CHAPTER 1

Wavefields

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1.1 INTRODUCTION

The theory of univariate stochastic processes [1, 2] forms the backbone of the analysis and processing of single sensor data. Stochastic processes $X(t)$ are parameterized by a single free variable t, which is a timelike variable. By performing some appropriate processing on samples from realizations of the process $X(t)$, one hopes to be able to infer physical properties about the source of the signal, about the transmission medium, or both. To aid in the development of processing techniques, it is customary to seek simplifying assumptions. A standard assumption is that of stationarity [3], which allows for drastic simplifications of moment functions and related quantities. Needless to say, the standard assumptions that we apply are often questionable for real-world situations.

However, our problems of interest in the physical world are often parameterized in four-dimensional space–time coordinates rather than in time alone. Examples of such are abundant; just think about any wave or fluctuation phenomena in physical space [4–7]. For engineering applications, the propagation and space–time distribution of radio waves is an important example [8–11], as is the propagation and space–time distribution of acoustic waves in audio applications [12]. In the geosciences and in oil-related prospecting and research, space–time statistics enter at all levels [13–15]. Such phenomena necessitate a generalization of the theory of univariate stochastic processes to a theory of multivariate stochastic wavefields [11, 16–18]. An appropriate notation for a space–time wavefield could then be $X(\mathbf{r}, t)$, where $\mathbf{r}$ is a three-dimensional position vector in some preferred coordinate system.

One would think that the generalization of stochastic processes to stochastic wavefields should be straightforward. On the surface, it may seem so, but in reality, it is not that simple. The reason is that for wave or fluctuation phenomena in physical media, the supporting medium imposes severe constraints on the waves and fluctuations that can be supported. This is the ubiquitous phenomenon of wave dispersion, which implies that the medium only allows certain combinations of frequencies and wavenumbers (spatial vector frequencies) to exist [19, 20]. While dispersion is well understood for classical deterministic space–time systems, it is poorly understood for stochastic systems.

Handbook on Array Processing and Sensor Networks. By Simon Haykin and K.J.R. Liu.

1.2 HARMONIZABLE STOCHASTIC PROCESSES

A recent and very welcome trend in signal processing is to abandon some of the standard simplifying and limiting assumptions from the early days of the field. This strategy would allow us to deal with situations that are of great practical importance. When dealing with stochastic processes, one often assumes that the process under study obeys some kind of stationarity or time invariance. However, in practice, crucial statistical parameters may be time variant, thus violating the stationarity assumption.

To be able to deal with time-variant statistics, we need to arm ourselves with a stringent theory, rather than qualitative and ad hoc descriptions. Luckily, a very systematic and useful theoretical framework for so-called harmonizable stochastic processes, developed mainly by Michel Loève and Harald Cramér in the 1940s, is at our disposal. We are of the opinion that engineering has now matured to such a degree that the theory of harmonizable stochastic processes should be part of the toolbox of any advanced engineer. In this chapter, we will briefly review this beautiful and by now classical theory, and we will emphasize an intuitive understanding rather than going to the deepest theoretical level.

In this chapter, we will invariably assume that the real valued stochastic process $X(t)$ belongs to the *harmonizable class* [21–24]. Any processes belonging to the harmonizable class has the following stochastic integral representation:

$$X(t) = \int \exp(j\omega t)\, dZ(\omega), \tag{1.1}$$

which should be understood in the mean-square sense, and where the integration is over the real axis, $\mathbb{R}$. Note that since $X(t)$ is a stochastic process, its conventional Fourier transform does not exist. Instead, in Eq. (1.1) the complex exponential $\exp(j\omega t)$ is integrated against a complex valued infinitesimal random process $dZ(\omega)$ parameterized by the radian frequency ω. This crucial quantity is the so-called *increment process* or the *generalized Fourier transform* for the stochastic process $X(t)$. Equation (1.1) is often called the *spectral representation* of $X(t)$, and this particular type of integral is often called a Fourier–Stieltjes integral.

From an engineering point of view, it is useful to realize that the Fourier–Stieltjes integral is nothing more exotic than an infinite sum of infinitesimal complex phasors, one phasor for each frequency ω. Depending on the distribution of the infinitesimal complex amplitudes, and the correlation among the phasors for different frequencies, a great variety of stochastic processes $X(t)$ can be constructed. We now understand that the statistical characteristics of the generalized Fourier transform $dZ(\omega)$ completely determines the behavior and properties of the observable stochastic process $X(t)$.

1.2.1 Moment Functions

The most important aspect of the harmonizable class is that nonstationarities are readily included in this formulation. To appreciate this fact, we start by defining the dual-time second-order autocorrelation function (ACF) by

$$M_{XX}(t, \tau) = E\left[X(t)X(t+\tau)\right], \tag{1.2}$$

where we understand that t is a global time variable, and τ is a local (or time shift) variable. For nonstationary processes, it is evident that we need to keep track of both time arguments, as the statistics may change with the global time t. Wide-sense stationary processes would be those processes for which the ACF is independent of global time t and where the mean process $E[X(t)]$ is a constant independent of t.

Inserting the spectral representation Eq. (1.1) into the dual-time ACF Eq. (1.2), we readily find that the ACF can be formulated as [25]

$$M_{XX}(t,\tau) = \int\int \exp\left[j(\nu t + \omega\tau)\right] S_{XX}(\omega,\nu)\frac{d\omega\, d\nu}{(2\pi)^2}, \tag{1.3}$$

where the dual-frequency complex-valued density $S_{XX}(\omega,\nu)$ is defined by

$$S_{XX}(\nu,\omega)\frac{d\omega\, d\nu}{(2\pi)^2} = E\left[dZ(\omega)\, dZ^*(\omega-\nu)\right]. \tag{1.4}$$

Here, $\omega \in \mathbb{R}$ is a global frequency variable, and $\nu \in \mathbb{R}$ is a local (or frequency shift) variable, and the asterisk denotes complex conjugation. The dual-frequency spectrum $S_{XX}(\nu,\omega)$ is often called the Loève spectrum. It is important to observe that the Loève spectrum is nothing but the correlation between the frequency components of the increment process that generates our stochastic process of interest. We also immediately notice that the dual-time ACF and the dual-frequency Loève spectrum are a two-dimensional Fourier transform pair, with the global time t transforming into the local frequency ν, and the local time τ transforming into the global frequency ω.

