Chapter 1

Section 1.1

1.1.2.

(a) $\mathscr{L}$ is linear.

(b) $\mathscr{L}$ is not linear, because of the uu_y term.

1.1.3.

(a) The equation is second-order, linear and inhomogeneous. It takes the form $\mathscr{L}(u) = -1$, where $\mathscr{L}(u) = u_t - u_{xx}$ is linear.

(c) The equation is third-order, nonlinear. The term uu_x is nonlinear.

1.1.5.

(a) The set is a vector space. Given any two vectors $[a_1, 0, c_1]$ and $[a_2, 0, c_2]$ in the set, their sum

$$[a_1, 0, c_1] + [a_2, 0, c_2] = [a_1 + a_2, 0, c_1 + c_2]$$

is also in the set, and the scalar multiple

$$c[a_1, 0, c_1] = [ca_1, 0, cc_1]$$

is in the set. Thus the set is a vector space.

(c) The set is not a vector space. Since both $[0, 1, 1]$ and $[1, 0, 3]$ are in the set, but their sum $[1, 1, 4]$ is not, since the product of the first two components is 1, not 0. So the set is not closed under addition. (It is closed under scalar multiplication, but that is irrelevant.)

1.1.7. The functions $1 + x, 1 - x$ and $1 + x + x^2$ are linearly independent. Suppose

$$a(1 + x) + b(1 - x) + c(1 + x + x^2) = 0$$

Then $cx^2 + (a - b + c)x + (a + b + c) = 0$. Since this must hold for every x, all three coefficients c, $a - b + c$ and $a + b + c$ must be zero. Since $c = 0$, the other equations imply $a = b$ and $a = -b$, which implies $a = b = 0$. So the only linear combination of $1 + x$, $1 - x$ and $1 + x + x^2$ which equals zero is the trivial combination.

1.1.10. Let u and v be any two solutions of the differential equation. Then

$$(u + v)''' - 3(u + v)'' + 4(u + v) = u''' - 3u'' + 4u + v''' - 3v'' + 4v = 0 + 0 = 0$$

and

$$(ku)''' - 3(ku)'' + 4(ku) = k(u''' - 3u'' + 4u) = k(0) = 0$$

so both $u + v$ and ku are solutions of the differential equation. Thus the set of solutions is closed under addition and scalar multiplication. The characteristic equation of the differential equation is $r^3 - 3r^2 + 4 = 0$, whose roots are $r = -1$ and $r = 2$ with multiplicity 2. Hence the solution set is spanned by $\{e^{-t}, e^{2t}, te^{2t}\}$. Since this spanning set is linearly independent, it is a basis for the set of solutions.

1.1.11. Since $u_x = f'(x)g(y)$, $u_y = f(x)g'(y)$ and $u_{xy} = f'(x)g'(y)$,

$$uu_{xy} = f(x)g(y)f'(x)g'(y) = f'(x)g(y)f(x)g'(y) = u_x u_y.$$

1.1.12. Since

$$\frac{\partial u_n}{\partial x} = n \cos nx \sinh ny, \qquad \frac{\partial^2 u_n}{\partial x^2} = -n^2 \sin nx \sinh ny$$

and

$$\frac{\partial u_n}{\partial y} = n \sin nx \cosh ny, \qquad \frac{\partial^2 u_n}{\partial y^2} = n^2 \sin nx \sinh ny$$

we have

$$\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} = -n^2 \sin nx \sinh ny + n^2 \sin nx \sinh ny = 0.$$

Section 1.2

1.2.1. The general solution is $u(x,t) = f(2x - 3t)$. The initial condition therefore implies $f(2x) = \sin x$, so $f(s) = \sin(s/2)$. Hence $u(x,t) = \sin(x - \frac{3}{2}t)$.

1.2.3. The equation is $\nabla u \cdot (1 + x^2, 1) = 0$, so the characteristics are solutions of

$$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

Integration yields $y = \arctan x + C$. Thus $u(x,y) = f(y - \arctan x)$, where f is any differentiable function. Some of the characteristics are shown in Figure 1.

1.2.7.

