness or inquisitiveness on the part of the student. In theoretical mathematics we are discouraged from following rote procedures in the hope of finding an answer without thought. To bypass thought would be to bypass mathematics itself. The student should not even consider such a route, just as he or she should not substitute a pill for a good meal.

We see by playing with the definition of absolute value that $|a|$ must be *equal* to either a or $-a$. This reminds us of what we observed when checking pairs of specific numbers of the same or opposite sign, as explained above. The playfulness appears when we choose to write this as $-|a| \leq a \leq |a|$ for all a, even though the truth of this double inequality hinges upon a being equal to either the left side or the right side. Then we do the same for b, recognizing that a and b do play symmetrical roles in the statement of the theorem. Then we add the two double inequalities, obtaining Equation (1.1). The remainder of the proof unfolds from considering that the value of $|a + b|$ hinges upon the sign of $a + b$.

This analysis of the proof of the triangle inequality is representative of what the student should do with each proof in this book, and with each proof presented in class by his or her professor. Take a fresh sheet of paper and write out a full analysis of the proof, including the perceived rationale for the course that it takes. Work on this until you are sure you understand correctly. If in doubt, ask your teacher! This is the way to learn advanced mathematics, and it is what the student must do to learn to prove theorems.

EXERCISES

1.1 Let $\epsilon > 0$. Determine how large $n \in \mathbb{N}$ must be to ensure that the given inequality is satisfied, and use the Archimedean Property to establish that such n exist.

a) $\frac{1}{n} < \epsilon$?
b) $\frac{1}{n^2} < \epsilon$?
c) $\frac{1}{\sqrt{n}} < \epsilon$? (Assume that $\sqrt{n}$ exists in $\mathbb{R}$.)

1.2 Prove the uniqueness of the additive inverse $-a$ of a. (Hint: Suppose that

$$x + a = 0 = y + a$$

and prove that $x = y$.)

1.3 Use the Axiom of Distributivity to prove that $a0 = 0$ for all $a \in \mathbb{R}$, and use this to prove that $(-1)(-1) = 1$.

1.4 Prove that $(-1)a = -a$ for all $a \in \mathbb{R}$.

1.5 Prove the uniqueness of the multiplicative inverse a^{-1} of a for all $a \neq 0$ in $\mathbb{R}$.

1.6 Prove: For all a and b in $\mathbb{R}$, $|ab| = |a||b|$. (Hint: Consider the three cases a and b both nonnegative, a and b both negative, and a and b of opposite sign.)

1.7 Prove: For all a, b, c in $\mathbb{R}$,

$$|a - c| \leq |a - b| + |b - c|.$$

(Hint: Use the triangle inequality.)

1.8 Let $\epsilon > 0$. Find a number $\delta > 0$ small enough so that $|a-b| < \delta$ and $|c-b| < \delta$ implies $|a - c| < \epsilon$.

1.9 † Prove: For all a and b in $\mathbb{R}$,

$$\big||a| - |b|\big| \leq |a - b|.$$

Intuitively, this says that $|a|$ and $|b|$ cannot be farther apart than a and b are. (Hint: Write $|a| = |(a - b) + b|$ and use the triangle inequality. Then do the same thing for $|b|$.)

1.10 Prove or give a counterexample:

a) If $a < b$ and $c < d$, then $a - c < b - d$.

b) If $a < b$ and $c < d$, then $a + c < b + d$.

1.11 † This exercise leads in three parts to a proof that there is no rational number the square of which is 2. The reader will need to know from another source that each rational number can be written in the form $\frac{m}{n}$ in *lowest terms*. This means that m and n have no common factors other than ± 1.

a) If $m \in \mathbb{Z}$ is odd, prove that m^2 is odd.

b) If $m \in \mathbb{Z}$ is such that m^2 is even, prove that m is even.

c) Suppose there exists $\frac{m}{n} \in \mathbb{Q}$, expressed in *lowest terms*, such that

$$\left(\frac{m}{n}\right)^2 = 2.$$

Prove that m and n are both even, resulting in a contradiction.

(Hint: For this problem, if the student has not taken any class in number theory, the following definitions may be helpful. A number n is called *even* if and only if it can be written as $n = 2k$ for some integer k. A number n is called *odd* if and only if it can be written as $n = 2k - 1$ for some integer k.)

1.2 LIMITS OF SEQUENCES & CAUCHY SEQUENCES

By a *sequence* x_n of elements of a set S we mean that to each natural number $n \in \mathbb{N}$ there is assigned an element $x_n \in S$. Unless otherwise stated, we will deal with sequences of real numbers. We can think of a sequence as an endless list of real numbers, or we could equivalently think of a sequence as being a *function* whose domain is $\mathbb{N}$ and whose range lies in $\mathbb{R}$. It is very important to define the concept of the *limit* of a sequence. Intuitively, we say that x_n *approaches* the *real* number L as n *approaches infinity*, written $x_n \to L \in \mathbb{R}$ as $n \to \infty$, provided we can force $|x_n - L|$ to become as small as we like just by making n sufficiently big. This is also written with the symbols $\lim_{n\to\infty} x_n = L$. The advantage of writing the definition symbolically as follows is that this definition provides inequalities that can be solved to determine whether or not $x_n \to L$.

Definition 1.2.1 *A sequence $x_n \to L \in \mathbb{R}$ as $n \to \infty$ if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$* corresponding to ϵ *such that*

$$n \geq N \Rightarrow |x_n - L| < \epsilon.$$

If there exists a number L such that $x_n \to L$, we say x_n is convergent. *Otherwise we say that x_n is* divergent.

See Exercise 1.12.

■ EXAMPLE 1.1

We claim that if $x_n = \frac{1}{n}$, then $x_n \to 0$.

Proof: Let $\epsilon > 0$. We need $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\left|\frac{1}{n} - 0\right| < \epsilon.$$

That is, we need to solve the inequality $\frac{1}{n} < \epsilon$. Multiplying both sides of this inequality by the positive number $\frac{n}{\epsilon}$, we see that $\frac{1}{\epsilon} < n$. That is, if we pick $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$, then

$$n \geq N \implies \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

We know that such an N exists in $\mathbb{N}$ since ϵ and 1 are both positive. Thus there exists $N \in \mathbb{N}$ such that $N1 = N > \frac{1}{\epsilon}$ by the Archimedean Principle. ■

The student should note that the value of N does indeed correspond to ϵ. If $\epsilon > 0$ is made smaller, then N must be chosen larger.

■ EXAMPLE 1.2

Let $|r| < 1$. We claim that $r^n \to 0$ as $n \to \infty$.

Let $\epsilon > 0$. We need to find $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|r^n - 0| = |r|^n < \epsilon.$$

In the special case in which $r = 0$, it would suffice to take $N = 1$. So suppose $r \neq 0$. Then we need to solve

$$\left(\frac{1}{|r|}\right)^n > \frac{1}{\epsilon}.$$

Note that we do not proceed by taking nth roots of both sides of this inequality, since we have not yet established the existence of such roots for all positive real numbers. Since $|r| < 1$, $\frac{1}{|r|} = 1 + p > 1$ for some $p > 0$. Thus

$$\begin{aligned}\left(\frac{1}{|r|}\right)^n &= (1+p)^n \\ &= (1+p)(1+p)\cdots(1+p) \\ &= 1^n + np + \cdots + p^n \\ &> np.\end{aligned}$$

By transitivity of inequalities, it would suffice to find $N \in \mathbb{N}$ such that $Np > \frac{1}{\epsilon}$. Such integers N exist because of the Archimedean property. So pick $N \in \mathbb{N}$ such $Np > \frac{1}{\epsilon}$ and we find that $n \geq N$ implies $np \geq Np > \frac{1}{\epsilon}$ so that $|r^n - 0| = |r|^n < \epsilon$.

Notice that if x_n is convergent, then after some finite number N of terms, all subsequent terms are bunched very close to one another: in fact, within ϵ of some number L. This motivates the following definition and theorem.

Definition 1.2.2 *A sequence x_n is called a* Cauchy *sequence if and only if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$,* corresponding to ϵ, *such that n and $m \geq N$ implies $|x_n - x_m| < \epsilon$.*

Theorem 1.2.1 *If x_n is any convergent sequence of real numbers, then x_n is a Cauchy sequence.*

Proof: Suppose x_n is convergent: say $x_n \to L$. Let $\epsilon > 0$. Then, since $\frac{\epsilon}{2} > 0$ as well, we see there exists $N \in \mathbb{N}$, *corresponding to ϵ*, such that $n \geq N$ implies $|x_n - L| < \frac{\epsilon}{2}$. Then, if n and $m \geq N$, we have

$$\begin{aligned}|x_n - x_m| &= |(x_n - L) + (L - x_m)| \\ &\leq |x_n - L| + |L - x_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

■

Remark 1.2.1 We make some remarks here to help the student to write his or her own detailed analysis of the proof of Theorem 1.2.1, as recommended in the introduction, on page xxiii. The student should begin with the intuitive understanding that if $x_n \to L$, then x_n will be very close to L for all sufficiently big n. The point is that we want both x_n and x_m to be so close to L that x_n and x_m must be within ϵ of one another. The student should use visualization to recognize that since x_n and x_m can be on opposite sides of L, we will need both x_n and x_m to be within $\frac{\epsilon}{2}$ of L. Then the triangle inequality for real numbers assures that x_n and x_m are no more than ϵ apart. The student should write a careful analysis of every proof in this course, whether proved in the text or by the professor in class.

■ EXAMPLE 1.3

We claim the sequence $x_n = (-1)^{n+1}$ is divergent.

In fact, if x_n *were* convergent, then x_n would have to be Cauchy. But $|x_n - x_{n+1}| \equiv 2$, for all n. Thus, if $0 < \epsilon \leq 2$, it is impossible to find $N \in \mathbb{N}$ such that n and $m \geq N$ implies $|x_n - x_m| < \epsilon$.

Definition 1.2.3 *A sequence x_n is called* bounded *if and only if there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$, for all $n \in \mathbb{N}$.*

Theorem 1.2.2 *If x_n is Cauchy, then x_n must be bounded.*

Remark 1.2.2 Observe that if x_n is convergent, then it is Cauchy, so this theorem implies that every convergent sequence is bounded.

