1

Fundamentals of Kinematics

Vectors and coordinate frames are human-made tools to study the motion of particles and rigid bodies. We introduce them in this chapter to review the fundamentals of kinematics.

1.1 COORDINATE FRAME AND POSITION VECTOR

To indicate the position of a point P relative to another point O in a three-dimensional (3D) space, we need to establish a coordinate frame and provide three relative coordinates. The three coordinates are scalar functions and can be used to define a position vector and derive other kinematic characteristics.

1.1.1 Triad

Take four non-coplanar points O, A, B, C and make three lines OA, OB, OC. The *triad* $OABC$ is defined by taking the lines OA, OB, OC as a rigid body. The position of A is arbitrary provided it stays on the same side of O. The positions of B and C are similarly selected. Now rotate OB about O in the plane OAB so that the angle AOB becomes 90 deg. Next, rotate OC about the line in AOB to which it is perpendicular until it becomes perpendicular to the plane AOB. The new triad $OABC$ is called an *orthogonal triad*.

Having an orthogonal triad $OABC$, another triad $OA'BC$ may be derived by moving A to the other side of O to make the *opposite triad* $OA'BC$. All orthogonal triads can be superposed either on the triad $OABC$ or on its opposite $OA'BC$.

One of the two triads $OABC$ and $OA'BC$ can be defined as being a *positive triad* and used as a *standard*. The other is then defined as a *negative triad*. It is immaterial which one is chosen as positive; however, usually the *right-handed convention* is chosen as positive. The right-handed convention states that the direction of rotation from OA to OB propels a *right-handed screw* in the direction OC. A right-handed or positive orthogonal triad cannot be superposed to a left-handed or negative triad. Therefore, there are only two essentially distinct types of triad. This is a property of 3D space.

We use an orthogonal triad $OABC$ with scaled lines OA, OB, OC to locate a point in 3D space. When the three lines OA, OB, OC have scales, then such a triad is called a *coordinate frame*.

Every moving body is carrying a *moving* or *body frame* that is attached to the body and moves with the body. A body frame accepts every motion of the body and may also be called a *local frame*. The position and orientation of a body with respect to other frames is expressed by the position and orientation of its local coordinate frame.

When there are several relatively moving coordinate frames, we choose one of them as a *reference frame* in which we express motions and measure kinematic information. The motion of a body may be observed and measured in different reference frames; however, we usually compare the motion of different bodies in the *global reference frame*. A global reference frame is assumed to be motionless and attached to the ground.

Example 1 Cyclic Interchange of Letters In any orthogonal triad $OABC$, cyclic interchanging of the letters ABC produce another orthogonal triad superposable on the original triad. Cyclic interchanging means relabeling A as B, B as C, and C as A or picking any three consecutive letters from $ABCABCABC \dots$.

Example 2 ★ Independent Orthogonal Coordinate Frames Having only two types of orthogonal triads in 3D space is associated with the fact that a plane has just two sides. In other words, there are two opposite normal directions to a plane. This may also be interpreted as: we may arrange the letters A, B, and C in just two orders when cyclic interchange is allowed:

$$ABC, \ ACB$$

In a 4D space, there are six cyclic orders for four letters A, B, C, and D:

$$ABCD, \ ABDC, \ ACBD, \ ACDB, \ ADBC, \ ADCB$$

So, there are six different tetrads in a 4D space.

In an nD space there are $(n-1)!$ cyclic orders for n letters, so there are $(n-1)!$ different coordinate frames in an nD space.

Example 3 Right-Hand Rule A right-handed triad can be identified by a right-hand rule that states: When we indicate the OC axis of an orthogonal triad by the thumb of the right hand, the other fingers should turn from OA to OB to close our fist.

The right-hand rule also shows the rotation of Earth when the thumb of the right hand indicates the north pole.

Push your right thumb to the center of a clock, then the other fingers simulate the rotation of the clock's hands.

Point your index finger of the right hand in the direction of an electric current. Then point your middle finger in the direction of the magnetic field. Your thumb now points in the direction of the magnetic force.

If the thumb, index finger, and middle finger of the right hand are held so that they form three right angles, then the thumb indicates the Z-axis when the index finger indicates the X-axis and the middle finger the Y-axis.

1.1.2 Coordinate Frame and Position Vector

Consider a positive orthogonal triad $OABC$ as is shown in Figure 1.1. We select a *unit length* and define a *directed line* $\hat{\imath}$ on OA with a unit length. A point P_1 on OA is at a distance x from O such that the directed line $\overrightarrow{OP_1}$ from O to P_1 is $\overrightarrow{OP_1} = x\hat{\imath}$. The

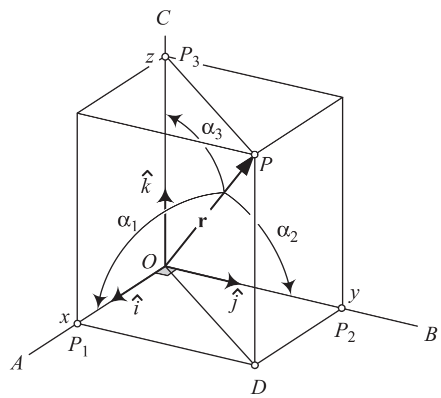

Figure 1.1 A positive orthogonal triad $OABC$, unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, and a position vector $\mathbf{r}$ with components x, y, z.

directed line $\hat{\imath}$ is called a *unit vector* on OA, the unit length is called the *scale*, point O is called the *origin*, and the real number x is called the $\hat{\imath}$*-coordinate* of P_1. The distance x may also be called the $\hat{\imath}$ *measure number* of $\overrightarrow{OP}_1$. Similarly, we define the unit vectors $\hat{\jmath}$ and $\hat{k}$ on OB and OC and use y and z as their coordinates, respectively. Although it is not necessary, we usually use the same scale for $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and refer to OA, OB, OC by $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and also by x, y, z.

The scalar coordinates x, y, z are respectively the length of projections of P on OA, OB, and OC and may be called the *components* of $\mathbf{r}$. The components x, y, z are independent and we may vary any of them while keeping the others unchanged.

A scaled positive orthogonal triad with unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ is called an *orthogonal coordinate frame*. The position of a point P with respect to O is defined by three coordinates x, y, z and is shown by a *position vector* $\mathbf{r} = \mathbf{r}_P$:

$$\mathbf{r} = \mathbf{r}_P = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1.1}$$

To work with multiple coordinate frames, we indicate coordinate frames by a capital letter, such as G and B, to clarify the coordinate frame in which the vector $\mathbf{r}$ is expressed. We show the name of the frame as a left superscript to the vector:

$${}^B\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1.2}$$

A vector $\mathbf{r}$ is expressed in a coordinate frame B only if its unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ belong to the axes of B. If necessary, we use a left superscript B and show the unit vectors as ${}^B\hat{\imath}$, ${}^B\hat{\jmath}$, ${}^B\hat{k}$ to indicate that $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ belong to B:

$${}^B\mathbf{r} = x\ {}^B\hat{\imath} + y\ {}^B\hat{\jmath} + z\ {}^B\hat{k} \tag{1.3}$$

We may drop the superscript B as long as we have just one coordinate frame.

The distance between O and P is a scalar number r that is called the *length*, *magnitude*, *modulus*, *norm*, or *absolute value* of the vector $\mathbf{r}$:

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \tag{1.4}$$

We may define a new unit vector $\hat{u}_r$ on $\mathbf{r}$ and show $\mathbf{r}$ by

$$\mathbf{r} = r\hat{u}_r \tag{1.5}$$

The equation $\mathbf{r} = r\hat{u}_r$ is called the *natural expression* of $\mathbf{r}$, while the equation $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is called the *decomposition* or *decomposed expression* of $\mathbf{r}$ over the axes $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$. Equating (1.1) and (1.5) shows that

$$\begin{aligned}\hat{u}_r &= \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{r} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} \\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\jmath} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}\end{aligned} \tag{1.6}$$

Because the length of $\hat{u}_r$ is unity, the components of $\hat{u}_r$ are the cosines of the angles α_1, α_2, α_3 between $\hat{u}_r$ and $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, respectively:

$$\cos\alpha_1 = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2+z^2}} \tag{1.7}$$

$$\cos\alpha_2 = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2+z^2}} \tag{1.8}$$

$$\cos\alpha_3 = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}} \tag{1.9}$$

The cosines of the angles α_1, α_2, α_3 are called the *directional cosines* of $\hat{u}_r$, which, as is shown in Figure 1.1, are the same as the directional cosines of any other vector on the same axis as $\hat{u}_r$, including $\mathbf{r}$.

Equations (1.7)–(1.9) indicate that the three directional cosines are related by the equation

$$\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^3\alpha_3 = 1 \tag{1.10}$$

Example 4 Position Vector of a Point P Consider a point P with coordinates $x = 3$, $y = 2$, $z = 4$. The position vector of P is

$$\mathbf{r} = 3\hat{\imath} + 2\hat{\jmath} + 4\hat{k} \tag{1.11}$$

The distance between O and P is

$$r = |\mathbf{r}| = \sqrt{3^2+2^2+4^2} = 5.3852 \tag{1.12}$$

and the unit vector $\hat{u}_r$ on $\mathbf{r}$ is

$$\begin{aligned}\hat{u}_r &= \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k} = \frac{3}{5.3852}\hat{\imath} + \frac{2}{5.3852}\hat{\jmath} + \frac{4}{5.3852}\hat{k} \\ &= 0.55708\hat{\imath} + 0.37139\hat{\jmath} + 0.74278\hat{k}\end{aligned} \tag{1.13}$$

The directional cosines of $\hat{u}_r$ are

$$\begin{aligned}\cos\alpha_1 &= \frac{x}{r} = 0.55708\\ \cos\alpha_2 &= \frac{y}{r} = 0.37139\\ \cos\alpha_3 &= \frac{z}{r} = 0.74278\end{aligned} \tag{1.14}$$

and therefore the angles between $\mathbf{r}$ and the x-, y-, z-axes are

$$\begin{aligned}\alpha_1 &= \cos^{-1}\frac{x}{r} = \cos^{-1}0.55708 = 0.97993\,\text{rad} \approx 56.146\,\text{deg}\\ \alpha_2 &= \cos^{-1}\frac{y}{r} = \cos^{-1}0.37139 = 1.1903\,\text{rad} \approx 68.199\,\text{deg}\\ \alpha_3 &= \cos^{-1}\frac{z}{r} = \cos^{-1}0.74278 = 0.73358\,\text{rad} \approx 42.031\,\text{deg}\end{aligned} \tag{1.15}$$

Example 5 Determination of Position Figure 1.2 illustrates a point P in a scaled triad $OABC$. We determine the position of the point P with respect to O by:

1. Drawing a line PD parallel OC to meet the plane AOB at D
2. Drawing DP_1 parallel to OB to meet OA at P_1

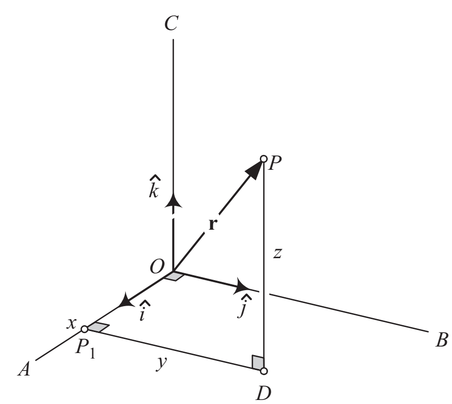

Figure 1.2 Determination of position.

The lengths OP_1, P_1D, DP are the coordinates of P and determine its position in triad $OABC$. The line segment OP is a diagonal of a parallelepiped with OP_1, P_1D, DP as three edges. The position of P is therefore determined by means of a parallelepiped whose edges are parallel to the legs of the triad and one of its diagonal is the line joining the origin to the point.

Example 6 Vectors in Different Coordinate Frames Figure 1.3 illustrates a globally fixed coordinate frame G at the center of a rotating disc O. Another smaller rotating disc with a coordinate frame B is attached to the first disc at a position ${}^{G}\mathbf{d}_{O}$. Point P is on the periphery of the small disc.

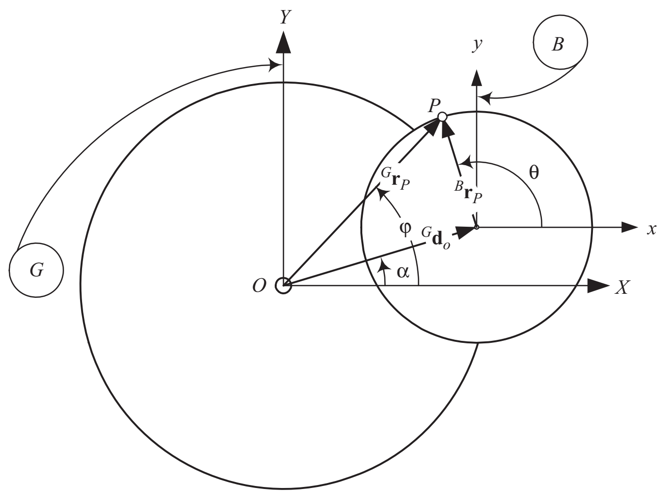

Figure 1.3 A globally fixed frame G at the center of a rotating disc O and a coordinate frame B at the center of a moving disc.

If the coordinate frame $G(OXYZ)$ is fixed and $B(oxyz)$ is always parallel to G, the position vectors of P in different coordinate frames are expressed by

$$ {}^{G}\mathbf{r}_{P} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k} = {}^{G}r_{P}\left(\cos\varphi\,\hat{\imath} + \sin\varphi\,\hat{\jmath}\right) \tag{1.16} $$

$$ {}^{B}\mathbf{r}_{P} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = {}^{B}r_{P}\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right) \tag{1.17} $$

The coordinate frame B in G may be indicated by a position vector ${}^{G}\mathbf{d}_{o}$:

$$ {}^{G}\mathbf{d}_{o} = d_{o}\left(\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}\right) \tag{1.18} $$

Example 7 Variable Vectors There are two ways that a vector can vary: length and direction. A variable-length vector is a vector in the natural expression where its magnitude is variable, such as

$$ \mathbf{r} = r(t)\,\hat{u}_{r} \tag{1.19} $$

The axis of a variable-length vector is fixed.

A variable-direction vector is a vector in its natural expression where the axis of its unit vector varies. To show such a variable vector, we use the decomposed expression of the unit vector and show that its directional cosines are variable:

$$ \mathbf{r} = r\,\hat{u}_{r}(t) = r\left(u_{1}(t)\hat{\imath} + u_{2}(t)\hat{\jmath} + u_{3}(t)\hat{k}\right) \tag{1.20} $$

$$ \sqrt{u_{1}^{2} + u_{2}^{2} + u_{3}^{2}} = 1 \tag{1.21} $$

The axis and direction characteristics are not fixed for a variable-direction vector, while its magnitude remains constant. The end point of a variable-direction vector slides on a sphere with a center at the starting point.

A variable vector may have both the length and direction variables. Such a vector is shown in its decomposed expression with variable components:

$$\mathbf{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k} \tag{1.22}$$

It can also be shown in its natural expression with variable length and direction:

$$\mathbf{r} = r(t)\,\hat{u}_r(t) \tag{1.23}$$

Example 8 Parallel and Perpendicular Decomposition of a Vector Consider a line l and a vector $\mathbf{r}$ intersecting at the origin of a coordinate frame such as shown is in Figure 1.4. The line l and vector $\mathbf{r}$ indicate a plane $(l, \mathbf{r})$. We define the unit vectors $\hat{u}_\parallel$ parallel to l and $\hat{u}_\perp$ perpendicular to l in the $(l, \mathbf{r})$-plane. If the angle between $\mathbf{r}$ and l is α, then the component of $\mathbf{r}$ parallel to l is

$$\mathbf{r}_\parallel = r\cos\alpha \tag{1.24}$$

and the component of $\mathbf{r}$ perpendicular to l is

$$\mathbf{r}_\perp = r\sin\alpha \tag{1.25}$$

These components indicate that we can decompose a vector $\mathbf{r}$ to its parallel and perpendicular components with respect to a line l by introducing the parallel and perpendicular unit vectors $\hat{u}_\parallel$ and $\hat{u}_\perp$:

$$\mathbf{r} = \mathbf{r}_\parallel\hat{u}_\parallel + \mathbf{r}_\perp\hat{u}_\perp = r\cos\alpha\,\hat{u}_\parallel + r\sin\alpha\,\hat{u}_\perp \tag{1.26}$$

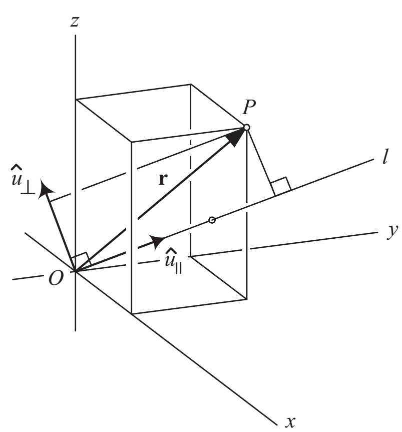

Figure 1.4 Decomposition of a vector $\mathbf{r}$ with respect to a line l into parallel and perpendicular components.

1.1.3 ★ Vector Definition

By a vector we mean any physical quantity that can be represented by a directed section of a line with a start point, such as O, and an end point, such as P. We may show a vector by an ordered pair of points with an arrow, such as $\overrightarrow{OP}$. The sign $\overrightarrow{PP}$ indicates a zero vector at point P.

Length and direction are necessary to have a vector; however, a vector may have five characteristics:

1. *Length*. The length of section OP corresponds to the magnitude of the physical quantity that the vector is representing.
2. *Axis*. A straight line that indicates the line on which the vector is. The vector axis is also called the *line of action*.
3. *End point*. A start or an end point indicates the point at which the vector is applied. Such a point is called the *affecting point*.
4. *Direction*. The direction indicates at what direction on the axis the vector is pointing.
5. *Physical quantity*. Any vector represents a physical quantity. If a physical quantity can be represented by a vector, it is called a *vectorial physical quantity*. The value of the quantity is proportional to the length of the vector. Having a vector that represents no physical quantity is meaningless, although a vector may be dimensionless.