(a) The equation is $\nabla u \cdot (y, x) = 0$, so the characteristics must solve

$$\frac{dy}{dx} = \frac{x}{y}.$$

Separating variables yields $y^2 = x^2 + C$, so the general solution is $u(x,y) = f(x^2 - y^2)$. The initial condition implies $f(-y^2) = e^{-y^2}$, so $f(s) = e^s$ for $s \le 0$. Thus one solution is $u(x,y) = e^{x^2 - y^2}$.

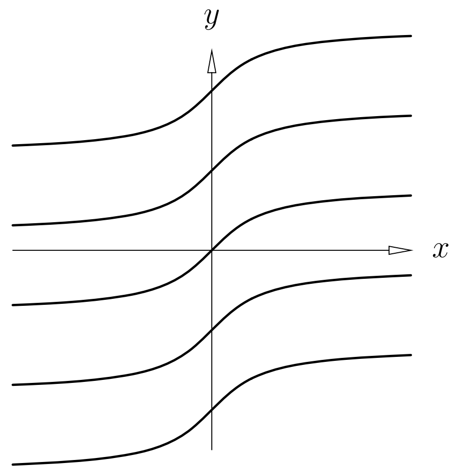

Figure 1: Characteristics of $(1+x^2)u_x + u_y = 0$.

(b) The auxiliary condition in part (a) determines $f(s)$ for $s \le 0$, and therefore the solution is determined uniquely only in the region $\{(x,y) \mid x^2 \le y^2\}$.

1.2.8. Making the change of variable $x' = ax + by$, $y' = bx - ay$ gives

$$u_x = au_{x'} + bu_{y'}, \quad u_y = bu_{x'} - au_{y'}$$

so the equation becomes

$$(a^2 + b^2)u_{x'} + cu = 0.$$

The solution of this ODE in x' is

$$u(x', y') = f(y')e^{-cx'/(a^2+b^2)},$$

where f is any differentiable function. In the original variables x and y,

$$u(x,y) = f(bx - ay)e^{-c(ax+by)/(a^2+b^2)}.$$

1.2.13. Let $x' = x + 2y$, $y' = 2x - y$. Then the equation becomes $5u_{x'} + y'u = x'y'$. This is a first-order linear ODE in x'. Using the integrating factor $\mu = e^{x'y'/5}$ it becomes

$$\frac{d}{dx'}\left(e^{x'y'/5}u\right) = \frac{1}{5}e^{x'y'/5}x'y'$$

so

$$e^{x'y'/5}u = \int \frac{1}{5}e^{x'y'/5}x'y'\,dx' = x'e^{x'y'/5} - \frac{5}{y'}e^{x'y'/5} + g(y'),$$

where g is an arbitrary differentiable function. This implies $u = x' - \frac{5}{y'} + e^{-x'y'/5}g(y')$. In the original variables (x,y) this becomes

$$u(x,y) = x + 2y - \frac{5}{2x-y} + e^{-(2x^2+3xy-2y^2)/5}g(2x-y).$$

Section 1.3

1.3.2. Proceeding as in the derivation of the wave equation, let ρ be the mass density of the chain and let g be the acceleration due to gravity. The horizontal component of the gravitational force is given by

$$\frac{u_x \rho g(l-x)}{\sqrt{1+u_x^2}}.$$

Assuming the oscillations are small this is approximately $\rho g(l-x)u_x$. Thus the net horizontal force on an interval $[x_0, x_1]$ is

$$\rho g(l-x)u_x\Big|_{x_0}^{x_1} = \rho g\left[(l-x_1)u_x(x_1,t) - (l-x_0)u_x(x_0,t)\right].$$

By Newton's Law this equals

$$\int_{x_0}^{x_1} \rho u_{tt}(x,t)\, dx.$$

Differentiating with respect to x_1 gives

$$\rho u_{tt}(x_1,t) = \rho g\left[(l-x_1)u_x(x_1,t)\right]_{x_1}.$$

Thus

$$u_{tt} = g((l-x)u_x)_x.$$

1.3.4 Since the concentration is uniform in x and y, it is described by a function $u(z,t)$. Consider fluid within the region $z_0 < z < z_1$. Then the mass of fluid in this region is

$$M(t) = \int_{z_0}^{z_1} u(z,t)\, dz, \qquad \frac{dM}{dt} = \int_{z_0}^{z_1} u_t(z,t)\, dz,$$

The new flow rate into the section is determined by both Fick's Law and the motion of the particles. By Fick's Law, the concentration gradient contributes a term $ku_z(z_1,t) - ku_z(z_0,t)$, while the motion of the fluid contributes $Vu(z_1,t) - Vu(z_0,t)$. Thus

$$\int_{z_0}^{z_1} u_t(z,t)\, dz = ku_z(z_1,t) - ku_z(z_0,t) + Vu(z_1,t) - Vu(z_0,t).$$

Differentiating with respect to z_1 gives $u_t = ku_{zz} + Vu_z$.