Proof: We will show that every Cauchy sequence is bounded. In fact, taking $\epsilon = 1$, we see that there exists $N \in \mathbb{N}$ such that n and $m \geq N$ implies $|x_n - x_m| < 1$. In particular, $n \geq N$ implies

$$|x_n| - |x_N| \leq \big||x_n| - |x_N|\big| \leq |x_n - x_N| < 1$$

so that $|x_n| < 1 + |x_N|$. If we let

$$M = \max\left\{|x_1|, \ldots, |x_{N-1}|, 1 + |x_N|\right\},$$

making M the largest element of the indicated set of N numbers, then $|x_n| \leq M$ for all $n \in \mathbb{N}$. ■

■ EXAMPLE 1.4

If $x_n = n$, then x_n is *not* convergent.

If x_n *were* convergent, then x_n would be bounded. But for all $M > 0$, there exists $n \in \mathbb{N}$, corresponding to M, such that $n > M$ by the Archimedean Property. So x_n is not bounded.

It is also convenient to define the concepts $x_n \to \infty$ and $x_n \to -\infty$. However, ∞ is not a real number, so we have not defined anything like $|x_n - \infty|$ and thus cannot prove such a difference is less than ϵ. (Compare this with the discussion on page 9.) We adopt the following definition.

Definition 1.2.4 *We write $x_n \to \infty$ if and only if for all $M > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n > M$. Similarly, we write $x_n \to -\infty$ if and only if for all $m < 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n < m$.*

EXERCISES

1.12 † Use Definition 1.2.1 to prove that the limit of a convergent sequence x_n is unique. That is, prove that if $x_n \to L$ and $x_n \to M$ then $L = M$.

1.13 Let

$$x_n = \begin{cases} 0 & \text{if } n < 100, \\ 1 & \text{if } n \geq 100. \end{cases}$$

Prove that x_n converges and find $\lim x_n$.

1.14 Let $x_n = \frac{n-1}{n}$. Prove x_n converges and find the limit.

1.15 Let $x_n = \frac{(-1)^n}{\sqrt{n}}$. Prove x_n converges and find the limit.

1.16 Let $x_n = \frac{1}{n^2}$. Prove x_n converges and find the limit.

1.17 Let $x_n = \frac{n^2-n}{n}$. Does x_n converge or diverge? Prove your claim.

1.18 Let $x_n = \frac{(-1)^n+1}{n}$. Does x_n converge or diverge? Prove your claim.

1.19 † Prove: If $s_n \leq t_n \leq u_n$ for all n and if both $s_n \to L$ and $u_n \to L$ then $t_n \to L$ as $n \to \infty$ as well. (This is sometimes called the *squeeze theorem* or the *sandwich theorem* for sequences.)

1.20 Prove or give a counterexample:

a) $x_n + y_n$ converges if and only if both x_n and y_n converge.
b) $x_n y_n$ converges if and only if both x_n and y_n converge.
c) If $x_n y_n$ converges, then $\lim x_n y_n = \lim x_n \lim y_n$.

1.21 Let $x_n = \frac{\sin n}{n}$. Prove x_n converges, and find the limit.

1.22 † Suppose $a \leq x_n \leq b$ for all n and suppose further that $x_n \to L$. Prove: $L \in [a, b]$. (Hint: If $L < a$ or if $L > b$, obtain a contradiction.)

1.23 Suppose $s_n \leq t_n \leq u_n$ for all n, $s_n \to a < b$, and $u_n \to b$. Prove or give a counterexample: $\lim_{n\to\infty} t_n \in [a, b]$.

1.24 For each of the following sequences:

i. Determine whether or not the sequence is Cauchy and explain why.

ii. Find $\lim_{n\to\infty} |x_{n+1} - x_n|$.

a) $x_n = (-1)^n n$
b) $x_n = n + \frac{1}{n}$
c) $x_n = \frac{1}{n^2}$

d) x_n is described as follows:

$$\mathbf{0}, \mathbf{1}, \frac{1}{2}, \mathbf{0}, \frac{1}{3}, \frac{2}{3}, \mathbf{1}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \mathbf{0}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \mathbf{1}, \ldots.$$

1.25 † Prove: The sequence x_n is *Cauchy* if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$, we have $|x_k - x_N| < \epsilon$.

1.26 Prove that if $x_n \to \infty$ then x_n is not Cauchy.

1.27 Let $x_n \neq 0$, for all $n \in \mathbb{N}$. Prove: $|x_n| \to \infty$ if and only if $\frac{1}{|x_n|} \to 0$.

1.3 THE COMPLETENESS AXIOM AND SOME CONSEQUENCES

Consider the following sequence of decimal approximations to $\sqrt{2}$:

$$x_1 = 1,\ x_2 = 1.4,\ x_3 = 1.41,\ x_4 = 1.414, \ldots.$$

Each x_k is a rational number, having only finitely many nonzero decimal places. For each k, the last nonzero decimal digit of x_k is selected in such a way that $x_k^2 < 2$ yet if that last digit were one bigger the square would be larger than 2. The number x_k^2 cannot equal 2, since there is no $\sqrt{2}$ in the rational number system. Naturally we hope for x_k to converge and for $\lim x_k = \sqrt{2}$. Indeed, x_k is a Cauchy sequence. We can see this by observing that if m and n are greater than or equal to N, then

$$|x_m - x_n| < \frac{1}{10^{N-1}}.$$

Since the sequence of successive powers of $\frac{1}{10}$ converges to 0, if $\epsilon > 0$ we can pick N large enough to ensure that $\frac{1}{10^{N-1}} < \epsilon$.

Since there is no $\sqrt{2}$ in $\mathbb{Q}$, there are Cauchy sequences in $\mathbb{Q}$ that have no limit in the set $\mathbb{Q}$ of rational numbers. It is reasonable, knowing from geometrical considerations that there *should* be a $\sqrt{2} \in \mathbb{R}$, to select the following axiom as the final axiom for the real number system.

Completeness Axiom of $\mathbb{R}$. Every Cauchy sequence of real numbers has a limit in the set $\mathbb{R}$ of real numbers.

In Example 1.10 we will see that in fact the completeness axiom does imply that there exists a $\sqrt{2}$ in $\mathbb{R}$.

Remark 1.3.1 In books that use a different but equivalent version of the Completeness Axiom, the statement that every Cauchy sequence of real numbers converges to a real number is called the *Cauchy Criterion for sequences.*

Definition 1.3.1 *The set* $\mathbb{R}$ *of* real numbers *is an Archimedean ordered field satisfying the Completeness Axiom.*

Thus a sequence of real numbers converges if and only if it is Cauchy. We remark that it can be proven, although we will not do so here, that any two complete

Archimedean ordered fields must be *isomorphic* in the sense of algebra. The interested reader can find a proof in the book [16] by Olmsted. On the other hand, the reader can find an *explicit construction* of a set having all the properties of a complete Archimedean ordered field, beginning from the natural numbers, in the book [12] by Landau.

In the next chapter, after studying the Intermediate Value Theorem, we will see easily that $\mathbb{R}$, with the Completeness Axiom, does possess an $\sqrt[n]{p}$ for each $p > 0$ and for all $n \in \mathbb{N}$. Most of the current chapter, however, will deal with other consequences of completeness, that we will begin exploring right now.

Definition 1.3.2 *A number M is called an* upper bound *for a set $A \subset \mathbb{R}$ if and only if for all $a \in A$ we have $a \leq M$. Similarly, a number m is called a* lower bound *for A if and only if for all $a \in A$ we have $a \geq m$. A set A of real numbers is called* bounded *provided that it has both an upper bound and a lower bound. A* least upper bound *for a set A is an upper bound L for A with the property that* no *number $L' < L$ is an upper bound of A. A least upper bound is denoted by* lub(A).

Note that not every subset of $\mathbb{R}$ has an upper or a lower bound. For example, $\mathbb{N}$ has no upper bound, and $\mathbb{Z}$ has neither an upper nor a lower bound. It is important to bear in mind also that many *bounded* sets of real numbers have neither a largest nor a smallest element. For example, this is true for the set of numbers in the *open* interval (0,1). The reader should prove this claim as an informal exercise.

Theorem 1.3.1 *If a nonempty set S has an upper bound, then S has a* least upper bound L.

Remark 1.3.2 If S has an upper bound, then its least upper bound is denoted by lub(S). If lub(S) exists, then it must have a unique value L. The reader should prove that no number greater or smaller than L could satisfy the definition of lub(S).

Proof: Since $S \neq \emptyset$, there exists $s \in S$. Select any number $a_1 < s$ so that a_1 is too small to be an upper bound for S. Let b_1 be any upper bound of S. We will use a process known as *interval halving*, in which we will cut the interval $[a_1, b_1]$ in half again and again without end. The midpoint between a_1 and b_1 is $\frac{a_1+b_1}{2}$.

i. If $\frac{a_1+b_1}{2}$ is an upper bound for S, then let $b_2 = \frac{a_1+b_1}{2}$ and let $a_2 = a_1$.

ii. But if $\frac{a_1+b_1}{2}$ is *not* an upper bound for S, then let $a_2 = \frac{a_1+b_1}{2}$ and let $b_2 = b_1$.

Thus we have chosen $[a_2, b_2]$ to be one of the two half-intervals of $[a_1, b_1]$, and we have done this in such a way that b_2 is again an upper bound of S and a_2 is too small to be an upper bound for S. Now we cut $[a_2, b_2]$ in half and select a half-interval of it to be $[a_3, b_3]$ in the same way we did for $[a_2, b_2]$. Note that

$$|b_N - a_N| = \frac{|b_1 - a_1|}{2^{N-1}} \to 0$$

as $N \to \infty$. Thus if $\epsilon > 0$, there exists $N \in \mathbb{N}$, *corresponding to* ϵ, such that $|b_N - a_N| < \epsilon$. But, if $n \geq N$, then a_n and $b_n \in [a_N, b_N]$, so n and $m \geq N$ implies $|a_n - a_m| < \epsilon$ and also $|b_n - b_m| < \epsilon$. Thus a_n and b_n are Cauchy sequences. Hence $a_n \to a$ and $b_n \to b$, for some real numbers a, b. By Exercise 1.22, a and b are in $[a_N, b_N]$, for all N. Thus $0 \leq |a - b| < \epsilon$, for all $\epsilon > 0$. Thus $|a - b| = 0$ and $a = b$. We claim that the number $L = a = b$ is the least upper bound of S. Note that for each k we have $a_k \leq L \leq b_k$, since for all $j \geq k$ we have a_j and $b_j \in [a_k, b_k]$.