Although it is quite obvious that the dual-frequency spectrum must contain important information about nonstationary stochastic processes, it has so far found surprisingly few uses in practice. It is, however, interesting to see that very advanced applications of dual-frequency spectra were discussed already in the early 1960s by Hagfors [9, 26, 27]. His analysis of radar returns from the lunar surface was instrumental for the Apollo missions and lunar landings in the 1960s and 1970s.

Since we have two time variables (t and τ) and two frequency variables (ω and ν) available, we may construct two more relevant second-order moment functions just by invoking partial Fourier transforms. First, by transforming the ACF with respect to τ, or by inverse transforming the Loève spectrum with respect to ν, we obtain the following important time–frequency spectrum [25]:

$$R_{XX}(t,\omega)\frac{d\omega}{2\pi} = E\left[X(t)\, dZ^*(\omega)\right] e^{-j\omega t}. \tag{1.5}$$

This is the so-called Kirkwood–Rihaczek (KR) time–frequency spectrum, which has a special status among all bilinear time–frequency density functions. This distribution first appeared in quantum mechanics in a work by John Kirkwood in 1933 [28], and it was later rediscovered by August Rihaczek in the context of signal theory in 1968 [29]. The KR spectrum enjoys many fundamental and important properties, and we see that it is basically the correlation between the process itself at time t and its generator, the generalized Fourier transform at frequency ω. As such, it is a measure of the similarity beween the stochastic process at time t with an infinitesimal stochastic phasor at frequency ω.

$$\begin{array}{ccc} M_{XX}(t,\tau) & \xrightarrow{\tau\to\omega} & R_{XX}(t,\omega) \\ \Big\downarrow{\scriptstyle t\to\nu} & & \Big\downarrow{\scriptstyle t\to\nu} \\ A_{XX}(\nu,\tau) & \xrightarrow[\tau\to\omega]{} & S_{XX}(\nu,\omega) \end{array}$$

Figure 1.1 Four-corners diagram: Fourier relations between the basic densities of harmonizable stochastic processes.

The fourth possibility results either from a partial Fourier transform of the dual-time ACF $M_{XX}(t,\tau)$ with respect to the global time t or a partial inverse Fourier transform of the dual-frequency Loève spectrum $S_{XX}(\nu,\omega)$ with respect to the global frequency ω. The resulting function $A_{XX}(\nu,\tau)$ is the so-called ambiguity function, which is a time–frequency spectrum in local coordinates.

It turns out that the four possible second-order moment functions can be arranged into a useful pattern often called a *four-corners diagram* [25]. In the four-corners diagram shown in Figure 1.1, functions in adjacent corners are one Fourier transform apart, while functions in opposite corners are two Fourier transforms apart.

1.2.2 Wide-Sense Stationarity

Note that the four-corners diagram is a direct generalization of the familiar Einstein–Wiener–Khinchine relation for wide-sense stationary (WSS) stochastic processes. Hence, for WSS processes, the four-corners diagram will collapse into the familiar single Fourier transform pair

$$\widetilde{M}_{XX}(\tau) \longleftrightarrow \widetilde{S}_{XX}(\omega), \tag{1.6}$$

where $\widetilde{M}_{XX}(\tau) = E\{X(t)X(t+\tau)\}$ is the conventional autocorrelation function that depends only on local time τ, and $\widetilde{S}_{XX}(\omega) = \int \widetilde{M}_{XX}(\tau)\exp(-j\omega\tau)\,d\tau$ is the conventional power spectral density.

In greater detail, we find that the conventional stationary power spectrum is always completely included in the dual-frequency spectrum Eq. (1.4), and that it can be extracted by inserting $\nu = 0$. For WSS processes, the dual-frequency spectrum has the particularly simple form

$$S_{XX}(\nu,\omega) = \widetilde{S}_{XX}(\omega)\delta(\nu), \tag{1.7}$$

that is, the conventional power spectral density $\widetilde{S}_{XX}(\omega)$ rides on a delta ridge concentrated on the single line $\nu = 0$. For this reason, the axis $\nu = 0$ is called the *stationary manifold*. In general, however, the Loève spectrum has contributions also outside the stationary manifold for an arbitrary nonstationary process.

Since the dual-frequency spectrum is a frequency correlation, we now understand that for a harmonizable process to be WSS, its frequency components are orthogonal, since for WSS,

$$E\left\{dZ(\omega_1)\,dZ^*(\omega_2)\right\} = 0 \qquad \forall\omega_1 \neq \omega_2. \tag{1.8}$$

We now understand that if at least one frequency pair of the generalized Fourier transform is nonorthogonal, then evidently the process will be nonstationary.

1.3 STOCHASTIC WAVEFIELDS

Stochastic wavefields are multidimensional generalizations of stochastic processes. In this chapter, we will specifically consider space–time wavefields; thus the time variable t must be augmented by a three-dimensional spatial vector variable $\mathbf{r} = [x, y, z]^{\mathrm{T}}$, where x, y, and z are Cartesian coordinates. Assume now that a scalar spatiotemporal stochastic wavefield $\xi(\mathbf{r}, t)$ is harmonizable in space and time, that is, that we can generalize Eq. (1.1) so that we can express $\xi(\mathbf{r}, t)$ by the following four-dimensional integral:

$$\xi(\mathbf{r}, t) = \int \exp\left[j(\boldsymbol{k}^{\mathrm{T}}\mathbf{r} - \omega t)\right] \, dZ(\boldsymbol{k}, \omega), \tag{1.9}$$

where $dZ(\boldsymbol{k}, \omega)$ is the increment processes or generalized Fourier transform for the random field, $\boldsymbol{k} = [k_x, k_y, k_y]^{\mathrm{T}}$ is a three-dimensional wavenumber vector variable with components k_x, k_y, and k_z, and ω is a scalar radian frequency variable.