1.3.9. First, we see that

$$\begin{aligned}
\iiint_D \nabla\cdot\mathbf{F}\,d\mathbf{x} &= \iiint_D (r^2x)_x + (r^2y)_y + (r^2z)_z\,d\mathbf{x} \\
&= 5\iiint_D x^2+y^2+z^2\,dx\,dy\,dz \\
&= 5\int_0^{\pi}\int_0^{2\pi}\int_0^a r^4\sin\theta\,dr\,d\phi\,d\theta \\
&= a^5\int_0^{\pi}\int_0^{2\pi}\sin\theta\,d\phi\,d\theta \\
&= 2\pi a^5\int_0^{\pi}\sin\theta\,d\theta \\
&= 4\pi a^5.
\end{aligned}$$

Next, using the fact that $\mathbf{n} = \frac{\mathbf{x}}{a}$ for $x \in \text{bdy } D$, we see that

$$\begin{aligned}
\iint_{\text{bdy } D}\mathbf{F}\cdot\mathbf{n}\,dS &= \iint_{\text{bdy } D} r^2\mathbf{x}\cdot\mathbf{n}\,dS \\
&= \iint_{\text{bdy } D} r^2\frac{|\mathbf{x}|^2}{a}\,dS \\
&= \iint_{\text{bdy } D}\frac{r^4}{a}\,dS \\
&= a^3\iint_{\text{bdy } D}dS.
\end{aligned}$$

Now using the fact that the surface area of a sphere of radius a is $4\pi a^2$, we conclude that

$$\iint_{\text{bdy } D}\mathbf{F}\cdot\mathbf{n}\,dS = 4\pi a^5.$$

Section 1.4

1.4.1. $u(x,t) = x^2 + 2t$

1.4.5. Consider the mass $M(t)$ of material from $z = a$ up to $z = z_1$. Due to the impermeability at $z = a$, there is no loss or gain of material there. Thus

$$\int_a^{z_1} u_t dz = Vu(z_1,t) + ku_z(z_1,t).$$

Putting $z_1 = a$ in this equation, we get the boundary condition $Vu + ku_z = 0$ at $z = a$.

1.4.6.

(a) The equilibrium will satisfy $u_{xx} = 0$ on each piece of the rod. Integrating twice gives $u_1 = c_1 x + d_1$ and $u_2 = c_2 x + d_2$. Since $u_1(0) = 0$, $d_1 = 0$, and since $u_2(L_1 + L_2) = T$ we have

$$d_2 = T - c_2(L_1 + L_2).$$

Thus the equilibrium is defined piecewise by

$$u(x) = \begin{cases} c_1 x & 0 \le x < L_1 \\ T + c_2(x - L_1 - L_2) & L_1 < x < L_1 + L_2. \end{cases}$$

To solve for c_1 and c_2 we use the fact that both u and κu_x are continuous at $x = L_1$ to get $c_1 L_1 = T - c_2 L_2$ and $\kappa_1 c_1 = \kappa_2 c_2$. Solving for c_1 and c_2 we get

$$c_1 = \frac{\kappa_2 T}{\kappa_1 L_2 + \kappa_2 L_1} \quad \text{and} \quad c_2 = \frac{\kappa_1 T}{\kappa_1 L_2 + \kappa_2 L_1}.$$

Thus

$$u(x) = \begin{cases} \dfrac{\kappa_2 T x}{\kappa_1 L_2 + \kappa_2 L_1} & 0 \le x < L_1 \\ T + \dfrac{\kappa_1 T(x - L_1 - L_2)}{\kappa_1 L_2 + \kappa_2 L_1} & L_1 < x < L_1 + L_2. \end{cases}$$

(b) When $\kappa_1 = 2$, $\kappa_2 = 1$, $L_1 = 3$, $L_2 = 2$ and $T = 10$ we have

$$u(x) = \begin{cases} \frac{10}{7} x & 0 \le x < 3 \\ 10 + \frac{20}{7}(x - 5) & 3 < x < 5. \end{cases}$$

The sketch of u is shown in Figure 2.