First, observe that if $s \in S$, then $s \leq L$. In fact, if we did have $s > L$, then, since $b_k \to L$, for some big enough value of k we would have $|b_k - L| < |s - L|$ and so $b_k < s$. But this is impossible, since b_k is an upper bound of S. Thus $s \leq L$ and L is an upper bound of S.

Finally, we claim L is the least upper bound of S. In fact, suppose $L' < L$. Then since $a_k \to L$, there exists k such that $L' < a_k$. But a_k is not an upper bound of S. Thus L' cannot be an upper bound of S. ∎

Remark 1.3.3 The proof of Theorem 1.3.1 is the most difficult proof presented thus far in this book. It proceeds by the method of *interval-halving*. This method can be likened to the way that a first baseman and a second baseman in a baseball game will attempt to tag a base-runner out by throwing the ball back and forth between them, steadily reducing the distance between them until one baseman is close enough to tag the runner. Interval halving is a very useful method of calculating roots of equations with a computer, provided it is possible to tell from the endpoints of each half-interval which half would need to contain the root. The student should take careful note of how the method of interval-halving produces two natural Cauchy sequences, a_n and b_n, corresponding to the left and right endpoints of the selected half-intervals.

Corollary 1.3.1 *If S is any* nonempty *set of real numbers that has a lower bound, then S has a greatest lower bound.*

For the *proof* see Exercise 1.28 in this section.

Remark 1.3.4 If S has a lower bound, then its greatest lower bound is denoted by glb(S).

Since not every subset $S \subset \mathbb{R}$ has either an upper or a lower bound, least upper bounds and greatest lower bounds do not exist in every case. Thus we introduce the concepts of the *supremum* and the *infimum* of an arbitrary set $S \subset \mathbb{R}$.

Definition 1.3.3 *Let S be any nonempty subset of* $\mathbb{R}$. *Define the* supremum *of S, denoted* $\sup(S)$, *to be the least upper bound of S if S is bounded above and define* $\sup(S) = \infty$ *if S has no upper bound. Similarly, define the* infimum *of S, denoted* $\inf(S)$, *to be the greatest lower bound of S if S is bounded below, and define* $\inf(S) = -\infty$ *if S has no lower bound.*

Thus

$$\sup(S) = \begin{cases} \text{lub}(S) & \text{if S is bounded above,} \\ \infty & \text{if S is not bounded above} \end{cases}$$

and

$$\inf(S) = \begin{cases} \text{glb}(S) & \text{if S is bounded below,} \\ -\infty & \text{if S is not bounded below.} \end{cases}$$

■ **EXAMPLE 1.5**

Let $S = \{x \mid 0 < x < 1\} = (0, 1)$. Then $\sup(S) = 1$ and $\inf(S) = 0$.

Proof: Clearly, 1 is an upper bound of S. But if $M < 1$, then there exists $x \in S \cap (M, 1)$. Thus M cannot be an upper bound of S. Hence 1 is the least upper bound of S. The argument for $\inf(S)$ is similar. ■

■ **EXAMPLE 1.6**

Observe that $\sup(\mathbb{N}) = \infty$ and $\inf(\mathbb{N}) = 1$. This follows because $\mathbb{N}$ has no upper bound, but $\mathbb{N}$ does have a least element, namely 1.

Definition 1.3.4 *We call a sequence x_n* increasing *provided $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and then we write this symbolically as*

$$x_n \nearrow .$$

Similarly, we call x_n a decreasing *sequence if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, which we denote by*

$$x_n \searrow .$$

In either case, we call x_n a monotone *sequence. Similarly, if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$, we write*

$$x_n \uparrow$$

and call x_n strictly *monotone increasing. And if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$, we write*

$$x_n \downarrow$$

and call x_n strictly *monotone decreasing.*

Theorem 1.3.2 *If x_k is an increasing sequence, then $x_k \to \sup\{x_n\}$. Similarly, if x_k is a decreasing sequence, then $x_k \to \inf\{x_n\}$.*

Remark 1.3.5 If $\sup\{x_n\} = L$, a real number, then this theorem says the increasing sequence $x_n \to L$ and this is an instance of convergence. But if $\sup\{x_n\} = \infty$, we write $x_n \to \infty$, but this is called *divergence to infinity.* We do not consider the latter circumstance as convergence because we cannot make $|x_n - \infty| < \epsilon$. In fact, $x_n - \infty$ is *meaningless*, since ∞ is not a real number and the arithmetic operations of real numbers are not defined for ∞. Similar remarks apply if x_n is a decreasing sequence.

Proof: Consider the case of x_n increasing. If $\{x_n\}$ is not bounded above, so that the supremum is infinite, we see that for all $M \in \mathbb{R}$ there exists N such that $x_N > M$. Then, $n \geq N$ implies $x_n \geq x_N > M$ too, and we call this divergence of x_n to infinity, denoted by $x_n \to \infty$.

Now suppose x_n is bounded above, so $\sup\{x_n\} = L$ is the least upper bound of the set of numbers $\{x_n\}$. We must show that $x_n \to L$. Let $\epsilon > 0$. Since $L - \epsilon < L$, $L - \epsilon$ cannot be an upper bound of $\{x_n\}$, so there exists N such that $L \geq x_N > L - \epsilon$. Thus for all $n \geq N$ we have

$$L - \epsilon < x_N \leq x_n \leq L,$$

so $n \geq N$ implies $|x_n - L| < \epsilon$; that is, $x_n \to L$.

The case in which x_n decreases is Exercise 1.29. ∎

Corollary 1.3.2 *A monotone sequence converges if and only if it is bounded.*

Proof: Exercise 1.30.

One inconvenience in the concept of limit is that $\lim x_n$ does not exist for every sequence x_n. One may not be sure in advance whether a given sequence is convergent or divergent. However, there are two related concepts called the ***Limit Superior***[4] and the ***Limit Inferior*** which are always defined.

Definition 1.3.5 *Let x_n be any sequence of real numbers. Denote $T_n = \{x_k \mid k \geq n\}$, which we call the nth* tail *of the sequence x_n.*

Note that

$$T_1 \supseteq T_2 \supseteq \ldots \supseteq T_n \supseteq \ldots.$$

Define

$$i_n = \inf(T_n) \text{ and } s_n = \sup(T_n).$$

It is easy to see that $i_n \leq s_n$, for all n. Moreover, as n increases, the set T_n of which one takes sup or inf shrinks to a subset of what it was the step before. Thus i_n increases and s_n decreases. Consequently, $i_k \to \sup\{i_n \mid n \in \mathbb{N}\}$ and $s_k \to \inf\{s_n \mid n \in \mathbb{N}\}$. Recall that this horizontal-arrow notation means convergence if the sequence is approaching a real number, but it indicates a special type of divergence if the sequence is approaching plus or minus infinity.

Definition 1.3.6 *We define the* limit superior *of x_n by*

$$\limsup x_n = \inf\{s_n \mid n \in \mathbb{N}\} = \inf\{\sup(T_n) \mid n \in \mathbb{N}\}$$

and we define the limit inferior *of x_n*

$$\liminf x_n = \sup\{i_n \mid n \in \mathbb{N}\} = \sup\{\inf(T_n) \mid n \in \mathbb{N}\},$$

[4]The lim sup and lim inf appear only occasionally in this book, but the concepts are presented because they are intrinsically interesting. Also they are very useful to know for further study in graduate courses. On the other hand, the sup, inf, lub, and glb appear often and are needed throughout this book.

where T_n is the nth tail of the sequence x_n.

Of course, lim sup and lim inf may be real numbers or they may be $\pm\infty$.

Theorem 1.3.3 *Let $L \in \mathbb{R}$ and let x_n be a sequence of real numbers. Then $x_n \to L$ if and only if* $\limsup x_n = L = \liminf x_n$.

Proof: First, suppose $x_n \to L$. Thus if $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - L| < \epsilon/2$, which implies $s_n = \sup(T_n) \leq L + \epsilon/2$ and $i_n = \inf(T_n) \geq L - \frac{\epsilon}{2}$. Thus

$$L - \frac{\epsilon}{2} \leq i_n \leq s_n \leq L + \frac{\epsilon}{2}$$

which implies that $|s_n - L| \leq \frac{\epsilon}{2} < \epsilon$ and $|i_n - L| \leq \frac{\epsilon}{2} < \epsilon$, for all $\epsilon > 0$. Thus $s_n \to L = \limsup x_n$ and $i_n \to L = \liminf x_n$.

For the opposite implication, suppose $\limsup x_n = \liminf x_n = L \in \mathbb{R}$. Thus there exists N_1 such that $n \geq N_1$ implies $\sup(T_n) \leq L + \frac{\epsilon}{2}$ and there exists N_2 such that $n \geq N_2$ implies $\inf(T_n) \geq L - \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$, and $n \geq N$ implies $|x_n - L| \leq \epsilon/2 < \epsilon$. Thus $x_n \to L$. ■

EXERCISES

1.28 † Prove Corollary 1.3.1. (Hint: Let $-S = \{-s \mid s \in S\}$. Which theorem can you apply to the set $-S$?)

1.29 † Prove the case in which x_n decreases in Theorem 1.3.2.

1.30 † Prove Corollary 1.3.2.

1.31 Find $\sup(S)$ and $\inf(S)$ for each set S below, and justify your conclusions.

- **a)** $S = \{(-1)^n \mid n \in \mathbb{N}\}$.
- **b)** $S = \{(-1)^n n \mid n \in \mathbb{N}\}$.
- **c)** $S = \{x \in \mathbb{R} \mid x^2 < 1\}$.

1.32 Suppose A and B are subsets of $\mathbb{R}$, both nonempty, with the special property that $a \leq b$ for all $a \in A$ and for all $b \in B$. Prove: $\sup(A) \leq \inf(B)$. (Hint: Every b is an upper bound of A. So how does the $\sup(A)$ relate to each $b \in B$?)