Depending on the physical quantity and application, there are seven types of vectors:

1. *Vecpoint*. When all of the vector characteristics—length, axis, end point, direction, and physical quantity—are specified, the vector is called a *bounded vector*, *point vector*, or *vecpoint*. Such a vector is fixed at a point with no movability.
2. *Vecline*. If the start and end points of a vector are not fixed on the vector axis, the vector is called a *sliding vector*, *line vector*, or *vecline*. A sliding vector is free to slide on its axis.
3. *Vecface*. When the affecting point of a vector can move on a surface while the vector displaces parallel to itself, the vector is called a *surface vector* or *vecface*. If the surface is a plane, then the vector is a *plane vector* or *veclane*.
4. *Vecfree*. If the axis of a vector is not fixed, the vector is called a *free vector*, *direction vector*, or *vecfree*. Such a vector can move to any point of a specified space while it remains parallel to itself and keeps its direction.
5. *Vecpoline*. If the start point of a vector is fixed while the end point can slide on a line, the vector is a *point-line vector* or *vecpoline*. Such a vector has a constraint variable length and orientation. However, if the start and end points of a vecpoline are on the sliding line, its orientation is constant.
6. *Vecpoface*. If the start point of a vector is fixed while the end point can slide on a surface, the vector is a *point-surface vector* or *vecpoface*. Such a vector has a constraint variable length and orientation. The start and end points of a vecpoface may both be on the sliding surface. If the surface is a plane, the vector is called a *point-plane vector* or *vecpolane*.

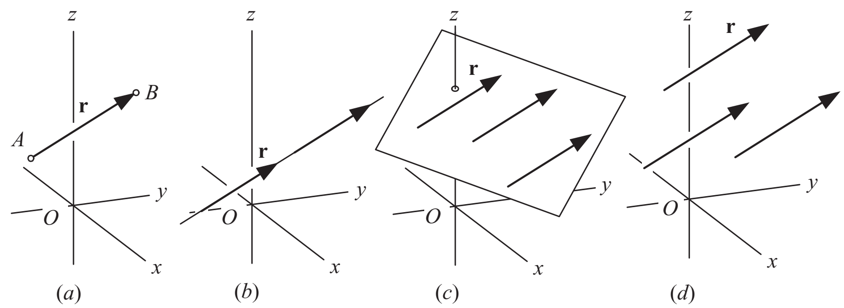

Figure 1.5 (*a*) A vecpoint, (*b*) a vecline, (*c*) a vecface, and (*d*) a vecfree.

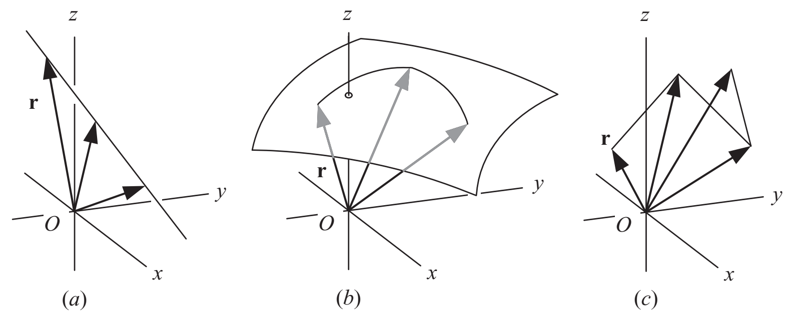

Figure 1.6 (*a*) a vecpoline, (*b*) vecpoface, (*c*) vecporee.

7. *Vecporee*. When the start point of a vector is fixed and the end point can move anywhere in a specified space, the vector is called a *point-free vector* or *vecporee*. Such a vector has a variable length and orientation.

Figure 1.5 illustrates a vecpoint, a vecline, vecface, and a vecfree and Figure 1.6 illustrates a vecpoline, a vecpoface, and a vecporee.

We may compare two vectors only if they represent the same physical quantity and are expressed in the same coordinate frame. Two vectors are equal if they are comparable and are the same type and have the same characteristics. Two vectors are equivalent if they are comparable and the same type and can be substituted with each other.

In summary, any physical quantity that can be represented by a directed section of a line with a start and an end point is a vector quantity. A vector may have five characteristics: length, axis, end point, direction, and physical quantity. The length and direction are necessary. There are seven types of vectors: vecpoint, vecline, vecface, vecfree, vecpoline, vecpoface, and vecporee. Vectors can be added when they are coaxial. In case the vectors are not coaxial, the decomposed expression of vectors must be used to add the vectors.

Example 9 Examples of Vector Types *Displacement* is a vecpoint. Moving from a point A to a point B is called the displacement. Displacement is equal to the difference of two position vectors. A *position vector* starts from the origin of a coordinate frame and ends as a point in the frame. If point A is at $\mathbf{r}_A$ and point B at $\mathbf{r}_B$, then displacement from A to B is

$$\mathbf{r}_{A/B} = {}_B\mathbf{r}_A = \mathbf{r}_A - \mathbf{r}_B \tag{1.27}$$

Force is a vecline. In Newtonian mechanics, a force can be applied on a body at any point of its axis and provides the same motion.

Torque is an example of vecfree. In Newtonian mechanics, a moment can be applied on a body at any point parallel to itself and provides the same motion.

A space curve is expressed by a vecpoline, a surface is expressed by a vecpoface, and a field is expressed by a vecporee.

Example 10 Scalars Physical quantities which can be specified by only a number are called *scalars*. If a physical quantity can be represented by a scalar, it is called a *scalaric physical quantity*. We may compare two scalars only if they represent the same physical quantity. Temperature, density, and work are some examples of scalaric physical quantities.

Two scalars are equal if they represent the same scalaric physical quantity and they have the same number in the same system of units. Two scalars are equivalent if we can substitute one with the other. Scalars must be equal to be equivalent.

1.2 VECTOR ALGEBRA

Most of the physical quantities in dynamics can be represented by vectors. Vector addition, multiplication, and differentiation are essential for the development of dynamics. We can combine vectors only if they are representing the same physical quantity, they are the same type, and they are expressed in the same coordinate frame.

1.2.1 Vector Addition

Two vectors can be *added* when they are *coaxial*. The result is another vector on the same axis with a component equal to the sum of the components of the two vectors. Consider two coaxial vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ in natural expressions:

$$\mathbf{r}_1 = r_1\hat{u}_r \qquad \mathbf{r}_2 = r_2\hat{u}_r \tag{1.28}$$

Their addition would be a new vector $\mathbf{r}_3 = r_3\hat{u}_r$ that is equal to

$$\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2 = (r_1 + r_2)\hat{u}_r = r_3\hat{u}_r \tag{1.29}$$

Because r_1 and r_2 are scalars, we have $r_1 + r_2 = r_1 + r_2$, and therefore, coaxial vector addition is *commutative*,

$$\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}_2 + \mathbf{r}_1 \tag{1.30}$$

and also *associative*,

$$\mathbf{r}_1 + (\mathbf{r}_2 + \mathbf{r}_3) = (\mathbf{r}_1 + \mathbf{r}_2) + \mathbf{r}_3 \tag{1.31}$$

When two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are not coaxial, we use their decomposed expressions

$$\mathbf{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} \qquad \mathbf{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k} \tag{1.32}$$

and add the coaxial vectors $x_1\hat{\imath}$ by $x_2\hat{\imath}$, $y_1\hat{\jmath}$ by $y_2\hat{\jmath}$, and $z_1\hat{k}$ by $z_2\hat{k}$ to write the result as the decomposed expression of $\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2$:

$$\begin{aligned}
\mathbf{r}_3 &= \mathbf{r}_1 + \mathbf{r}_2 \\
&= \left(x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}\right) + \left(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}\right) \\
&= \left(x_1\hat{\imath} + x_2\hat{\imath}\right) + \left(y_1\hat{\jmath} + y_2\hat{\jmath}\right) + \left(z_1\hat{k} + z_2\hat{k}\right) \\
&= (x_1 + x_2)\,\hat{\imath} + (y_1 + y_2)\,\hat{\jmath} + (z_1 + z_2)\,\hat{k} \\
&= x_3\hat{\imath} + y_3\hat{\jmath} + z_3\hat{k}
\end{aligned} \tag{1.33}$$

So, the sum of two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is defined as a vector $\mathbf{r}_3$ where its components are equal to the sum of the associated components of $\mathbf{r}_1$ and $\mathbf{r}_2$. Figure 1.7 illustrates vector addition $\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2$ of two vecpoints $\mathbf{r}_1$ and $\mathbf{r}_2$.

Subtraction of two vectors consists of adding to the minuend the subtrahend with the opposite sense:

$$\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1 + (-\mathbf{r}_2) \tag{1.34}$$

The vectors $-\mathbf{r}_2$ and $\mathbf{r}_2$ have the same axis and length and differ only in having opposite direction.

If the coordinate frame is known, the decomposed expression of vectors may also be shown by column matrices to simplify calculations:

$$\mathbf{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \tag{1.35}$$

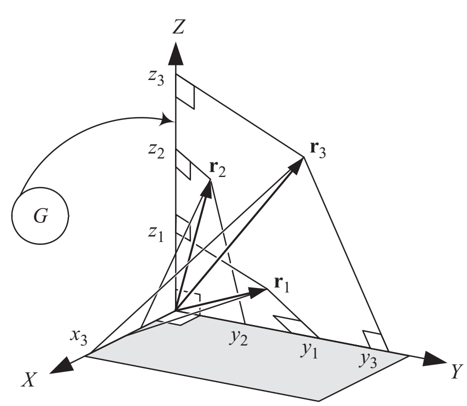

Figure 1.7 Vector addition of two vecpoints $\mathbf{r}_1$ and $\mathbf{r}_2$.

$$\mathbf{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \tag{1.36}$$

$$\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \tag{1.37}$$

Vectors can be added only when they are expressed in the same frame. Thus, a vector equation such as

$$\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2 \tag{1.38}$$

is meaningless without indicating that all of them are expressed in the same frame, such that

$$^B\mathbf{r}_3 = {}^B\mathbf{r}_1 + {}^B\mathbf{r}_2 \tag{1.39}$$

The three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are coplanar, and $\mathbf{r}_3$ may be considered as the diagonal of a parallelogram that is made by $\mathbf{r}_1$, $\mathbf{r}_2$.

Example 11 Displacement of a Point Point P moves from the origin of a global coordinate frame G to a point at $(1, 2, 0)$ and then moves to $(4, 3, 0)$. If we express the first displacement by a vector $\mathbf{r}_1$ and its final position by $\mathbf{r}_3$, the second displacement is $\mathbf{r}_2$, where

$$\mathbf{r}_2 = \mathbf{r}_3 - \mathbf{r}_1 = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \tag{1.40}$$

Example 12 Vector Interpolation Problem Having two digits n_1 and n_2 as the start and the final interpolants, we may define a controlled digit n with a variable q such that

$$n = \begin{cases} n_1 & q = 0 \\ n_2 & q = 1 \end{cases} \qquad 0 \le q \le 1 \tag{1.41}$$

Defining or determining such a controlled digit is called the interpolation problem. There are many functions to be used for solving the interpolation problem. Linear interpolation is the simplest and is widely used in engineering design, computer graphics, numerical analysis, and optimization:

$$n = n_1(1 - q) + n_2 q \tag{1.42}$$

The control parameter q determines the weight of each interpolants n_1 and n_2 in the interpolated n. In a linear interpolation, the weight factors are proportional to the distance of q from 1 and 0.

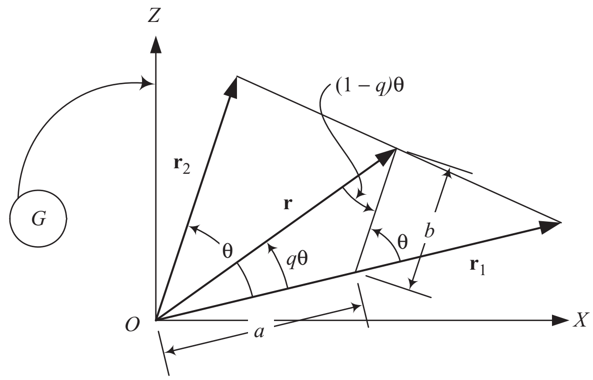

Figure 1.8 Vector linear interpolation.

Employing the linear interpolation technique, we may define a vector $\mathbf{r} = \mathbf{r}(q)$ to interpolate between the interpolant vectors $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{r} = (1-q)\mathbf{r}_1 + q\mathbf{r}_2 = \begin{bmatrix} x_1(1-q) + qx_2 \\ y_1(1-q) + qy_2 \\ z_1(1-q) + qz_2 \end{bmatrix} \tag{1.43}$$

In this interpolation, we assumed that equal steps in q results in equal steps in $\mathbf{r}$ between $\mathbf{r}_1$ and $\mathbf{r}_2$. The tip point of $\mathbf{r}$ will move on a line connecting the tip points of $\mathbf{r}_1$ and $\mathbf{r}_2$, as is shown in Figure 1.8.

We may interpolate the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ by interpolating the angular distance θ between $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{r} = \frac{\sin\left[(1-q)\theta\right]}{\sin\theta}\mathbf{r}_1 + \frac{\sin(q\theta)}{\sin\theta}\mathbf{r}_2 \tag{1.44}$$

To derive Equation (1.44), we may start with

$$\mathbf{r} = a\mathbf{r}_1 + b\mathbf{r}_2 \tag{1.45}$$

and find a and b from the following trigonometric equations:

$$a\sin(q\theta) - b\sin\left[(1-q)\theta\right] = 0 \tag{1.46}$$

$$a\cos(q\theta) + b\cos\left[(1-q)\theta\right] = 1 \tag{1.47}$$

Example 13 Vector Addition and Linear Space Vectors and adding operation make a *linear space* because for any vectors $\mathbf{r}_1$, $\mathbf{r}_2$ we have the following properties:

1. Commutative:

$$\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}_2 + \mathbf{r}_1 \tag{1.48}$$

2. Associative:

$$\mathbf{r}_1 + (\mathbf{r}_2 + \mathbf{r}_3) = (\mathbf{r}_1 + \mathbf{r}_2) + \mathbf{r}_3 \tag{1.49}$$

3. Null element:

$$\mathbf{0} + \mathbf{r} = \mathbf{r} \tag{1.50}$$

4. Inverse element:

$$\mathbf{r} + (-\mathbf{r}) = \mathbf{0} \tag{1.51}$$

Example 14 Linear Dependence and Independence The n vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_n$ are *linearly dependent* if there exist n scalars $c_1, c_2, c_3, \ldots, c_n$ not all equal to zero such that a linear combination of the vectors equals zero:

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 + \cdots + c_n\mathbf{r}_n = 0 \tag{1.52}$$

The vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_n$ are *linearly independent* if they are not linearly dependent, and it means the n scalars $c_1, c_2, c_3, \ldots, c_n$ must all be zero to have Equation (1.52):

$$c_1 = c_2 = c_3 = \cdots = c_n = 0 \tag{1.53}$$

Example 15 Two Linearly Dependent Vectors Are Colinear Consider two linearly dependent vectors $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 = 0 \tag{1.54}$$

If $c_1 \neq 0$, we have

$$\mathbf{r}_1 = -\frac{c_2}{c_1}\mathbf{r}_2 \tag{1.55}$$

and if $c_2 \neq 0$, we have

$$\mathbf{r}_2 = -\frac{c_1}{c_2}\mathbf{r}_1 \tag{1.56}$$

which shows $\mathbf{r}_1$ and $\mathbf{r}_2$ are colinear.

Example 16 Three Linearly Dependent Vectors Are Coplanar Consider three linearly dependent vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$,

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = 0 \tag{1.57}$$

where at least one of the scalars c_1, c_2, c_3, say c_3, is not zero; then

$$\mathbf{r}_3 = -\frac{c_1}{c_3}\mathbf{r}_1 - \frac{c_2}{c_3}\mathbf{r}_2 \tag{1.58}$$

which shows $\mathbf{r}_3$ is in the same plane as $\mathbf{r}_1$ and $\mathbf{r}_2$.

1.2.2 Vector Multiplication

There are three types of vector multiplications for two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$:

1. *Dot, Inner, or Scalar Product*

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$
$$= r_1 r_2 \cos \alpha \tag{1.59}$$

The inner product of two vectors produces a scalar that is equal to the product of the length of individual vectors and the cosine of the angle between them. The vector inner product is *commutative* in orthogonal coordinate frames,

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_2 \cdot \mathbf{r}_1 \tag{1.60}$$

The inner product is dimension free and can be calculated in n-dimensional spaces. The inner product can also be performed in nonorthogonal coordinate systems.

2. *Cross, Outer, or Vector Product*

$$\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - y_2 z_1 \\ x_2 z_1 - x_1 z_2 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$
$$= (r_1 r_2 \sin \alpha)\, \hat{u}_{r_3} = r_3 \hat{u}_{r_3} \tag{1.61}$$
$$\hat{u}_{r_3} = \hat{u}_{r_1} \times \hat{u}_{r_2} \tag{1.62}$$

The outer product of two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ produces another vector $\mathbf{r}_3$ that is perpendicular to the plane of $\mathbf{r}_1$, $\mathbf{r}_2$ such that the cycle $\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3$ makes a right-handed triad. The length of $\mathbf{r}_3$ is equal to the product of the length of individual vectors multiplied by the sine of the angle between them. Hence r_3 is numerically equal to the area of the parallelogram made up of $\mathbf{r}_1$ and $\mathbf{r}_2$. The vector inner product is *skew commutative* or *anticommutative*:

$$\mathbf{r}_1 \times \mathbf{r}_2 = -\mathbf{r}_2 \times \mathbf{r}_1 \tag{1.63}$$

The outer product is defined and applied only in 3D space. There is no outer product in lower or higher dimensions than 3. If any vector of $\mathbf{r}_1$and $\mathbf{r}_2$ is in a lower dimension than 3D, we must make it a 3D vector by adding zero components for missing dimensions to be able to perform their outer product.

3. *Quaternion Product*

$$\mathbf{r}_1\mathbf{r}_2 = \mathbf{r}_1 \times \mathbf{r}_2 - \mathbf{r}_1 \cdot \mathbf{r}_2 \tag{1.64}$$

We will talk about the quaternion product in Section 5.3.

