CHAPTER 1

THE COMPLEX PLANE AND THE SPACE $L^2(\mathbb{R})$

We make extensive use of complex numbers throughout the book. Thus for the purposes of making the book self-contained, this chapter begins with a review of the complex plane and basic operations with complex numbers. To build wavelet functions, we need to define the proper space of functions in which to perform our constructions. The space $L^2(\mathbb{R})$ lends itself well to this task, and we introduce this space in Section 1.2.

We discuss the inner product in $L^2(\mathbb{R})$ in Section 1.3, as well as vector spaces and subspaces. In Section 1.4 we talk about bases for $L^2(\mathbb{R})$. The construction of wavelet functions requires the decomposition of $L^2(\mathbb{R})$ into nested subspaces. We frequently need to approximate a function $f(t) \in L^2(\mathbb{R})$ in these subspaces. The tool we use to form the approximation is the *projection* operator. We discuss (orthogonal) projections in Section 1.4.

1.1 COMPLEX NUMBERS AND BASIC OPERATIONS

Any discussion of the complex plane starts with the definition of the *imaginary unit*:

$$i = \sqrt{-1}$$

Wavelet Theory: An Elementary Approach with Applications. By D. K. Ruch and P. J. Van Fleet

We immediately see that

$$i^2 = (\sqrt{-1})^2 = -1, \quad i^3 = i^2 \cdot i = -i, \quad i^4 = (-1) \cdot (-1) = 1$$

In Problem 1.1 you will compute i^n for any integer n.

A *complex number* is any number of the form $z = a + bi$ where $a, b \in \mathbb{R}$. The number a is called the *real part* of z and b is called the *imaginary part* of z. The set of complex numbers will be denoted by $\mathbb{C}$. It is easy to see that $\mathbb{R} \subseteq \mathbb{C}$ since real numbers are those complex numbers with the imaginary part equal zero.

We can use the *complex plane* to envision complex numbers. The complex plane is a two-dimensional plane where the horizontal axis is used for the real part of complex numbers and the vertical axis is used for the imaginary part of complex numbers. To plot the number $z = a + bi$, we simply plot the ordered pair (a, b). In Figure 1.1 we plot some complex numbers.

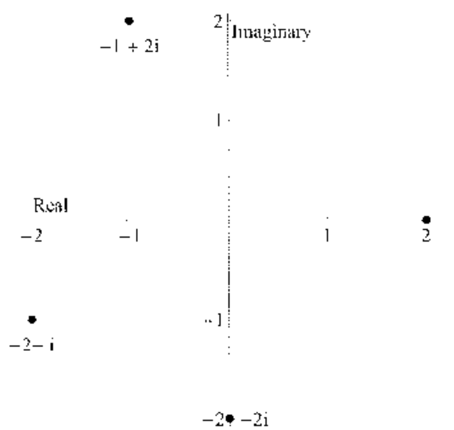

Figure 1.1 Some complex numbers in the complex plane.

Complex Addition and Multiplication

Addition and subtraction of complex numbers is a straightforward process. Addition of two complex numbers $u = a + bi$ and $v = c + di$ is defined as $y = u + v = (a + c) + (b + d)i$. Subtraction is similar: $z = u - v = (a - c) + (b - d)i$.

To multiply the complex numbers $u = a + bi$ and $v = c + di$, we proceed just as we would if $a + bi$ and $c + di$ were binomials:

$$u \cdot v = (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Example 1.1 (Complex Arithmetic) *Let $u = 2 + i$, $v = -1 - i$, $y = 2i$, and $z = 3 + 2i$. Compute $u + v$, $z - v$, $u \cdot y$, and $v \cdot z$.*

Solution

$$\begin{aligned} u+v &= (2-1)+(1-1)i = 1 \\ z-v &= (3-(-1))+(2-(-1))i = 4+3i \\ u\cdot y &= (2+i)\cdot 2i = 4i+2i^2 = -2+4i \\ v\cdot z &= (-1-i)\cdot(3+2i) = (3(-1)-(-1)2)+(3(-1)+2(-1))i = -1-5i \end{aligned}$$

■

Complex Conjugation

One of the most important operations used to work with complex numbers is *conjugation.*

Definition 1.1 (Conjugate of a Complex Numbers) *Let* $z = a + bi \in \mathbb{C}$. *The* conjugate *of* z*, denoted by* $\overline{z}$*, is defined by*

$$\overline{z} = a - bi$$

■

Conjugation is used to divide two complex numbers and also has a natural relation to the length of a complex number.

To plot $z = a + bi$, we plot the ordered pair (a, b) in the complex plane. For the conjugate $\overline{z} = a - bi$, we plot the ordered pair $(a, -b)$. So geometrically speaking, the conjugate $\overline{z}$ of z is simply the reflection of z over the real axis. In Figure 1.2 we have plotted several complex numbers and their conjugates.

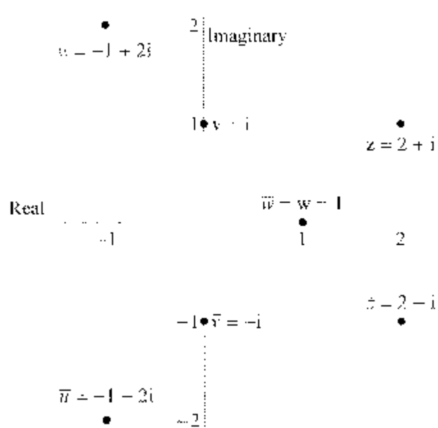

Figure 1.2 Complex numbers and their conjugates in the complex plane.

A couple of properties of the conjugation operator are immediate and we state them in the proposition below. The proof is left as Problem 1.3.

Proposition 1.1 (Properties of the Conjugation Operator) *Let $z = a + bi$ be a complex number. Then*

(a) $\overline{\overline{z}} = z$

(b) $z \subset \mathbb{R}$ *if and only if* $\overline{z} = z$

■

Proof: Problem 1.3. ■

Note that if we graph the points $z = \cos\theta + i\sin\theta$ as θ ranges from 0 to 2π, we trace a circle with center (0,0) with radius 1 in a counterclockwise manner. Note that if we produce the graph of $\overline{z} = \cos\theta - i\sin\theta$ as θ ranges from 0 to 2π, we get the same picture, but the points are drawn in a clockwise manner. Figure 1.3 illustrates this geometric interpretation of the conjugation operator.

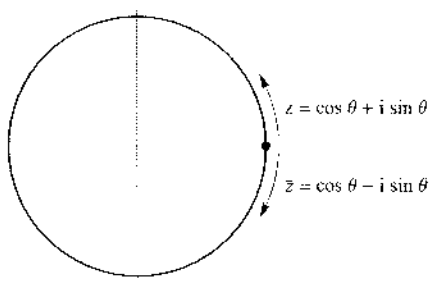

Figure 1.3 A circle is traced in two ways. Both start at $\theta = 0$. As θ ranges from 0 to 2π, the points z trace the circle in a counterclockwise manner while the points $\overline{z}$ trace the circle in a clockwise manner.

Modulus of a Complex Number

We can use the distance formula to determine how far the point $z = a + bi$ is away from $0 = 0 + 0i$ in the complex plane. The distance is $\sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$. This computation gives rise to the following definition.

Definition 1.2 (Modulus of a Complex Number) *The* modulus *of the complex number $z = a + bi$ is denoted by $|z|$ and is defined as*

$$|z| = \sqrt{a^2 + b^2}$$

■

Other names for the value $|z|$ are *length*, *absolute value*, and *norm* of z.

There is a natural relationship between $|z|$ and $\overline{z}$. If we compute the product $z \cdot \overline{z}$ where $z = a + bi$, we obtain

$$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$$

The right side of the equation above is simply $|z|^2$ so we have the following useful identity:

$$\boxed{|z|^2 = z \cdot \overline{z}} \tag{1.1}$$

In Problem 1.5 you are asked to compute the norms of some complex numbers.

Division of Complex Numbers

We next consider division of complex numbers. That is, given $z = a + bi$ and $y = c + di \neq 0$, how do we express the quotient z/y as a complex number? We proceed by multiplying both the numerator and denominator of the quotient by $\overline{y}$:

$$\frac{z}{y} = \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i$$

PROBLEMS

1.1 Let n be any integer. Find a closed formula for i^n.

1.2 Plot the numbers $3 - i$, $5i$, -1, and $\cos\theta + i\sin\theta$ for $\theta = 0, \pi/4, \pi/2, 5\pi/6, \pi$ in the complex plane.

1.3 Prove Proposition 1.1.

1.4 Compute the following values.

(a) $(3 - i) + (2 + i)$

(b) $(1 + i) - \overline{(3 + i)}$

(c) $-i^3 \cdot (-2 + 3i)$

(d) $\overline{(2 + 5i)} \cdot (4 - i)$

(e) $\overline{(2 + 5i) \cdot (4 - i)}$

(f) $(2 - i) \div i$

(g) $(1 + i) \div (1 - i)$

1.5 For each complex number z, compute $|z|$.

(a) $z = 2 + 3i$

(b) $z = 5$

(c) $z = -4i$

(d) $z = \tan\theta + i$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(e) z satisfies $z \cdot \overline{z} = 6$

1.6 Let $z = a + bi$ and $y = c + di$. For parts (a) – (d) show that:

(a) $\overline{y \cdot z} = \overline{y} \cdot \overline{z}$

(b) $|z| = |\overline{z}|$

(c) $|y \cdot z| = |y| \cdot |z|$

(d) $\overline{y + z} = \overline{y} + \overline{z}$

(e) Find the real and imaginary parts of $z^{-1} = \dfrac{1}{z}$.