1.33 Prove that every real number $M \in \mathbb{R}$ is both an upper bound and a lower bound of the empty set, $\emptyset$.

1.34 Let $x_n = \frac{n-1}{n}$. Show that x_n is convergent and find $\lim x_n$. Justify your conclusions.

1.35 Let $x_n = (1.5)^n$, for all $n \in \mathbb{N}$. Find $\sup(T_n)$ and $\inf(T_n)$, where T_n is the nth *tail* of the sequence, and explain. Find $\liminf x_n$ and $\limsup x_n$.

1.36 Prove or give a counterexample: If x_n increases and y_n increases, then $(x_n + y_n)$ is monotone.

1.37 Prove or give a counterexample: if x_n increases and y_n increases then $(x_n - y_n)$ is monotone.

1.38 Prove or give a counterexample: if x_n increases and y_n increases then the product $(x_n y_n)$ is monotone.

1.39 Prove: If x_n is a *constant* sequence if and only if x_n is *both* monotone increasing and monotone decreasing.

1.40 Let $x_n = \frac{(-1)^n}{n}$. Find $\inf(T_n)$, $\sup(T_n)$, $\limsup x_n$, and $\liminf x_n$. Does $\lim_{n\to\infty} x_n$ exist? (Hint: T_n is the nth *tail* of the sequence x_n.)

1.41 Let $x_n = (-1)^n + \frac{1}{n}$. Find $\limsup x_n$ and $\liminf x_n$. Does $\lim_{n\to\infty} x_n$ exist?

1.42 Give an example of a sequence $x_n \to \infty$ for which x_n is *not monotone*.

1.43 Let x_n be any sequence of real numbers. Prove: x_n diverges to ∞ if and only if $\liminf x_n = \limsup x_n = \infty$.

1.44 Let x_n be any sequence of real numbers. Prove: x_n diverges to $-\infty$ if and only if $\liminf x_n = \limsup x_n = -\infty$.

1.45 Prove that $\liminf x_n \leq \limsup x_n$, for every *bounded* sequence x_n of real numbers. (Hint: The result of problem 5 may help.)

1.46 Let x_n be any *unbounded* sequence of real numbers. Let s_n and i_n be defined as in the proof of Theorem 1.3.3.

a) If $\{x_n \mid n \in \mathbb{N}\}$ has no upper bound, prove $s_n = \infty$ for all n, so that $\limsup x_n = \infty$.

b) If $\{x_n \mid n \in \mathbb{N}\}$ has no lower bound, prove $i_n = -\infty$ for all n, so that $\liminf x_n = -\infty$.

c) In either of the two cases above, conclude that

$$\liminf x_n \leq \limsup x_n.$$

1.4 ALGEBRAIC COMBINATIONS OF SEQUENCES

If s_n is some algebraic combination of other sequences, then we may be able to determine whether or not s_n converges if we know the behavior of the other sequences of which s_n is composed.

Theorem 1.4.1 *Suppose x_n and y_n both converge, with $x_n \to L$ and $y_n \to M$ as $n \to \infty$. Then*

i. $x_n + y_n \to L + M$.

ii. $x_n - y_n \to L - M$.

iii. $x_n y_n \to LM$.

iv. $\frac{x_n}{y_n} \to \frac{L}{M}$, provided that $M \neq 0$ *and* $y_n \neq 0$, *for all* $n \in \mathbb{N}$.

In order to prove this four-part theorem, it is helpful first to introduce the following definition and the two lemmas that follow it.

Definition 1.4.1 *A sequence that converges to zero is called a* null sequence.

Lemma 1.4.1 *The sequence* $x_n \to L \in \mathbb{R}$ *if and only if* $x_n - L \to 0$.

Proof of Lemma. We remark that in words we are proving that $x_n \to L$ if and only if $x_n - L$ is a null sequence. By definition, $x_n \to L \in \mathbb{R}$ if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$, *corresponding to* ϵ, such that $n \geq N$ implies $|x_n - L| < \epsilon$. This is equivalent to $|(x_n - L) - 0| < \epsilon$, which is equivalent to the statement that $(x_n - L) \to 0$, since $|x_n - L| = |(x_n - L) - 0|$. ■

Lemma 1.4.2 *If* $s_n \to 0$ *and if* t_n *is bounded, then* $s_n t_n \to 0$.

Proof: We are proving that a null sequence times a bounded sequence must be a null sequence. There exists $M > 0$ such that $|t_n| \leq M$, for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Since $s_n \to 0$, there exists N such that $n \geq N$ implies $|s_n - 0| = |s_n| < \frac{\epsilon}{M}$. Now, $n \geq N$ implies

$$|s_n t_n - 0| = |s_n t_n| = |s_n||t_n| < \frac{\epsilon}{M} M = \epsilon.$$

■

With the preceding definition and two lemmas in hand, we proceed to the main task of proving the theorem.

Proof:

i. Let $\epsilon > 0$. There exists N_1 such that $n \geq N_1$ implies $|x_n - L| < \epsilon/2$, and there exists N_2 such that $n \geq N_2$ implies $|y_n - M| < \epsilon/2$. Now let $N = \max\{N_1, N_2\}$. Then $n \geq N$ implies

$$|(x_n + y_n) - (L + M)| \leq |x_n - L| + |y_n - M| < \epsilon.$$

ii. This proof is almost identical to the preceding case.

iii. Since y_n converges, y_n is bounded. And

$$\begin{aligned} x_n y_n - LM &= x_n y_n - L y_n + L y_n - LM \\ &= (x_n - L) y_n + L(y_n - M) \\ &\to 0 + 0 = 0 \end{aligned}$$

using the two lemmas and the first part, proven above.

iv. Because of the third part, proven above, it suffices to prove that $\frac{1}{y_n} \to \frac{1}{M}$. But

$$\left|\frac{1}{y_n} - \frac{1}{M}\right| = \frac{|y_n - M|}{|y_n M|} = |y_n - M|\frac{1}{|y_n M|}.$$

Since $|y_n - M| \to 0$, it suffices to show $\frac{1}{|y_n M|}$ is bounded. There exists N such that $n \geq N$ implies $|y_n - M| < \frac{|M|}{2}$. Thus $|y_n| > \frac{|M|}{2}$ and $\frac{1}{|y_n M|} < \frac{2}{|M|^2}$. Thus $\frac{1}{|y_n M|}$ is bounded by $\max\left\{\frac{1}{|y_1 M|}, \ldots, \frac{1}{|y_{N-1} M|}, \frac{2}{|M|^2}\right\}$.

■

EXERCISES

1.47 Give examples of divergent sequences x_n and y_n such that $x_n + y_n$ converges.

1.48 Let $a \in \mathbb{R}$ be arbitrary. Give examples of sequences $x_n \to \infty$ and $y_n \to \infty$ such that $x_n - y_n \to a$.

1.49 Give examples of divergent sequences x_n and y_n such that $x_n y_n$ converges.

1.50 Let the real number $a \geq 0$ be arbitrary. Give examples of sequences $x_n \to \infty$ and $y_n \to \infty$ such that $\frac{x_n}{y_n} \to a$.

1.51 Prove or else give a counterexample: If $x_n + y_n$ converges and if $x_n - y_n$ converges, then x_n converges and y_n converges.

1.52 Prove or else give a counterexample: If $ad - bc \neq 0$ and if

$$ax_n + by_n \to L \text{ and } cx_n + dy_n \to M$$

as $n \to \infty$, then x_n converges and y_n converges.

1.53 Suppose for all $n \in \mathbb{N}$ we have $y_n \neq 0$. Prove or else give a counterexample: If both $x_n y_n$ and $\frac{x_n}{y_n}$ converge, then x_n converges and y_n converges.

1.54 Prove or else give a counterexample:

a) A bounded sequence times a convergent sequence must be convergent.

b) A null sequence times a bounded sequence must be a null sequence.

1.55

a) If $q(n) = b_k n^k + b_{k-1} n^{k-1} + \cdots + b_1 n + b_0$ is a polynomial in the variable $n \in \mathbb{N}$ with $b_k \neq 0$, show that there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $q(n) \neq 0$.

b) Show that

$$\lim_{n\to\infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0}{b_k n^k + b_{k-1} n^{k-1} + \cdots + b_1 n + b_0} = \frac{a_k}{b_k},$$

provided that $b_k \neq 0$ and k is a positive integer.

1.56 ◇ †[5] Define the nth Cesàro mean of a sequence x_n by

$$\sigma_n = \frac{1}{n}(x_1 + \ldots + x_n)$$

for all $n \in \mathbb{N}$.

a) Suppose $x_n \to L$ as $n \to \infty$. Prove: $\sigma_n \to L$ as $n \to \infty$. (Hint: Write $|\sigma_n - L| = \left|\sum_{k=1}^{n} \frac{x_k - L}{n}\right|$.)

b) Give an example of a *divergent* sequence x_n for which σ_n *converges*.

1.57 Let x_n and y_n be any two bounded sequences of real numbers. Prove that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

Give an example in which strict inequality occurs.

1.58 Let x_n and y_n be any two bounded sequences of real numbers. Prove that

$$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n.$$

Give an example in which strict inequality occurs.

1.5 THE BOLZANO–WEIERSTRASS THEOREM

A *subsequence* of a sequence x_n is a sequence consisting of some (but not necessarily all) of the terms of the sequence x_n. The terms appear in the same order as they appeared in x_n, but with omissions. We formalize this concept in the following definition.

Definition 1.5.1 *Let n_k be any* strictly *increasing sequence of natural numbers, so that*

$$n_1 < n_2 < \cdots < n_k < \cdots .$$

Then we call x_{n_k} a subsequence *of x_n.*

We remark that since $n_1 \geq 1$, it follows that $n_2 \geq 2$, ..., and $n_k \geq k$, for all k. An alternative way to think about and to notate subsequences is to write that $n_k = \phi(k)$, where $\phi : \mathbb{N} \to \mathbb{N}$ is a *strictly increasing* function, in the sense that $j < k \implies \phi(j) < \phi(k)$. Then we could alternatively write x_{n_k} as $x_{\phi(k)}$.

■ **EXAMPLE 1.7**

Let $x_n = n^2$, for all $n \in \mathbb{N}$. If $n_k = 2k$, then $x_{n_k} = (2k)^2$ is the sequence of squares of even natural numbers.