Depending on the application, the stochastic wavefield may be a scalar, a vector, or a tensor function of space–time. For example, a scalar field may be used to represent acoustic fluctuations, a vector field may be used to represent electric and magnetic field vectors in electromagnetics, whereas tensor fields may be used to model mechanical stress and strain, or electric conductance in anisotropic media. In the present chapter, we will only consider scalar wavefields.

By reading some of the statistics literature (e.g., [30, 31]), one may be left with the impression that it is trivial to extend the harmonizable class of random processes to random fields. According to these sources, one would simply have to replace the "time" variable with a "time" vector, and hence all defining integrals would inherit the dimensionality of the time vector, and ω would become a corresponding frequency vector. If these statements were true, it would be straightforward to generalize the concept of nonstationary processes to inhomogeneous random fields.

In reality, however, physics dictates the dynamics of random fields. In short, random fields evolving in space–time have to respond to the actual physical medium in which they operate. It should be obvious that any physical medium supporting a fluctuation or disturbance of some kind, puts severe constraints on the fluctuations. The most important physical constraint is that of *dispersion*, which among other things implies that components with different frequencies (or wavenumbers) in general have different propagation speeds. In more technical terms, this means that a given frequency can only correspond to a single (or a certain set of) wavenumbers (spatial frequencies).

It has proven difficult to combine inhomogenity and dispersion in a single description for random fields. In this chapter, we will outline how such a stringent and systematic theory may be built within the class of harmonizable random fields. We expect that much of the theory and practice of array processing would benefit from being generalized and rewritten in terms of this theory.

1.3.1 Second-Order Moment Functions

Paralleling the discussion for stochastic processes, we seek a systematic theory for the second-order moment functions of stochastic wavefields. Depending on which variables one chooses to emphasize, one may formulate the second-order moments in space–time, frequency–wavenumber, and combinations thereof. By a proper

generalization of the theory for harmonizable random processes to random wavefields, we will be able to deal with nonstationary and inhomogeneous random fields. Among the attractive properties of the resulting formulation is that we will be able to retain our understanding of frequencies and wavenumber vectors, even in the presence of nonstationary and inhomogeneous fields.

1.3.1.1 Spatiotemporal Correlation Functions By autocorrelating the wavefield at two different position vectors ($\boldsymbol{r}$ and $\boldsymbol{r}+\boldsymbol{\rho}$) and two different time instants (t and $t+\tau$), we define the following space–time correlation function:

$$M_{\xi\xi}(\boldsymbol{r},\boldsymbol{\rho};t,\tau)=E\left\{\xi(\boldsymbol{r},t)\xi(\boldsymbol{r}+\boldsymbol{\rho},t+\tau)\right\}. \tag{1.10}$$

Here, $\boldsymbol{r}$ and t are global spatial and temporal coordinates, and $\boldsymbol{\rho}$ and τ are the corresponding local coordinates.

1.3.1.2 Frequency–Wavenumber Spectrum By inserting the four-dimensional generalized Fourier description from Eq. (1.9) into the definition of the second-order space–time correlation function, Eq. (1.10), we find that the space–time correlation function can be formulated through the following eight-dimensional integral:

$$M_{\xi\xi}(\boldsymbol{r},\boldsymbol{\rho};t,\tau)=\int \exp\left[j(\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\rho}+\boldsymbol{\kappa}^{\mathrm{T}}\boldsymbol{r}-\omega\tau-\nu t)\right] S_{\xi\xi}(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu)\frac{d\boldsymbol{k}\,d\boldsymbol{\kappa}\,d\omega\,d\nu}{(2\pi)^8}, \tag{1.11}$$

where the complex-valued dual-wavenumber, dual-frequency density is defined by

$$S_{\xi\xi}(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu)\frac{d\boldsymbol{k}\,d\boldsymbol{\kappa}\,d\omega\,d\nu}{(2\pi)^8}=E\left[dZ(\boldsymbol{k},\omega)\,dZ^{*}(\boldsymbol{k}-\boldsymbol{\kappa},\omega-\nu)\right]. \tag{1.12}$$

Basically, this important density is the nonnormalized correlation or inner product between the stochastic generator of the wavefield, at a pair of wavenumber vectors, and at a pair of frequencies.

1.3.1.3 Spatiotemporal Frequency–Wavenumber Spectrum: Global Variables When dealing with wave propagation and stochastic fluctuations, it is often desirable to formulate the wavefields simultaneously in space, wavenumber, frequency, and time. Such a formulation can be derived from either of the two preceding second-order moment functions. For example, if we Fourier transform the dual-space, dual-time correlation function with respect to local space and local time, we obtain a function that is defined in terms of global space, global wavenumber, global time, and global frequency. The same function can be derived by an inverse Fourier transform of the dual-wavenumber, dual-frequency correlation function, with respect to the local wavenumber and local time variables. In both cases, the following interesting complex-valued correlation function appears:

$$R_{\xi\xi}(\boldsymbol{k},\boldsymbol{r};\omega,t)=\int S_{\xi\xi}(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu)\exp[j(\nu t-\boldsymbol{\kappa}^{\mathrm{T}}\boldsymbol{r})]\frac{d\nu\,d\boldsymbol{\kappa}}{(2\pi)^4}. \tag{1.13}$$

The resulting spatiotemporal frequency–wavenumber spectrum is a direct generalization of the Kirkwood–Rihaczek time–frequency spectrum to wavefields.

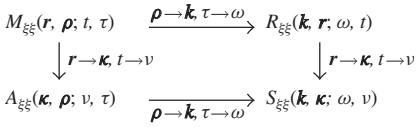

Figure 1.2 Fourier relations between the basic spectral densities of nonstationary and inhomogeneous harmonizable wavefields.