★**1.7** Suppose $z = a + bi$ with $|z| = 1$. Show that $\overline{z} = z^{-1}$.

★**1.8** We can generalize Problem 1.6(d). Suppose that $z_k = a_k + b_k i$, for $k = 1, \ldots, n$. Show that

$$\overline{\sum_{k=1}^{n} z_k} = \sum_{k=1}^{n} \overline{z_k} = \sum_{k=1}^{n} a_k - i \sum_{k=1}^{n} b_k$$

★**1.9** Suppose that $\sum\limits_{k \in \mathbb{Z}} a_k$ and $\sum\limits_{k \in \mathbb{Z}} b_k$ are convergent series where $a_k, b_k \in \mathbb{R}$. For $z_k = a_k + ib_k$, $k \in \mathbb{Z}$, show that

$$\overline{\sum_{k=1}^{\infty} z_k} = \sum_{k=1}^{\infty} \overline{z_k} = \sum_{k=1}^{\infty} a_k - i \sum_{k=1}^{\infty} b_k$$

★**1.10** The identity in this problem is key to the development of the material in Section 6.1. Suppose that $z, w \in \mathbb{C}$ with $|z| = 1$. Show that

$$\left|(z - w)\left(z - 1/\overline{w}\right)\right| = |w|^{-1} |z - w|^2$$

The following steps will help you organize your work:

(a) Using the fact that $|z| = 1$, expand $|z - w|^2 = (z - w)\overline{(z - w)}$ to obtain

$$|z - w|^2 = 1 + |w|^2 - w\overline{z} - \overline{w}z$$

(b) Factor $-\overline{w}z^{-1}$ from the right side of the identity in part (a) and use Problem 1.7 to show that

$$|z - w|^2 = -\overline{w}z^{-1}\left(z^2 - \frac{1+|w|^2}{\overline{w}}z + \frac{w}{\overline{w}}\right)$$

(c) Show that the quadratic on the right-hand side of part (b) can be factored as $(z - w)(z - 1/\overline{w})$.

(d) Take norms of both sides of the identity obtained in part (c) and simplify the result to complete the proof.

1.2 THE SPACE $L^2(\mathbb{R})$

In order to create a mathematical model with which to build wavelet transforms, it is important that we work in a vector space that lends itself to applications in digital imaging and signal processing. Unlike $\mathbb{R}^N$, where elements of the space are N-tuples $\mathbf{v} = (v_1, \ldots, v_N)^T$, elements of our space will be functions. We can view a digital image as a function of two variables where the function value is the gray-level intensity, and we can view audio signals as functions of time where the function values are the frequencies of the signal. Since audio signals and digital images can have abrupt changes, we will not require functions in our space to necessarily be continuous. Since audio signals are constructed of sines and cosines and these functions are defined over all real numbers, we want to allow our space to hold functions that are supported (the notion of support is formally provided in Definition 1.5) on $\mathbb{R}$. Since rows or columns of digital images usually are of finite dimension and audio signals taper off, we want to make sure that the functions $f(t)$ in our space decay sufficiently fast as $t \to \pm\infty$. The rate of decay must be fast enough to ensure that the energy of the signal is finite. (We will soon make precise what we mean by the *energy* of a function.) Finally, it is desirable from a mathematical standpoint to use a space where the inner product of a function with itself is related to the size (norm) of the function. For this reason, we will work in the space $L^2(\mathbb{R})$. We define it now.

$L^2(\mathbb{R})$ Defined

Definition 1.3 (The Space $\mathbf{L^2(\mathbb{R})}$) *We define the space $L^2(\mathbb{R})$ to be the set*

$$\boxed{L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} |f(t)|^2 \, dt < \infty \right\}} \tag{1.2}$$

■

Note: A reader with some background in analysis will understand that a rigorous definition of $L^2(\mathbb{R})$ requires knowledge of the Lebesgue integral and sets of measure zero. If the reader is willing to accept some basic properties obeyed by Lebesgue integrals, then Definition 1.3 will suffice.

We define the *norm* of a function in $L^2(\mathbb{R})$ as follows:

Definition 1.4 (The $\mathbf{L^2(\mathbb{R})}$ Norm) *Let $f(t) \in L^2(\mathbb{R})$. Then the* norm *of $f(t)$ is*

$$\boxed{\|f(t)\| = \left(\int_{\mathbb{R}} |f(t)|^2 \, \mathrm{d}t\right)^{\frac{1}{2}}} \tag{1.3}$$

■

The norm of the function is also referred to as the *energy* of the function. There are several properties that the norm should satisfy. Since it is a measure of energy or size, it should be nonnegative. Moreover, it is natural to expect that the only function for which $\|f(t)\| = 0$ is $f(t) = 0$. Some clarification of this property is in order before we proceed.

If $f(t) = 0$ for all $t \in \mathbb{R}$, then certainly $|f(t)|^2 = 0$, so that $\|f(t)\| = 0$. But what about the function that is 0 everywhere except, say, for a finite number of values? It is certainly possible that a signal might have such abrupt changes at a finite set of points. We learned in calculus that such a finite set of points has no bearing on the integral. That is, for $a < c < b$, $f(c)$ might not even be defined, but

$$\int_a^b f(t) \, \mathrm{d}t = \lim_{L \to c^-} \int_a^L f(t) \, \mathrm{d}t + \lim_{L \to c^+} \int_L^b f(t) \, \mathrm{d}t$$

could very well exist. This is certainly the case when $f(t) = 0$ except at a finite number of values.

This idea is generalized using the notion of *measurable sets*. Intervals (a, b) are measured by their length $b - a$, and in general, sets are measured by writing them as a limit of the union of nonintersecting intervals. The measure of a single point a is 0, since for an arbitrarily small positive measure $\epsilon > 0$, we can find an interval that contains a and has measure less than ϵ (the interval $(a - \epsilon/4, a + \epsilon/4)$ with measure $\epsilon/2$ works). We can generalize this argument to claim that a finite set of points has measure 0 as well. The general definition of sets of measure 0 is typically covered in an analysis text (see Rudin [48], for example).

The previous discussion leads us to the notion of *equivalent functions*. Two functions $f(t)$ and $g(t)$ are said to be equivalent if $f(t) = g(t)$ except on a set of measure 0.

We state the following proposition without proof.

Proposition 1.2 (Functions for Which $\|\mathbf{f(t)}\| = 0$) *Suppose that $f(t) \in L^2(\mathbb{R})$. Then $\|f(t)\| = 0$ if and only if $f(t) = 0$ except on a set of measure 0.* ■

Examples of Functions in $L^2(\mathbb{R})$

Our first example of elements of $L^2(\mathbf{R})$ introduces functions that are used throughout the book.

Example 1.2 (The Box $\sqcap(t)$, Triangle $\wedge(t)$, and Sinc Functions) *We define the* box function

$$\boxed{\sqcap(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \textit{otherwise} \end{cases}} \tag{1.4}$$

the triangle function

$$\boxed{\wedge(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \\ 0, & \textit{otherwise} \end{cases}} \tag{1.5}$$

and the sinc function

$$\boxed{\operatorname{sinc}(t) = \begin{cases} 1, & t = 0 \\ \dfrac{\sin(t)}{t}, & \textit{otherwise} \end{cases}} \tag{1.6}$$

These functions are plotted in Figure 1.4.

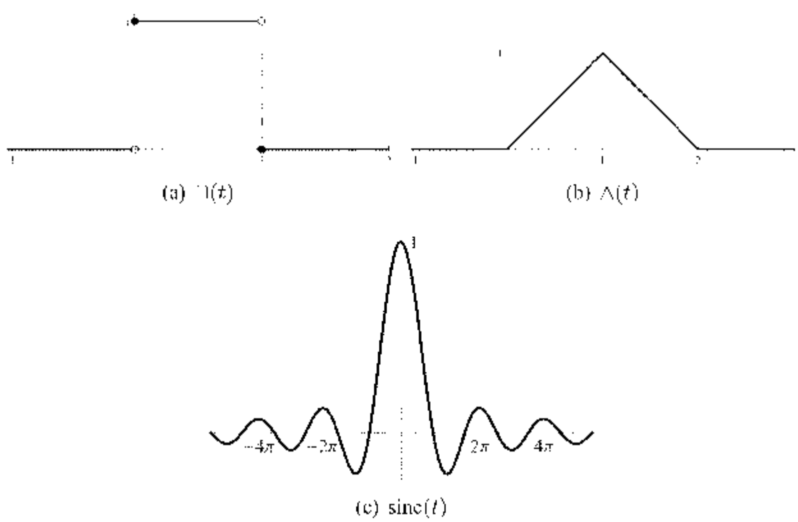

Figure 1.4 The functions $\sqcap(t)$, $\wedge(t)$, and $\operatorname{sinc}(t)$.