[5] This exercise is used to develop the Fejer kernel for Fourier series in Exercise 6.47.

Theorem 1.5.1 *If x_n converges to the limit L as $n \to \infty$, then every subsequence $x_{n_k} \to L$ as $k \to \infty$.*

Proof: Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - L| < \epsilon$. Since $n_k \geq k$ for all k, it follows that $k \geq N \implies |x_{n_k} - L| < \epsilon$. ∎

Corollary 1.5.1 *If x_n has two subsequences that converge to different limits, then x_n is not convergent.*

Theorem 1.5.1 should be compared carefully with the following example.

■ EXAMPLE 1.8

Let $x_n = (-1)^{n+1}$. The sequence x_n is bounded but is not convergent. The subsequences $x_{2k-1} \to 1$ and $x_{2k} \to -1$ as $k \to \infty$,

We have learned previously that every convergent sequence is bounded. Although the student has seen several examples of bounded sequences that are *not* convergent, we do have the following very important theorem.

Theorem 1.5.2 (Bolzano–Weierstrass) *Let x_n be any* bounded *sequence of real numbers, so that there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$ for all n. Then there exists a convergent subsequence x_{n_k} of x_n. That is, there exists a subsequence x_{n_k} that converges to some $L \in [-M, M]$.*

Proof: We will use the method of interval-halving introduced previously to prove the existence of least upper bounds. Let $a_1 = -M$ and $b_1 = M$. So $x_n \in [a_1, b_1]$, for all $n \in \mathbb{N}$. Let $x_{n_1} = x_1$. Now divide $[a_1, b_1]$ in half using the midpoint $\frac{a_1+b_1}{2} = 0$.

i. If there exist ∞-many values of n such that $x_n \in [a_1, 0]$, then let $a_2 = a_1$ and $b_2 = 0$.

ii. But if there do not exist ∞-many such terms in $[a_1, 0]$, then there exist ∞-many such terms in $[0, b_1]$. In that case let $a_2 = 0$ and $b_2 = b_1$.

Now since there exist ∞-many terms of x_n in $[a_2, b_2]$, pick any $n_2 > n_1$ such that $x_{n_2} \in [a_2, b_2]$. Next divide $[a_2, b_2]$ in half and pick one of the halves $[a_3, b_3]$ having ∞-many terms of x_n in it. Then pick $n_3 > n_2$ such that $x_{n_3} \in [a_3, b_3]$. Observe that

$$|b_k - a_k| = \frac{2M}{2^{k-1}} \to 0$$

as $k \to \infty$. So if $\epsilon > 0$, there exists K such that $k \geq K$ implies $|b_k - a_k| < \epsilon$. Thus if j and $k \geq K$, we have $|x_{n_j} - x_{n_k}| < \epsilon$ as well. Hence x_{n_k} is a Cauchy sequence and must converge. Since $[-M, M]$ is a closed interval, we know from a previous exercise that $x_{n_k} \to L$ as $k \to \infty$ for some $L \in [-M, M]$. ∎

EXERCISES

1.59 Give an example of a bounded sequence that does not converge.

1.60 Use Corollary 1.5.1 to prove that the sequence $x_n = (-1)^n + \frac{1}{n}$ does not converge.

1.61 Suppose $x_n \to \infty$. Prove that every subsequence $x_{n_k} \to \infty$ as $k \to \infty$ as well. (Hint: The sequence x_n is divergent, so it is not enough to quote Theorem 1.5.1.)

1.62 Use the following steps to prove that the sequence x_n has no convergent subsequences if and only if $|x_n| \to \infty$ as $n \to \infty$.

- **a)** Suppose that the sequence x_n has no convergent subsequences. Let $M > 0$. Prove that there exist at most finitely many values of n such that $x_n \in [-M, M]$. Explain why this implies $|x_n| \to \infty$ as $n \to \infty$.
- **b)** Suppose $|x_n| \to \infty$ as $n \to \infty$. Show that x_n has no convergent subsequence. (Hint: Exercise 1.61 may help.)

1.63 Give an example in which $y_j > 0$ for all j and $y_j \to 0$ yet y_j is *not* monotone.

1.64 The following questions provide an easy, alternative proof of the Bolzano–Weierstrass Theorem.

- **a)** Use the following steps to prove that *every* sequence x_n of real numbers has a monotone subsequence. Denote the nth tail of the sequence by $T_n = \{x_j \mid j \geq n\}$.

  (i) Suppose the following special condition is satisfied: For each $n \in \mathbb{N}$, T_n has a smallest element. Prove that there exists an increasing subsequence x_{n_j}.

  (ii) Suppose the condition above fails, so that there exists $N \in \mathbb{N}$ such that T_N has no smallest element. Prove that there exists a decreasing subsequence x_{n_j}.

- **b)** Give an easy alternative proof of the Bolzano–Weierstrass Theorem.

1.65 Prove: A sequence $x_n \to L \in \mathbb{R}$ if and only if *every* subsequence x_{n_i} possesses a sub-subsequence $x_{n_{i_j}}$ that converges to L as $j \to \infty$. (Hint: To prove the *if* part, suppose false and write out the logical negation of convergence of x_n to L.)

1.66 *Prove or Give a Counterexample*: A sequence $x_n \in \mathbb{R}$ converges if and only if *every* subsequence x_{n_i} possesses a sub-subsequence $x_{n_{i_j}}$ that converges as $j \to \infty$.

1.6 THE NESTED INTERVALS THEOREM

Having used the method of interval-halving twice already, it is natural to consider the following theorem.

Theorem 1.6.1 (Nested Intervals Theorem) *Suppose*

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_k, b_k] \supseteq \cdots$$

is a decreasing nest *of closed finite intervals. Suppose also that*

$$b_k - a_k \to 0 \text{ as } k \to \infty.$$

Then there exists exactly one *point* $L \in \bigcap_{k=1}^{\infty}[a_k, b_k]$. *Moreover,* $a_k \to L$ *and* $b_k \to L$ *as* $k \to \infty$.

Proof: Let $\epsilon > 0$. Then there exists K such that $k \geq K$ implies $|b_k - a_k| < \epsilon$. But, for all $k \geq K$, $a_k \in [a_K, b_K]$. Thus $j, k \geq K$ implies $|a_j - a_k| < \epsilon$. Hence the sequence a_k is a Cauchy sequence so there exists a point L such that $a_k \to L$. Since $k \geq n$ implies for all n that $a_k \in [a_n, b_n]$, it follows that $L \in [a_n, b_n]$ for all n and that

$$L \in \bigcap_{k=1}^{\infty} [a_k, b_k].$$

Now, if

$$L' \in \bigcap_{k=1}^{\infty} [a_k, b_k]$$

also, then $|L - L'| \leq |b_k - a_k| \to 0$, which implies $L = L'$. Hence the point L is unique. Observe that $|b_k - L| \leq |b_k - a_k| \to 0$ so that $b_k \to L$ as claimed. ∎

The reader is aware that there are real numbers that are not rational. For example, we will prove that there is a square root of 2 in $\mathbb{R}$ in Example 1.10. Yet we know that no rational number can be a square root of 2 as was shown in Exercise 1.11. Despite the fact that not every real number is rational, every finitely long decimal expansion represents a rational number, and common sense tells us that we may approximate any real number as closely as we wish by using a suitable but finitely long decimal expansion. This observation gives rise to the following definition of what it means for a subset $S \subseteq \mathbb{R}$ to be *dense* in $\mathbb{R}$.

Definition 1.6.1 *A subset* $S \subseteq \mathbb{R}$ *is called* dense *in* $\mathbb{R}$ *if and only if for all* $x \in \mathbb{R}$, *there exists a sequence* s_k *of elements of* S *such that* $s_k \to x$.

■ EXAMPLE 1.9

We will show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: Let $x \in \mathbb{R}$. If $x \in \mathbb{Q}$, we could simply let $s_k \equiv x$ so that $s_k \to x$, being a constant sequence.

So suppose $x \notin \mathbb{Q}$, so that x is irrational. Then there exists $n \in \mathbb{Z}$ such that $n < x < n + 1$. Let $a_1 = n$ and $b_1 = n + 1$, both rational numbers. Then the midpoint is also a rational number, and x must lie in one half-interval but not the other. Let $[a_2, b_2]$ be the half-interval containing x. Now cut $[a_2, b_2]$ in half again and select $[a_3, b_3]$ containing x again. Note that

$$|b_k - a_k| = \frac{1}{2^{k-1}} \to 0.$$

Since $x \in [a_k, b_k]$ for all k, $|a_k - x| \to 0$, so $a_k \to x$, and $a_k \in \mathbb{Q}$ for all k. Thus we have a sequence of rational numbers converging to x in this case as well. (Note that b_k would have served just as well as a_k.) ■

Remark. Because $\mathbb{R}$ is complete and because the set $\mathbb{Q} \subset \mathbb{R}$ is dense in $\mathbb{R}$, it follows that any set of numbers that contains limits for all its Cauchy sequences and that contains $\mathbb{Q}$ must also contain $\mathbb{R}$. For this reason $\mathbb{R}$ is called the *completion* of $\mathbb{Q}$.

■ **EXAMPLE 1.10**

We will show that $\sqrt{2}$ exists in $\mathbb{R}$.

Proof: Recall that in the first paragraph of Section 1.3 we constructed an increasing sequence x_k as follows:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1.4 \\ x_3 &= 1.41 \\ x_4 &= 1.414 \\ &\vdots \end{aligned}$$

Here x_k is the largest *k-digit* decimal greater than 1 such that $x_k^2 < 2$. We could have constructed also a decreasing sequence y_k by letting y_k be the smallest k-digit decimal such that $y_k^2 > 2$. Thus

$$|y_k - x_k| \leq \frac{1}{10^{k-1}} \to 0$$

as $k \to \infty$ We see that the intervals $[x_k, y_k]$ satisfy the hypotheses of Theorem 1.6.1. Thus there exists a unique

$$L \in \bigcap_{k=1}^{\infty} [x_k, y_k]$$

such that $x_k \to L$ and $y_k \to L$. Hence $x_k^2 \to L^2$, so that $L^2 \leq 2$, and $y_k^2 \to L^2$, so that $L^2 \geq 2$. Thus $L^2 = 2$ and $L = \sqrt{2}$ exists in $\mathbb{R}$. ■

EXERCISES

1.67 Give an example of a decreasing nest of nonempty open finite intervals

$$(a_1, b_1) \supseteq (a_2, b_2) \supseteq \cdots$$

such that $\bigcap_{k=1}^{\infty}(a_k, b_k) = \emptyset$, the *empty set.*

1.68 Give an example of a decreasing nest of open intervals

$$(a_1, b_1) \supseteq (a_2, b_2) \supseteq \cdots$$

such that $b_k - a_k \to 0$ yet $\bigcap_{k=1}^{\infty}(a_k, b_k) \neq \emptyset$.