1.3.1.4 Spatiotemporal Frequency–Wavenumber Spectrum: Local Variables To derive the spatiotemporal frequency–wavenumber spectrum in terms of global variables, we performed an inverse Fourier transform of the dual-frequency dual-wavenumber spectrum $S_{\xi\xi}(\boldsymbol{k}, \boldsymbol{\kappa}; \omega, \nu)$ with respect to the local variables ($\boldsymbol{\kappa}$ and ν). However, one more possibility remains. If we perform an inverse Fourier transform of $S_{\xi\xi}(\boldsymbol{k}, \boldsymbol{\kappa}; \omega, \nu)$ with respect to the global variables $\boldsymbol{k}$ and t, we obtain a direct generalization of the well-known ambiguity function. The resulting spatiotemporal frequency–wavenumber spectrum is expressed exclusively in terms of local variables, and takes the form

$$A_{\xi\xi}(\boldsymbol{\kappa}, \boldsymbol{\rho}; \nu, \tau) = \int S_{\xi\xi}(\boldsymbol{k}, \boldsymbol{\kappa}; \omega, \nu) \exp[j(\omega\tau - \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\rho})] \frac{d\boldsymbol{k}\, d\omega}{(2\pi)^4}. \tag{1.14}$$

This quantity is a direct generalization of the ambiguity function that often appears in sonar and radar problems. However, the present generalization includes both temporal and spatial ambiguity.

1.3.1.5 Stationary and Homogeneous Manifold When dealing with stochastic processes and time series, a standard simplifying assumption is that of wide-sense stationarity. Wide-sense stationarity implies that the second-order temporal moment function is invariant with respect to shifts in global time, which implies that the dual-frequency spectrum is invariant with respect to shifts in the local frequency. Thus, we may formulate the stationary manifold of stochastic processes as being those processes for which the dual frequency is confined to the manifold:

$$\{(f, \nu) \mid f \in \mathbb{R}, \nu = 0\}. \tag{1.15}$$

Recall also from Section 1.2 that assuming wide-sense stationarity is equivalent to the assumption that the frequency components of the process are orthogonal.

For spatiotemporal wavefields, one must also generalize the spatial statistics so that the dual-wavenumber part becomes invariant with respect to shifts in the local wavenumber vector. We thus define the stationary and homogeneous manifold to be the subspace:

$$\left\{(\boldsymbol{k}, \boldsymbol{\kappa}; f, \nu) \mid \boldsymbol{k} \in \mathbb{R}^3, \boldsymbol{\kappa} = \boldsymbol{0}, f \in \mathbb{R}, \nu = 0\right\}. \tag{1.16}$$

The orthogonality argument carries directly over to the wavefields, implying that for stationary and homogeneous wavefields we must have

$$S_{\xi\xi}(\boldsymbol{k}, \boldsymbol{\kappa}; \omega, \nu) = \widetilde{S}_{\xi\xi}(\boldsymbol{k}, \omega)\delta(\nu)\delta(\boldsymbol{\kappa}) \tag{1.17}$$

for some nonnegative frequency–wavenumber spectral density function $\widetilde{S}_{\xi\xi}(\boldsymbol{k}, \omega)$. It is important to realize that $\widetilde{S}_{\xi\xi}(\boldsymbol{k}, \omega)$ is the conventional frequency–wavenumber

spectrum. It is now evident that the conventional stationary and homogeneous frequency–wavenumber spectrum resides on a four-dimensional subspace (defined by $\boldsymbol{\kappa} = \mathbf{0}$, $\nu = 0$) in $\mathbb{R}^8$.

1.3.2 Hilbert Space and Generalized Coherences

To gain some insight into the meaning of the dual-frequency dual-wavenumber spectrum for nonstationary and inhomogeneous stochastic wavefields, we will now take a closer look at the geometry of this quantity. First, we note that the spectrum in Eq. (1.12) can be written as

$$S_{\xi\xi}(\boldsymbol{k}, \boldsymbol{\kappa}; \omega, \nu) \frac{d\boldsymbol{k}\, d\boldsymbol{\kappa}\, d\omega\, d\nu}{(2\pi)^8} = \langle dZ(\boldsymbol{k}, \omega), dZ(\boldsymbol{k} - \boldsymbol{\kappa}, \omega - \nu) \rangle \,, \tag{1.18}$$

where the Hilbert space inner product between two complex-valued stochastic variables U and V is defined by $\langle U, V \rangle \equiv E\{UV^*\}$.

Any inner product can be associated with the cosine of an angle between subspaces. In our case, we can define a magnitude-squared dual-frequency dual-wavenumber generalized coherence function as [17, 18]

$$\gamma^2_{\xi\xi}(\boldsymbol{k}, \boldsymbol{\kappa}; \omega, \nu) = \cos^2 \psi_{\xi\xi}(\boldsymbol{k}, \boldsymbol{\kappa}, \omega, \nu) = \frac{|E\{dZ(\boldsymbol{k}, \omega)\, dZ^*(\boldsymbol{k} - \boldsymbol{\kappa}, \omega - \nu)\}|^2}{E\left\{|dZ(\boldsymbol{k}, \omega)|^2\right\} E\left\{|dZ(\boldsymbol{k} - \boldsymbol{\kappa}, \omega - \nu)|^2\right\}}. \tag{1.19}$$

By employing the Cauchy–Schwartz inequality, one can show that

$$0 \leq \gamma^2_{\xi\xi}(\boldsymbol{k}, \boldsymbol{\kappa}; \omega, \nu) \leq 1. \tag{1.20}$$

Full coherence [$\gamma^2_{\xi\xi}(\boldsymbol{k}, \boldsymbol{\kappa}; \omega, \nu) \equiv 1$] is achieved whenever [17, 18]

$$dZ(\boldsymbol{k} - \boldsymbol{\kappa}, \omega - \nu) = \alpha\, dZ(\boldsymbol{k}, \omega), \tag{1.21}$$

for some $\alpha \in \mathbb{R}$. If we express the stochastic infinitesimal wavefield generator by its magnitude and phase,

$$dZ(\boldsymbol{k}, \omega) = |dZ(\boldsymbol{k}, \omega)| \exp\left[j\phi_\xi(\boldsymbol{k}, \omega)\right], \tag{1.22}$$

we reach the conclusion that full coherence is obtained for those $\boldsymbol{k}, \boldsymbol{\kappa}, \omega, \nu$ for which

$$\phi_\xi(\boldsymbol{k} - \boldsymbol{\kappa}, \omega - \nu) = \phi_\xi(\boldsymbol{\kappa}, \omega) + n\pi, \tag{1.23}$$

where $n \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the signed integers. This fundamental result shows that full coherence corresponds to the case where the phases of the components at $(\boldsymbol{k} - \boldsymbol{\kappa}, \omega - \nu)$ and $(\boldsymbol{\kappa}, \omega)$ are identical modulo π.