In summary, there are three types of vector multiplication: inner, outer, and quaternion products, of which the inner product is the only one with commutative property.

Example 17 Geometric Expression of Inner Products Consider a line l and a vector $\mathbf{r}$ intersecting at the origin of a coordinate frame as is shown in Figure 1.9. If the angle between $\mathbf{r}$ and l is α, the parallel component of $\mathbf{r}$ to l is

$$\mathbf{r}_{\|} = \overline{OA} = r\cos\alpha \tag{1.65}$$

This is the length of the projection of $\mathbf{r}$ on l. If we define a unit vector $\hat{u}_l$ on l by its direction cosines β_1, β_2, β_3,

$$\hat{u}_l = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos\beta_1 \\ \cos\beta_2 \\ \cos\beta_3 \end{bmatrix} \tag{1.66}$$

then the inner product of $\mathbf{r}$ and $\hat{u}_l$ is

$$\mathbf{r}\cdot\hat{u}_l = \mathbf{r}_{\|} = r\cos\alpha \tag{1.67}$$

We may show $\mathbf{r}$ by using its direction cosines α_1, α_2, α_3,

$$\mathbf{r} = r\hat{u}_r = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = r\begin{bmatrix} x/r \\ y/r \\ z/r \end{bmatrix} = r\begin{bmatrix} \cos\alpha_1 \\ \cos\alpha_2 \\ \cos\alpha_3 \end{bmatrix} \tag{1.68}$$

Then, we may use the result of the inner product of $\mathbf{r}$ and $\hat{u}_l$,

$$\begin{aligned} \mathbf{r}\cdot\hat{u}_l &= r\begin{bmatrix} \cos\alpha_1 \\ \cos\alpha_2 \\ \cos\alpha_3 \end{bmatrix} \cdot \begin{bmatrix} \cos\beta_1 \\ \cos\beta_2 \\ \cos\beta_3 \end{bmatrix} \\ &= r\left(\cos\beta_1\cos\alpha_1 + \cos\beta_2\cos\alpha_2 + \cos\beta_3\cos\alpha_3\right) \end{aligned} \tag{1.69}$$

to calculate the angle α between $\mathbf{r}$ and l based on their directional cosines:

$$\cos\alpha = \cos\beta_1\cos\alpha_1 + \cos\beta_2\cos\alpha_2 + \cos\beta_3\cos\alpha_3 \tag{1.70}$$

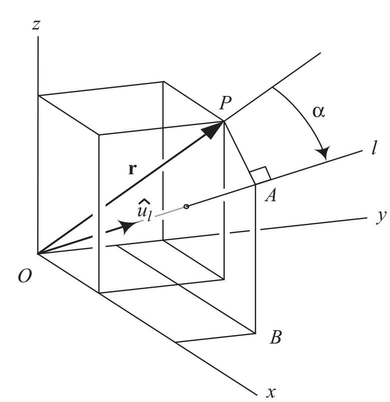

Figure 1.9 A line l and a vector $\mathbf{r}$ intersecting at the origin of a coordinate frame.

So, the inner product can be used to find the projection of a vector on a given line. It is also possible to use the inner product to determine the angle α between two given vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ as

$$\cos\alpha = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2} = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{\sqrt{\mathbf{r}_1 \cdot \mathbf{r}_1}\sqrt{\mathbf{r}_2 \cdot \mathbf{r}_2}} \tag{1.71}$$

Example 18 Power 2 of a Vector By writing a vector $\mathbf{r}$ to a power 2, we mean the inner product of $\mathbf{r}$ to itself:

$$\mathbf{r}^2 = \mathbf{r} \cdot \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2 + y^2 + z^2 = r^2 \tag{1.72}$$

Using this definition we can write

$$(\mathbf{r}_1 + \mathbf{r}_2)^2 = (\mathbf{r}_1 + \mathbf{r}_2) \cdot (\mathbf{r}_1 + \mathbf{r}_2) = \mathbf{r}_1^2 + 2\mathbf{r}_1 \cdot \mathbf{r}_2 + \mathbf{r}_2^2 \tag{1.73}$$

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 + \mathbf{r}_2) = \mathbf{r}_1^2 - \mathbf{r}_2^2 \tag{1.74}$$

There is no meaning for a vector with a negative or positive odd exponent.

Example 19 Unit Vectors and Inner and Outer Products Using the set of unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ of a positive orthogonal triad and the definition of inner product, we conclude that

$$\hat{\imath}^2 = 1 \qquad \hat{\jmath}^2 = 1 \qquad \hat{k}^2 = 1 \tag{1.75}$$

Furthermore, by definition of the vector product we have

$$\hat{\imath} \times \hat{\jmath} = -\left(\hat{\jmath} \times \hat{\imath}\right) = \hat{k} \tag{1.76}$$

$$\hat{\jmath} \times \hat{k} = -\left(\hat{k} \times \hat{\jmath}\right) = \hat{\imath} \tag{1.77}$$

$$\hat{k} \times \hat{\imath} = -\left(\hat{\imath} \times \hat{k}\right) = \hat{\jmath} \tag{1.78}$$

It might also be useful if we have these equalities:

$$\hat{\imath} \cdot \hat{\jmath} = 0 \qquad \hat{\jmath} \cdot \hat{k} = 0 \qquad \hat{k} \cdot \hat{\imath} = 0 \tag{1.79}$$

$$\hat{\imath} \times \hat{\imath} = 0 \qquad \hat{\jmath} \times \hat{\jmath} = 0 \qquad \hat{k} \times \hat{k} = 0 \tag{1.80}$$

Example 20 Vanishing Dot Product If the inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is zero,

$$\mathbf{a} \cdot \mathbf{b} = 0 \tag{1.81}$$

then either $\mathbf{a} = 0$ or $\mathbf{b} = 0$, or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Example 21 Vector Equations Assume $\mathbf{x}$ is an unknown vector, k is a scalar, and $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three constant vectors in the following vector equation:

$$k\mathbf{x} + (\mathbf{b} \cdot \mathbf{x})\, \mathbf{a} = \mathbf{c} \tag{1.82}$$

To solve the equation for $\mathbf{x}$, we dot product both sides of (1.82) by $\mathbf{b}$:

$$k\mathbf{x} \cdot \mathbf{b} + (\mathbf{x} \cdot \mathbf{b})\, (\mathbf{a} \cdot \mathbf{b}) = \mathbf{c} \cdot \mathbf{b} \tag{1.83}$$

This is a linear equation for $\mathbf{x} \cdot \mathbf{b}$ with the solution

$$\mathbf{x} \cdot \mathbf{b} = \frac{\mathbf{c} \cdot \mathbf{b}}{k + \mathbf{a} \cdot \mathbf{b}} \tag{1.84}$$

provided

$$k + \mathbf{a} \cdot \mathbf{b} \neq 0 \tag{1.85}$$

Substituting (1.84) in (1.82) provides the solution $\mathbf{x}$:

$$\mathbf{x} = \frac{1}{k}\mathbf{c} - \frac{\mathbf{c} \cdot \mathbf{b}}{k\, (k + \mathbf{a} \cdot \mathbf{b})}\mathbf{a} \tag{1.86}$$

An alternative method is decomposition of the vector equation along the axes $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ of the coordinate frame and solving a set of three scalar equations to find the components of the unknown vector.

Assume the decomposed expression of the vectors $\mathbf{x}$, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{1.87}$$

Substituting these expressions in Equation (1.82),

$$k \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{1.88}$$

provides a set of three scalar equations

$$\begin{bmatrix} k + a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & k + a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & k + a_3 b_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{1.89}$$

that can be solved by matrix inversion:

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} k + a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & k + a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & k + a_3 b_3 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{kc_1 - a_1 b_2 c_2 + a_2 b_2 c_1 - a_1 b_3 c_3 + a_3 b_3 c_1}{k\, (k + a_1 b_1 + a_2 b_2 + a_3 b_3)} \\ \dfrac{kc_2 + a_1 b_1 c_2 - a_2 b_1 c_1 - a_2 b_3 c_3 + a_3 b_3 c_2}{k\, (k + a_1 b_1 + a_2 b_2 + a_3 b_3)} \\ \dfrac{kc_3 + a_1 b_1 c_3 - a_3 b_1 c_1 + a_2 b_2 c_3 - a_3 b_2 c_2}{k\, (k + a_1 b_1 + a_2 b_2 + a_3 b_3)} \end{bmatrix} \end{aligned} \tag{1.90}$$

Solution (1.90) is compatible with solution (1.86).

Example 22 Vector Addition, Scalar Multiplication, and Linear Space Vector addition and scalar multiplication make a linear space, because

$$k_1 (k_2\mathbf{r}) = (k_1k_2)\,\mathbf{r} \tag{1.91}$$

$$(k_1 + k_2)\,\mathbf{r} = k_1\mathbf{r} + k_2\mathbf{r} \tag{1.92}$$

$$k\,(\mathbf{r}_1 + \mathbf{r}_2) = k\mathbf{r}_1 + k\mathbf{r}_2 \tag{1.93}$$

$$1 \cdot \mathbf{r} = \mathbf{r} \tag{1.94}$$

$$(-1) \cdot \mathbf{r} = -\mathbf{r} \tag{1.95}$$

$$0 \cdot \mathbf{r} = \mathbf{0} \tag{1.96}$$

$$k \cdot \mathbf{0} = \mathbf{0} \tag{1.97}$$

Example 23 Vanishing Condition of a Vector Inner Product Consider three non-coplanar constant vectors **a**, **b**, **c** and an arbitrary vector **r**. If

$$\mathbf{a} \cdot \mathbf{r} = 0 \qquad \mathbf{b} \cdot \mathbf{r} = 0 \qquad \mathbf{c} \cdot \mathbf{r} = 0 \tag{1.98}$$

then

$$\mathbf{r} = 0 \tag{1.99}$$

Example 24 Vector Product Expansion We may prove the result of the inner and outer products of two vectors by using decomposed expression and expansion:

$$\begin{aligned}
\mathbf{r}_1 \cdot \mathbf{r}_2 &= \left(x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}\right) \cdot \left(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}\right) \\
&= x_1x_2\hat{\imath}\cdot\hat{\imath} + x_1y_2\hat{\imath}\cdot\hat{\jmath} + x_1z_2\hat{\imath}\cdot\hat{k} \\
&\quad + y_1x_2\hat{\jmath}\cdot\hat{\imath} + y_1y_2\hat{\jmath}\cdot\hat{\jmath} + y_1z_2\hat{\jmath}\cdot\hat{k} \\
&\quad + z_1x_2\hat{k}\cdot\hat{\imath} + z_1y_2\hat{k}\cdot\hat{\jmath} + z_1z_2\hat{k}\cdot\hat{k} \\
&= x_1x_2 + y_1y_2 + z_1z_2
\end{aligned} \tag{1.100}$$

$$\begin{aligned}
\mathbf{r}_1 \times \mathbf{r}_2 &= \left(x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}\right) \times \left(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}\right) \\
&= x_1x_2\hat{\imath}\times\hat{\imath} + x_1y_2\hat{\imath}\times\hat{\jmath} + x_1z_2\hat{\imath}\times\hat{k} \\
&\quad + y_1x_2\hat{\jmath}\times\hat{\imath} + y_1y_2\hat{\jmath}\times\hat{\jmath} + y_1z_2\hat{\jmath}\times\hat{k} \\
&\quad + z_1x_2\hat{k}\times\hat{\imath} + z_1y_2\hat{k}\times\hat{\jmath} + z_1z_2\hat{k}\times\hat{k} \\
&= (y_1z_2 - y_2z_1)\,\hat{\imath} + (x_2z_1 - x_1z_2)\,\hat{\jmath} + (x_1y_2 - x_2y_1)\,\hat{k}
\end{aligned} \tag{1.101}$$

We may also find the outer product of two vectors by expanding a determinant and derive the same result as Equation (1.101):

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \tag{1.102}$$

Example 25 bac−cab Rule If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three vectors, we may expand their triple cross product and show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b}) \tag{1.103}$$

because

$$\begin{aligned}
&\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) \\
&= \begin{bmatrix} a_2\,(b_1c_2 - b_2c_1) + a_3\,(b_1c_3 - b_3c_1) \\ a_3\,(b_2c_3 - b_3c_2) - a_1\,(b_1c_2 - b_2c_1) \\ -a_1\,(b_1c_3 - b_3c_1) - a_2\,(b_2c_3 - b_3c_2) \end{bmatrix} \\
&= \begin{bmatrix} b_1\,(a_1c_1 + a_2c_2 + a_3c_3) - c_1\,(a_1b_1 + a_2b_2 + a_3b_3) \\ b_2\,(a_1c_1 + a_2c_2 + a_3c_3) - c_2\,(a_1b_1 + a_2b_2 + a_3b_3) \\ b_3\,(a_1c_1 + a_2c_2 + a_3c_3) - c_3\,(a_1b_1 + a_2b_2 + a_3b_3) \end{bmatrix}
\end{aligned} \tag{1.104}$$

Equation (1.103) may be referred to as the *bac−cab rule*, which makes it easy to remember. The bac−cab rule is the most important in 3D vector algebra. It is the key to prove a great number of other theorems.

Example 26 Geometric Expression of Outer Products Consider the free vectors $\mathbf{r}_1$ from A to B and $\mathbf{r}_2$ from A to C, as are shown in Figure 1.10:

$$\mathbf{r}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \sqrt{10} \begin{bmatrix} -0.31623 \\ 0.94868 \\ 0 \end{bmatrix} \tag{1.105}$$

$$\mathbf{r}_2 = \begin{bmatrix} -1 \\ 0 \\ 2.5 \end{bmatrix} = 2.6926 \begin{bmatrix} -0.37139 \\ 0 \\ 0.92847 \end{bmatrix} \tag{1.106}$$

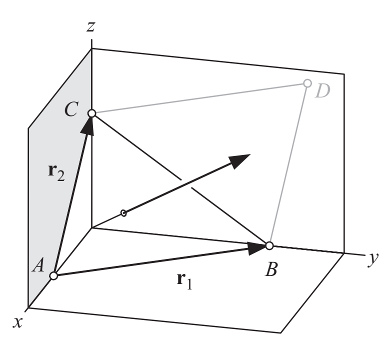

Figure 1.10 The cross product of the two free vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and the resultant $\mathbf{r}_3$.

The cross product of the two vectors is $\mathbf{r}_3$:

$$\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2 = \begin{bmatrix} 7.5 \\ 2.5 \\ 3 \end{bmatrix} = 8.4558 \begin{bmatrix} 0.88697 \\ 0.29566 \\ 0.35479 \end{bmatrix}$$

$$= r_3 \hat{u}_{r_3} = (r_1 r_2 \sin\alpha)\, \hat{u}_{r_3} \tag{1.107}$$

$$\hat{u}_{r_3} = \hat{u}_{r_1} \times \hat{u}_{r_2} = \begin{bmatrix} 0.88697 \\ 0.29566 \\ 0.35479 \end{bmatrix} \tag{1.108}$$

where $r_3 = 8.4558$ is numerically equivalent to the area A of the parallelogram $ABCD$ made by the sides AB and AC:

$$A_{ABCD} = |\mathbf{r}_1 \times \mathbf{r}_2| = 8.4558 \tag{1.109}$$

The area of the triangle ABC is $A/2$. The vector $\mathbf{r}_3$ is perpendicular to this plane and, hence, its unit vector $\hat{u}_{r_3}$ can be used to indicate the plane $ABCD$.

Example 27 Scalar Triple Product The dot product of a vector $\mathbf{r}_1$ with the cross product of two vectors $\mathbf{r}_2$ and $\mathbf{r}_3$ is called the *scalar triple product* of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. The scalar triple product can be shown and calculated by a determinant:

$$\mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3) = \mathbf{r}_1 \cdot \mathbf{r}_2 \times \mathbf{r}_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \tag{1.110}$$

Interchanging two rows (or columns) of a matrix changes the sign of its determinant. So, we may conclude that the scalar triple product of three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ is also equal to

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{r}_2 \times \mathbf{r}_3 &= \mathbf{r}_2 \cdot \mathbf{r}_3 \times \mathbf{r}_1 = \mathbf{r}_3 \cdot \mathbf{r}_1 \times \mathbf{r}_2 \\ &= \mathbf{r}_1 \times \mathbf{r}_2 \cdot \mathbf{r}_3 = \mathbf{r}_2 \times \mathbf{r}_3 \cdot \mathbf{r}_1 = \mathbf{r}_3 \times \mathbf{r}_1 \cdot \mathbf{r}_2 \\ &= -\mathbf{r}_1 \cdot \mathbf{r}_3 \times \mathbf{r}_2 = -\mathbf{r}_2 \cdot \mathbf{r}_1 \times \mathbf{r}_3 = -\mathbf{r}_3 \cdot \mathbf{r}_2 \times \mathbf{r}_1 \\ &= -\mathbf{r}_1 \times \mathbf{r}_3 \cdot \mathbf{r}_2 = -\mathbf{r}_2 \times \mathbf{r}_1 \cdot \mathbf{r}_3 = -\mathbf{r}_3 \times \mathbf{r}_2 \cdot \mathbf{r}_1 \end{aligned} \tag{1.111}$$

Because of Equation (1.111), the scalar triple product of the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ can be shown by the short notation $[\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3]$:

$$[\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3] = \mathbf{r}_1 \cdot \mathbf{r}_2 \times \mathbf{r}_3 \tag{1.112}$$

This notation gives us the freedom to set the position of the dot and cross product signs as required.

If the three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are position vectors, then their scalar triple product geometrically represents the volume of the parallelepiped formed by the three vectors. Figure 1.11 illustrates such a parallelepiped for three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$.

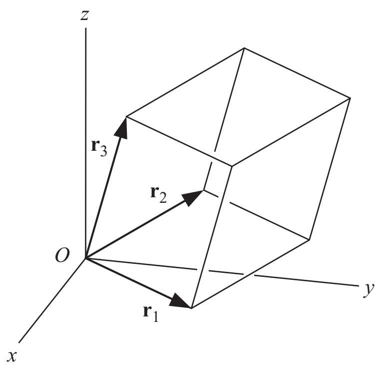

Figure 1.11 The parallelepiped made by three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$.