1.69 Give an example of a decreasing nest of *infinite* intervals with empty intersection.

1.70 Prove or give a counterexample: If $a_n \uparrow$, $b_n \downarrow$, and (a_n, b_n) is a decreasing nest of finite open intervals, then there exists $L \in \mathbb{R}$ such that

$$\bigcap_{n=1}^{\infty}(a_n, b_n) = \{L\}.$$

1.71 Show that every open interval $(a, b) \subset \mathbb{R}$, with $0 < b - a$ but no matter how small, must contain a rational number. (Hint: Apply Example 1.9.)

1.72 † Let I denote the set of all irrational numbers. The following steps will lead to the conclusion that I is dense in $\mathbb{R}$. (You may assume it is known that $\sqrt{2} \in I$.) Let $x \in \mathbb{R}$. We must show there exists a sequence s_k of elements of I converging to x.

- **a)** Show that if $\frac{m}{n}$ is any nonzero rational number then $\frac{m}{n}\sqrt{2}$ is irrational. (Hint: Suppose the claim is false, and deduce a contradiction.)
- **b)** Now suppose x is any real number. Explain why there exists a sequence t_k of *nonzero* elements of $\mathbb{Q}$ converging to $\frac{x}{\sqrt{2}}$. Define a sequence s_k of elements of I converging to x.

1.73 Show that every open interval (a,b), with $b - a > 0$ but no matter how small, must contain an irrational number. (Hint: Use the result of Exercise 1.72.)

1.74 Is the set $\left\{\frac{m}{2^n} \,\middle|\, m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ dense in $\mathbb{R}$? Prove your conclusion.

1.75 ◇ Let $D \neq \emptyset$ be a subset of the set of strictly positive real numbers, and let $S = \{nd \mid n \in \mathbb{Z}, d \in D\}$. Prove: S is dense in $\mathbb{R}$ if and only if $\inf(D) = 0$.

1.7 THE HEINE–BOREL COVERING THEOREM

Although the study of continuous functions belongs to the next chapter, let us think in advance on an intuitive level about this concept. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be everywhere continuous provided that for each point $p \in \mathbb{R}$, $f(x)$ remains very close to $f(p)$ provided that x is kept sufficiently close to p. For example, the set

$$S = \{x \,|\, |f(x) - f(p)| < \epsilon\}$$

should contain some sufficiently small open interval around p, although S may also include points far away from p.

Consider next an open interval (a, b) that is contained in the set of values achieved by f. Let $O = \{x \mid f(x) \in (a, b)\}$. For each $p \in O$ there will be a corresponding small number $\epsilon > 0$ such that

$$(f(p) - \epsilon, f(p) + \epsilon) \subseteq (a, b).$$

Because f is continuous at p, there will be a small open interval around p that is contained in O. This example motivates the concept of an *open* set, which generalizes the familiar notion of an open interval.

Definition 1.7.1 *A set $O \subseteq \mathbb{R}$ is called an* open *subset of $\mathbb{R}$ provided for each $x \in O$ there exists $r_x > 0$ such that*

$$(x - r_x, x + r_x) \subseteq O.$$

Thus O is called open provided that each $x \in O$ has some (perhaps very small) open interval of *radius* $r_x > 0$ around it that is entirely in O.

■ **EXAMPLE 1.11**

We claim that every open interval (a, b) is an open set. In fact, if $x \in (a, b)$, then $a < x < b$ and we can let

$$r_x = \min\{|x - a|, |x - b|\}.$$

Then $(x - r_x, x + r_x) \subseteq (a, b)$.

Theorem 1.7.1 *Every open subset $O \subseteq \mathbb{R}$ is a union of (perhaps infinitely many) open intervals. Moreover, every union of open sets is an open set.*

Proof: Let $O \subseteq \mathbb{R}$ be open. Then, using the notation of Definition 1.7.1, you will show in Exercise 1.78 that

$$O = \bigcup_{x \in O} (x - r_x, x + r_x).$$

To prove the second conclusion, let $O = \bigcup_{\alpha \in A} O_\alpha$ be any union[6] of open sets. Let $x \in O$. We know there exists $\alpha_0 \in A$ such that $x \in O_{\alpha_0}$, which is open. Thus there exists $r_x > 0$ such that $(x - r_x, x + r_x) \subseteq O_{\alpha_0} \subseteq O$. Thus O is open. ■

Definition 1.7.2 *An* open cover *of a set $S \subseteq \mathbb{R}$ is a collection*

$$\mathcal{O} = \{O_\alpha \mid \alpha \in A\}$$

[6]When denoting an *arbitrary* union of open sets, it is customary to use a so-called *index set*, such as the set A used here. One should think of A as being a set of labels, or names, used to tag, or identify the sets of which the union is being formed. One cannot always index sets by means of natural numbers, because there exist sets so large that they cannot be uniquely indexed by natural numbers. Even the infinite set $\mathbb{N}$ is too small. The reader will learn more about this in Theorem 1.15.

of (perhaps infinitely many) open sets O_α, where α ranges over some index set A, *such that* $S \subseteq \bigcup_{\alpha \in A} O_\alpha$.

In analysis, it is often necessary to try to control small-scale local variations of some structure defined on a domain D. Under suitable conditions, one can control variations by restricting ones view to a very small open set surrounding each given point of D. Then in the large we *cover* the whole domain D with a family $\mathcal{O}$ of these (possibly small) open sets whose *union* contains D. Usually $\mathcal{O}$ will have infinitely many open sets as members, or elements of itself. Within each one of the open sets that are elements of $\mathcal{O}$ the fine structure varies only slightly. We hope for the availability of a finite *sub*cover, consisting of only finitely many of the open sets belonging to $\mathcal{O}$, so as to produce *uniform* controls on fine-scale variations for the entire large domain D. Below, we show an example of an open covering of a set for which there is no finite subcover. This will motivate the Heine–Borel Theorem which follows.

■ EXAMPLE 1.12

Consider the set $S = (0,2)$, a finite open interval. We claim that

$$S \subseteq \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right).$$

In fact, for each $x \in (0,2)$ there exists $n \in \mathbb{N}$ such that $x \in \left(\frac{1}{n}, 2\right)$. (Make sure you see why this is so.) Thus $\mathcal{O} = \left\{\left(\frac{1}{n}, 2\right) \mid n \in \mathbb{N}\right\}$ is an open cover of S. However, it is impossible to select any *finite* subset of $\mathcal{O}$ that covers S. The reason is that any finite subset of $\mathcal{O}$ would have a largest value n_0 of n for which $\frac{1}{n}$ would be the left hand endpoint of an interval belonging to the chosen finite subset of $\mathcal{O}$. Thus the finite subset would fail to cover any points to the left of $\frac{1}{n_0}$.

Remark 1.7.1 Note that the term *finite interval* means an interval of finite *length*. Any finite interval with strictly positive length has infinitely many distinct points within it. Thus the word *finite* in *finite interval* means the same thing as *bounded*. On the other hand, a *finite set* means a set with *finitely many elements*. In Example 1.12, a finite subset of a set of intervals means a collection of finitely many of those intervals. This does *not* mean that the intervals in question have finitely many points.

The Heine–Borel theorem is one of the most important in advanced calculus. But it is the most abstract theorem presented thus far in this book, and the reader will need time and experience to absorb fully its significance. It is recommended to consider Exercise 1.80 below after reading the statement of the theorem.

Theorem 1.7.2 (Heine–Borel) *Suppose the closed finite interval*

$$[a,b] \subseteq \bigcup_{\alpha \in A} O_\alpha,$$

where $\mathcal{O} = \{O_\alpha \mid \alpha \in A\}$ *is an* open cover *of* $[a, b]$. *Then there exists a finite set* $F = \{\alpha_1, \ldots, \alpha_n\} \subseteq A$ *such that*

$$[a, b] \subseteq \bigcup_{\alpha \in F} O_\alpha = \bigcup_{i=1}^{n} O_{\alpha_i}.$$

The collection $\{O_{\alpha_1}, \ldots, O_{\alpha_n}\} \subseteq \mathcal{O}$ *is called a* finite subcover *of* $[a, b]$.

Proof: We suppose the theorem were false. We will deduce a logical self-contradiction from that supposition. This will prove the theorem. So suppose the Heine–Borel theorem were false: Thus we can assume the given cover does not admit a finite subcover of [a,b].

Let $a_1 = a$ and $b_1 = b$, and let $c = \frac{a_1+b_1}{2}$. Then each of the intervals $[a_1, c]$ and $[c, b_1]$ is covered by $\bigcup_{\alpha \in A} O_\alpha$. If *both* of these half-intervals had finite subcovers, then the whole interval [a,b] would have a finite subcover since the union of two finite families is still finite. Since we are supposing [a,b] has no finite subcover, pick a half-interval $[a_2, b_2]$ that has *no* finite subcover. Now cut $[a_2, b_2]$ in half and reason the same way for $[a_2, b_2]$ as we did for $[a_1, b_1]$. We obtain a decreasing next of intervals

$$[a_1, b_1] \supseteq \cdots \supseteq [a_k, b_k] \supseteq \cdots$$

such that each $[a_k, b_k]$ is covered by $\bigcup_{\alpha \in A} O_\alpha$ but has no finite subcover.