The box function is an element of $L^2(\mathbb{R})$. Since $\sqcap^2(t) = \sqcap(t)$, we have

$$\int_{\mathbb{R}} \sqcap^2(t)\,\mathrm{d}t = \int_{\mathbb{R}} \sqcap(t)\,\mathrm{d}t = \int_0^1 1 \cdot \mathrm{d}t = 1$$

To show that $\wedge(t) \in L^2(\mathbb{R})$, *we first note that*

$$\wedge(t)^2 = \begin{cases} t^2, & 0 \le t < 1 \\ (2-t)^2, & 1 \le t < 2 \\ 0, & \textit{otherwise} \end{cases}$$

so that

$$\int_{\mathbb{R}} \wedge(t)^2\,\mathrm{d}t = \int_0^1 t^2\,\mathrm{d}t + \int_1^2 (2-t)^2\,\mathrm{d}t = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

To see that $\operatorname{sinc}(t) \in L^2(\mathbb{R})$, *we consider the integral*

$$\int_{\mathbb{R}} \operatorname{sinc}^2(t)\,\mathrm{d}t = \int_{-\infty}^0 \frac{\sin^2(t)}{t^2}\,\mathrm{d}t + \int_0^\infty \frac{\sin^2(t)}{t^2}\,\mathrm{d}t \tag{1.7}$$

We split the second integral in (1.7) as follows:

$$\int_0^\infty \frac{\sin^2(t)}{t^2}\,\mathrm{d}t = \int_0^1 \frac{\sin^2(t)}{t^2}\,\mathrm{d}t + \int_1^\infty \frac{\sin^2(t)}{t^2}\,\mathrm{d}t \tag{1.8}$$

Note that the second integral in (1.8) is certainly nonnegative and we can bound the integral above by 1*:*

$$\begin{aligned} 0 \le \int_1^\infty \frac{\sin^2(t)}{t^2}\,\mathrm{d}t &= \lim_{L\to\infty} \int_1^L \frac{\sin^2(t)}{t^2}\,\mathrm{d}t \\ &\le \lim_{L\to\infty} \int_1^L \frac{1}{t^2}\,\mathrm{d}t \\ &= \lim_{L\to\infty} -\frac{1}{t}\Big|_1^L \\ &= \lim_{L\to\infty} -\frac{1}{L} + 1 = 1 \end{aligned} \tag{1.9}$$

Now we analyze the first integral on the right side of (1.8). By L'Hôpital's rule, we know that $\lim\limits_{t\to 0} \frac{\sin^2(t)}{t^2} = 1$ *so* $\operatorname{sinc}^2(t)$ *is a continuous function on* $[0,1]$. *From calculus, we recall that a continuous function on a closed interval achieves a maximum value, and it can be shown that the maximum value of* $\operatorname{sinc}^2(t)$ *on* $[0,1]$ *is* 1 *(see Problem 1.12). Thus*

$$0 \le \int_0^1 \frac{\sin^2(t)}{t^2}\,\mathrm{d}t \le \int_0^1 1 \cdot \mathrm{d}t = 1 \tag{1.10}$$

Combining (1.9) and (1.10) gives

$$\int_0^\infty \operatorname{sinc}^2(t)\,\mathrm{d}t = \int_0^\infty \frac{\sin^2(t)}{t^2}\,\mathrm{d}t \le 2$$

In a similar manner we can bound the first integral in (1.7) by 2 as well so that $\operatorname{sinc}(t) \in L^2(\mathbb{R})$. ∎

Let's look at some other functions in $L^2(\mathbb{R})$.

Example 1.3 (Functions in $\mathbf{L^2(\mathbb{R})}$) *Determine whether or not the following functions are in $L^2(\mathbb{R})$. For those functions in $L^2(\mathbb{R})$, compute their norm.*

(a) $f_1(t) = t^n$, *where* $n = 0, 1, 2, \ldots$

(b) $f_2(t) = t^2 \sqcap (t/4)$, *where* $\sqcap(t)$ *is the box function defined in Example 1.2*

(c) $f_3(t) = \begin{cases} \dfrac{i}{\sqrt{t}}, & t \geq 1 \\ 0, & \text{otherwise} \end{cases}$

(d) $f_4(t) = \begin{cases} \dfrac{1}{t}, & t \geq 1 \\ 0, & \text{otherwise} \end{cases}$

Solution

For $f_1(t)$ *we have* $|f_1(t)|^2 = t^{2n}$, *so that*

$$\int_{\mathbb{R}} t^{2n}\,dt = \lim_{L\to\infty} \int_{-L}^{0} t^{2n}\,dt + \lim_{L\to\infty} \int_{0}^{L} t^{2n}\,dt$$

Both of these integrals diverge — in particular

$$\lim_{L\to\infty} \int_0^L t^{2n}\,dt = \lim_{L\to\infty} \frac{1}{2n+1} L^{2n+1} \to \infty$$

Thus we see that no monomials are elements of $L^2(\mathbb{R})$. We could generalize part (a) to easily show that polynomials are not elements of $L^2(\mathbb{R})$.

Since $\sqcap(t/4)$ *is 0 whenever* $t \notin [0,4)$, *we can write* $f_2(t)$ *as*

$$f_2(t) = \begin{cases} t^2, & 0 \leq t < 4 \\ 0, & \text{otherwise} \end{cases}$$

so that

$$\int_{\mathbb{R}} |f_2(t)|^2\,dt = \int_0^4 (t^2)^2\,dt = \int_0^4 t^4\,dt = \frac{1024}{5}$$

Thus $f_2(t) \in L^2(\mathbb{R})$ *and* $\|f_2(t)\| = \frac{32\sqrt{5}}{5}$. *Computing the modulus of* $f_3(t)$ *gives*

$$|f_3(t)|^2 = \begin{cases} \dfrac{1}{t}, & t \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{\mathbb{R}} |f_3(t)|^2\,|dt = \int_1^\infty \frac{1}{t}\,dt = \lim_{L\to\infty} \int_1^L t^{-1}\,dt = \lim_{L\to\infty} \ln(L) \to \infty$$

Thus $f_3 \notin L^2(\mathbb{R})$. Finally, $|f_4(t)|^2 = t^{-2}$ so integrating over $t > 1$ gives

$$\int_{\mathbb{R}} |f_4(t)|^2 \, dt = \int_1^{\infty} t^{-2} \, dt = \lim_{L \to \infty} \int_1^L t^{-2} \, dt = \lim_{L \to \infty} 1 - 1/L = 1$$

So we see that $f_4(t) \in L^2(\mathbb{R})$ and $\|f_4(t)\| = 1$. ■

The Support of a Function

As we will learn in subsequent chapters, the support of a function plays an important role in the theory we develop. We define it now:

Definition 1.5 (Support of a Function) *Suppose that $f(t) \in L^2(\mathbb{R})$. We define the* support of f, *denoted supp(f), to be the set*

$$\text{supp}(f) = \{t \in \mathbb{R} \mid f(t) \neq 0\} \tag{1.11}$$

■

Here are some examples to better illustrate Definition 1.5.

Example 1.4 (Examples of Function Support) *Find the support of each of the following functions:*

(a) $f(t) = \dfrac{1}{1 + |t|}$

(b) $\sqcap(t)$

(c) $\wedge(t)$

(d) $g(t) = \sum\limits_{k=0}^{\infty} c_k \wedge (t - 2k)$*, where* $c_k \neq 0$, $k \in \mathbb{Z}$*, and* $\sum\limits_{k=0}^{\infty} c_k^2 < \infty$

Solution

We observe that $f(t) > 0$, so supp$(f) = \mathbb{R}$. *In Problem 1.15 you will show that $f(t) \in L^2(\mathbb{R})$. The box function $\sqcap(t)$ is supported on the interval* $[0,1)$. *The triangle function $\wedge(t)$ is supported on the interval* $(0,2)$.

The final function is a linear combination of even-integer translates of triangle functions. It is zero on the interval $(-\infty, 0]$ and $g(2k) = 0$, $k = 1,2,\ldots$ So

$$\text{supp}(g) = (0,2) \cup (2,4) \cup (4,6) \cup \cdots = \bigcup_{k=0}^{\infty} (2k, 2k+2)$$

In Problem 1.17 you will show that $g(t) \in L^2(\mathbb{R})$. ■

The support of the functions in Example 1.4(b) and (c) were finite-length intervals. We are often interested in functions whose supports are contained in a finite-length interval and we define this type of support below.

Definition 1.6 (Functions of Compact Support) *Let $f(t) \in L^2(\mathbb{R})$. We say that f is* compactly supported *if* $\text{supp}(f)$ *is contained in a closed interval of finite length. In this case we say that the* compact support of f *is the* <u>*smallest*</u> *closed interval $[a, b]$ such that* $\text{supp}(f) \subseteq [a, b]$. *This interval is denoted by* $\overline{\text{supp}(f)}$. ■

We know from Example 1.4 that $\text{supp}(\wedge) = (0, 2)$. The compact support of $\wedge(t)$ is $\overline{\text{supp}(\wedge)} = [0, 2]$. In a similar manner, $\overline{\text{supp}(\sqcap)} = [0, 1]$.