Generalizations to complex stochastic processes and to time–frequency coherences are discussed in [32, 33].

1.3.3 Linear Spatiotemporal Systems

Wavefields can be temporally, spatially, or spatiotemporally filtered [16]. If the wavefield $\eta(\boldsymbol{r}, t)$ is filtered by a spatiotemporal impulse response $h(\boldsymbol{r}, t)$, then the linear spatiotemporal output wavefield $\xi(\boldsymbol{r}, t)$ is defined by the following four-dimensional convolution integral:

$$\xi(\boldsymbol{r}, t) = \int h(\boldsymbol{r}', t')\eta(\boldsymbol{r} - \boldsymbol{r}', t - t')\, d\boldsymbol{r}'\, dt'. \tag{1.24}$$

If the input process $\eta(\boldsymbol{r}, t)$ is stationary and spatially homogeneous, then the output filtered wavefield $\xi(\boldsymbol{r}, t)$ is also stationary and homogeneous. In frequency–wavenumber space, we now readily find the conventional $(\boldsymbol{k}, \omega)$ spectrum of the filtered wavefield to be given by

$$\widetilde{S}_{\xi\xi}(\boldsymbol{k}, \omega) = |H(\boldsymbol{k}, \omega)|^2\, \widetilde{S}_{\eta\eta}(\boldsymbol{k}, \omega). \tag{1.25}$$

Here, $H(\boldsymbol{k}, \omega) = \int h(\boldsymbol{r}, t) \exp\left[-j(\boldsymbol{k}^{\mathrm{T}}\boldsymbol{r} - \omega t)\right] d\boldsymbol{r}\, dt/(2\pi)^4$ is the frequency–wavenumber response function of the linear spatiotemporal system.

1.3.4 Estimation

The state-of-the-art nonparametric estimator for conventional power spectral densities is David J. Thomson's multitaper estimator [34]. Thomson's estimator utilizes a set of orthonormal data tapers called discrete prolate spheroidal sequences. The beauty of the estimator is that through the tuning of a single-frequency bandwidth parameter, one attains control over spectral leakage, and one stabilizes the estimate through an inherent variance reduction offered by the estimator. The superiority of Thomson's multitaper technique has been demonstrated in numerous publications, and it has become a widespread estimator among practitioners with demanding data.

Thomson's technique can readily be extended to higher dimensional estimation problems, as has been demonstrated among others by [17, 18, 35]. We have long-term experience with higher dimensional extensions of the multitaper estimators, and our recommendation to other users is to employ the multitaper class.

1.4 WAVE DISPERSION

While a substantial amount of literature has been devoted to the correlation theory of stationary and homogeneous random waves and fluctuations (e.g., [6, 8, 10, 11, 16, 36]), one would have a hard time identifying contributions dealing with systems that are simultaneously nonstationary and inhomogeneous. Similarly, while dispersive effects are well treated for nonrandom space–time systems [7, 37], the combination of random fluctuations and dispersion seems to be totally lacking in the literature. In this chapter, we will take the first steps toward a full theory for random, nonstationary, inhomogeneous, and dispersive fluctuations.

1.4.1 Linear Systems Approach

Consider the following governing dynamical model for a random scalar spatiotemporal quantity (i.e., a scalar random field) $\xi(\mathbf{r}, t)$:

$$L\{\xi(\mathbf{r}, t)\} = \eta(\mathbf{r}, t). \tag{1.26}$$

Here, $L\{\cdot\}$ is a linear spatiotemporal differential operator; $\mathbf{r} = [x, y, z]^{\mathrm{T}}$ is a three-dimensional spatial vector variable with Cartesian components x, y, and z; t is a temporal variable, and $\eta(\mathbf{r}, t)$ is a random scalar space–time driving force field. (Note that the theory presented in this chapter could readily be generalized to vector and tensor fields.)

Assume that the two random fields in Eq. (1.26) are harmonizable in space and time, that is, that we can generalize Eq. (1.1) so that we can express $\xi(\mathbf{r}, t)$ and $\eta(\mathbf{r}, t)$ by the following four-dimensional integrals:

$$\xi(\mathbf{r}, t) = \int \exp\left[j(\boldsymbol{k}^{\mathrm{T}}\mathbf{r} - \omega t)\right] dZ(\boldsymbol{k}, \omega) \tag{1.27}$$

and

$$\eta(\mathbf{r}, t) = \int \exp\left[j(\boldsymbol{k}^{\mathrm{T}}\mathbf{r} - \omega t)\right] dW(\boldsymbol{k}, \omega), \tag{1.28}$$

respectively, where $dZ(\boldsymbol{k}, \omega)$ and $dW(\boldsymbol{k}, \omega)$ are the increment processes or generalized Fourier transforms for the two random fields, $\boldsymbol{k} = [k_x, k_y, k_y]^{\mathrm{T}}$ is a three-dimensional wavenumber vector variable with components k_x, k_y, and k_z, and ω is a scalar radian frequency variable.

A physically meaningful and sufficiently general form of the linear partial differential operator $L\{\cdot\}$ is

$$L\{\cdot\} = \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{p=0}^{P}\sum_{q=0}^{Q} A_{m,n,p,q}\frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n}\frac{\partial^p}{\partial y^p}\frac{\partial^q}{\partial z^q}, \tag{1.29}$$

where $A_{m,n,p,q}$ are coefficients, and M, N, P, and Q are the respective orders of the four differential operators. By inserting Eqs. (1.27)–(1.29) into Eq. (1.26), we find that the partial differential equation can be recast into a wavenumber–frequency integral equation

$$\int D(\boldsymbol{k}, \omega)\exp\left[j(\boldsymbol{k}^{\mathrm{T}}\mathbf{r} - \omega t)\right] dZ(\boldsymbol{k}, \omega) = \int \exp\left[j(\boldsymbol{k}^{\mathrm{T}}\mathbf{r} - \omega t)\right] dW(\boldsymbol{k}, \omega) \tag{1.30}$$

where

$$D(\boldsymbol{k}, \omega) = \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{p=0}^{P}\sum_{q=0}^{Q} A_{m,n,p,q}(-j\omega)^m(jk_x)^n(jk_y)^p(jk_z)^q \tag{1.31}$$

is the wavenumber–frequency version of the differential operator. The function $D(\boldsymbol{k}, \omega)$ is directly connected to wave dispersion, normal modes, damping, and physical resonances.