Example 28 Vector Triple Product The cross product of a vector $\mathbf{r}_1$ with the cross product of two vectors $\mathbf{r}_2$ and $\mathbf{r}_3$ is called the *vector triple product* of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. The *bac−cab* rule is always used to simplify a vector triple product:

$$\mathbf{r}_1 \times (\mathbf{r}_2 \times \mathbf{r}_3) = \mathbf{r}_2 (\mathbf{r}_1 \cdot \mathbf{r}_3) - \mathbf{r}_3 (\mathbf{r}_1 \cdot \mathbf{r}_2) \tag{1.113}$$

Example 29 ★ Norm and Vector Space Assume $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are arbitrary vectors and c, c_1, c_3 are scalars. The *norm* of a vector $\|\mathbf{r}\|$ is defined as a real-valued function on a vector space v such that for all $\{\mathbf{r}_1, \mathbf{r}_2\} \in V$ and all $c \in \mathbb{R}$ we have:

1. Positive definition: $\|\mathbf{r}\| > 0$ if $\mathbf{r} \neq 0$ and $\|\mathbf{r}\| = 0$ if $\mathbf{r} = 0$.
2. Homogeneity: $\|c\mathbf{r}\| = \|c\| \, \|\mathbf{r}\|$.
3. Triangle inequality: $\|\mathbf{r}_1 + \mathbf{r}_2\| = \|\mathbf{r}_1\| + \|\mathbf{r}_2\|$.

The definition of norm is up to the investigator and may vary depending on the application. The most common definition of the norm of a vector is the length:

$$\|\mathbf{r}\| = |\mathbf{r}| = \sqrt{r_1^2 + r_2^2 + r_3^2} \tag{1.114}$$

The set v with vector elements is called a *vector space* if the following conditions are fulfilled:

1. Addition: If $\{\mathbf{r}_1, \mathbf{r}_2\} \in V$ and $\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}$, then $\mathbf{r} \in V$.
2. Commutativity: $\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}_2 + \mathbf{r}_1$.
3. Associativity: $\mathbf{r}_1 + (\mathbf{r}_2 + \mathbf{r}_3) = (\mathbf{r}_1 + \mathbf{r}_2) + \mathbf{r}_3$ and $c_1 (c_2\mathbf{r}) = (c_1c_2) \mathbf{r}$.
4. Distributivity: $c (\mathbf{r}_1 + \mathbf{r}_2) = c\mathbf{r}_1 + c\mathbf{r}_2$ and $(c_1 + c_2) \mathbf{r} = c_1\mathbf{r} + c_2\mathbf{r}$.
5. Identity element: $\mathbf{r} + \mathbf{0} = \mathbf{r}$, $1\mathbf{r} = \mathbf{r}$, and $\mathbf{r} - \mathbf{r} = \mathbf{r} + (-1) \mathbf{r} = \mathbf{0}$.

Example 30 ★ Nonorthogonal Coordinate Frame It is possible to define a coordinate frame in which the three scaled lines OA, OB, OC are nonorthogonal. Defining three unit vectors $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_3$ along the nonorthogonal non-coplanar axes OA, OB, OC, respectively, we can express any vector $\mathbf{r}$ by a linear combination of the three non-coplanar unit vectors $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_3$ as

$$\mathbf{r} = r_1\hat{b}_1 + r_2\hat{b}_2 + r_3\hat{b}_3 \tag{1.115}$$

where, r_1, r_2, and r_3 are constant.

Expression of the unit vectors $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$ and vector $\mathbf{r}$ in a Cartesian coordinate frame is

$$\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1.116}$$

$$\hat{b}_1 = b_{11}\hat{\imath} + b_{12}\hat{\jmath} + b_{13}\hat{k} \tag{1.117}$$

$$\hat{b}_2 = b_{21}\hat{\imath} + b_{22}\hat{\jmath} + b_{23}\hat{k} \tag{1.118}$$

$$\hat{b}_3 = b_{31}\hat{\imath} + b_{32}\hat{\jmath} + b_{33}\hat{k} \tag{1.119}$$

Substituting (1.117)–(1.119) in (1.115) and comparing with (1.116) show that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{1.120}$$

The set of equations (1.120) may be solved for the components r_1, r_2, and r_3:

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1.121}$$

We may also express them by vector scalar triple product:

$$r_1 = \frac{1}{\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}} \begin{vmatrix} x & y & z \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \frac{\mathbf{r} \cdot \hat{b}_2 \times \hat{b}_3}{\hat{b}_1 \cdot \hat{b}_2 \times \hat{b}_3} \tag{1.122}$$

$$r_2 = \frac{1}{\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}} \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ x & y & z \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \frac{\mathbf{r} \cdot \hat{b}_3 \times \hat{b}_1}{\hat{b}_1 \cdot \hat{b}_2 \times \hat{b}_3} \tag{1.123}$$

$$r_3 = \frac{1}{\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}} \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ x & y & z \end{vmatrix} = \frac{\mathbf{r} \cdot \hat{b}_1 \times \hat{b}_2}{\hat{b}_1 \cdot \hat{b}_2 \times \hat{b}_3} \tag{1.124}$$

The set of equations (1.120) is solvable provided $\hat{b}_1 \cdot \hat{b}_2 \times \hat{b}_3 \neq 0$, which means $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$ are not coplanar.

1.2.3 ★ Index Notation

Whenever the components of a vector or a vector equation are structurally similar, we may employ the summation sign, $\sum$, and show only one component with an index to be changed from 1 to 2 and 3 to indicate the first, second, and third components. The axes and their unit vectors of the coordinate frame may also be shown by x_1, x_2, x_3 and $\hat{u}_1, \hat{u}_2, \hat{u}_3$ instead of x, y, z and $\hat{\imath}, \hat{\jmath}, \hat{k}$. This is called *index notation* and may simplify vector calculations.

There are two symbols that may be used to make the equations even more concise:

1. *Kronecker delta* δ_{ij}:

$$\delta_{ij} = \left\{ \begin{array}{ll} 1 & i = j \\ 0 & i \neq j \end{array} \right\} = \delta_{ji} \tag{1.125}$$

It states that $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$.

2. *Levi-Civita symbol* ϵ_{ijk}:

$$\epsilon_{ijk} = \frac{1}{2}(i - j)(j - k)(k - i) \qquad i, j, k = 1, 2, 3 \tag{1.126}$$

It states that $\epsilon_{ijk} = 1$ if i, j, k is a cyclic permutation of 1, 2, 3, $\epsilon_{ijk} = -1$ if i, j, k is a cyclic permutation of 3, 2, 1, and $\epsilon_{ijk} = 0$ if at least two of i, j, k are equal. The Levi-Civita symbol is also called the *permutation symbol*.

The Levi-Civita symbol ϵ_{ijk} can be expanded by the Kronecker delta δ_{ij}:

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \tag{1.127}$$

This relation between ϵ and δ is known as the $e{-}delta$ or $\epsilon{-}delta$ identity.

Using index notation, the vectors **a** and **b** can be shown as

$$\mathbf{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = \sum_{i=1}^{3} a_i\hat{u}_i \tag{1.128}$$

$$\mathbf{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k} = \sum_{i=1}^{3} b_i\hat{u}_i \tag{1.129}$$

and the inner and outer products of the unit vectors of the coordinate system as

$$\hat{u}_j \cdot \hat{u}_k = \delta_{jk} \tag{1.130}$$

$$\hat{u}_j \times \hat{u}_k = \epsilon_{ijk}\hat{u}_i \tag{1.131}$$

Example 31 Fundamental Vector Operations and Index Notation Index notation simplifies the vector equations. By index notation, we show the elements r_i, $i = 1, 2, 3$ instead of indicating the vector **r**. The fundamental vector operations by index notation are:

1. Decomposition of a vector **r**:

$$\mathbf{r} = \sum_{i=1}^{3} r_i \hat{u}_i \tag{1.132}$$

2. Orthogonality of unit vectors:

$$\hat{u}_i \cdot \hat{u}_j = \delta_{ij} \qquad \hat{u}_i \times \hat{u}_j = \epsilon_{ijk} \hat{u}_k \tag{1.133}$$

3. Projection of a vector **r** on $\hat{u}_i$:

$$\mathbf{r} \cdot \hat{u}_j = \sum_{i=1}^{3} r_i \hat{u}_i \cdot \hat{u}_j = \sum_{i=1}^{3} r_i \delta_{ij} = r_j \tag{1.134}$$

4. Scalar, dot, or inner product of vectors **a** and **b**:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \sum_{i=1}^{3} a_i \hat{u}_i \cdot \sum_{j=1}^{3} b_j \hat{u}_j = \sum_{j=1}^{3} \sum_{i=1}^{3} a_i b_j \left(\hat{u}_i \cdot \hat{u}_j\right) = \sum_{j=1}^{3} \sum_{i=1}^{3} a_i b_j \delta_{ij} \\ &= \sum_{i=1}^{3} a_i b_i \end{aligned} \tag{1.135}$$

5. Vector, cross, or outer product of vectors **a** and **b**:

$$\mathbf{a} \times \mathbf{b} = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \hat{u}_i a_j b_k \tag{1.136}$$

6. Scalar triple product of vectors **a**, **b**, and **c**:

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = [\mathbf{abc}] = \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{i=1}^{3} \epsilon_{ijk} a_j b_j c_k \tag{1.137}$$

Example 32 Levi-Civita Density and Unit Vectors The *Levi-Civita symbol* ϵ_{ijk}, also called the "*e*" tensor, *Levi-Civita density*, and *permutation tensor* and may be defined by the clearer expression

$$\epsilon_{ijk} = \begin{cases} 1 & ijk = 123, 231, 312 \\ 0 & i = j \ or \ j = k \ or \ k = 1 \\ -1 & ijk = 321, 213, 132 \end{cases} \tag{1.138}$$

can be shown by the scalar triple product of the unit vectors of the coordinate system,

$$\epsilon_{ijk} = \left[\hat{u}_i \hat{u}_j \hat{u}_k\right] = \hat{u}_i \cdot \hat{u}_j \times \hat{u}_k \tag{1.139}$$

and therefore,

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{kji} = -\epsilon_{jik} = -\epsilon_{ikj} \tag{1.140}$$

The product of two Levi-Civita densities is

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \qquad i, j, k, l, m, n = 1, 2, 3 \tag{1.141}$$

If $k = l$, we have

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{mnk} = \begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \tag{1.142}$$

and if also $j = n$, then

$$\sum_{k=1}^{3}\sum_{j=1}^{3} \epsilon_{ijk}\epsilon_{mjk} = 2\delta_{im} \tag{1.143}$$

and finally, if also $i = m$, we have

$$\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ijk} = 6 \tag{1.144}$$

Employing the permutation symbol ϵ_{ijk}, we can show the vector scalar triple product as

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} a_i b_j c_k = \sum_{i,j,k=1}^{3} \epsilon_{ijk} a_i b_j c_k \tag{1.145}$$

Example 33 ★ Einstein Summation Convention The *Einstein summation convention* implies that we may not show the summation symbol if we agree that there is a hidden summation symbol for every repeated index over all possible values for that index. In applied kinematics and dynamics, we usually work in a 3D space, so the range of summation symbols are from 1 to 3. Therefore, Equations (1.135) and (1.136) may be shown more simply as

$$d = a_i b_i \tag{1.146}$$

$$c_i = \epsilon_{ijk} a_j b_k \tag{1.147}$$

and the result of $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ as

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} &= \sum_{i=1}^{3} a_i \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} b_j c_k = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} a_i b_j c_k \\ &= \epsilon_{ijk} a_i b_j c_k \end{aligned} \tag{1.148}$$

The repeated index in a term must appear only twice to define a summation rule. Such an index is called a *dummy index* because it is immaterial what character is used for it. As an example, we have

$$a_i b_i = a_m b_m = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.149}$$

Example 34 ★ A Vector Identity We may use the index notation and verify vector identities such as

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{c}\,(\mathbf{d} \cdot \mathbf{a} \times \mathbf{b}) - \mathbf{d}\,(\mathbf{c} \cdot \mathbf{a} \times \mathbf{b}) \tag{1.150}$$

Let us assume that

$$\mathbf{a} \times \mathbf{b} = \mathbf{p} = p_i \hat{u}_i \tag{1.151}$$

$$\mathbf{c} \times \mathbf{d} = \mathbf{q} = q_i \hat{u}_i \tag{1.152}$$

The components of these vectors are

$$p_i = \epsilon_{ijk} a_j b_k \tag{1.153}$$

$$q_i = \epsilon_{ijk} c_j d_k \tag{1.154}$$

and therefore the components of $\mathbf{p} \times \mathbf{q}$ are

$$\mathbf{r} = \mathbf{p} \times \mathbf{q} = r_i \hat{u}_i \tag{1.155}$$

$$\begin{aligned}
r_i &= \epsilon_{ijk} p_j q_k = \epsilon_{ijk} \epsilon_{jmn} \epsilon_{krs} a_m b_n c_r d_s \\
&= \epsilon_{ijk} \epsilon_{rsk} \epsilon_{jmn} a_m b_n c_r d_s \\
&= \left(\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}\right) \epsilon_{jmn} a_m b_n c_r d_s \\
&= \epsilon_{jmn} \left((c_r \delta_{ir}) \left(d_s \delta_{js}\right) a_m b_n - \left(c_r \delta_{jr}\right) (d_s \delta_{is})\, a_m b_n \right) \\
&= \epsilon_{jmn} \left(a_m b_n c_i d_j - a_m b_n c_j d_i \right) \\
&= c_i \left(\epsilon_{jmn} d_j a_m b_n \right) - d_i \left(\epsilon_{jmn} c_j a_m b_n \right)
\end{aligned} \tag{1.156}$$

so we have

$$\mathbf{r} = \mathbf{c}\,(\mathbf{d} \cdot \mathbf{a} \times \mathbf{b}) - \mathbf{d}\,(\mathbf{c} \cdot \mathbf{a} \times \mathbf{b}) \tag{1.157}$$

Example 35 ★ bac–cab Rule and ϵ–Delta Identity Employing the ϵ–delta identity (1.127), we can prove the *bac–cab* rule (1.103):

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \epsilon_{ijk} a_i b_k c_m \epsilon_{njm} \hat{u}_n = \epsilon_{ijk} \epsilon_{jmn} a_i b_k c_m \hat{u}_n \\
&= (\delta_{im} \delta_{kn} - \delta_{in} \delta_{km})\, a_i b_k c_m \hat{u}_n \\
&= a_m b_n c_m \hat{u}_n - a_n b_m c_m \hat{u}_n \\
&= a_m c_m \mathbf{b} - b_m c_m \mathbf{c} = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})
\end{aligned} \tag{1.158}$$

Example 36 ★ Series Solution for Three-Body Problem Consider three point masses m_1, m_2, and m_3 each subjected to Newtonian gravitational attraction from the other two particles. Let us indicate them by position vectors $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ with respect to their mass center C. If their position and velocity vectors are given at a time t_0, how will the particles move? This is called the *three-body problem*.

This is one of the most celebrated unsolved problems in dynamics. The three-body problem is interesting and challenging because it is the smallest n-body problem that cannot be solved mathematically. Here we present a series solution and employ index notation to provide concise equations. We present the expanded form of the equations in Example 177.

The equations of motion of m_1, m_2, and m_3 are

$$\ddot{\mathbf{X}}_i = -G\sum_{j=1}^{3} m_j \frac{\mathbf{X}_i - \mathbf{X}_j}{\left|\mathbf{X}_{ji}\right|^3} \qquad i = 1, 2, 3 \tag{1.159}$$

$$\mathbf{X}_{ij} = \mathbf{X}_j - \mathbf{X}_i \tag{1.160}$$

Using the mass center as the origin implies

$$\sum_{i=1}^{3} G_i \mathbf{X}_i = 0 \qquad G_i = Gm_i \qquad i = 1, 2, 3 \tag{1.161}$$

$$G = 6.67259 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}} \tag{1.162}$$

Following Belgium-American mathematician Roger Broucke (1932–2005), we use the relative position vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ to derive the most symmetric form of the three-body equations of motion:

$$\mathbf{x}_i = \epsilon_{ijk}\left(\mathbf{X}_k - \mathbf{X}_j\right) \qquad i = 1, 2, 3 \tag{1.163}$$

Using $\mathbf{x}_i$, the kinematic constraint (1.161) reduces to

$$\sum_{i=1}^{3} \mathbf{x}_i = 0 \tag{1.164}$$

The absolute position vectors in terms of the relative positions are

$$m\mathbf{X}_i = \epsilon_{ijk}\left(m_k \mathbf{x}_{jj} - m_j \mathbf{x}_k\right) \qquad i = 1, 2, 3 \tag{1.165}$$

$$m = m_1 + m_2 + m_3 \tag{1.166}$$

Substituting Equation (1.165) in (1.161), we have

$$\ddot{\mathbf{x}}_i = -Gm\frac{\mathbf{x}_i}{\left|\mathbf{x}_i\right|^3} + G_i \sum_{j=1}^{3} \frac{\mathbf{x}_j}{\left|\mathbf{x}_j\right|^3} \qquad i = 1, 2, 3 \tag{1.167}$$

We are looking for a series solution of Equations (1.167) in the following form:

$$\mathbf{x}_i\left(t\right) = \mathbf{x}_{i_0} + \dot{\mathbf{x}}_{i_0}\left(t - t_0\right) + \ddot{\mathbf{x}}_{i_0}\frac{\left(t - t_0\right)^2}{2!} + \dddot{\mathbf{x}}_{i_0}\frac{\left(t - t_0\right)^3}{3!} + \cdots \tag{1.168}$$

$$\mathbf{x}_{i_0} = \mathbf{x}_i\left(t_0\right) \qquad \dot{\mathbf{x}}_{i_0} = \dot{\mathbf{x}}_i\left(t_0\right) \qquad i = 1, 2, 3 \tag{1.169}$$

Let us define $\mu = Gm$ along with an ε-set of parameters

$$\mu = Gm \qquad \varepsilon_i = \frac{1}{\left|\mathbf{x}_i\right|^3} \qquad i = 1, 2, 3 \tag{1.170}$$

to rewrite Equations (1.167) as

$$\ddot{\mathbf{x}}_i = -\mu\varepsilon_i\mathbf{x}_i + G_i\sum_{j=1}^{3}\varepsilon_j\mathbf{x}_j \qquad i = 1, 2, 3 \tag{1.171}$$

We also define three new sets of parameters

$$a_{ijk} = \frac{\mathbf{x}_i\cdot\mathbf{x}_j}{|\mathbf{x}_k|^2} \qquad b_{ijk} = \frac{\dot{\mathbf{x}}_i\cdot\mathbf{x}_j}{|\mathbf{x}_k|^2} \qquad c_{ijk} = \frac{\dot{\mathbf{x}}_i\cdot\dot{\mathbf{x}}_j}{|\mathbf{x}_k|^2} \tag{1.172}$$

where

$$a_{iii} = 1 \qquad a_{ijk} = a_{jik} \qquad c_{ijk} = c_{jik} \tag{1.173}$$

The time derivatives of the ε-set, a-set, b-set, and c-set are

$$\dot{\varepsilon}_i = -3b_{iii}\varepsilon_i \tag{1.174}$$

$$\dot{a}_{ijk} = -2b_{kkk}a_{ijk} + b_{ijk} + b_{jik} \qquad \dot{a}_{iii} = 0 \tag{1.175}$$

$$\dot{b}_{ijk} = -2b_{kkk}b_{ijk} + c_{ijk} - \mu\varepsilon_i a_{ijk} + G_i\sum_{r=1}^{3}\varepsilon_r a_{rjk} \tag{1.176}$$

$$\begin{aligned}\dot{c}_{ijk} &= -2b_{kkk}c_{ijk} - \mu\left(\varepsilon_i b_{jik} + \varepsilon_j b_{ijk}\right)\\ &\quad + G_i\sum_{r=1}^{3}\varepsilon_r b_{jrk} + G_i\sum_{s=1}^{3}\varepsilon_s a_{isk}\end{aligned} \tag{1.177}$$

The ε-set, a-set, b-set, and c-set make 84 fundamental parameters that are independent of coordinate systems. Their time derivatives are expressed only by themselves. Therefore, we are able to find the coefficients of series (1.168) to develop the series solution of the three-body problem.