$L^2(\mathbb{R})$ Functions at $\pm\infty$

When motivating the definition of $L^2(\mathbb{R})$, we stated that we want functions that tend to 0 as $t \to \pm\infty$ in such a way that $\|f\|^2 < \infty$. The following proposal shows a connection between the rate of decay and the finite energy of a function in $L^2(\mathbb{R})$.

Proposition 1.3 (Integrating the "Tails" of an $\mathbf{L}^2(\mathbb{R})$ Function) *Suppose that $f(t) \in L^2(\mathbb{R})$ and let $\epsilon > 0$. Then there exists a real number $L > 0$ such that*

$$\int_{-\infty}^{-L} |f(t)|^2\,dt + \int_{L}^{\infty} |f(t)|^2\,dt = \int_{|t|>L} |f(t)|^2\,dt < \epsilon$$

■

Before we prove Proposition 1.3, let's understand what it is saying. Consider the $L^2(\mathbb{R})$ function $f_4(t)$ from Example 1.3(d). Let's pick $\epsilon = 10^{-16}$. Then Proposition 1.3 says that for some $L > 0$, the "tail" $\int\limits_L^\infty |f_4(t)|^2\,dt$ of the integral $\int\limits_{\mathbb{R}} |f_4(t)|^2\,dt$ will satisfy

$$\int_L^\infty \frac{1}{t^2}\,dt < 10^{-16} = .0000000000000001$$

Proof: This proof requires some ideas from infinite series in calculus. Suppose that $\epsilon > 0$ and let's first consider the integral $\int_0^\infty |f(t)|^2\,dt$. Since $f(t) \in L^2(\mathbb{R})$, we know that this integral converges to some value s. Now write the integral as a limit. For $N \in \mathbb{N}$ we write

$$\begin{aligned}
\int_0^\infty |f(t)|^2\,dt &= \lim_{N\to\infty} \int_0^N |f(t)|^2\,dt \\
&= \lim_{N\to\infty} \int_0^1 |f(t)|^2\,dt + \int_1^2 |f(t)|^2\,dt + \cdots + \int_{N-1}^N |f(t)|^2\,dt \\
&= \lim_{N\to\infty} \sum_{k=0}^{N} \int_k^{k+1} |f(t)|^2\,dt \\
&= \lim_{N\to\infty} \sum_{k=0}^{N} a_k
\end{aligned}$$

where $a_k = \int\limits_k^{k+1} |f(t)|^2\,\mathrm{d}t \geq 0$. Thus we can view the integral as the sum of the series $\sum\limits_{k=0}^{\infty} a_k$ and we know that the series converges to s.

Recall from calculus that a series converges to s if its sequence of partial sums $s_N = \sum\limits_{k=0}^{N} a_k$ converges to s. The formal definition of a convergent sequence says that for all $\epsilon > 0$, there exists an $N_0 \in \mathbb{N}$ such that whenever $N \geq N_0$, we have $|s_N - s| < \epsilon$. We use this formal definition with $\epsilon/2$. That is, for $\frac{\epsilon}{2} > 0$, there exists an $N_0 \in \mathbb{N}$ such that $N > N_0$ implies that $|s_N - s| < \frac{\epsilon}{2}$.

In particular, for $N = N_0$ we have

$$|s_{N_0} - s| < \frac{\epsilon}{2} \tag{1.12}$$

But

$$\begin{aligned}
|s_{N_0} - s| &= \left| \sum_{k=0}^{N_0} a_k - \int_0^{\infty} |f(t)|^2\,\mathrm{d}t \right| \\
&= \left| \sum_{k=0}^{N_0} \int_k^{k+1} |f(t)|^2\,\mathrm{d}t - \int_0^{\infty} |f(t)|^2\,\mathrm{d}t \right| \\
&= \left| \int_0^{N_0} |f(t)|^2\,\mathrm{d}t - \int_0^{\infty} |f(t)|^2\,\mathrm{d}t \right|
\end{aligned}$$

Now $\int\limits_0^{\infty} |f(t)|^2\,\mathrm{d}t \geq \int\limits_0^{N_0} |f(t)|^2\,\mathrm{d}t$, so that we can drop the absolute-value symbols and rewrite the last identity as

$$\begin{aligned}
|s_{N_0} - s| &= \left| \int_0^{N_0} |f(t)|^2\,\mathrm{d}t - \int_0^{\infty} |f(t)|^2\,\mathrm{d}t \right| \\
&= \int_0^{\infty} |f(t)|^2\,\mathrm{d}t - \int_0^{N_0} |f(t)|^2\,\mathrm{d}t \\
&= \int_{N_0}^{\infty} |f(t)|^2\,\mathrm{d}t
\end{aligned} \tag{1.13}$$

Combining (1.12) and (1.13) gives

$$\int_{N_0}^{\infty} |f(t)|^2\,\mathrm{d}t < \frac{\epsilon}{2}$$

In a similar manner (see Problem 1.19), we can find $N_1 > 0$ such that

$$\int_{-\infty}^{-N_1} |f(t)|^2\,\mathrm{d}t < \frac{\epsilon}{2}$$

Now choose L to be the bigger of N_0 and N_1. Then $L > N_0$, $-L < -N_1$, and from this we can write

$$\int_L^\infty |f(t)|^2 \, \mathrm{d}t < \int_{N_0}^\infty |f(t)|^2 \, \mathrm{d}t < \frac{\epsilon}{2} \tag{1.14}$$

and

$$\int_{-\infty}^{-L} |f(t)|^2 \, \mathrm{d}t \leq \int_{-\infty}^{-N_1} |f(t)|^2 \, \mathrm{d}t < \frac{\epsilon}{2} \tag{1.15}$$

Combining (1.14) and (1.15) gives

$$\int_L^\infty |f(t)|^2 \, \mathrm{d}t + \int_{-\infty}^{L} |f(t)|^2 \, \mathrm{d}t < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and the proof is complete. ■

Convergence in $L^2(\mathbb{R})$

In an elementary calculus class, you undoubtedly talked about sequences and limits. We are also interested in looking at sequences in $L^2(\mathbb{R})$. We need to be a little careful when describing convergence. It does not make sense to measure convergence at a point since equivalent functions might disagree on some set of measure 0. Since we are measuring everything using the norm, it is natural to view convergence in this light as well.

Definition 1.7 (Convergence in $\mathbf{L^2(\mathbb{R})}$) *Suppose that $f_1(t), f_2(t), \ldots$ is a sequence of functions in $L^2(\mathbb{R})$. We say that $\{f_n(t)\}_{n\in\mathbb{N}}$ converges to $f(t) \in L^2(\mathbb{R})$ if for all $\epsilon > 0$, there exists $L > 0$ such that whenever $n > L$, we have $\|f_n(t) - f(t)\| < \epsilon$.* ■

If a sequence of functions $f_n(t)$, $n = 1, 2, \ldots$ converges in $L^2(\mathbb{R})$ to $f(t)$, we also say that the sequence of functions *converges in norm* to $f(t)$.

Other than the use of the norm, the definition should look similar to that of the formal definition of the limit of a sequence, which is covered in many calculus books (see Stewart [55], for example). The idea is that no matter how small a distance ϵ we pick, we are guaranteed that we can find an $N > 0$ so that $n > N$ ensures that the distance between $f_n(t)$ and $f(t)$ is smaller than ϵ.

Example 1.5 (Convergence in $\mathbf{L^2(\mathbb{R})}$) *Let $f_n(t) = t^n \sqcap(t)$. Show that $f_n(t)$ converges (in an $L^2(\mathbb{R})$ sense) to 0.*

Solution

Let $\epsilon > 0$. We note that $\sqcap(t)^2 = \sqcap(t)$ and compute

$$\begin{aligned} \|f_n(t) - 0\| &= \left(\int_{\mathbb{R}} t^{2n} \sqcap(t) \, \mathrm{d}t \right)^{1/2} \\ &= \left(\int_0^1 t^{2n} \, \mathrm{d}t \right)^{1/2} \\ &= 1/\sqrt{2n+1} \end{aligned}$$

Now, if we can say how to choose L so that $1/\sqrt{2n+1} < \epsilon$ whenever $n > L$, we are done. The natural thing to do is to solve the inequality $1/\sqrt{2n+1} < \epsilon$ for n. We obtain

$$n > \frac{1}{2}\left(\frac{1}{\epsilon^2} - 1\right)$$

Now the right-hand side is negative when $\epsilon > 1$, and in this case we can choose any $L \geq 0$. Then $n > L$ gives $\frac{1}{\sqrt{2n+1}} < 1 < \epsilon$.

When $\epsilon \leq 1$, we take $L = \frac{1}{2}\left(\frac{1}{\epsilon^2} - 1\right)$ to complete the proof. ■

In a real analysis class, we study *pointwise* convergence of sequences of functions. Convergence in norm is quite different, and in Problem 1.20 you will investigate the pointwise convergence properties of the sequence of functions from Example 1.5.