However,

$$|b_k - a_k| = \frac{b-a}{2^{k-1}} \to 0$$

as $k \to \infty$. By the nested intervals theorem, there exists a unique

$$x \in \bigcap_{k=1}^{\infty} [a_k, b_k] \subseteq [a, b].$$

Since $x \in [a, b]$, there exists $\alpha \in A$ such that $x \in O_\alpha$. So there exists $r_x > 0$ such that $(x - r_x, x + r_x) \subseteq O_\alpha$. Now pick k big enough so that $b_k - a_k < r_x$. Thus

$$x \in [a_k, b_k] \subset (x - r_x, x + r_x) \subseteq O_\alpha$$

and we have covered $[a_k, b_k]$ with a *single* open set O_α from the original cover. This is a (very small) finite subcover. This contradicts the statement that $[a_k, b_k]$ could not have a finite subcover. This contradiction proves the Heine–Borel theorem. ■

EXERCISES

1.76 Show that a closed finite interval [a,b] is *not* an open set.

1.77 Show that a half-closed finite interval (a,b] is *not* an open set.

1.78 Let O be any open subset of $\mathbb{R}$, and for each $x \in O$ let r_x be defined as in the proof of Theorem 1.7.1. Complete the proof of that theorem by showing that $O = \bigcup_{x \in O}(x - r_x, x + r_x)$.

1.79 The empty set $\emptyset$ satisfies the definition of being open. Explain.

1.80 Find an open cover of the interval $(-1, 1)$ that has no finite subcover. Justify your claims.

1.81 Find an open cover of the interval $(-\infty, \infty)$ that has no finite subcover. Justify your claims.

1.82 Let $E \subseteq \mathbb{R}$ be any *unbounded* set. Find an open cover of E that has no finite subcover. Prove that you have chosen an open cover and that it has no finite subcover.

1.83 Let $E = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. Find an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E that has no finite subcover, and prove that $\mathcal{O}$ is an open cover and that $\mathcal{O}$ has no finite subcover.

1.84 $\diamondsuit$ We call p a *cluster point* of E, provided that for all $\epsilon > 0$ there exists $e \in E$ such that

$$0 < |e - p| < \epsilon.$$

(See Defintion 2.1.1.) Let $E \subset \mathbb{R}$ be any set with the property that there is a *cluster point* p of E such that $p \notin E$. Show that there exists an open cover of E that has no finite subcover. Justify your claims. (Note: Exercise 1.83 is an example of the claim of this exercise.)

1.85 True or False: Finitely many of the open sets in the collection

$$\left\{ \left(\frac{x}{2}, \frac{3x}{2} \right) \,\middle|\, x \in [0, 1] \right\}$$

would suffice to cover $[0, 1]$.

1.86 Prove or give a counterexample: Every open cover of a finite subset of $\mathbb{R}$ has a finite subcover. (Note: For the real line, the phrase *finite subset* does *not* mean the same thing as *finite interval*.)

1.8 COUNTABILITY OF THE RATIONAL NUMBERS

Definition 1.8.1 *A set S is called* countable *if it is an* infinite *set for which it is possible to arrange all the elements of S into a sequence. That is, S is countable if $S = \{s_1, s_2, \ldots, s_k, \ldots\}$ with each element of S listed exactly once in the sequence.*

Equivalently, we may say that S is countable if and only if there exists a *function* $s : \mathbb{N} \to S$ that is both one-to-one, which is also called *injective*, and onto S. Onto maps are often called *surjective*. The term s_n in the definition above would be $s(n)$ in this notation.

EXAMPLE 1.13

Let E denote the set of all *even* natural numbers. Thus $E \subsetneq \mathbb{N}$. We claim that E is countable. In fact, the elements of E can be arranged into a sequence by means of a function $s : n \to 2n$ that is both an injection and a surjection of E onto $\mathbb{N}$. That is, the sequence is given by $s_n = 2n$. It may surprise the reader that the elements of an infinite set can be paired one-to-one with those of a *proper* subset.

EXAMPLE 1.14

We will prove the surprising and useful fact that the set $\mathbb{Q}$ of all rational numbers is countable. It is important to understand that if a sequence s_n is to include *all* the rational numbers, then these numbers cannot be listed in *size places*. That is, if $s_n < s_{n+1}$, both in $\mathbb{Q}$, then $\frac{s_n+s_{n+1}}{2}$ lies between them and is again rational. Hence there is no *next smallest* rational number after s_n.

We can explain how to list the rational numbers in a sequence, disregarding the order relation, as follows. We are going to consider a table of numbers with infinitely many rows. The entry in the m^{th} row and nth column will be the fraction $\frac{n}{m}$. Here $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. Thus there will be a first row, in which each denominator is understood to be 1, but no last row. Each row will extend endlessly to left and to the right. We can draw only part of this table below.

$$\begin{array}{ccccccccccc}
\cdots & -4 & -3 & -2 & -1 & \mathbf{0} & 1 & 2 & 3 & 4 & \cdots \\
\cdots & -\frac{4}{2} & -\frac{3}{2} & -\frac{2}{2} & -\frac{1}{2} & \frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\cdots & -\frac{4}{3} & -\frac{3}{3} & -\frac{2}{3} & -\frac{1}{3} & \frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\cdots & -\frac{4}{4} & -\frac{3}{4} & -\frac{2}{4} & -\frac{1}{4} & \frac{0}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

We will describe a systematic *expanding search pattern* that reaches each term on the infinite table after some finite number of terms in the sequence described below. We will list side-by-side those terms $\frac{n}{m}$ for which

$$|m| + |n| = k$$

beginning with $k = 1$, $k = 2$, and so on. If parentheses are placed around a number, we are skipping that number because it was already listed previously. Here is the resulting list:

$$0, -1, \left(\frac{0}{2} = 0\right), 1, -2, -\frac{1}{2}, \left(\frac{0}{3} = 0\right), \frac{1}{2}, 2, -3, \left(-\frac{2}{2} = -1\right), -\frac{1}{3},$$

$$\left(\frac{0}{4} = 0\right), \frac{1}{3}, \left(\frac{2}{2} = 1\right), 3, \ldots .$$

It is clear that this expanding search pattern eventually reaches any rational number $\frac{n}{m}$ that one might choose, and each rational number is listed exactly once in the resulting endless sequence. The first several terms of the sequence s_n, corresponding to $k = 0, 1, 2, 3, 4$, are

$$0, -1, 1, -2, -\frac{1}{2}, \frac{1}{2}, 2, -3, -\frac{1}{3}, \frac{1}{3}, 3, \dots .$$

We will see several applications of the countability of $\mathbb{Q}$ in this book. However, for now we describe a startling example.

EXAMPLE 1.15

We will describe a set O that is *both open* and *dense* in $\mathbb{R}$, yet which is *quite small.*

Let $\epsilon > 0$, a small positive number. Consider the line segment $[0, \epsilon]$ of length ϵ. We will construct a sequence of intervals (a_k, b_k), each of length $\frac{\epsilon}{2^k}$. That is, the first interval, (a_1, b_1) will have length $\frac{\epsilon}{2}$. This leaves half of $[0, \epsilon]$ remaining. But for (a_2, b_2) we will use only half that remainder: namely, $\frac{\epsilon}{4}$. $b_3 - a_3$ will be taken to be $\frac{\epsilon}{8}$, or half of the remaining $\frac{\epsilon}{4}$ from the original interval $[0, \epsilon]$.

Let $\mathbb{Q} = \{s_1, s_2, \dots, s_k, \dots\}$, which can be arranged since $\mathbb{Q}$ is countable, as explained above. Let (a_1, b_1) be centered around s_1, (a_2, b_2) centered around s_2, and in general (a_k, b_k) will be centered around the point s_k. For any finite subcollection of the intervals (a_k, b_k), $k = 1, 2, 3, \cdots$, the sum of the lengths of each of the finitely many intervals chosen must be less than ϵ. That is because the whole infinite sequence of intervals is chosen by cutting ϵ in half again and again without end.

Now consider that $\mathbb{Q}$ is dense in $\mathbb{R}$. But if we let $O = \bigcup_{k=1}^{\infty}(a_k, b_k)$, then $\mathbb{Q} \subset O$ and so O is also dense in $\mathbb{R}$. Moreover, O is *open* by Theorem 1.7.1.

We claim that O is a small set in the following sense. Let [a,b] be *any* closed finite interval of length $\geq \epsilon$. We claim it is impossible for $[a, b] \subseteq O$. In fact, if [a,b] were a subset of O, then

$$[a, b] \subseteq \bigcup_{k=1}^{\infty}(a_k, b_k),$$

an open cover of [a,b]. By the Heine–Borel theorem, there must be a finite number of intervals from among the (a_k, b_k)'s that cover [a,b]. Yet the sum of the lengths of these finitely many intervals must be less than $\epsilon \leq b - a$. This is impossible.

It is interesting to compare the preceding example with Exercise 1.91. The interested student can learn much more about surprising subsets of the line in the book by Gelbaum and Olmsted [7].

It is natural to wonder at this point whether or not perhaps every infinite set is countable. The answer is *no*, as is shown by the following surprising theorem.

Theorem 1.8.1 (Cantor) *The set $\mathbb{R}$ of real numbers is* uncountable. *(That is, it is impossible to include all the real numbers in a sequence.)*

Proof: We begin by noting that the (possibly endless) decimal expansions of real numbers are not unique, because an infinite *tail* of 9's can always be replaced by an expansion ending in an infinite *tail* of 0's. For example,

$$0.999\ldots = 1.000\ldots$$

This is understood in the sense that if $x_n = 0.999\ldots 9$ with n 9's then

$$|x_n - 1| = \frac{1}{10^n} \to 0$$

as $n \to \infty$. But if we agree not to allow endless tails of 9's, then decimal expansions of real numbers are unique. Moreover, *every infinite decimal representation corresponds to a real number.* The reason for this fact is as follows. Consider any infinite decimal expression. It could be written in terms of a whole number K in the form

$$K + 0.d_1 d_2 d_3 \ldots d_n \ldots,$$

where d_n is the nth digit to the right of the decimal point. Then let

$$x_n = K + 0.d_1 d_2 \ldots d_n.$$

It follows that if m and n are both greater than N, then

$$|x_n - x_m| < \frac{1}{10^N} \to 0$$

as $N \to \infty$. Hence the sequence x_n of *truncations* of the endless decimal expression to n digits is itself a Cauchy sequence. By the *completeness axiom* this sequence x_n must converge to a limit $x \in \mathbb{R}$. That is why we say that the endless decimal *represents* x.