1.3 ORTHOGONAL COORDINATE FRAMES

Orthogonal coordinate frames are the most important type of coordinates. It is compatible to our everyday life and our sense of dimensions. There is an orthogonality condition that is the principal equation to express any vector in an orthogonal coordinate frame.

1.3.1 Orthogonality Condition

Consider a coordinate system $(Ouvw)$ with unit vectors $\hat{u}_u$, $\hat{u}_v$, $\hat{u}_w$. The condition for the coordinate system $(Ouvw)$ to be orthogonal is that $\hat{u}_u$, $\hat{u}_v$, $\hat{u}_w$ are mutually perpendicular and hence

$$\begin{aligned}\hat{u}_u\cdot\hat{u}_v &= 0\\ \hat{u}_v\cdot\hat{u}_w &= 0\\ \hat{u}_w\cdot\hat{u}_u &= 0\end{aligned} \tag{1.178}$$

In an orthogonal coordinate system, every vector $\mathbf{r}$ can be shown in its decomposed description as

$$\mathbf{r} = (\mathbf{r} \cdot \hat{u}_u)\hat{u}_u + (\mathbf{r} \cdot \hat{u}_v)\hat{u}_v + (\mathbf{r} \cdot \hat{u}_w)\hat{u}_w \tag{1.179}$$

We call Equation (1.179) the *orthogonality condition* of the coordinate system $(Ouvw)$. The orthogonality condition for a Cartesian coordinate system reduces to

$$\mathbf{r} = (\mathbf{r} \cdot \hat{\imath})\hat{\imath} + (\mathbf{r} \cdot \hat{\jmath})\hat{\jmath} + (\mathbf{r} \cdot \hat{k})\hat{k} \tag{1.180}$$

Proof: Assume that the coordinate system $(Ouvw)$ is an orthogonal frame. Using the unit vectors $\hat{u}_u$, $\hat{u}_v$, $\hat{u}_w$ and the components u, v, and w, we can show any vector $\mathbf{r}$ in the coordinate system $(Ouvw)$ as

$$\mathbf{r} = u\,\hat{u}_u + v\,\hat{u}_v + w\,\hat{u}_w \tag{1.181}$$

Because of orthogonality, we have

$$\hat{u}_u \cdot \hat{u}_v = 0 \qquad \hat{u}_v \cdot \hat{u}_w = 0 \qquad \hat{u}_w \cdot \hat{u}_u = 0 \tag{1.182}$$

Therefore, the inner product of $\mathbf{r}$ by $\hat{u}_u$, $\hat{u}_v$, $\hat{u}_w$ would be equal to

$$\begin{aligned}
\mathbf{r} \cdot \hat{u}_u &= \left(u\,\hat{u}_u + v\,\hat{u}_v + w\,\hat{u}_w\right) \cdot \left(1\hat{u}_u + 0\hat{u}_v + 0\hat{u}_w\right) = u \\
\mathbf{r} \cdot \hat{u}_v &= \left(u\,\hat{u}_u + v\,\hat{u}_v + w\,\hat{u}_w\right) \cdot \left(0\hat{u}_u + 1\hat{u}_v + 0\hat{u}_w\right) = v \\
\mathbf{r} \cdot \hat{u}_v &= \left(u\,\hat{u}_u + v\,\hat{u}_v + w\,\hat{u}_w\right) \cdot \left(0\hat{u}_u + 0\hat{u}_v + 1\hat{u}_w\right) = w
\end{aligned} \tag{1.183}$$

Substituting for the components u, v, and w in Equation (1.181), we may show the vector $\mathbf{r}$ as

$$\mathbf{r} = (\mathbf{r} \cdot \hat{u}_u)\hat{u}_u + (\mathbf{r} \cdot \hat{u}_v)\hat{u}_v + (\mathbf{r} \cdot \hat{u}_w)\hat{u}_w \tag{1.184}$$

If vector $\mathbf{r}$ is expressed in a Cartesian coordinate system, then $\hat{u}_u = \hat{\imath}$, $\hat{u}_v = \hat{\jmath}$, $\hat{u}_w = \hat{k}$, and therefore,

$$\mathbf{r} = (\mathbf{r} \cdot \hat{\imath})\hat{\imath} + (\mathbf{r} \cdot \hat{\jmath})\hat{\jmath} + (\mathbf{r} \cdot \hat{k})\hat{k} \tag{1.185}$$

The orthogonality condition is the most important reason for defining a coordinate system $(Ouvw)$ orthogonal. ■

Example 37 ★ Decomposition of a Vector in a Nonorthogonal Frame Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be any three non-coplanar, nonvanishing vectors; then any other vector $\mathbf{r}$ can be expressed in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

$$\mathbf{r} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c} \tag{1.186}$$

provided u, v, and w are properly chosen numbers. If the coordinate system $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a Cartesian system $(\hat{I}, \hat{J}, \hat{K})$, then

$$\mathbf{r} = (\mathbf{r} \cdot \hat{I})\hat{I} + (\mathbf{r} \cdot \hat{J})\hat{J} + (\mathbf{r} \cdot \hat{K})\hat{K} \tag{1.187}$$

To find u, v, and w, we dot multiply Equation (1.186) by $\mathbf{b} \times \mathbf{c}$:

$$\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) = u\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + v\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) + w\mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) \tag{1.188}$$

Knowing that $\mathbf{b} \times \mathbf{c}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$, we find

$$\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) = u\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \tag{1.189}$$

and therefore,

$$u = \frac{[\mathbf{rbc}]}{[\mathbf{abc}]} \tag{1.190}$$

where $[\mathbf{abc}]$ is a shorthand notation for the scalar triple product

$$[\mathbf{abc}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{1.191}$$

Similarly, v and w would be

$$v = \frac{[\mathbf{rca}]}{[\mathbf{abc}]} \qquad w = \frac{[\mathbf{rab}]}{[\mathbf{abc}]} \tag{1.192}$$

Hence,

$$\mathbf{r} = \frac{[\mathbf{rbc}]}{[\mathbf{abc}]}\mathbf{a} + \frac{[\mathbf{rca}]}{[\mathbf{abc}]}\mathbf{b} + \frac{[\mathbf{rab}]}{[\mathbf{abc}]}\mathbf{c} \tag{1.193}$$

which can also be written as

$$\mathbf{r} = \left(\mathbf{r} \cdot \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]}\right)\mathbf{a} + \left(\mathbf{r} \cdot \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]}\right)\mathbf{b} + \left(\mathbf{r} \cdot \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{abc}]}\right)\mathbf{c} \tag{1.194}$$

Multiplying (1.194) by $[\mathbf{abc}]$ gives the symmetric equation

$$[\mathbf{abc}]\,\mathbf{r} - [\mathbf{bcr}]\,\mathbf{a} + [\mathbf{cra}]\,\mathbf{b} - [\mathbf{rab}]\,\mathbf{c} = 0 \tag{1.195}$$

If the coordinate system $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a Cartesian system $(\hat{I}, \hat{J}, \hat{K})$, then

$$\left[\hat{I}\hat{J}\hat{K}\right] = 1 \tag{1.196}$$

$$\hat{I} \times \hat{J} = \hat{K} \qquad \hat{J} \times \hat{I} = \hat{K} \qquad \hat{K} \times \hat{I} = \hat{J} \tag{1.197}$$

and Equation (1.194) becomes

$$\mathbf{r} = \left(\mathbf{r} \cdot \hat{I}\right)\hat{I} + \left(\mathbf{r} \cdot \hat{J}\right)\hat{J} + \left(\mathbf{r} \cdot \hat{K}\right)\hat{K} \tag{1.198}$$

This example may considered as a general case of Example 30.

1.3.2 Unit Vector

Consider an orthogonal coordinate system $(Oq_1q_2q_3)$. Using the orthogonality condition (1.179), we can show the position vector of a point P in this frame by

$$\mathbf{r} = (\mathbf{r} \cdot \hat{u}_1)\hat{u}_1 + (\mathbf{r} \cdot \hat{u}_2)\hat{u}_2 + (\mathbf{r} \cdot \hat{u}_3)\hat{u}_3 \tag{1.199}$$

where q_1, q_2, q_3 are the coordinates of P and $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ are the unit vectors along q_1, q_2, q_3 axes, respectively. Because the unit vectors $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ are orthogonal and independent, they respectively show the direction of change in $\mathbf{r}$ when q_1, q_2, q_3 are positively varied. Therefore, we may define the unit vectors $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ by

$$\hat{u}_1 = \frac{\partial \mathbf{r}/\partial q_1}{|\partial \mathbf{r}/\partial q_1|} \qquad \hat{u}_2 = \frac{\partial \mathbf{r}/\partial q_2}{|\partial \mathbf{r}/\partial q_2|} \qquad \hat{u}_3 = \frac{\partial \mathbf{r}/\partial q_3}{|\partial \mathbf{r}/\partial q_3|} \tag{1.200}$$

Example 38 Unit Vector of Cartesian Coordinate Frames If a vector $\mathbf{r}$ given as

$$\mathbf{r} = q_1\hat{u}_1 + q_2\hat{u}_2 + q_3\hat{u}_3 \tag{1.201}$$

is expressed in a Cartesian coordinate frame, then

$$q_1 = x \qquad q_2 = y \qquad q_3 = z \tag{1.202}$$

and the unit vectors would be

$$\begin{aligned} \hat{u}_1 = \hat{u}_x &= \frac{\partial \mathbf{r}/\partial x}{|\partial \mathbf{r}/\partial x|} = \frac{\hat{\imath}}{1} = \hat{\imath} \\ \hat{u}_2 = \hat{u}_y &= \frac{\partial \mathbf{r}/\partial y}{|\partial \mathbf{r}/\partial y|} = \frac{\hat{\jmath}}{1} = \hat{\jmath} \\ \hat{u}_3 = \hat{u}_z &= \frac{\partial \mathbf{r}/\partial z}{|\partial \mathbf{r}/\partial z|} = \frac{\hat{k}}{1} = \hat{k} \end{aligned} \tag{1.203}$$

Substituting $\mathbf{r}$ and the unit vectors in (1.199) regenerates the orthogonality condition in Cartesian frames:

$$\mathbf{r} = (\mathbf{r} \cdot \hat{\imath})\hat{\imath} + (\mathbf{r} \cdot \hat{\jmath})\hat{\jmath} + (\mathbf{r} \cdot \hat{k})\hat{k} \tag{1.204}$$

Example 39 Unit Vectors of a Spherical Coordinate System Figure 1.12 illustrates an option for spherical coordinate system. The angle φ may be measured from the equatorial plane or from the Z-axis. Measuring φ from the equator is used in geography and positioning a point on Earth, while measuring φ from the Z-axis is an applied method in geometry. Using the latter option, the spherical coordinates r, θ, φ are related to the Cartesian system by

$$x = r\cos\theta\sin\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\varphi \tag{1.205}$$

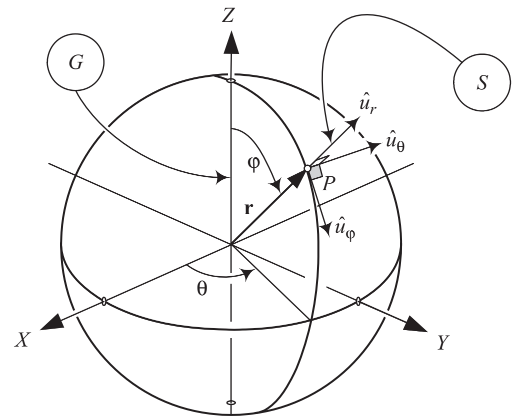

Figure 1.12 An optional spherical coordinate system.

To find the unit vectors $\hat{u}_r$, $\hat{u}_\theta$, $\hat{u}_\varphi$ associated with the coordinates r, θ, φ, we substitute the coordinate equations (1.205) in the Cartesian position vector,

$$\begin{aligned}\mathbf{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ &= (r\cos\theta\sin\varphi)\,\hat{\imath} + (r\sin\theta\sin\varphi)\,\hat{\jmath} + (r\cos\varphi)\,\hat{k} \qquad (1.206)\end{aligned}$$

and apply the unit vector equation (1.203):

$$\begin{aligned}\hat{u}_r &= \frac{\partial\mathbf{r}/\partial r}{|\partial\mathbf{r}/\partial r|} = \frac{(\cos\theta\sin\varphi)\,\hat{\imath} + (\sin\theta\sin\varphi)\,\hat{\jmath} + (\cos\varphi)\,\hat{k}}{1} \\ &= \cos\theta\sin\varphi\hat{\imath} + \sin\theta\sin\varphi\hat{\jmath} + \cos\varphi\hat{k} \qquad (1.207) \\ \hat{u}_\theta &= \frac{\partial\mathbf{r}/\partial\theta}{|\partial\mathbf{r}/\partial\theta|} = \frac{(-r\sin\theta\sin\varphi)\,\hat{\imath} + (r\cos\theta\sin\varphi)\,\hat{\jmath}}{r\sin\varphi} \\ &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} \qquad (1.208) \\ \hat{u}_\varphi &= \frac{\partial\mathbf{r}/\partial\varphi}{|\partial\mathbf{r}/\partial\varphi|} = \frac{(r\cos\theta\cos\varphi)\,\hat{\imath} + (r\sin\theta\cos\varphi)\,\hat{\jmath} + (-r\sin\varphi)\,\hat{k}}{r} \\ &= \cos\theta\cos\varphi\hat{\imath} + \sin\theta\cos\varphi\hat{\jmath} - \sin\varphi\hat{k} \qquad (1.209)\end{aligned}$$

where $\hat{u}_r$, $\hat{u}_\theta$, $\hat{u}_\varphi$ are the unit vectors of the spherical system expressed in the Cartesian coordinate system.

Example 40 Cartesian Unit Vectors in Spherical System The unit vectors of an orthogonal coordinate system are always a linear combination of Cartesian unit vectors and therefore can be expressed by a matrix transformation. Having unit vectors of an orthogonal coordinate system B_1 in another orthogonal system B_2 is enough to find the unit vectors of B_2 in B_1.

Based on Example 39, the unit vectors of the spherical system shown in Figure 1.12 can be expressed as

$$\begin{bmatrix} \hat{u}_r \\ \hat{u}_\theta \\ \hat{u}_\varphi \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -\sin\theta & \cos\theta & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} \tag{1.210}$$

So, the Cartesian unit vectors in the spherical system are

$$\begin{aligned} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} &= \begin{bmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -\sin\theta & \cos\theta & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{bmatrix}^{-1} \begin{bmatrix} \hat{u}_r \\ \hat{u}_\theta \\ \hat{u}_\varphi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\sin\varphi & -\sin\theta & \cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \cos\theta & \cos\varphi\sin\theta \\ \cos\varphi & 0 & -\sin\varphi \end{bmatrix} \begin{bmatrix} \hat{u}_r \\ \hat{u}_\theta \\ \hat{u}_\varphi \end{bmatrix} \end{aligned} \tag{1.211}$$

1.3.3 Direction of Unit Vectors

Consider a moving point P with the position vector $\mathbf{r}$ in a coordinate system $(Oq_1q_2q_3)$. The unit vectors $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ associated with q_1, q_2, q_3 are tangent to the curve traced by $\mathbf{r}$ when the associated coordinate varies.