PROBLEMS

1.11 Suppose that $\int_{\mathbb{R}} f(t)\,dt < \infty$. Show that:

(a) $\int_{\mathbb{R}} f(t - a)\,dt = \int_{\mathbb{R}} f(t)\,dt$ for any $a \in \mathbb{R}$

(b) $\int_{\mathbb{R}} f(mt + b)\,dt = \frac{1}{m}\int_{\mathbb{R}} f(t)\,dt$ for $m, b \in \mathbb{R}$, with $m \neq 0$

1.12 Use the definition (1.6) of $\operatorname{sinc}(t)$ in Example 1.2 and L'Hôpital's rule to show that $f(t) = \operatorname{sinc}^2(t)$ is continuous for $t \in \mathbb{R}$ and then show that the maximum value of $f(t)$ is 1.

1.13 Determine whether or not the following functions are elements of $L^2(\mathbb{R})$. For those functions that are in $L^2(\mathbb{R})$, compute their norms.

(a) $f(t) = e^{-|t|}$

(b) $r(t) = \begin{cases} 1, & t \geq 1 \\ 0, & t < 1 \end{cases}$

(c) $g(t) = (1 + t^2)^{-1/2}$

(d) $h(t) = \sqcap(t)\,(\cos(2\pi t) + i\sin(2\pi t))$

1.14 Give examples of $L^2(\mathbb{R})$ functions f such that $\|f(t)\| = 0$ and $f(t) \neq 0$ at

(a) a single point

(b) five points

(c) an infinite number of points

★**1.15** Show that the function $f(t) = \dfrac{1}{1 + |t|} \in L^2(\mathbb{R})$.

★**1.16** Consider the function

$$f(t) = \begin{cases} \frac{1}{2}t^3 - \frac{1}{2}t + 3, & -2 < t < 1 \\ 2t - 4, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

Show that $f(t) \in L^2(\mathbb{R})$.

1.17 Show that the function $g(t)$ from Example 1.4(d) is in $L^2(\mathbb{R})$.

1.18 Find the support of each of the following functions. For those functions f that are compactly supported, identify $\overline{\text{supp}(f)}$.

(a) $f_1(t) = e^{-t^2}$

(b) $f_2(t) = \sqcap(2t - k), k \in \mathbb{Z}$

(c) $f_3(t) = \sqcap(2^j t - k), j, k \in \mathbb{Z}$

(d) $f_4(t) = \text{sinc}(t)$

(e) $f_5(t) = \sqcap\left(\frac{t}{n\pi}\right) \text{sinc}(\pi t)$, where n is a positive integer

(f) $f_6(t) = \sum\limits_{k=0}^{n} c_k v(t - 2k)$, where $c_k \neq 0$, $k \in \mathbb{Z}$,

$$v(t) = \begin{cases} |t|, & -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

and n is a positive integer

1.19 Complete the proof of Proposition 1.3. That is, show that if $f(t) \in L^2(\mathbb{R})$ and $\epsilon > 0$, there exists an $N_1 > 0$ such that

$$\int_{-\infty}^{-N_1} |f(t)|^2 \, dt < \epsilon/2$$

1.20 In Example 1.5 we showed that $f_n(t) = t^n \sqcap(t)$ converges in norm to $f(t) = 0$. Does $f_n(t)$ converge pointwise to 0 for all $t \in \mathbb{R}$?

1.21 Consider the sequence of functions

$$g_n(t) = \begin{cases} t^n, & -1 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $g_n(t)$ converges in norm to 0.

(b) Show that $\lim\limits_{n \to \infty} g_n(a) = 0$ for $a \neq \pm 1$.

(c) Compute $\lim\limits_{n \to \infty} g_n(1)$ and $\lim\limits_{n \to \infty} g_n(-1)$.

1.3 INNER PRODUCTS

Recall that for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, with $\mathbf{u} = (u_1, u_2, \ldots, u_N)^T$ and $\mathbf{v} = (v_1, v_2, \ldots, v_N)^T$, we define the inner product as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{N} u_k v_k$$

Inner Products Defined

We can also define an inner product for functions in $L^2(\mathbb{R})$. In some sense it can be viewed as an extension of the definition above (see Van Fleet [60]). We state the formal definition of the inner product of two functions $f(t), g(t) \in L^2(\mathbb{R})$ at this time:

Definition 1.8 (The Inner Product in $\mathbf{L^2}(\mathbb{R})$) *Let $f(t), g(t) \in L^2(\mathbb{R})$. Then we define the* inner product *of $f(t)$ and $g(t)$ as*

$$\langle f(t), g(t)\rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}\,\mathrm{d}t \tag{1.16}$$

■

Here are some examples of the inner products:

Example 1.6 (Computing $\mathbf{L^2}(\mathbb{R})$ Inner Products) *Compute the following inner products:*

(a) The triangle function $f(t) = \wedge(t)$ and the box function $g(t) = \sqcap(t)$ (see Example 1.2)

(b) $f(t) = \begin{cases} t^{-1}, & t \geq 1 \\ 0, & t < 1 \end{cases}$ *and* $g(t) = \begin{cases} it^{-2}, & t \geq 1 \\ 0, & t < 1 \end{cases}$

(c) $f(t) = \wedge(t+1)$ *and* $g(t) = \begin{cases} \sin(2\pi t), & -1 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$

Solution

It is easy to verify that both functions in part (a) are in $L^2(\mathbb{R})$ and we also know that $\overline{g(t)} = g(t)$ since $g(t)$ is real-valued. Thus we can compute

$$\langle \wedge(t), \sqcap(t)\rangle = \int_{\mathbb{R}} \wedge(t)\sqcap(t)\,\mathrm{d}t = \int_0^1 \wedge(t)\,\mathrm{d}t = \int_0^1 t\,\mathrm{d}t = \frac{1}{2}$$

For part (b), $\overline{g(t)} = -g(t)$. We have

$$\langle f(t), g(t)\rangle = -i\int_1^{\infty} t^{-1}t^{-2}\,\mathrm{d}t = -i\int_1^{\infty} t^{-3}\,\mathrm{d}t = -i\lim_{L\to\infty} -\frac{1}{2t^2}\bigg|_1^L = -\frac{i}{2}$$

For the final inner product, both functions are real-valued and compactly supported on $[-1,1]$. Note that $\wedge(t+1)$ is even and $g(t)$ is odd, so that the product $\wedge(t+1)g(t)$ is odd. Recall from calculus (see Stewart [55]) that the integral of an odd function over any finite interval symmetric about zero is zero. Thus $\langle \wedge(t+1), g(t)\rangle = 0$. ■

Properties of Inner Products

Example 1.6(b) illustrates an important point about the inner product that we have defined. You should verify that if we had reversed the order of $f(t)$ and $g(t)$, the resulting inner product would be $\frac{i}{2}$. On the other hand, in Example 1.6(a) it is easy to verify that $\langle \wedge(t), \sqcap(t)\rangle = \langle \sqcap(t), \wedge(t)\rangle = \frac{1}{2}$. Thus we see that the inner product is not necessarily a commutative operation. The following proposition describes exactly what happens if we commute functions and then compute their inner product.

Proposition 1.4 (The $\mathbf{L^2(\mathbb{R})}$ Inner Product Is Not Commutative) *If $f(t), g(t)$ are functions in $L^2(\mathbb{R})$, then their inner product (1.16) satisfies*

$$\boxed{\langle g(t), f(t)\rangle = \overline{\langle f(t), g(t)\rangle}}$$

■

Proof: The proof of this proposition is left as Problem 1.24. ■

Since many of the inner products we will compute involve only real-valued functions. We have the following simple corollary:

Corollary 1.1 (Inner Products of Real-Valued $\mathbf{L^2(\mathbb{R})}$ Functions Commute) *Suppose $f(t), g(t) \in L^2(\mathbb{R})$ and further assume $f(t)$ and $g(t)$ are real-valued functions. Then*

$$\boxed{\langle f(t), g(t)\rangle = \langle g(t), f(t)\rangle}$$

■

Proof: This simple proof is left as Problem 1.25. ■

One of the nice properties of $L^2(\mathbb{R})$ is the relationship between the inner product and the norm. We have the following proposition.

Proposition 1.5 (Relationship Between the $\mathbf{L^2(\mathbb{R})}$ Norm and Inner Product) *For $f(t) \in L^2(\mathbb{R})$, we have*

$$\boxed{\|f\|^2 = \langle f(t), f(t)\rangle} \tag{1.17}$$

■

Proof: Let $f(t) \in L^2(\mathbb{R})$. We use (1.1) to write $f(t)\overline{f(t)} = |f(t)|^2$. Integrating both sides over $\mathbb{R}$ gives

$$\int_{\mathbb{R}} f(t)\overline{f(t)}\,\mathrm{d}t = \int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t$$

The left-hand side of this identity is $\langle f(t), f(t)\rangle$ and the right-hand side is $\|f(t)\|^2$. ■

The next result describes how the inner product is affected by scalar multiplication.