1.4.2 Dispersion Relation, Phase, and Group Velocities

From the previous section, we see directly that the increment processes themselves are related through the relation

$$D(\boldsymbol{k}, \omega)\, dZ(\boldsymbol{k}, \omega) = dW(\boldsymbol{k}, \omega). \tag{1.32}$$

For the undriven case, $\eta(\mathbf{x}, t) = 0$, so $dW(\boldsymbol{k}, \omega) = 0$, and we obtain the dispersion relation, or the equation for the normal modes as [6, 7]

$$D(\boldsymbol{k}, \omega) = 0. \tag{1.33}$$

The solutions to Eq. (1.33) can be written either as $\omega = \Omega(\boldsymbol{k})$ or $\boldsymbol{k} = \boldsymbol{k}(\omega)$, for some scalar function $\Omega(\cdot)$ and some vector function $\boldsymbol{k}(\cdot)$. Note that many fluctuating media or systems are such that several roots (or modes) result from the dispersion relation [6, 7].

Two important concepts from classical wave propagation theory are phase velocity and group velocity [4, 5, 7, 37, 38]. The phase velocity vector is defined by

$$\boldsymbol{v}_\phi(\boldsymbol{k}) = \frac{\omega(\boldsymbol{k})}{||\boldsymbol{k}||}\boldsymbol{u}_k, \tag{1.34}$$

where the unit wave vector is defined by $\boldsymbol{u}_k = \boldsymbol{k}/||\boldsymbol{k}||$. The group velocity vector is defined as the gradient of $\omega(\boldsymbol{k})$ in wavenumber space:

$$\boldsymbol{v}_g(\boldsymbol{k}) = \frac{\partial \omega(\boldsymbol{k})}{\partial \boldsymbol{k}}, \tag{1.35}$$

where the wavenumber space gradient operator is defined by

$$\frac{\partial}{\partial \boldsymbol{k}} \equiv \boldsymbol{u}_x \frac{\partial}{\partial k_x} + \boldsymbol{u}_y \frac{\partial}{\partial k_y} + \boldsymbol{u}_z \frac{\partial}{\partial k_z}, \tag{1.36}$$

and where $\boldsymbol{u}_x$, $\boldsymbol{u}_y$, and $\boldsymbol{u}_z$ are the unit vectors along the x, y, and z axes, respectively.

When the phase velocity vector $\boldsymbol{v}_\phi(\boldsymbol{k})$ and the group velocity vector $\boldsymbol{v}_g(\boldsymbol{k})$ are equal and independent of the wavenumber $\boldsymbol{k}$, the disturbance propagates dispersionless, that is, a pulse will not change its shape during propagation. In general, however, one must expect that wave dispersion occurs.

While the group velocity defines the propagation velocity of wave groups in non-absorptive media, and as such corresponds to a physical propagation, it must be borne in mind that the phase velocity does not correspond to any actual physical propagation.

1.4.3 Moment Functions

For random fields, the relation for the increment processes, Eq. (1.32), is by itself of little value. Instead, we must consider relevant moment functions. The most important moments are of second order, and we now construct a second-order moment function by multiplying (1.32) by its complex conjugate at a wavenumber–frequency pair $(\boldsymbol{k}', \omega')$

and subsequently taking the ensemble average. This yields the following important relation:

$$E\left\{dZ(\boldsymbol{k},\omega)\,dZ^*(\boldsymbol{k}',\omega')\right\} = \frac{E\left\{dW(\boldsymbol{k},\omega)\,dW^*(\boldsymbol{k}',\omega')\right\}}{D(\boldsymbol{k},\omega)D^*(\boldsymbol{k}',\omega')}. \tag{1.37}$$

Note that the moment functions $E\left\{dZ(\boldsymbol{k},\omega)\,dZ^*(\boldsymbol{k}',\omega')\right\}$ and $E\left\{dW(\boldsymbol{k},\omega)\,dW^*(\boldsymbol{k}',\omega')\right\}$ are direct generalizations of the standard Loève spectrum in Eq. (1.4) to dual frequency/dual wavenumber. From Eq. (1.37) we understand that this is in fact a generalized resonance theory for systems that may be simultaneously nonstationary in time, inhomogeneous in space, and include linear dispersive effects.

It is often more convenient to work in terms of global and local space and time coordinates, and global and local frequency and wavenumber coordinates. Let $\mathbf{x}$ and $\boldsymbol{\rho}$ be global and local spatial positions, respectively; let t and τ be global and local time, respectively; let $\boldsymbol{k}$ and $\boldsymbol{\kappa}$ be global and local wavenumber vectors, respectively; and let ω and ν be global and local frequencies, respectively. Then we can rewrite the dual-frequency/dual-wavenumber spectrum in Eq. (1.37) as

$$S_\xi(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu) = \frac{S_\eta(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu)}{Q(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu)}, \tag{1.38}$$

where

$$\begin{aligned} S_\xi(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu)\frac{d\boldsymbol{k}\,d\boldsymbol{\kappa}\,d\omega\,d\nu}{(2\pi)^8} &= E\left[dZ(\boldsymbol{k},\omega)\,dZ^*(\boldsymbol{k}-\boldsymbol{\kappa},\omega-\nu)\right], \\ S_\eta(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu)\frac{d\boldsymbol{k}\,d\boldsymbol{\kappa}\,d\omega\,d\nu}{(2\pi)^8} &= E\left[dW(\boldsymbol{k},\omega)\,dW^*(\boldsymbol{k}-\boldsymbol{\kappa},\omega-\nu)\right], \end{aligned}$$

and

$$Q(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu) = D(\boldsymbol{k},\omega)D^*(\boldsymbol{k}-\boldsymbol{\kappa},\omega-\nu).$$