Proof: Consider a coordinate system $\left(Oq_1q_2q_3\right)$ that has the following relations with Cartesian coordinates:

$$\begin{aligned} x &= f\left(q_1, q_2, q_3\right) \\ y &= g\left(q_1, q_2, q_3\right) \\ z &= h\left(q_1, q_2, q_3\right) \end{aligned} \tag{1.212}$$

The unit vector $\hat{u}_1$ given as

$$\hat{u}_1 = \frac{\partial\mathbf{r}/\partial q_1}{|\partial\mathbf{r}/\partial q_1|} \tag{1.213}$$

associated with q_1 at a point $P\,(x_0, y_0, z_0)$ can be found by fixing q_2, q_3 to q_{2_0}, q_{3_0} and varying q_1. At the point, the equations

$$\begin{aligned} x &= f\left(q_1, q_{2_0}, q_{3_0}\right) \\ y &= g\left(q_1, q_{2_0}, q_{3_0}\right) \\ z &= h\left(q_1, q_{2_0}, q_{3_0}\right) \end{aligned} \tag{1.214}$$

provide the parametric equations of a space curve passing through (x_0, y_0, z_0). From (1.228) and (1.358), the tangent line to the curve at point P is

$$\frac{x - x_0}{dx/dq_1} = \frac{y - y_0}{dy/dq_1} = \frac{z - z_0}{dz/dq_1} \tag{1.215}$$

and the unit vector on the tangent line is

$$\hat{u}_1 = \frac{dx}{dq_1}\hat{\imath} + \frac{dy}{dq_1}\hat{\jmath} + \frac{dz}{dq_1}\hat{k} \tag{1.216}$$

$$\left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2 + \left(\frac{dz}{dq}\right)^2 = 1 \tag{1.217}$$

This shows that the unit vector $\hat{u}_1$ (1.213) associated with q_1 is tangent to the space curve generated by varying q_1. When q_1 is varied positively, the direction of $\hat{u}_1$ is called positive and vice versa.

Similarly, the unit vectors $\hat{u}_2$ and $\hat{u}_3$ given as

$$\hat{u}_2 = \frac{\partial \mathbf{r}/\partial q_2}{|\partial \mathbf{r}/\partial q_2|} \qquad \hat{u}_3 = \frac{\partial \mathbf{r}/\partial q_3}{|\partial \mathbf{r}/\partial q_3|} \tag{1.218}$$

associated with q_2 and q_3 are tangent to the space curve generated by varying q_2 and q_3, respectively. ■

Example 41 Tangent Unit Vector to a Helix Consider a helix

$$x = a\cos\varphi \qquad y = a\sin\varphi \qquad z = k\varphi \tag{1.219}$$

where a and k are constant and φ is an angular variable. The position vector of a moving point P on the helix

$$\mathbf{r} = a\cos\varphi\,\hat{\imath} + a\sin\varphi\,\hat{\jmath} + k\varphi\,\hat{k} \tag{1.220}$$

may be used to find the unit vector $\hat{u}_\varphi$:

$$\begin{aligned} \hat{u}_\varphi &= \frac{\partial \mathbf{r}/\partial q_1}{|\partial \mathbf{r}/\partial q_1|} = \frac{-a\sin\varphi\,\hat{\imath} + a\cos\varphi\,\hat{\jmath} + k\,\hat{k}}{\sqrt{(-a\sin\varphi)^2 + (a\cos\varphi)^2 + (k)^2}} \\ &= -\frac{a\sin\varphi}{\sqrt{a^2+k^2}}\,\hat{\imath} + \frac{a\cos\varphi}{\sqrt{a^2+k^2}}\,\hat{\jmath} + \frac{k}{\sqrt{a^2+k^2}}\,\hat{k} \end{aligned} \tag{1.221}$$

The unit vector $\hat{u}_\varphi$ at $\varphi = \pi/4$ given as

$$\hat{u}_\varphi = -\frac{\sqrt{2}a}{2\sqrt{a^2+k^2}}\,\hat{\imath} + \frac{\sqrt{2}a}{2\sqrt{a^2+k^2}}\,\hat{\jmath} + \frac{k}{\sqrt{a^2+k^2}}\,\hat{k} \tag{1.222}$$

is on the tangent line (1.255).

1.4 DIFFERENTIAL GEOMETRY

Geometry is the world in which we express kinematics. The path of the motion of a particle is a curve in space. The analytic equation of the space curve is used to determine the vectorial expression of kinematics of the moving point.

1.4.1 Space Curve

If the position vector ${}^G\mathbf{r}_P$ of a moving point P is such that each component is a function of a variable q,

$$ {}^G\mathbf{r} = {}^G\mathbf{r}(q) = x(q)\,\hat{\imath} + y(q)\,\hat{\jmath} + z(q)\,\hat{k} \tag{1.223}$$

then the end point of the position vector indicates a curve C in G, as is shown in Figure 1.13. The curve ${}^G\mathbf{r} = {}^G\mathbf{r}(q)$ reduces to a point on C if we fix the parameter q. The functions

$$ x = x(q) \qquad y = y(q) \qquad z = z(q) \tag{1.224}$$

are the parametric equations of the curve.

When the parameter q is the arc length s, the infinitesimal arc distance ds on the curve is

$$ ds^2 = d\mathbf{r} \cdot d\mathbf{r} \tag{1.225}$$

The arc length of a curve is defined as the limit of the diagonal of a rectangular box as the length of the sides uniformly approach zero.

When the space curve is a straight line that passes through point $P(x_0, y_0, z_0)$ where $x_0 = x(q_0)$, $y_0 = y(q_0)$, $z_0 = z(q_0)$, its equation can be shown by

$$ \frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta} = \frac{z - z_0}{\gamma} \tag{1.226}$$

$$ \alpha^2 + \beta^2 + \gamma^2 = 1 \tag{1.227}$$

where α, β, and γ are the directional cosines of the line.

The equation of the tangent line to the space curve (1.224) at a point $P(x_0, y_0, z_0)$ is

$$ \frac{x - x_0}{dx/dq} = \frac{y - y_0}{dy/dq} = \frac{z - z_0}{dz/dq} \tag{1.228}$$

$$ \left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2 + \left(\frac{dz}{dq}\right)^2 = 1 \tag{1.229}$$

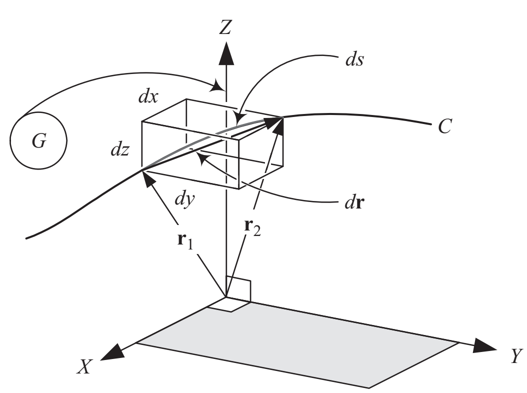

Figure 1.13 A space curve and increment arc length *ds*

Proof: Consider a position vector ${}^G\mathbf{r} = {}^G\mathbf{r}(s)$ that describes a space curve using the length parameter s:

$$ {}^G\mathbf{r} = {}^G\mathbf{r}(s) = x(s)\,\hat{\imath} + y(s)\,\hat{\jmath} + z(s)\,\hat{k} \tag{1.230} $$

The arc length s is measured from a fixed point on the curve. By a very small change ds, the position vector will move to a very close point such that the increment in the position vector would be

$$ d\mathbf{r} = dx(s)\,\hat{\imath} + dy(s)\,\hat{\jmath} + dz(s)\,\hat{k} \tag{1.231} $$

The length of $d\mathbf{r}$ and ds are equal for infinitesimal displacement:

$$ ds = \sqrt{dx^2 + dy^2 + dz^2} \tag{1.232} $$

The arc length has a better expression in the square form:

$$ ds^2 = dx^2 + dy^2 + dz^2 = d\mathbf{r} \cdot d\mathbf{r} \tag{1.233} $$

If the parameter of the space curve is q instead of s, the increment arc length would be

$$ \left(\frac{ds}{dq}\right)^2 = \frac{d\mathbf{r}}{dq} \cdot \frac{d\mathbf{r}}{dq} \tag{1.234} $$

Therefore, the arc length between two points on the curve can be found by integration:

$$ s = \int_{q_1}^{q_2} \sqrt{\frac{d\mathbf{r}}{dq} \cdot \frac{d\mathbf{r}}{dq}}\,dq \tag{1.235} $$

$$ = \int_{q_1}^{q_2} \sqrt{\left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2 + \left(\frac{dz}{dq}\right)^2}\,dq \tag{1.236} $$

Let us expand the parametric equations of the curve (1.224) at a point $P(x_0, y_0, z_0)$,

$$ \begin{aligned} x &= x_0 + \frac{dx}{dq}\triangle q + \frac{1}{2}\frac{d^2x}{dq^2}\triangle q^2 + \cdots \\ y &= y_0 + \frac{dy}{dq}\triangle q + \frac{1}{2}\frac{d^2y}{dq^2}\triangle q^2 + \cdots \\ z &= z_0 + \frac{dz}{dq}\triangle q + \frac{1}{2}\frac{d^2z}{dq^2}\triangle q^2 + \cdots \end{aligned} \tag{1.237} $$

and ignore the nonlinear terms to find the tangent line to the curve at P:

$$ \frac{x - x_0}{dx/dq} = \frac{y - y_0}{dy/dq} = \frac{z - z_0}{dz/dq} = \triangle q \tag{1.238} $$

■

Example 42 Arc Length of a Planar Curve A planar curve in the (x, y)-plane

$$y = f(x) \tag{1.239}$$

can be expressed vectorially by

$$\mathbf{r} = x\hat{\imath} + y(x)\,\hat{\jmath} \tag{1.240}$$

The displacement element on the curve

$$\frac{d\mathbf{r}}{dx} = \hat{\imath} + \frac{dy}{dx}\hat{\jmath} \tag{1.241}$$

provides

$$\left(\frac{ds}{dx}\right)^2 = \frac{d\mathbf{r}}{dx}\cdot\frac{d\mathbf{r}}{dx} = 1 + \left(\frac{dy}{dx}\right)^2 \tag{1.242}$$

Therefore, the arc length of the curve between $x = x_1$ and $x = x_2$ is

$$s = \int_{x_1}^{x_2}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \tag{1.243}$$

In case the curve is given parametrically,

$$x = x(q) \qquad y = y(q) \tag{1.244}$$

we have

$$\left(\frac{ds}{dq}\right)^2 = \frac{d\mathbf{r}}{dq}\cdot\frac{d\mathbf{r}}{dq} = \left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2 \tag{1.245}$$

and hence,

$$s = \int_{q_1}^{q_2}\left|\frac{d\mathbf{r}}{dq}\right| = \int_{q_1}^{q2}\sqrt{\left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2}\,dq \tag{1.246}$$

As an example, we may show a circle with radius R by its polar expression using the angle θ as a parameter:

$$x = R\cos\theta \qquad y = R\sin\theta \tag{1.247}$$

The circle is made when the parameter θ varies by 2π. The arc length between $\theta = 0$ and $\theta = \pi/2$ would then be one-fourth the perimeter of the circle. The equation for calculating the perimeter of a circle with radius R is

$$\begin{aligned} s &= 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = R\int_0^{\pi/2}\sqrt{\sin^2\theta + \cos^2\theta}\,d\theta \\ &= 4R\int_0^{\pi/2} d\theta = 2\pi R \end{aligned} \tag{1.248}$$

Example 43 Alternative Space Curve Expressions We can represent a space curve by functions

$$y = y(x) \qquad z = z(x) \tag{1.249}$$

or vector

$$\mathbf{r}(q) = x\hat{\imath} + y(x)\,\hat{\jmath} + z(x)\,\hat{k} \tag{1.250}$$

We may also show a space curve by two relationships between x, y, and z,

$$f(x, y, z) = 0 \qquad g(x, y, z) = 0 \tag{1.251}$$

where $f(x, y, z) = 0$ and $g(x, y, z) = 0$ represent two surfaces. The space curve would then be indicated by intersecting the surfaces.

Example 44 Tangent Line to a Helix Consider a point P that is moving on a helix with equation

$$x = a\cos\varphi \qquad y = a\sin\varphi \qquad z = k\varphi \tag{1.252}$$

where a and k are constant and φ is an angular variable. To find the tangent line to the helix at $\varphi = \pi/4$,

$$x_0 = \frac{\sqrt{2}}{2}a \qquad y_0 = \frac{\sqrt{2}}{2}a \qquad z_0 = k\frac{\pi}{4} \tag{1.253}$$

we calculate the required derivatives:

$$\begin{aligned}
\frac{dx}{d\varphi} &= -a\sin\varphi = -\frac{\sqrt{2}}{2}a \\
\frac{dy}{d\varphi} &= a\cos\varphi = \frac{\sqrt{2}}{2}a \\
\frac{dz}{d\varphi} &= k
\end{aligned} \tag{1.254}$$

So, the equation of the tangent line is

$$-\frac{\sqrt{2}}{a}\left(x - \frac{1}{2}\sqrt{2}a\right) = \frac{\sqrt{2}}{a}\left(y - \frac{1}{2}\sqrt{2}a\right) = \frac{1}{k}\left(z - \frac{1}{4}\pi k\right) \tag{1.255}$$

Example 45 Parametric Form of a Line The equation of a line that connects two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_3)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \tag{1.256}$$

This line may also be expressed by the following parametric equations:

$$\begin{aligned} x &= x_1 + (x_2 - x_1)\, t \\ y &= y_1 + (y_2 - y_1)\, t \\ z &= z_1 + (z_2 - z_1)\, t \end{aligned} \tag{1.257}$$

Example 46 Length of a Roller Coaster Consider the roller coaster illustrated later in Figure 1.22 with the following parametric equations:

$$\begin{aligned} x &= (a + b \sin\theta) \cos\theta \\ y &= (a + b \sin\theta) \sin\theta \\ z &= b + b \cos\theta \end{aligned} \tag{1.258}$$

for

$$a = 200\,\text{m} \qquad b = 150\,\text{m} \tag{1.259}$$

The total length of the roller coaster can be found by the integral of ds for θ from 0 to 2π:

$$\begin{aligned} s &= \int_{\theta_1}^{\theta_2} \sqrt{\frac{d\mathbf{r}}{d\theta} \cdot \frac{d\mathbf{r}}{d\theta}} d\theta = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial x}{\partial \theta}\right)^2} d\theta \\ &= \int_0^{2\pi} \frac{\sqrt{2}}{2} \sqrt{2a^2 + 3b^2 - b^2 \cos 2\theta + 4ab \sin\theta} d\theta \\ &= 1629.367\,\text{m} \end{aligned} \tag{1.260}$$

Example 47 Two Points Indicate a Line Consider two points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ in a coordinate frame. The condition for a point P with position vector $\mathbf{r}$ to lie on the line AB is that r $-$ $\mathbf{a}$ and $\mathbf{b} - \mathbf{a}$ be parallel. So,

$$\mathbf{r} - \mathbf{a} = c\,(\mathbf{b} - \mathbf{a}) \tag{1.261}$$

where c is a parameter. The outer product of Equation (1.261) by $\mathbf{b} - \mathbf{a}$ provides

$$(\mathbf{r} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a}) = 0 \tag{1.262}$$

which is the equation of the line AB.

Example 48 Line through a Point and Parallel to a Given Line Consider a point A with position vector $\mathbf{a}$ and a line l that is indicated by a unit vector $\hat{u}_l$. To determine the equation of the parallel line to $\hat{u}_l$ that goes over A, we employ the condition that $\mathbf{r} - \mathbf{a}$ and $\hat{u}_l$ must be parallel:

$$\mathbf{r} = \mathbf{a} + c\hat{u}_l \tag{1.263}$$

We can eliminate the parameter c by the outer product of both sides with $\hat{u}_l$:

$$\mathbf{r} \times \hat{u}_l = \mathbf{a} \times \hat{u}_l \tag{1.264}$$

1.4.2 Surface and Plane

A plane is the locus of the tip point of a position vector

$$\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1.265}$$

such that the coordinates satisfy a linear equation

$$Ax + By + Cz + D = 0 \tag{1.266}$$

A space surface is the locus of the tip point of the position vector (1.265) such that its coordinates satisfy a nonlinear equation:

$$f(x, y, z) = 0 \tag{1.267}$$

Proof: The points P_1, P_2, and P_3 at $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$,

$$\mathbf{r}_1 = \begin{bmatrix} -\frac{D}{A} \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} 0 \\ -\frac{D}{B} \\ 0 \end{bmatrix} \qquad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ -\frac{D}{C} \end{bmatrix} \tag{1.268}$$

satisfy the equations of the plane (1.266). The position of P_2 and P_3 with respect to P_1 are shown by ${}_1\mathbf{r}_2$ and ${}_1\mathbf{r}_3$ or $\mathbf{r}_{2/1}$ and $\mathbf{r}_{3/1}$:

$${}_1\mathbf{r}_2 = \mathbf{r}_2 - \mathbf{r}_1 = \begin{bmatrix} \frac{D}{A} \\ -\frac{D}{B} \\ 0 \end{bmatrix} \qquad {}_1\mathbf{r}_3 = \mathbf{r}_3 - \mathbf{r}_1 = \begin{bmatrix} \frac{D}{A} \\ 0 \\ -\frac{D}{C} \end{bmatrix} \tag{1.269}$$

The cross product of ${}_1\mathbf{r}_2$ and ${}_1\mathbf{r}_3$ is a normal vector to the plane:

$${}_1\mathbf{r}_2 \times {}_1\mathbf{r}_3 = \begin{bmatrix} \frac{D}{A} \\ -\frac{D}{B} \\ 0 \end{bmatrix} \times \begin{bmatrix} \frac{D}{A} \\ 0 \\ -\frac{D}{C} \end{bmatrix} = \begin{bmatrix} \frac{D^2}{BC} \\ \frac{D^2}{AC} \\ \frac{D^2}{AB} \end{bmatrix} \tag{1.270}$$

The equation of the plane is the locus of any point P,

$$\mathbf{r}_P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1.271}$$

where its position with respect to P_1,

$$ {}_1\mathbf{r}_P = \mathbf{r}_P - \mathbf{r}_1 = \begin{bmatrix} x + \dfrac{D}{A} \\ y \\ z \end{bmatrix} \tag{1.272} $$

is perpendicular to the normal vector:

$$ {}_1\mathbf{r}_P \cdot ({}_1\mathbf{r}_2 \times {}_1\mathbf{r}_3) = D + Ax + By + Cz = 0 \tag{1.273} $$

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Example 49 Plane through Three Points Every three points indicate a plane. Assume that (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are the coordinates of three points P_1, P_2, and P_3. The plane made by the points can be found by

$$ \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \tag{1.274} $$

The points P_1, P_2, and P_3 satisfy the equation of the plane

$$ \begin{aligned} Ax_1 + By_1 + Cz_1 + D &= 0 \\ Ax_2 + By_2 + Cz_2 + D &= 0 \\ Ax_3 + By_3 + Cz_3 + D &= 0 \end{aligned} \tag{1.275} $$

and if P with coordinates (x, y, z) is a general point on the surface,

$$ Ax + By + Cz + D = 0 \tag{1.276} $$

then there are four equations to determine A, B, C, and D:

$$ \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.277} $$

The determinant of the equations must be zero, which determines the equation of the plane.