Proposition 1.6 (Scalar Multiplication and the Inner Product) *Suppose that $f(t)$ and $g(t)$ are functions in $L^2(\mathbb{R})$, and assume that $c \in \mathbb{C}$. Then*

$$\langle cf(t), g(t)\rangle = c\langle f(t), g(t)\rangle \tag{1.18}$$

and

$$\langle f(t), cg(t)\rangle = \overline{c}\,\langle f(t), g(t)\rangle \tag{1.19}$$

■

Proof: For (1.18), we have

$$\langle cf(t), g(t)\rangle = \int_{\mathbb{R}} cf(t)\overline{g(t)}\,\mathrm{d}t = c\int_{\mathbb{R}} f(t)\overline{g(t)}\,\mathrm{d}t = c\langle f(t), g(t)\rangle$$

and for (1.19), we have

$$\langle f(t), c\,g(t)\rangle = \int_{\mathbb{R}} f(t)\overline{c\,g(t)}\,\mathrm{d}t = \overline{c}\int_{\mathbb{R}} f(t)\overline{g(t)}\,\mathrm{d}t = \overline{c}\,\langle f(t), g(t)\rangle$$

■

In the sequel we frequently compute inner products of the form $\langle f(t-k), g(t-\ell)\rangle$ or $\langle f(2t-k), g(2t-\ell)\rangle$, where $k, \ell \in \mathbb{Z}$ and $f(t), g(t) \in L^2(\mathbb{R})$. The following proposition gives reformulations of these inner products.

Proposition 1.7 (Translates and Dilates in Inner Products) *Suppose that $f(t)$ and $g(t)$ are functions in $L^2(\mathbb{R})$ and $k, \ell, m \in \mathbb{Z}$. Then*

$$\langle f(t-k), g(t-\ell)\rangle = \langle f(t), g(t-(\ell-k))\rangle \tag{1.20}$$

and

$$\langle f(2^m t-k), g(2^m t-\ell)\rangle = 2^{-m}\,\langle f(t), g(t-(\ell-k))\rangle \tag{1.21}$$

■

Proof: The proof of this proposition is straightforward and is left as Problem 1.29. ■

The Cauchy–Schwarz Inequality

Recall for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, the *Cauchy–Schwarz* inequality (see Strang [56]) states that

$$\mathbf{u}\cdot\mathbf{v} \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

Functions in $L^2(\mathbb{R})$ also satisfy the *Cauchy–Schwarz inequality*. Before stating and proving the Cauchy–Schwarz inequality, we need to return once again to the concept of equivalent functions and their role in computing inner products. If $f(t), g(t) \in L^2(\mathbb{R})$ are equivalent, we would expect their inner products with any other function $h(t) \in L^2(\mathbb{R})$ to be the same. The following proposition, stated without proof, confirms this fact.[1]

Proposition 1.8 (Integrals of Equivalent Functions) *Suppose that $f(t)$, $g_1(t)$, and $g_2(t)$ are functions in $L^2(\mathbb{R})$ with $g_1(t) = g_2(t)$ except on a set of measure zero. Then $\langle f(t), g_1(t)\rangle = \langle f(t), g_2(t)\rangle$.* ■

We are now ready to state the Cauchy–Schwarz inequality for functions in $L^2(\mathbb{R})$.

Proposition 1.9 (Cauchy–Schwarz Inequality) *Suppose that $f(t), g(t) \in L^2(\mathbb{R})$. Then*

$$\boxed{|\langle f(t), g(t)\rangle| \le \|f(t)\| \cdot \|g(t)\|} \tag{1.22}$$

■

Proof: First, suppose that

$$\|g(t)\|^2 = \int_{\mathbb{R}} |g(t)|^2 \, dt = \langle g(t), g(t)\rangle = 0$$

Then by Proposition 1.2 (with $g_1(t) = g(t)$ and $g_2(t) = 0$), $g(t) = 0$ except on a set of measure 0. We next employ Proposition 1.8 to see that $\langle f(t), g(t)\rangle = 0$ and the result holds.

Now assume that $\|g(t)\|^2 > 0$ and for any $z \in \mathbb{C}$, consider

$$0 \le \|f(t) + zg(t)\|^2 = \langle f(t) + zg(t), f(t) + zg(t)\rangle$$

We can expand the right-hand side of this identity to write

$$0 \le \langle f(t), f(t)\rangle + \langle f(t), zg(t)\rangle + \langle zg(t), f(t)\rangle + \langle zg(t), zg(t)\rangle$$

Using Proposition 1.1, Proposition 1.5, and (1.19), we can rewrite the preceding equation as

$$0 \le \|f(t)\|^2 + \overline{z}\langle f(t), g(t)\rangle + z\overline{\langle f(t), g(t)\rangle} + |z|^2\|g(t)\|^2 \tag{1.23}$$

We now make a judicious choice for z. Since $\|g(t)\| \ne 0$, we take

$$z = -\frac{\langle f(t), g(t)\rangle}{\|g(t)\|^2}$$

The second term in (1.23) becomes

$$\overline{z}\langle f(t), g(t)\rangle = -\frac{\overline{\langle f(t), g(t)\rangle}}{\|g(t)\|^2}\langle f(t), g(t)\rangle = -\frac{|\langle f(t), g(t)\rangle|^2}{\|g(t)\|^2} \tag{1.24}$$

[1] The reader interested in the proof of Proposition 1.8 should consult an analysis text such as Rudin [48].

and the third term in (1.23) is

$$z\overline{\langle f(t), g(t)\rangle} = -\frac{\langle f(t), g(t)\rangle}{\|g(t)\|^2}\overline{\langle f(t), g(t)\rangle} = -\frac{|\langle f(t), g(t)\rangle|^2}{\|g(t)\|^2} \tag{1.25}$$

The last term in (1.23) can be written as

$$|z|^2\|g(t)\|^2 = \frac{|\langle f(t), g(t)\rangle|^2}{\|g(t)\|^4}\|g(t)\|^2 = \frac{|\langle f(t), g(t)\rangle|^2}{\|g(t)\|^2} \tag{1.26}$$

Inserting (1.24), (1.25), and (1.26) into (1.23) gives

$$\begin{aligned} 0 &\le \|f(t)\|^2 - 2\frac{|\langle f(t), g(t)\rangle|^2}{\|g(t)\|^2} + \frac{|\langle f(t), g(t)\rangle|^2}{\|g(t)\|^2} \\ &= \|f(t)\|^2 - \frac{|\langle f(t), g(t)\rangle|^2}{\|g(t)\|^2} \end{aligned}$$

Adding the second term on the right side to both sides of the identity above gives

$$\frac{|\langle f(t), g(t)\rangle|^2}{\|g(t)\|^2} \le \|f(t)\|^2$$

Finally, we multiply both sides of this inequality by $\|g(t)\|^2$ to obtain

$$|\langle f(t), g(t)\rangle|^2 \le \|f(t)\|^2 \cdot \|g(t)\|^2$$

Taking square roots of this last inequality gives the desired result. ■

We have referred to the space $\mathbb{R}^N$ several times in this chapter. The space $\mathbb{R}^N$ is a standard example of a *vector space*. Basically, a vector space is a space where addition and scalar multiplication obey fundamental properties. We require the sum of any two vectors, or the product of a scalar and a vector, to remain in the space. We also want addition to be commutative, associative, and distributive over scalar multiplication. We want the space to contain a zero element and additive inverses for all elements in the space. Scalar multiplication should be associative as well as distributive over vectors, and we also require a multiplicative identity. Certainly, $\mathbb{R}^N$ is an example of a vector space (see Strang [56] for more details on vector spaces) with standard vector addition and real numbers as scalars.

Vector Spaces

We summarize these properties in the following formal definition of a vector space.

Definition 1.9 (Vector Space) *Let $\mathcal{V}$ be any space such that addition of elements of $\mathcal{V}$ and multiplication of elements by scalars from a set $\mathcal{F}$ are well defined.*[2] *Then $\mathcal{V}$ is called a* vector space *or* linear space *over $\mathcal{F}$ if*

[2] The set of scalars $\mathcal{F}$ for $\mathcal{V}$ is called a *field*. For our purposes, our set of scalars will be either the real or the complex numbers.

(a) *$\mathcal{V}$ is closed under addition. That is, for all* $\mathbf{u}, \mathbf{v} \in \mathcal{V}$*, we have* $\mathbf{u} + \mathbf{v} \in \mathcal{V}$.

(b) *Addition is commutative and associative. That is, for all* $\mathbf{u}, \mathbf{v} \in \mathcal{V}$*, we have*

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

and for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$*, we have*

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

(c) *There exists an additive identity* $\mathbf{0} \in \mathcal{V}$ *so that for all* $\mathbf{v} \in \mathcal{V}$*, we have*

$$\mathbf{v} + \mathbf{0} = \mathbf{v}$$

(d) *For each* $\mathbf{v} \in \mathcal{V}$*, there exists an additive inverse* $-\mathbf{v} \in \mathcal{V}$ *so that for all* $\mathbf{v} \in \mathcal{V}$*, we have*

$$\mathbf{v} + -\mathbf{v} = \mathbf{0}$$

(e) *$\mathcal{V}$ is closed under scalar multiplication. That is, for* $c \in \mathcal{F}$ *and* $\mathbf{v} \in \mathcal{V}$*, we have* $c\mathbf{v} \in \mathcal{V}$.