Now we suppose that Cantor's theorem were false and deduce a contradiction. Suppose therefore that all real numbers could be placed into a sequence. Then there would be a subsequence x_n containing *all* the real numbers in [0,1). We denote the decimal expansions of the numbers x_n in a vertical column below.

$$\begin{aligned}
x_1 &= .d_{11}d_{12}d_{13}\ldots d_{1k}\ldots \\
x_2 &= .d_{21}d_{22}d_{23}\ldots d_{2k}\ldots \\
x_3 &= .d_{31}d_{32}d_{33}\ldots d_{3k}\ldots \\
&\vdots \\
x_n &= .d_{n1}d_{n2}d_{n3}\ldots d_{nk}\ldots \\
&\vdots
\end{aligned}$$

Now we obtain a contradiction by constructing a number $x \in [0, 1)$ that is *not* in the sequence x_n. We define x by the digits d_k in its decimal expansion. If $d_{11} \neq 0$, we let $d_1 = 0$. If $d_{11} = 0$, let $d_1 = 1$. If $d_{22} \neq 0$, we let $d_2 = 0$. If $d_{22} = 0$, we let $d_2 = 1$. In general, if $d_{kk} \neq 0$, we let $d_k = 0$, but if $d_{kk} = 0$, then we let $d_k = 1$. We observe that $x = .d_1 d_2 d_3 \ldots d_k \ldots \in [0, 1)$, yet $x \notin \{x_n\}$ since for all n, x differs from x_n in the nth decimal digit. ■

EXERCISES

1.87 †

a) If A and B are each countable sets, show that $A \cup B$ is countable. (Hint: For each set, consider a sequence of all elements, and show how to *splice* the sequences together to make one sequence. Remember that the sets need not be disjoint.)

b) Prove that the union of countably many finite sets is either countable or finite.

1.88 † If A_n is a countable set for each $n \in \mathbb{N}$, show that

$$A = \bigcup_{n=1}^{\infty} A_n,$$

is again a countable set. (Hint: Explain why each set A_n can be written in the form

$$A_n = \{a_{nk} \mid k \in \mathbb{N}\}$$

but these sets need not be *disjoint* from one another. Consider an array similar to that displayed in the proof of Cantor's Theorem in this section, but reason in a manner similar to that in Example 1.14.)

1.89 Is the set $\mathbb{Z}$ of integers countable? Why or why not? How about the set of all odd positive integers? Even integers?

1.90 Show that the set I of all *irrational* numbers must be uncountable. (Hint: Use Exercise 1.87.)

1.91 A subset $E \subset \mathbb{R}$ is called *closed* if and only if its complement $\mathbb{R} \setminus E$ is *open*. (For example, $\mathbb{R}$ itself is a closed set since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is an open set.) *Prove* that a closed set E that is also *dense* in $\mathbb{R}$ must be all of $\mathbb{R}$. (Hint: Suppose the claim were false, so that $\mathbb{R} \setminus E$ is a nonempty open set. Deduce a contradiction.)

1.92 Referring to the definition in Exercise 1.91, answer the following questions.

a) Prove that every closed finite interval $[a, b]$ is a closed set.

b) Give an example of subset $E \subset \mathbb{R}$ for which E is *neither* open *nor* closed. Justify your example.

c) Give an example of a set $S \subseteq \mathbb{R}$ that is *both* open and closed.

1.93 ◇ Prove that every open set $S \subseteq \mathbb{R}$ can be expressed as the union of a *countable* set of open intervals. *Hint*: Let $S \cap \mathbb{Q} = \{q_n \mid n \in \mathbb{N}\}$ be a sequence listing all the rational numbers in S. Let

$$r_n = \sup\{r \mid (q_n - r, q_n + r) \subseteq S\}$$

1.94 Prove that every subset E of $\mathbb{R}$ is the union of some family of *closed* sets. Can every subset E of $\mathbb{R}$ be the union of a family of *open* sets? Prove your answer.

1.95 Let $S = \mathbb{Q} \cap [0,1]$. Then S is countable, so we can write

$$S = \{s_n \mid n \in \mathbb{N}\}.$$

We follow the model of Example 1.15 using $\epsilon = 1/2$. Thus, for each n, (a_n, b_n) is an open interval centered about s_n and $b_n - a_n = \frac{1}{2^{n+1}}$.

a) Show that $O = \bigcup_{n=1}^{\infty}(a_n, b_n)$ is an *open* subset of $\mathbb{R}$ and that every point of $[0,1]$ is the limit of a sequence of points from O.

b) Use the Heine–Borel Theorem to prove that $\mathcal{O} = \{(a_n, b_n) \mid n \in \mathbb{N}\}$ is not an open cover of $[0,1]$.

1.96 $\diamondsuit$ A real number a is called an *algebraic number* provided there exists a polynomial equation $p(x) = 0$ with integer coefficients such that $p(a) = 0$.

a) Let $P_{N,n}$ denote the set of all polynomials with *integer* coefficients of the form $p(x) = a_n x^n + \cdots + a_1 x + a_0$ for which the sum of the absolute values of the coefficients is bounded by N. That is

$$\sum_{k=0}^{n} |a_k| \leq N.$$

Show that $P_{N,n}$ is a finite set.

b) Prove that the set of algebraic numbers is countable. (Hint: Consider first the set of those numbers that are roots of a polynomial equation of degree n with integer coefficients.)

1.97 A real number is called *transcendental* provided that it is not algebraic. Prove that the set of all transcendental numbers is uncountable.

Remark 1.8.1 The method of proof employed in Cantor's theorem is known as the *Cantor diagonalization process* after its inventor, Georg Cantor (1845–1918). The discovery that some infinite sets are significantly larger than others, as uncountable sets are larger than countable ones, led to the invention of the subject of transfinite arithmetic. The student who is curious to learn more about this may enjoy the classic book by E. Kamke [11].

It is interesting to note that Cantor embarked upon his study of transfinite sets with particular applications to analysis in mind. So-called trigonometric series, or Fourier series, are representations of suitable functions as sums of perhaps infinitely many sine and cosine waves of various periods. Such representations had been shown by Fourier to be very useful for the solution of the heat equation in physics. There were, however, major difficulties regarding the uniqueness of these representations and the actual pointwise convergence of the sums of sine and cosine waves to the function under study. In the long run, it turned out that a different development undertaken by Henri Lebesgue (the Lebesgue integral) was more effective than set theory for this application. However, Cantor's research cast a new light upon the whole of mathematics, far beyond the applications that motivated the initial study. This is a good example of how investigation of an interesting question can lead to vast and totally unanticipated branches of mathematical knowledge.

The interested reader can find this and many other historical topics in Mathematics at the website of the MacTutor History of Mathematics archive[7] at the University of St. Andrews in Scotland.

1.9 TEST YOURSELF

Test Yourself sections, found at the end of each chapter, contain short questions to check your understanding of basic concepts and examples. Proofs are not tested in these sections, since proofs must be read individually by the student's teacher or teaching assistant.

EXERCISES

1.98 $\epsilon = \frac{1}{100}$. Find a *number* $\delta > 0$ small enough so that $|a-b| < \delta$ and $|c-b| < \delta$ implies $|a-c| < \epsilon$.

1.99 The sequence x_n begins as follows: $\mathbf{0}, \mathbf{1}, \frac{3}{2}, \mathbf{2}, \frac{7}{3}, \frac{8}{3}, \mathbf{3}, \frac{13}{4}, \frac{14}{4}, \frac{15}{4}, \mathbf{4}, \ldots$ and continues according to the same pattern.

a) True or False: $\lim_{n\to\infty} |x_n - x_{n+1}| = 0$.

b) True or False: x_n is a Cauchy sequence.

1.100 Give an example of two sequences, x_n and $y_n \neq 0$ such that $x_n y_n$ converges, $\frac{x_n}{y_n}$ converges, but *neither* x_n *nor* y_n converges.

1.101 Let $x_n = ((-1)^n + 1) + \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Find both $\liminf x_n$ and $\limsup x_n$.

1.102 Give an example of *two sequences* of *real numbers* x_n and y_n for which $\liminf(x_n + y_n) = 0$ but $\liminf x_n = -\infty = \liminf y_n$.

1.103 State *True* or *Give a Counterexample*: If x_n is an *unbounded* sequence, then x_n has no convergent subsequences.

1.104 Give an example of a *decreasing nest* of *nonempty* open intervals (a_n, b_n) such that $b_n - a_n \to 0$ but $\bigcap_{i=1}^{\infty}(a_n, b_n) = \emptyset$.

1.105 True or False: The set $S = \left\{ \frac{m}{2^n} \;\middle|\; m \in \mathbb{Z}, n \in \mathbb{N} \right\}$ is dense in $\mathbb{R}$.

1.106 Let $E = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$. Find an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E that has no finite subcover.

1.107 True or False: The set $\mathbb{Q}$ is closed in the real line $\mathbb{R}$.

1.108 True or False: The set $S = \{0.d_1 d_2 \ldots d_n \mid n \in \mathbb{N}\}$ of all *finitely long* decimal expansions (with each d_i an integer between 0 and 9) is countable.

1.109 True or False: The set $S = \left\{ \frac{p}{q}\sqrt{2} \;\middle|\; \frac{p}{q} \in \mathbb{Q} \right\}$ is *uncountable*.

[7] http://www-history.mcs.st-andrews.ac.uk/history/

1.110 Let $x_n = 1 + \frac{(-1)^n}{\sqrt{n}}$. If $\epsilon > 0$ find a $N \in \mathbb{N}$ sufficiently big so that $n \geq N$ implies $|x_n - 1| < \epsilon$.

1.111 True or Give a Counterexample: A bounded sequence times a convergent sequence must be a convergent sequence.

1.112 Find $\bigcap_{n=1}^{\infty}(-\infty, -n]$.

1.113 Give an example of an open cover $\mathcal{O} = \{O_n \mid |n \in \mathbb{N}\}$ of the set

$$S = \left\{ \frac{1}{n} \middle| n \geq 2,\, n \in \mathbb{N} \right\}$$

such that S has no finite subcover from $\mathcal{O}$.

1.114 Let

$$E = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\} \cup \{0\}.$$

True or False: The set $\mathbb{R} \setminus E$, that is the complement of E, is an open subset of $\mathbb{R}$.