It is now possible to formulate a spatiotemporal frequency–wavenumber spectrum that includes linear dispersion and that directly generalizes the Kirkwood–Rihaczek spectrum. By a Fourier transformation over local frequency and local wavenumber space, we can write

$$R_\xi(\boldsymbol{k},\mathbf{r};\omega,t) = \int\int \frac{S_\eta(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu)}{Q(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu)} \exp\left[j(\nu t - \boldsymbol{\kappa}^{\mathrm{T}}\mathbf{r})\right] \frac{d\nu\,d\boldsymbol{\kappa}}{(2\pi)^4}. \tag{1.39}$$

We see that the resonances of the system, that is, where $Q(\boldsymbol{k},\boldsymbol{\kappa};\omega,\nu) \simeq 0$, are of special importance for the time–frequency/space–wavenumber spectrum. Often, the integration may be carried out by means of classical (but multidimensional) complex residue calculation.

1.4.3.1 Stationary and Homogeneous Driving Force Field The stationary manifold discussed in [25] can now be generalized to a *stationary and homogeneous manifold* described by $\nu = 0$ and $\boldsymbol{\kappa} = \mathbf{0}$. In terms of the second-order moment function for the driving term, this translates to the relation

$$E\left\{|dZ(\boldsymbol{k},\omega)|^2\right\} = \frac{E\left\{|dW(\boldsymbol{k},\omega)|^2\right\}}{|D(\boldsymbol{k},\omega)|^2}. \tag{1.40}$$

Here, $E\left\{|dZ(\boldsymbol{k},\omega)|^2\right\}$ and $E\left\{|dW(\boldsymbol{k},\omega)|^2\right\}$ are the convential wavenumber–frequency spectra for stationary and homogeneous fields.

1.4.4 Applications

By choosing the appropriate linear differential operator $L\{\cdot\}$ in the random dynamical model, one can derive the crucial function $D(\boldsymbol{k},\omega)$, which lies at the heart in the description of the second-order dual-frequency/dual-wavenumber and the time–frequency/space–wavenumber moment functions. In the following, we derive $D(\boldsymbol{k},\omega)$ for a few relevant and physically important one-dimensional models. Armed with this function, one can in principle calculate the generalized Loéve spectrum, and the generalized Kirkwood–Rihaczek spectrum for dispersive, nonstationary, and inhomogeneous random fields.

1.4.4.1 Nondispersive Wave Equation The ubiquitous (one-dimensional) linear wave equation (e.g., [4, 6, 7]) has the second-order differential operator

$$L = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}, \tag{1.41}$$

where c is a constant wave propagation speed. This model appears in a number of linearized acoustic, electromagnetic, and mechanical systems. This operator leads to dispersionless propagation of disturbances. However, it is still of great practical interest to solve for the resulting fluctuation spectra using the theory of this chapter since inhomogeneities driving fields would be included in our description.

The orders and coefficients of the basic linear operator in Eq. (1.29) are $p = q = 2$, and $A_{0,0} = A_{1,0} = A_{0,1} = A_{1,1} = A_{2,1} = A_{1,2} = A_{2,2} = 0$, and $A_{2,0} = 1$, $A_{0,2} = c^2$. The corresponding wavenumber–frequency operator becomes

$$D(k,\omega) = c^2k^2 - \omega^2. \tag{1.42}$$

By inserting this operator into Eqs. (1.38) and (1.39), we can explicitly derive the second-order properties of the inhomogeneous random field for nondispersive media.

1.4.4.2 Euler–Bernoulli Beam Equation A classical linear model for a deflecting beam under axial loading is the Euler–Bernoulli beam equation. Using Newton's second law, one can show that one useful form of the beam equation leads to the fourth-order differential operator [39]

$$L = \frac{\partial^2}{\partial t^2} + \gamma^2\frac{\partial^4}{\partial x^4} - \beta^2\frac{\partial^4}{\partial t^2\,\partial x^2}, \tag{1.43}$$

where γ^2 and β^2 are bulk mechanical parameters for the beam. This is evidently a dispersive model, with associated wavenumber–frequency operator

$$D(k,\omega) = -\omega^2 - \gamma^2k^4 - \beta^2\omega^2k^2. \tag{1.44}$$

It will now be possible to study the nonstationary and inhomogeneous characteristics for a dispersive beam under nonstationary and inhomogeneous loading using the framework described in this chapter.

1.4.4.3 Long Water Waves Two interesting linear models that appear in approximate theories of long water waves are the linear Korteweg–de Vries (KdV) equation and the linear Boussinesq equation [7]. The partial differential operator for the linear KdV equation is

$$L = \frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x} + \beta \frac{\partial^3}{\partial x^3}, \tag{1.45}$$

with associated wavenumber–frequency operator

$$D(k, \omega) = j(c_0 k - \beta k^3 - \omega), \tag{1.46}$$

where c_0 and β are constant parameters.

The partial differential operator for the linear Boussinesq equation reads

$$L = \frac{\partial^2}{\partial t^2} - \alpha^2 \frac{\partial^2}{\partial x^2} - \beta^2 \frac{\partial^4}{\partial x^2 \, \partial t^2}, \tag{1.47}$$

with an associated wavenumber–frequency operator

$$D(k, \omega) = \alpha k^2 - \beta^2 \omega^2 k^2 - \omega^2, \tag{1.48}$$

where α and β are constant parameters.

1.4.4.4 Plasma Waves A plasma is a partially ionized gas with very rich dynamics. Since a plasma will be governed mainly by electromagnetic effects, it can support a host of wave modes that do not even exist in neutral gases or fluids. As the plasma models soon become very complicated, we will not list any models in terms of their partial differential equations. We will, however, list the wavenumber–frequency operators for a few linear but dispersive modes, to indicate what kind of models one can expect. To derive these operators, one would have to treat the plasma as a dielectric medium, and take the full set of Maxwell's equations as the starting dynamical model. Depending on the assumptions made, one may derive a variety of different wave equations.