(f) *Scalar multiplication is associative. That is, for* $c, d \in \mathcal{F}$ *and* $\mathbf{v} \in \mathcal{V}$*, we have*

$$(cd)\mathbf{v} = c(d\mathbf{v})$$

(g) *For all* $\mathbf{v} \in \mathcal{V}$*, we have* $1 \cdot \mathbf{v} = \mathbf{v}$.

(h) *Addition distributes over scalar multiplication and scalar multiplication distributes over addition. That is, for* $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ *and* $c, d \in \mathcal{F}$*, we have*

$$\begin{aligned} c(\mathbf{u} + \mathbf{v}) &= c\mathbf{u} + c\mathbf{v} \\ (c + d)\mathbf{v} &= c\mathbf{v} + d\mathbf{v} \end{aligned}$$

■

If we review the properties listed in Definition 1.9, it is easy to verify that $L^2(\mathbb{R})$ satisfies properties (b)–(h). You are asked to do so in Problem 1.30. The most difficult property to verify is the fact that $L^2(\mathbb{R})$ is closed under addition. The triangle inequality, stated and proved below, ensures that $L^2(\mathbb{R})$ is closed under addition and is thus a vector space.

Proposition 1.10 (The Triangle Inequality) *Suppose that* $f(t), g(t) \in L^2(\mathbb{R})$*. Then*

$$\boxed{\|f(t) + g(t)\| \le \|f(t)\| + \|g(t)\|} \tag{1.27}$$

■

Proof:

Let $f(t), g(t) \in L^2(\mathbb{R})$. We begin by computing

$$\begin{aligned}\|f(t)+g(t)\|^2 &= \langle f(t)+g(t), f(t)+g(t)\rangle \\ &= \|f(t)\|^2 + \langle f(t), g(t)\rangle + \langle g(t), f(t)\rangle + \|g(t)\|^2 \\ &\le \|f(t)\|^2 + |\langle f(t), g(t)\rangle| + |\langle g(t), f(t)\rangle| + \|g(t)\|^2\end{aligned}$$

Now we use the Cauchy–Schwarz inequality on the two inner products in the previous line. We have

$$\begin{aligned}\|f(t)+g(t)\|^2 &\le \|f(t)\|^2 + \|f(t)\|\cdot\|g(t)\| + \|g(t)\|\cdot\|f(t)\| + \|g(t)\|^2 \\ &= \|f(t)\|^2 + 2\|f(t)\|\cdot\|g(t)\| + \|g(t)\|^2 \\ &= (\|f(t)\| + \|g(t)\|)^2\end{aligned}$$

Taking square roots of both sides of the previous inequality gives the desired result. ■

Subspaces

In many applications we are interested in special subsets of a vector space $\mathcal{V}$. We will insist that these subsets carry all the properties of a vector space. Such subsets are known as *subspaces*.

Definition 1.10 (Subspace) *Suppose that $\mathcal{W}$ is any nonempty subset of a vector space $\mathcal{V}$. $\mathcal{W}$ is called a* subspace *of $\mathcal{V}$ if whenever $\mathbf{u}, \mathbf{v} \in \mathcal{W}$ and c, d are any scalars, we have $c\mathbf{u} + d\mathbf{v} \in \mathcal{W}$.* ■

In Problem 1.31 you will show that if $\mathcal{W}$ is a subspace of vector space $\mathcal{V}$, then $\mathcal{W}$ is a vector space. Below we give an example of a subspace.

Example 1.7 (An Example of a Subspace) *Consider the vector space $L^2(\mathbb{R})$ and let $\mathcal{W}$ be the set of all piecewise constant functions with possible breakpoints at the integers with the added condition that each function in $\mathcal{W}$ is zero outside the interval $[-N, N)$ for N some positive integer. Show that $\mathcal{W}$ is a subspace of $\mathcal{V}$.*

Solution

Suppose that $f(t), g(t) \in \mathcal{W}$. Then f and g must be linear combinations of some integer translates of the box function $\sqcap(t)$. That is, there exist scalars (complex numbers) a_k, b_k, $k = -N, \ldots, N-1$ such that

$$f(t) = \sum_{k=-N}^{N-1} a_k \sqcap(t-k) \quad \textit{and} \quad g(t) = \sum_{k=-N}^{N-1} b_k \sqcap(t-k)$$

Now for $c, d \in \mathbb{C}$, we form the function

$$
\begin{aligned}
h(t) &= cf(t) + dg(t) \\
&= c\sum_{k=-N}^{N-1} a_k \sqcap(t-k) + d\sum_{k=-N}^{N-1} b_k \sqcap(t-k) \\
&= \sum_{k=-N}^{N-1} (ca_k + db_k) \sqcap(t-k)
\end{aligned}
$$

and note that $h(t)$ is a piecewise constant function with possible breakpoints at the integers and $h(t) = 0$ for $t \notin [-N, N)$. Since $h(t) \in W$ we have $W \neq \emptyset$ and by Definition 1.10, we see that W is a subspace of $L^2(\mathbb{R})$. ■

PROBLEMS

1.22 Compute the following inner products $\langle f(t), g(t)\rangle$:

(a) $f(t) = \wedge(t)$ and $g(t) = (1+t^2)^{-1/2}$

(b) $f(t) = g(t) = e^{-|t|}$ (Compare your answer with Problem 1.13(a))

(c) $f(t) = \sqcap(t-k)$ and $g(t) = \wedge(t)$ where $k \in \mathbb{Z}$

1.23 In this problem we illustrate the result of Proposition 1.8. Let $h(t) = \sqcap\left(\frac{t}{2}\right)$. Define the functions

$$
f(t) = \begin{cases} e^{-t}, & t > 0 \\ 0, & t \leq 0 \end{cases}
$$

and

$$
g(t) = \begin{cases} f(t), & t \notin \mathbb{Z} \\ 0, & t \in \mathbb{Z} \end{cases}
$$

(a) Verify that $f(t)$, $g(t)$, and $h(t)$ are in $L^2(\mathbb{R})$ and that $f(t)$ and $g(t)$ are equivalent functions.

(b) Show that $\langle f(t), h(t)\rangle = \langle g(t), h(t)\rangle$.

1.24 Prove Proposition 1.4.

1.25 Prove Corollary 1.1.

1.26 Let $f, g \in L^2(\mathbb{R})$. Show that $\|f(t)+g(t)\|^2 = \|f(t)\|^2 + \|g(t)\|^2$ if and only if $\langle f(t), g(t)\rangle = 0$.

1.27 Verify the Cauchy–Schwarz inequality for $f(t)$ and $h(t)$ in Problem 1.23.

1.28 Find two functions $f(t)$ and $g(t)$ that satisfy the equality part of the Cauchy–Schwarz inequality. That is, find $f(t)$ and $g(t)$ so that $|\langle f(t), g(t)\rangle| = \|f(t)\| \cdot \|g(t)\|$.

Can you state conditions in general that guarantee equality in the Cauchy–Schwarz inequality?

1.29 Prove Proposition 1.7. (*Hint:* Substitute for $t - k$ in the first identity and $2^m t - k$ in the second identity.)

1.30 Show that $L^2(\mathbb{R})$ satisfies properties (b)–(h) of Definition 1.9. Since $L^2(\mathbb{R})$ also satisfies the triangle inequality (1.27), we see that $L^2(\mathbb{R})$ is a vector space.

1.31 Suppose that $\mathcal{W}$ is a subspace of a vector space V over a set of scalars $\mathcal{F} = \mathbb{R}$ or $\mathcal{F} = \mathbb{C}$. Show that $\mathcal{W}$ is a vector space over $\mathcal{F}$.

1.4 BASES AND PROJECTIONS

Our derivation of wavelets depends heavily on the construction of orthonormal bases for subspaces of $L^2(\mathbb{R})$. Recall (see Strang [56]) that a *basis* for the vector space $\mathbb{R}^N$ is a set of N linearly independent vectors $\{\mathbf{u}^1, \ldots, \mathbf{u}^N\}$ that span $\mathbb{R}^N$. This basis is called *orthonormal* if $\mathbf{u}^j \cdot \mathbf{u}^k = 0$ whenever $j \neq k$, and $\mathbf{u}^j \cdot \mathbf{u}^j = 1$.

Bases

We can easily extend the ideas of basis and orthonormal basis to $L^2(\mathbb{R})$.