Example 50 Normal Vector to a Plane A plane may be expressed by the linear equation

$$ Ax + By + Cz + D = 0 \tag{1.278} $$

or by its intercept form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{1.279}$$

$$a = -\frac{D}{A} \qquad b = -\frac{D}{B} \qquad c = -\frac{D}{C} \tag{1.280}$$

In either case, the vector

$$\mathbf{n}_1 = A\hat{\imath} + B\hat{\jmath} + C\hat{k} \tag{1.281}$$

or

$$\mathbf{n}_2 = a\hat{\imath} + b\hat{\jmath} + c\hat{k} \tag{1.282}$$

is normal to the plane and may be used to represent the plane.

Example 51 Quadratic Surfaces A quadratic relation between x, y, z is called the quadratic form and is an equation containing only terms of degree 0, 1, and 2 in the variables x, y, z. Quadratic surfaces have special names:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \textit{Ellipsoid} \tag{1.283}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \textit{Hyperboloid of one sheet} \tag{1.284}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \textit{Hyperboloid of two sheets} \tag{1.285}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1 \qquad \textit{Imaginary ellipsoid} \tag{1.286}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2nz \qquad \textit{Elliptic paraboloid} \tag{1.287}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2nz \qquad \textit{Hyperbolic paraboloid} \tag{1.288}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad \textit{Real quadratic cone} \tag{1.289}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0 \qquad \textit{Real imaginary cone} \tag{1.290}$$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \pm 1 \quad y^2 = 2px \qquad \textit{Quadratic cylinders} \tag{1.291}$$

1.5 MOTION PATH KINEMATICS

The derivative of vector functions is based on the derivative of scalar functions. To find the derivative of a vector, we take the derivative of its components in a decomposed Cartesian expression.

1.5.1 Vector Function and Derivative

The derivative of a vector is possible only when the vector is expressed in a Cartesian coordinate frame. Its derivative can be found by taking the derivative of its components. The Cartesian unit vectors are invariant and have zero derivative with respect to any parameter.

A vector $\mathbf{r} = \mathbf{r}(t)$ is called a *vector function* of the *scalar variable* t if there is a definite vector for every value of t from a certain set $T = [\tau_1, \tau_2]$. In a Cartesian coordinate frame G, the specification of the vector function $\mathbf{r}(t)$ is equivalent to the specification of three scalar functions $x(t)$, $y(t)$, $z(t)$:

$$ {}^{G}\mathbf{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k} \tag{1.292}$$

If the vector $\mathbf{r}$ is expressed in Cartesian decomposition form, then the derivative $d\mathbf{r}/dt$ is

$$ \frac{{}^{G}d}{dt}\,{}^{G}\mathbf{r} = \frac{dx(t)}{dt}\hat{\imath} + \frac{dy(t)}{dt}\hat{\jmath} + \frac{dz(t)}{dt}\hat{k} \tag{1.293}$$

and if $\mathbf{r}$ is expressed in its natural form

$$ {}^{G}\mathbf{r} = r\hat{u}_r = r(t)\left[u_1(t)\,\hat{\imath} + u_2(t)\,\hat{\jmath} + u_3(t)\,\hat{k}\right] \tag{1.294}$$

then, using the chain rule, the derivative $d\mathbf{r}/dt$ is

$$ \begin{aligned} \frac{{}^{G}d}{dt}\,{}^{G}\mathbf{r} &= \frac{dr}{dt}\hat{u}_r + r\frac{d}{dt}\hat{u}_r \\ &= \frac{dr}{dt}\left(u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}\right) + r\left(\frac{du_1}{dt}\hat{\imath} + \frac{du_2}{dt}\hat{\jmath} + \frac{du_3}{dt}\hat{k}\right) \\ &= \left(\frac{dr}{dt}u_1 + r\frac{du_1}{dt}\right)\hat{\imath} + \left(\frac{dr}{dt}u_2 + r\frac{du_2}{dt}\right)\hat{\jmath} + \left(\frac{dr}{dt}u_3 + r\frac{du_3}{dt}\right)\hat{k} \end{aligned} \tag{1.295}$$

When the independent variable t is time, an overdot $\dot{\mathbf{r}}(t)$ is used as a shorthand notation to indicate the time derivative.

Consider a moving point P with a continuously varying position vector $\mathbf{r} = \mathbf{r}(t)$. When the starting point of $\mathbf{r}$ is fixed at the origin of G, its end point traces a continuous curve C as is shown in Figure 1.14. The curve C is called a *configuration path* that describes the motion of P, and the vector function $\mathbf{r}(t)$ is its vector representation. At each point of the continuously smooth curve $C = \{\mathbf{r}(t), t \in [\tau_1, \tau_2]\}$ there exists a tangent line and a derivative vector $d\mathbf{r}(t)/dt$ that is directed along the tangent line and directed toward increasing the parameter t. If the parameter is the arc length s of

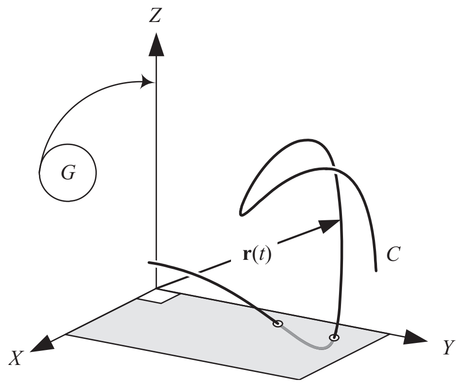

Figure 1.14 A space curve is the trace point of a single variable position vector.

the curve that is measured from a convenient point on the curve, the derivative of ${}^G\mathbf{r}$ with respect to s is the tangential unit vector $\hat{u}_t$ to the curve at ${}^G\mathbf{r}$:

$$\frac{{}^G d}{ds}\,{}^G\mathbf{r} = \hat{u}_t \tag{1.296}$$

Proof: The position vector ${}^G\mathbf{r}$ in its decomposed expression

$$ {}^G\mathbf{r}\,(t) = x\,(t)\,\hat{\imath} + y\,(t)\,\hat{\jmath} + z\,(t)\,\hat{k} \tag{1.297}$$

is a combination of three variable-length vectors $x\,(t)\hat{\imath}$, $y\,(t)\,\hat{\jmath}$, and $z\,(t)\,\hat{k}$. Consider the first one that is a multiple of a scalar function $x\,(t)$ and a constant unit vector $\hat{\imath}$. If the variable is time, then the time derivative of this variable-length vector in the same frame in which the vector is expressed is

$$\frac{{}^G d}{dt}\left(x\,(t)\,\hat{\imath}\right) = \dot{x}\,(t)\,\hat{\imath} + x\,(t)\,\frac{{}^G d}{dt}\hat{\imath} = \dot{x}\,(t)\,\hat{\imath} \tag{1.298}$$

Similarly, the time derivatives of $y\,(t)\,\hat{\jmath}$ and $z\,(t)\,\hat{k}$ are $\dot{y}\,(t)\,\hat{\jmath}$ and $\dot{z}\,(t)\,\hat{k}$, and therefore, the time derivative of the vector ${}^G\mathbf{r}\,(t)$ can be found by taking the derivative of its components

$$ {}^G\mathbf{v} = \frac{{}^G d}{dt}\,{}^G\mathbf{r}\,(t) = \dot{x}\,(t)\,\hat{\imath} + \dot{y}\,(t)\,\hat{\jmath} + \dot{z}\,(t)\,\hat{k} \tag{1.299}$$

If a variable vector ${}^G\mathbf{r}$ is expressed in a natural form

$$ {}^G\mathbf{r} = r\,(t)\,\hat{u}_r\,(t) \tag{1.300}$$

we express the unit vector $\hat{u}_r\,(t)$ in its decomposed form

$$\begin{aligned} {}^G\mathbf{r} &= r\,(t)\,\hat{u}_r\,(t) \\ &= r\,(t)\left[u_1\,(t)\,\hat{\imath} + u_2\,(t)\,\hat{\jmath} + u_3\,(t)\,\hat{k}\right] \end{aligned} \tag{1.301}$$

and take the derivative using the chain rule and variable-length vector derivative:

$$\begin{aligned} {}^{G}\mathbf{v} &= \frac{{}^{G}d}{dt}\,{}^{G}\mathbf{r} = \dot{r}\hat{u}_r + r\frac{{}^{G}d}{dt}\hat{u}_r \\ &= \dot{r}\left(u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}\right) + r\left(\dot{u}_1\hat{\imath} + \dot{u}_2\hat{\jmath} + \dot{u}_3\hat{k}\right) \\ &= \left(\dot{r}u_1 + r\dot{u}_1\right)\hat{\imath} + \left(\dot{r}u_2 + r\dot{u}_2\right)\hat{\jmath} + \left(\dot{r}u_3 + r\dot{u}_3\right)\hat{k} \end{aligned} \tag{1.302}$$

■

Example 52 Geometric Expression of Vector Derivative Figure 1.15 depicts a configuration path C that is the trace of a position vector $\mathbf{r}(t)$ when t varies. If $\Delta t > 0$, then the vector $\Delta\mathbf{r}$ is directed along the secant AB of the curve C toward increasing values of the parameter t. The derivative vector $d\mathbf{r}(t)/dt$ is the limit of $\Delta\mathbf{r}$ when $\Delta t \to 0$:

$$\frac{d}{ds}\mathbf{r}(t) = \lim_{\Delta t \to 0}\frac{\Delta\mathbf{r}}{\Delta t} \tag{1.303}$$

where $d\mathbf{r}(t)/dt$ is directed along the tangent line to C.

Let us show the unit vectors along $\Delta\mathbf{r}$ and $d\mathbf{r}(t)/dt$ by $\Delta\mathbf{r}/\Delta r$ and $\hat{u}_t$ to get

$$\hat{u}_t = \lim_{\Delta r \to 0}\frac{\Delta\mathbf{r}}{\Delta r} = \lim_{\Delta t \to 0}\frac{\Delta\mathbf{r}/\Delta t}{\Delta r/\Delta t} = \frac{d\mathbf{r}(t)/dt}{dr/dt} \tag{1.304}$$

The tangent unit vector $\hat{u}_t$ to the curve C is called the *orientation* of the curve C. When $\Delta t \to 0$, the length of $\Delta\mathbf{r}$ approaches the arc length Δs between A and B. So, Equation (1.304) can also be written as

$$\hat{u}_t = \lim_{\Delta s \to 0}\frac{\Delta\mathbf{r}}{\Delta s} = \lim_{\Delta t \to 0}\frac{\Delta\mathbf{r}/\Delta t}{\Delta s/\Delta t} = \frac{d\mathbf{r}(t)/dt}{ds(t)/dt} \tag{1.305}$$

If $\Delta t < 0$, then the vector $\Delta\mathbf{r}$ is directed toward decreasing values of t.

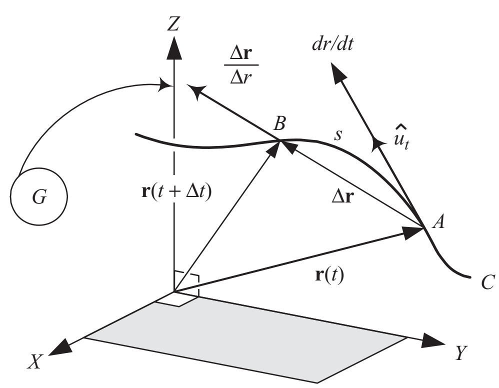

Figure 1.15 The increment vector $\Delta\mathbf{r}$ for $\Delta t > 0$ of a position vector $\mathbf{r}(t)$ is directed along the increasing secant AB of the curve configuration path C.

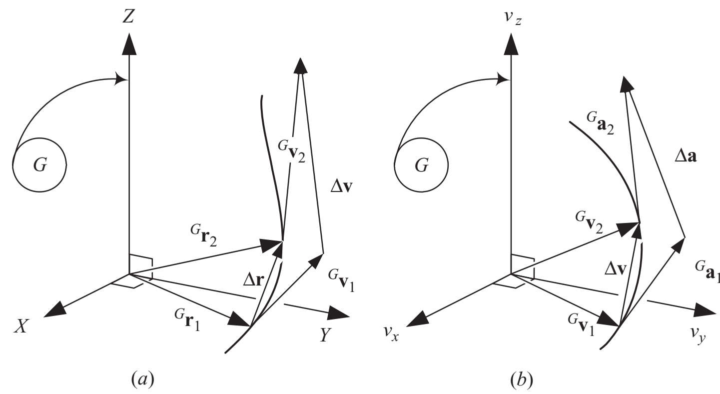

Figure 1.16 (a) Path of a position vector $\mathbf{r}$. (b) Path of the velocity vector $\mathbf{v} = d\mathbf{r}/dt$.

Consider a moving point P in a coordinate frame $G(x, y, z)$, with a continuously varying position vector $\mathbf{r} = \mathbf{r}\,(t)$ from a fixed origin, as is shown in Figure 1.16(a). The end point of the vector describes a path C when time t varies. Assume that $\mathbf{r} = {}^G\mathbf{r}_1$ is the position vector at a time $t = t_1$ and $\mathbf{r} = {}^G\mathbf{r}_2$ is the position vector at a time $t = t_2$. The difference vector

$$ {}^G\Delta\mathbf{r} = {}^G\mathbf{r}_2 - {}^G\mathbf{r}_1 \tag{1.306} $$

becomes smaller by shortening the time duration:

$$ \Delta t = t_2 - t_1 \tag{1.307} $$

The quotient $\Delta\mathbf{r}/\Delta t$ is the average rate of change of $\mathbf{r}$ in the interval Δt. Following the method of calculus, the limit of this quotient when $\Delta t \to 0$ by moving t_2 toward t_1 is the derivative of $\mathbf{r}$ at t_1:

$$ \lim_{\Delta t \to 0} \frac{{}^G\Delta\mathbf{r}}{\Delta t} = \frac{{}^Gd}{dt}\,{}^G\mathbf{r} = {}^G\mathbf{v} \tag{1.308} $$

where ${}^G\mathbf{v}$ is a tangent vector to the path C at the position ${}^G\mathbf{r}_1$ and is called the *velocity* of P.

We may express the velocity vector in a new orthogonal coordinate frame $G(v_x, v_y, v_z)$. The tip point of the velocity vector traces a path in the velocity coordinate frame called a *velocity hodograph*. Employing the same method, we can define the velocity $\mathbf{v} = {}^G\mathbf{v}_1$ at time $t = t_1$ and the velocity $\mathbf{v} = {}^G\mathbf{v}_2$ at time $t = t_2$. The difference vector

$$ {}^G\Delta\mathbf{v} = {}^G\mathbf{v}_2 - {}^G\mathbf{v}_1 \tag{1.309} $$

becomes smaller by shortening the time duration

$$ \Delta t = t_2 - t_1 \tag{1.310} $$

The quotient $\Delta\mathbf{v}/\Delta t$ is the average rate of change of $\mathbf{v}$ in the interval Δt. The limit of this quotient is the derivative of $\mathbf{v}$ that makes the acceleration of P:

$$ \lim_{\Delta t \to 0} \frac{{}^G\Delta\mathbf{v}}{\Delta t} = \frac{{}^Gd}{dt}\,{}^G\mathbf{v} = {}^G\mathbf{a} \tag{1.311} $$

Example 53 A Moving Point on a Helix Consider the point P in Figure 1.17 with position vector ${}^G\mathbf{r}(\varphi)$,

$$ {}^G\mathbf{r}(\varphi) = a\cos\varphi\,\hat{\imath} + a\sin\varphi\,\hat{\jmath} + k\varphi\,\hat{k} \tag{1.312} $$

that is moving on a helix with equation

$$ x = a\cos\varphi \qquad y = a\sin\varphi \qquad z = k\varphi \tag{1.313} $$

where a and k are constant and φ is an angular variable. The first, second, and third derivatives of ${}^G\mathbf{r}(\varphi)$ with respect to φ are

$$ \frac{{}^Gd}{d\varphi}\mathbf{r}(\varphi) = \mathbf{r}'(\varphi) = -a\sin\varphi\,\hat{\imath} + a\cos\varphi\,\hat{\jmath} + k\,\hat{k} \tag{1.314} $$

$$ \frac{{}^Gd^2}{d\varphi^2}\mathbf{r}(\varphi) = \mathbf{r}''(\varphi) = -a\cos\varphi\,\hat{\imath} - a\sin\varphi\,\hat{\jmath} \tag{1.315} $$

$$ \frac{{}^Gd^3}{d\varphi^3}\mathbf{r}(\varphi) = \mathbf{r}'''(\varphi) = a\sin\varphi\,\hat{\imath} - a\cos\varphi\,\hat{\jmath} \tag{1.316} $$

If the angle φ is a function of time t, then the first, second, and third derivatives of ${}^G\mathbf{r}(\varphi)$ with respect to t are

$$ \frac{{}^Gd}{dt}\mathbf{r}(t) = -a\dot{\varphi}\sin\varphi\,\hat{\imath} + a\dot{\varphi}\cos\varphi\,\hat{\jmath} + k\dot{\varphi}\,\hat{k} \tag{1.317} $$

$$ \begin{aligned} \frac{{}^Gd^2}{dt^2}\mathbf{r}(t) &= \left(-a\ddot{\varphi}\sin\varphi - a\dot{\varphi}^2\cos\varphi\right)\hat{\imath} \\ &\quad + \left(a\ddot{\varphi}\cos\varphi - a\dot{\varphi}^2\sin\varphi\right)\hat{\jmath} + k\ddot{\varphi}\,\hat{k} \end{aligned} \tag{1.318} $$

$$ \begin{aligned} \frac{{}^Gd^3}{dt^3}\mathbf{r}(t) &= \left(-a\dddot{\varphi}\sin\varphi - 3a\dot{\varphi}\ddot{\varphi}\cos\varphi + a\dot{\varphi}^3\sin\varphi\right)\hat{\imath} \\ &\quad + \left(a\dddot{\varphi}\cos\varphi - 3a\dot{\varphi}\ddot{\varphi}\sin\varphi - a\dot{\varphi}^3\cos\varphi\right)\hat{\jmath} + k\dddot{\varphi}\,\hat{k} \end{aligned} \tag{1.319} $$

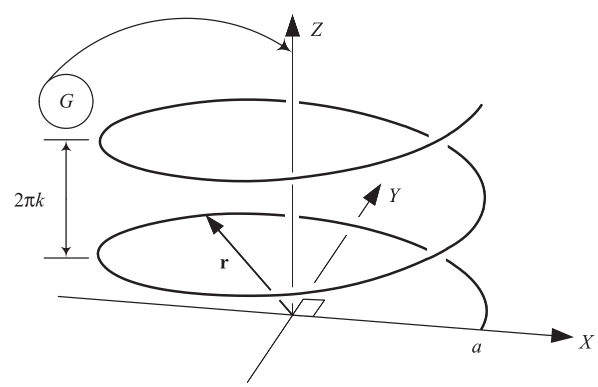

Figure 1.17 Helical path of a moving point.