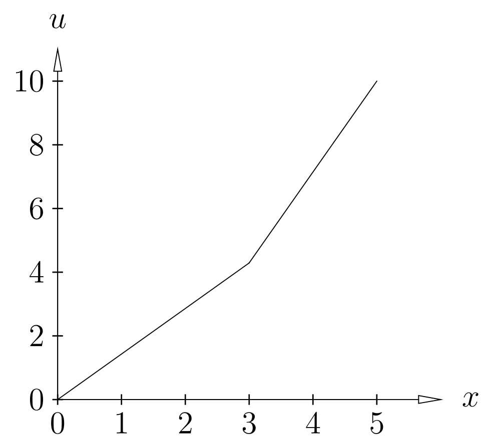

Figure 2: Solution of Exercise 1.4.6(b).

Section 1.5

1.5.1. Any function of the form $A\cos x + B\sin x$ solves the ODE. The condition $u(0) = 0$ implies $A = 0$, and the condition $u(L) = 0$ then implies $B\sin(L) = 0$. Thus, if L is any integer multiple of π, any function of the form $B\sin x$ is a solution, so solutions are not unique. Otherwise, if L is not an integer multiple of π, we must have $B = 0$, in which case the zero solution is the only solution.

1.5.2.

(a) Suppose u and v are both solutions. Let $w = u - v$. Then w satisfies

$$\begin{aligned}&w'' + w' = 0\\&w'(0) = w(0) = \tfrac{1}{2}[w'(l) + w(l)].\end{aligned}$$

Solutions of the original problem are unique if and only if $w = 0$ is the only solution of this problem. This ODE has general solution $w = C_1 + C_2e^{-x}$. Since $w(0) = C_1 + C_2$ and $w'(0) = -C_2$, the first part of the boundary condition implies $C_1 = -2C_2$ and therefore $w = C_2(-2 + e^{-x})$. Next,

$$\frac{1}{2}[w'(l) + w(l)] = \frac{1}{2}(-C_2e^{-l} - 2C_2 + C_2e^{-l}) = -C_2,$$

which equals $w(0)$ and $w'(0)$. Hence any multiple of $-2 + e^{-x}$ is a solution of the above problem, so solutions of the original problem are not unique.

(b) Integrating the equation from 0 to l and using the boundary conditions gives

$$\begin{aligned}\int_0^l f(x)\,dx &= \int_0^l u''(x) + u'(x)\,dx = \Big[u'(x) + u(x)\Big]_0^l\\&= u'(l) + u(l) - (u'(0) + u(0)) = 0.\end{aligned}$$

Hence a solution can exist only if the integral of f over $[0, l]$ is zero.

Section 1.6

1.6.1.

(a) Since $a_{12}^2 - a_{11}a_{22} = 2^2 - 1\cdot 1 = 3 > 0$, the equation is hyperbolic.

1.6.5. Let $u = ve^{\alpha x+\beta y}$. Then

$$\begin{aligned}u_x &= e^{\alpha x+\beta y}(\alpha v + v_x)\\u_y &= e^{\alpha x+\beta y}(\beta v + v_y)\\u_{xx} &= e^{\alpha x+\beta y}(\alpha^2 v + 2\alpha v_x + v_{xx})\\u_{yy} &= e^{\alpha x+\beta y}(\beta^2 v + 2\beta v_y + v_{yy}).\end{aligned}$$

Substituting into the equation gives

$$e^{\alpha x+\beta y}(v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v) = 0.$$

To eliminate the v_x and v_y terms, let $\alpha = 1$ and $\beta = -4$. Then the new equation is

$$v_{xx} + 3v_{yy} - 44v = 0.$$

Letting $y' = \frac{1}{\sqrt{3}}y$ it follows that $v_{y'y'} = 3v_{yy}$, so $v_{xx} + v_{y'y'} - 44v = 0$.