Definition 1.11 (Basis and Orthonormal Basis in $\mathbf{L}^2(\mathbb{R})$) *Suppose that $\mathcal{W}$ is a subspace of $L^2(\mathbb{R})$ and suppose that $\{e_k(t)\}_{k\in\mathbb{Z}}$ is a set of functions in $\mathcal{W}$. We say that $\{e_k(t)\}_{k\in\mathbb{Z}}$ is a* basis *for $\mathcal{W}$ if the functions span $\mathcal{W}$ and are linearly independent. We say that $\{e_k(t)\}_{k\in\mathbb{Z}}$ is an* orthonormal basis *for $\mathcal{W}$ if*

$$\langle e_j(t), e_k(t)\rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \tag{1.28}$$

■

Suppose that $\mathcal{W}$ is a subspace of $L^2(\mathbb{R})$ and $\{e_k(t)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{W}$. Since $\{e_k(t)\}_{k\in\mathbb{Z}}$ is a basis for $\mathcal{W}$, we can write any $f(t) \in \mathcal{W}$ as

$$f(t) = \sum_{k\in\mathbb{Z}} a_k e_k(t) \tag{1.29}$$

where $a_k \in \mathbb{C}$. It would be quite useful to know more about the coefficients a_k, $k \in \mathbb{Z}$. We compute the inner product of both sides of (1.29) with $e_j(t)$ to obtain

$$\begin{aligned}\langle f(t), e_j(t)\rangle &= \Big\langle \sum_{k\in\mathbb{Z}} a_k e_k(t), e_j(t)\Big\rangle \\ &= \sum_{k\in\mathbb{Z}} a_k \langle e_k(t), e_j(t)\rangle \\ &= \sum_{k\in\mathbb{Z}} a_k \int_{\mathbb{R}} e_k(t)\overline{e_j(t)}\,\mathrm{d}t\end{aligned}$$

Since $\{e_k(t)\}_{k\in\mathbb{Z}}$ is an orthonormal basis, the integral in each term of the last identity is 0 unless $j = k$. In the case where $j = k$, the integral is 1 and the right-hand side of the last identity reduces to the single term:

$$\boxed{a_j = \langle f(t), e_j(t)\rangle = \int_{\mathbb{R}} f(t)\overline{e_j(t)}\,\mathrm{d}t} \tag{1.30}$$

Projections

An orthonormal basis gives us a nice representation of the coefficients a_k, $k \in \mathbb{Z}$. Now suppose that $g(t)$ is an arbitrary function in $L^2(\mathbb{R})$. The representation (1.30) suggests a way that we can approximate $g(t)$ in the subspace W. We begin by defining a *projection*.

Definition 1.12 (Projection) *Let W be a subspace of $L^2(\mathbb{R})$. Then $P: L^2(\mathbb{R}) \to W$ is a* projection *from $L^2(\mathbb{R})$ into W if for all $f(t) \in W$, we have $P(f(t)) = f(t)$.* ■

Thus a projection is any linear transformation from $L^2(\mathbb{R})$ to subspace W that is an identity operator for $f(t) \subset W$. If you have taken a multivariable calculus class, you probably learned how to project vectors from $\mathbb{R}^2$ into the subspace $W = \{c\mathbf{a} \mid c \in \mathbb{R}\}$ where $\mathbf{a}$ is some nonzero vector in $\mathbb{R}^2$ (see Stewart [55]). This is an example of a projection (using the vector space $\mathbb{R}^2$ instead of $L^2(\mathbb{R})$) with $P(\mathbf{v}) = \left(\frac{\mathbf{a}^T\mathbf{v}}{\|\mathbf{a}\|^2}\right)\mathbf{a}$.

A useful way to project $g(t) \in L^2(\mathbb{R})$ into a subspace W is to take an orthonormal basis $\{e_k(t)\}_{k\in\mathbb{Z}}$ and write

$$P(g(t)) = \sum_{k\in\mathbb{Z}} \langle g(t), e_k(t)\rangle e_k(t) \tag{1.31}$$

We need to show that (1.31) is a projection from $L^2(\mathbb{R})$ into W. To do so, we need the following auxiliary results. The proofs of both results are outlined as exercises.

Proposition 1.11 (The Norm of $\mathbf{P(g(t))}$) *Let W be a subspace of $L^2(\mathbb{R})$ with orthonormal basis $\{e_k(t)\}_{k\in\mathbb{Z}}$. For $g(t) \in L^2(\mathbb{R})$, the function $P(g(t))$ defined in (1.31) satisfies*

$$\|P(g(t))\|^2 = \sum_{k\in\mathbb{Z}} |\langle g(t), e_k(t)\rangle|^2 \tag{1.32}$$

■

Proof: Problem 1.34. ■

Proposition 1.12 (An Upper Bound for $\|\mathbf{P(g(t))}\|$) *Let W be a subspace of $L^2(\mathbb{R})$ with orthonormal basis $\{e_k(t)\}_{k\in\mathbb{Z}}$. Then $\|P(g(t))\|$, where $P(g(t))$ is defined in (1.31), satisfies*

$$\|P(g(t))\| \le \|g(t)\| \tag{1.33}$$

■

Proof: Problem 1.35. ■

Proposition 1.12 tells us that if $g(t) \in L^2(\mathbb{R})$, then so is $P(g(t))$. This fact is required to establish the following result.

Proposition 1.13 (A Projection from $\mathbf{L}^2(\mathbb{R})$ into $\mathcal{W}$) *Let $\mathcal{W}$ be a subspace of $L^2(\mathbb{R})$ with orthonormal basis $\{e_k(t)\}_{k\in\mathbb{Z}}$. Then the function $P(g(t))$ defined by (1.31) is a projection from $L^2(\mathbb{R})$ into $\mathcal{W}$.* ■

Proof: From Proposition 1.12, we know that $P(g(t)) \in L^2(\mathbb{R})$ whenever $g(t) \in L^2(\mathbb{R})$. We need to show that for any $f(t) \in \mathcal{W}$, we have $P(f(t)) = f(t)$.

Since $f(t) \in \mathcal{W}$, we can write it as a linear combination of basis functions:

$$f(t) = \sum_{j\in\mathbb{Z}} a_j e_j(t)$$

Then

$$\begin{aligned} P(f(t)) &= \sum_{k\in\mathbb{Z}} \langle f(t), e_k(t)\rangle\, e_k(t) \\ &= \sum_{k\in\mathbb{Z}} \Big\langle \sum_{j\in\mathbb{Z}} a_j e_j(t), e_k(t) \Big\rangle\, e_k(t) \\ &= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} a_j \langle e_j(t), e_k(t)\rangle\, e_k(t) \end{aligned}$$

Since $\{e_k(t)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{W}$, the inner product $\langle e_j(t), e_k(t)\rangle$ is nonzero only when $j = k$. In this case the inner product is 1. Thus the double sum in the last identity reduces to a single sum with j replaced by k. We have

$$P(f(t)) = \sum_{k\in\mathbb{Z}} a_k\, e_k(t) = f(t)$$

and the proof is complete. ■

PROBLEMS

1.32 Suppose that $\{e_k(t)\}$ is an orthonormal basis for $L^2(\mathbb{R})$. For $L \in \mathbb{Z}$, $L > 0$, let $f_L(t) = \sum\limits_{k=-L}^{L} a_k e_k(t)$. Show that

$$\|f_L(t)\|^2 = \sum_{k=-L}^{L} a_k^2$$

1.33 Suppose that P is a projection from vector space $\mathcal{V}$ into subspace $\mathcal{W}$. Show that $P^2 = P$. That is, show that for all $\mathbf{v} \in \mathcal{V}$, we have $P^2\mathbf{v} = P\mathbf{v}$.

1.34 Expand the right-hand side of

$$\begin{aligned}\|P(g(t))\|^2 &= \langle P(g(t)), P(g(t))\rangle \\ &= \left\langle \sum_{k\in\mathbb{Z}} \langle g(t), e_k(t)\rangle\, e_k(t), \sum_{k\in\mathbb{Z}} \langle g(t), e_k(t)\rangle\, e_k(t)\right\rangle\end{aligned}$$

to provide a proof of Proposition 1.11.

1.35 In this problem you will prove Proposition 1.12. The following steps will help you organize the proof. Let $g(t) \in L^2(\mathbb{R})$ and suppose that $\{e_k(t)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for subspace $\mathcal{W}$.

(a) Let $g_L(t) = \sum\limits_{k=-L}^{L} \langle g(t), e_k(t)\rangle\, e_k(t)$. Show that

$$\langle g(t), g_L(t)\rangle = \langle g_L(t), g(t)\rangle = \sum_{k=-L}^{L} |\langle g(t), e_k(t)\rangle|^2$$

(b) Use part (a) to show that

$$\|g(t) - g_L(t)\|^2 = \|g(t)\|^2 - 2\sum_{k=-L}^{L} |\langle g(t), e_k(t)\rangle|^2 + \|g_L(t)\|^2$$

(c) Show that $\|g_L(t)\|^2 = \sum\limits_{k=-L}^{L} |\langle g(t), e_k(t)\rangle|^2$. This is a special case of Problem 1.32.

(d) Use parts (b) and (c) to write

$$\|g(t) - g_L(t)\|^2 + \sum_{k=-L}^{L} |\langle g(t), e_k(t)\rangle|^2 = \|g(t)\|^2$$

and thus infer that

$$\sum_{k=-L}^{L} |\langle g(t), e_k(t)\rangle|^2 \le \|g(t)\|^2$$

(e) Use Proposition 1.11 and let $L \to \infty$ in part (d) to complete the proof.