If the direction of the fluctuating electric field is the same as the wavenumber, one calls the fluctuations "electrostatic" (note that there is nothing "static" about these waves—they are highly dynamic). The high-frequency electrostatic plasma waves can be shown to have the following wavenumber–frequency operator (e.g., [40]):

$$D(k, \omega) = \omega^2 - \omega_p^2 - 3c_{\text{th}}^2 k^2, \tag{1.49}$$

where ω_p is the so-called plasma frequency, and c_{th} is electron thermal speed.

Another often discussed operator is the one that describes the so-called ion-acoustic waves in the plasma (e.g., [40])

$$D(k, \omega) = \omega^2 (1 + k^2 \lambda_D^2) - c_s^2 k^2, \tag{1.50}$$

where c_s is the ion-acoustic speed, and λ_D is the Debye length or screening length of the plasma.

1.4.5 Example

As an illustrative analytical demonstration of how a time–frequency/space–wavenumber spectrum may be derived from this theory, we make two assumptions. First, we assume that the linear partial differential operator has the form

$$L = \frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial x}, \tag{1.51}$$

where $\gamma > 0$ is a constant. Then, the corresponding operator in the transform domain reads

$$D(k, \omega) = j\gamma k - \omega^2. \tag{1.52}$$

Second, we assume that the spectrum of the source random field $\eta(x, t)$ is stationary in time. Then we may formulate the source Loève spectrum as

$$S_\eta(k, \kappa; \omega, \nu) = 2\pi \widetilde{S}_\eta(k, \kappa; \omega)\delta(\nu), \tag{1.53}$$

for a function $\widetilde{S}_\eta(k, \kappa; \omega)$, which is independent of the local frequency ν.

With these assumptions, it is easy to carry out the integral over ν in Eq. (1.39), so that the two-dimensional integral for the response field reduces to the following one-dimensional integral over κ:

$$R_\xi(k, x; \omega, t) = \int \frac{\widetilde{S}_\eta(k, \kappa; \omega) \exp(-j\kappa x)}{(j\gamma k - \omega^2)[j\gamma(\kappa - k) - \omega^2]} \frac{d\kappa}{2\pi}. \tag{1.54}$$

To carry out the integral in Eq. (1.54), we use ordinary residue integration in the complex plane. We see that the integrand has a simple pole at $\kappa = k - j\omega^2/\gamma$. Then, through the residue theorem (e.g., [38]), we can readily complete the κ integral to obtain the closed-form solution:

$$R_\xi(k, x; \omega, t) = \frac{\widetilde{S}_\eta(k, k - j(\omega^2/\gamma); \omega)}{j\gamma k - \omega^2} e^{jkx + j\pi/2 - \omega^2 x/\gamma}. \tag{1.55}$$

We observe that the spatiotemporal frequency–wavenumber spectrum for the response random field in this case becomes independent of global time t, due to the stationarity assumption for the source field $\eta(x, t)$. Still, the spectrum is a nontrivial function of space, wavenumber, and frequency. Note also that the complex residue integration caused the wavenumber–frequency dispersion operator to appear explicitly as an argument in the source random field.

The generalized Kirkwood–Rihaczek spectrum we found in Eq. (1.55) is complex valued, as expected. In practice, however, one often displays the modulus of $R_\xi(k, x; \omega, t)$, which in our case simplifies to

$$\left|R_\xi(k, x; \omega, t)\right| = \frac{\left|\widetilde{S}_\eta(k, k - j\frac{\omega^2}{\gamma}; \omega)\right|}{\sqrt{\gamma^2 k^2 + \omega^4}} e^{-\omega^2 x/\gamma}. \tag{1.56}$$

Now, assume that the spatial inhomogeneity of the source is such that all the source wavenumbers are equally correlated (implying a very inhomogeneous source), then we may formulate the source spectrum as

$$\widetilde{S}_\eta\left(k, k - j\frac{\omega^2}{\gamma}; \omega\right) = \alpha F(\omega), \tag{1.57}$$

for some constant α independent of k, and some source frequency spectrum function $F(\omega)$. For this particular example, the modulus of the spatiotemporal frequency–wavenumber spectrum collapses to the following relatively simple form:

$$\left|R_\xi(k, x; \omega, t)\right| = \frac{|\alpha|\, F(\omega)}{\sqrt{\gamma^2 k^2 + \omega^4}} e^{-\omega^2 x/\gamma}. \tag{1.58}$$

We see that for this particular example, the spectral decay is much faster in the frequency variable ω than in the wavenumber variable k. This is a property inherited from properties of the wavenumber–frequency operator $D(k, \omega)$, and it shows us how a perturbation of the dynamical system is influenced by dispersion and damping.

1.5 CONCLUSIONS

We have presented a physical-statistical review of the theory for harmonizable stochastic wavefields. Our approach emphasized the wavelike aspects of the wavefields. In particular, we derived a systematic way of dealing with simultaneously nonstationary and inhomogeneous stochastic wavefields.

As demonstrated, we could generalize the conventional Einstein–Wiener–Khinchine relation to a relation between four different second-order moment functions. We found it beneficial and intuitively appealing to express the moment functions in terms of global and local spatial, temporal, wavenumber, and frequency variables. The relevant moment functions apparent from our reasoning were (1) a dual-space dual-time correlation, (2) a spatiotemporal frequency–wavenumber correlation, generalizing the conventional Kirkwood–Rihaczek time–frequency spectrum, (3) a dual-wavenumber dual-frequency correlation, generalizing the conventional Loève dual-frequency spectrum, and (4) another spatiotemporal frequency–wavenumber spectrum generalizing the conventional ambiguity function.

We presented a general and formalistic framework for how to handle linear wave dispersion in conjunction with stochastic system theory. As shown, the linear dispersion could readily be included in the theory for stochastic wavefields, and we were able to formulate all the relevant second-order moment functions in a rather elegant fashion. We believe that in array processing applications, our framework could be used to improve beamforming, delay estimation, signal estimation, and imaging in nonstationary, inhomogeneous, and dispersive environments. Spatiotemporal frequency–wavenumber adaptive algorithms generalizing the results from [41] seems desirable for numerous array processing tasks and applications in nonstationary, inhomogeneous, and dispersive environments.

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