Example 54 Vector Function If the magnitude of a vector **r** and/or direction of **r** in a reference frame B depends on a scalar variable, say q, then **r** is called a *vector function* of q in B. A vector may be a function of a variable in one coordinate frame but be independent of this variable in another coordinate frame.

1.5.2 Velocity and Acceleration

If the vector $\mathbf{r} = {}^G\mathbf{r}(t)$ is a position vector in a coordinate frame G, then its time derivative is a *velocity* vector ${}^G\mathbf{v}$. It shows the speed and the direction of motion of the tip point of ${}^G\mathbf{r}$:

$$ {}^G\mathbf{v} = \frac{{}^Gd}{dt}\,{}^G\mathbf{r}(t) = \dot{x}(t)\,\hat{\imath} + \dot{y}(t)\,\hat{\jmath} + \dot{z}(t)\,\hat{k} \tag{1.320} $$

The time derivative of a velocity vector ${}^G\mathbf{v}$ is called the *acceleration* ${}^G\mathbf{a}$,

$$ {}^G\mathbf{a} = \frac{{}^Gd}{dt}\,{}^G\mathbf{v}(t) = \ddot{x}(t)\,\hat{\imath} + \ddot{y}(t)\,\hat{\jmath} + \ddot{z}(t)\,\hat{k} \tag{1.321} $$

and the time derivative of an acceleration vector ${}^G\mathbf{a}$ is called the *jerk* ${}^G\mathbf{j}$,

$$ {}^G\mathbf{j} = \frac{{}^Gd}{dt}\,{}^G\mathbf{a}(t) = \dddot{x}(t)\,\hat{\imath} + \dddot{y}(t)\,\hat{\jmath} + \dddot{z}(t)\,\hat{k} \tag{1.322} $$

Example 55 Velocity, Acceleration, and Jerk of a Moving Point on a Helix Consider a moving point P with position vector in a coordinate frame G as

$$ {}^G\mathbf{r}(t) = \cos(\omega t)\,\hat{\imath} + \sin(\omega t)\,\hat{\jmath} + 2t\,\hat{k} \tag{1.323} $$

Such a path is called a *helix* or *screw*. The helix is uniformly turning on a circle in the (x, y)-plane while the circle is moving with a constant speed in the z-direction.

Taking the derivative shows that the velocity, acceleration, and jerk of the point P are

$$ {}^G\mathbf{v}(t) = -\omega\sin(\omega t)\,\hat{\imath} + \omega\cos(\omega t)\,\hat{\jmath} + 2\,\hat{k} \tag{1.324} $$

$$ {}^G\mathbf{a}(t) = -\omega^2\cos(\omega t)\,\hat{\imath} - \omega^2\sin(\omega t)\,\hat{\jmath} \tag{1.325} $$

$$ {}^G\mathbf{j}(t) = \omega^3\sin(\omega t)\,\hat{\imath} - \omega^3\cos(\omega t)\,\hat{\jmath} \tag{1.326} $$

Example 56 ★ Flight of a Bug Consider two cars A and B that are initially 15 km apart. The cars begin moving toward each other. The speeds of cars A and B are 10 and 5 km/ h, respectively. The instant they started a bug on the bumper of car A starts flying with speed 12 km/ h straight toward car B. As soon as it reaches the other car it turns and flies back. The bug flies back and forth from one car to the other until the two cars meet. The total length that the bug flies would be 12 km.

To calculate the total length of the bug's motion, let us show the velocities of the cars by $\mathbf{v}_A$ and $\mathbf{v}_B$ and the velocity of the bug by $\mathbf{v}_F$. Figure 1.18 illustrates the position of the cars and the bug at a time $t > 0$. Their positions are

$$X_A = v_A t \qquad X_B = l - v_B t \qquad X_F = v_F t \tag{1.327}$$

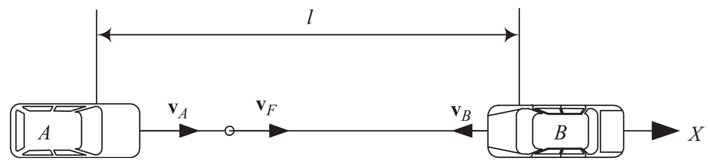

Figure 1.18 Two cars A and B moving toward each other and a bug F flying from one car to the other.

The bug reaches B at time t_1 after flying the distance d_1:

$$t_1 = \frac{l}{v_B + v_F} \qquad d_1 = v_F t_1 = \frac{v_F}{v_B + v_F} l \tag{1.328}$$

At this time, cars A, B and the bug are at

$$X_{A_1} = \frac{v_A}{v_B + v_F} l \tag{1.329}$$

$$X_{B_1} = \left(1 - \frac{v_B}{v_B + v_F}\right) l \tag{1.330}$$

$$X_{F_1} = X_{B_1} \tag{1.331}$$

so their positions when the bug is flying back are

$$X_A = X_{A_1} + v_A t = \frac{v_A}{v_B + v_F} l + v_A t \tag{1.332}$$

$$X_B = X_{B_1} - v_B t = \left(1 - \frac{v_B}{v_B + v_F}\right) l - v_B t \tag{1.333}$$

$$X_F = X_{F_1} - v_F t = \left(1 - \frac{v_B}{v_B + v_F}\right) l - v_F t \tag{1.334}$$

The bug reaches A at time t_2 after flying the distance d_2:

$$t_2 = \frac{l}{v_B + v_F} \frac{v_F - v_A}{v_A + v_F} \qquad d_2 = \frac{v_F}{v_B + v_F} \frac{v_A}{v_A + v_F} l \tag{1.335}$$

At this time cars A, B and the bug are at

$$X_{A_2} = 2 \frac{v_F}{v_B + v_F} \frac{v_A}{v_A + v_F} l \tag{1.336}$$

$$X_{B_2} = \left(1 - \frac{v_B}{v_B + v_F} + \frac{v_B}{v_B + v_F} \frac{v_F - v_A}{v_A + v_F}\right) l \tag{1.337}$$

$$X_{F_2} = X_{A_2} \tag{1.338}$$

so their positions when the bug is flying forward are

$$X_A = X_{A_2} + v_A t = 2\frac{v_A}{v_B + v_F}\frac{v_F}{v_A + v_F}l + v_A t \tag{1.339}$$

$$X_B = X_{B_2} - v_B t = l - \frac{v_B}{v_B + v_F}l\left(1 - \frac{v_F - v_A}{v_A + v_F}\right) - v_B t \tag{1.340}$$

$$X_F = X_{F_2} + v_F t = 2\frac{v_A}{v_B + v_F}\frac{v_F}{v_A + v_F}l - v_F t \tag{1.341}$$

By repeating this procedure, we can find the next times and distances and determine the total time t and distance d as

$$t = t_1 + t_2 + t_3 + \cdots \tag{1.342}$$

$$d = d_1 + d_2 + d_3 + \cdots \tag{1.343}$$

However, there is a simpler method to analyze this problem. The total time t at which the cars meet is

$$t = \frac{l}{v_A + v_B} \tag{1.344}$$

At this time, the bug can fly a distance d:

$$d = v_F t = \frac{v_F}{v_A + v_B}l \tag{1.345}$$

Therefore, if the speeds of the cars are $v_A = 10\,\text{km/h}$ and $v_B = 5\,\text{km/h}$ and their distance is $d = 15\,\text{km}$, it takes an hour for the cars to meet. The bug with a speed of $v_F = 12\,\text{km/h}$ can fly $d = 12\,\text{km}$ in an hour.

Example 57 ★ Jerk, Snap, and Other Derivatives The derivative of acceleration or the third time derivative of the position vector $\mathbf{r}$ is called the *jerk* $\mathbf{j}$; in England the word *jolt* is used instead of jerk. The third derivative may also wrongly be called pulse, impulse, bounce, surge, shock, or superacceleration.

In engineering, jerk is important for evaluating the destructive effects of motion on a moving object. For instance, high jerk is a reason for the discomfort of passengers in a vehicle. Jerk is the reason for liquid splashing from an open container. The movement of fragile objects, such as eggs, needs to be kept within specified limits of jerk to avoid damage. It is required that engineers keep the jerk of public transportation vehicles less than $2\,\text{m/s}^3$ for passenger comfort. There is an instrument in the aerospace industry called a *jerkmeter* that measures jerk.

There are no universally accepted names for the fourth and higher derivatives of a position vector $\mathbf{r}$. However, the terms *snap* $\mathbf{s}$ and *jounce* $\mathbf{s}$ have been used for derivatives of jerk. The fifth derivative of $\mathbf{r}$ is *crackle* $\mathbf{c}$, the sixth derivative is *pop* $\mathbf{þ}$, the seventh derivative is *larz* $\mathbf{z}$, the eight derivative is *bong* $\mathbf{b}$, the ninth derivative is *jeeq* $\mathbf{q}$, and the tenth derivative is *sooz* $\mathbf{u}$.

1.5.3 ★ Natural Coordinate Frame

Consider a space curve

$$x = x(s) \qquad y = y(s) \qquad z = z(s) \tag{1.346}$$

where s is the arc length of the curve from a fixed point on the curve. At the point there are three important planes: the *perpendicular plane* to the curve,

$$(x - x_0)\frac{dx}{ds} + (y - y_0)\frac{dy}{ds} + (z - z_0)\frac{dz}{ds} = 0 \tag{1.347}$$

the *osculating plane*,

$$\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right)(x - x_0) + \left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right)(y - y_0) + \left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right)(z - z_0) = 0 \tag{1.348}$$

and the *rectifying plane*,

$$(x - x_0)\frac{d^2x}{ds^2} + (y - y_0)\frac{d^2y}{ds^2} + (z - z_0)\frac{d^2z}{ds^2} = 0 \tag{1.349}$$

The osculating plane is the plane that includes the tangent line and the curvature center of the curve at P. The rectifying plane is perpendicular to both the osculating and normal planes.

The curvature of the curve at P is

$$\kappa = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2} \tag{1.350}$$

and the radius of curvature is

$$\rho = \frac{1}{\kappa} \tag{1.351}$$

The radius of curvature indicates the center of curvature in the osculating plane. Figure 1.19 illustrates a space curve and the three planes at a point P. The unit vectors $\hat{u}_t$, $\hat{u}_n$, and $\hat{u}_b$ are indicators of the rectifying, perpendicular, and osculating planes and make an orthogonal triad. This triad can be used to express the velocity and acceleration of the moving point P along the space curve C:

$$\mathbf{v} = \dot{s}\hat{u}_t \tag{1.352}$$

$$\mathbf{a} = \ddot{s}\hat{u}_t + \frac{\dot{s}^2}{\rho}\hat{u}_n \tag{1.353}$$

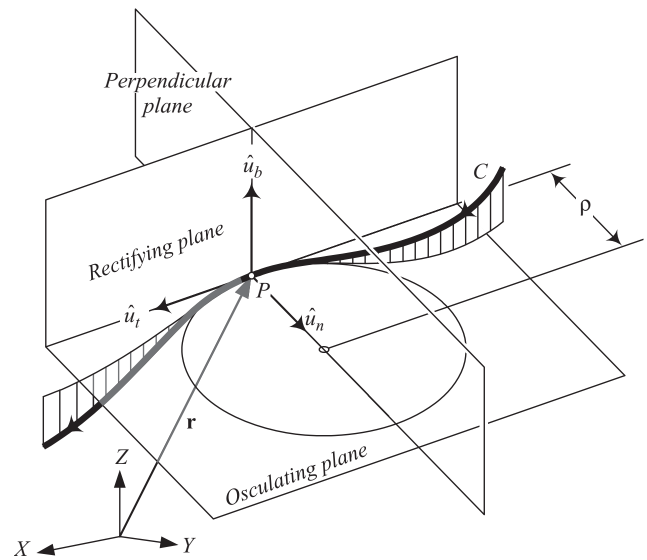

Figure 1.19 A space curve C and the three associated planes to the natural coordinates at a point P.

The orthogonal triad $\hat{u}_t$, $\hat{u}_n$, $\hat{u}_b$ is called the *natural triad* or *natural coordinate frame*:

$$\hat{u}_t = \frac{d\mathbf{r}}{ds} \tag{1.354}$$

$$\hat{u}_n = \frac{1}{\left|d^2\mathbf{r}/ds^2\right|}\frac{d^2\mathbf{r}}{ds^2} \tag{1.355}$$

$$\hat{u}_b = \frac{1}{\left|(d\mathbf{r}/ds) \times (d^2\mathbf{r}/ds^2)\right|}\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \tag{1.356}$$

Proof: Consider the tangent line (1.228) to the space curve (1.346) at point $P\ (x_0, y_0, z_0)$:

$$\frac{x - x_0}{dx/ds} = \frac{y - y_0}{dy/ds} = \frac{z - z_0}{dz/ds} \tag{1.357}$$

The unit vector along the tangent line l_t is

$$\hat{u}_t = \frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath} + \frac{dz}{ds}\hat{k} = \frac{d\mathbf{r}}{ds} \tag{1.358}$$

because $dx/ds, dy/ds, dz/ds$ are the directional cosines of the tangent line. A perpendicular plane to this vector is

$$\frac{dx}{ds}x + \frac{dy}{ds}y + \frac{dz}{ds}z = c \tag{1.359}$$

where c is a constant. The coordinates of $P\,(x_0, y_0, z_0)$ must satisfy the equation of the plane

$$\frac{dx}{ds}x_0 + \frac{dy}{ds}y_0 + \frac{dz}{ds}z_0 = c \tag{1.360}$$

and the perpendicular plane to the space curve at $P\,(x_0, y_0, z_0)$ is

$$(x - x_0)\frac{dx}{ds} + (y - y_0)\frac{dy}{ds} + (z - z_0)\frac{dz}{ds} = 0 \tag{1.361}$$

The equation of any plane that includes $P\,(x_0, y_0, z_0)$ is

$$A\,(x - x_0) + B\,(y - y_0) + C\,(z - z_0) = 0 \tag{1.362}$$

It also includes the tangent line (1.357) if

$$A\frac{dx}{ds} + B\frac{dy}{ds} + C\frac{dz}{ds} = 0 \tag{1.363}$$

and includes the space curve up to $\triangle s^2$ if

$$A\frac{d^2x}{ds^2} + B\frac{d^2y}{ds^2} + C\frac{d^2z}{ds^2} = 0 \tag{1.364}$$

Eliminating A, B, and C provides

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ \dfrac{dx}{ds} & \dfrac{dy}{ds} & \dfrac{dz}{ds} \\ \dfrac{d^2x}{ds^2} & \dfrac{d^2y}{ds^2} & \dfrac{d^2z}{ds^2} \end{vmatrix} = 0 \tag{1.365}$$

which is the equation of the osculating plane (1.348). The osculating plane can be identified by its unit vector $\hat{u}_b$, called the *bivector*:

$$\hat{u}_b = \frac{1}{u_b}\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right)\hat{\imath} + \frac{1}{u_b}\left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right)\hat{\jmath} + \frac{1}{u_b}\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right)\hat{k} = \frac{1}{u_b}\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \tag{1.366}$$

$$u_b^2 = \left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right)^2 + \left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right)^2 + \left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right)^2 = \left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right)^2 \tag{1.367}$$

The line of intersection of the osculating plane (1.348) and the perpendicular plane (1.361) is called the *principal normal line* to the curve at P. From (1.361) and (1.348) the equation of the principal normal is

$$\frac{x - x_0}{d^2x/ds^2} = \frac{y - y_0}{d^2y/ds^2} = \frac{z - z_0}{d^2z/ds^2} \tag{1.368}$$

The plane through P and perpendicular to the principal normal is called the *rectifying* or *tangent plane*, which has the equation

$$(x - x_0)\frac{d^2x}{ds^2} + (y - y_0)\frac{d^2y}{ds^2} + (z - z_0)\frac{d^2z}{ds^2} = 0 \tag{1.369}$$

The intersection of the rectifying plane and the perpendicular plane is a line that is called the *binormal* line:

$$\frac{x - x_0}{\dfrac{dy}{ds}\dfrac{d^2z}{ds^2} - \dfrac{dz}{ds}\dfrac{d^2y}{ds^2}} = \frac{y - y_0}{\dfrac{dz}{ds}\dfrac{d^2x}{ds^2} - \dfrac{dx}{ds}\dfrac{d^2z}{ds^2}} = \frac{z - z_0}{\dfrac{dx}{ds}\dfrac{d^2y}{ds^2} - \dfrac{dy}{ds}\dfrac{d^2x}{ds^2}} \tag{1.370}$$

The bivector (1.366) is along the binormal line (1.370). The unit vector perpendicular to the rectifying plane is called the normal vector $\hat{u}_n$, which is in the osculating plane and in the direction of the center of curvature of the curve at P:

$$\hat{u}_n = \frac{1}{u_n}\frac{d^2x}{ds^2}\hat{\imath} + \frac{1}{u_n}\frac{d^2y}{ds^2}\hat{\jmath} + \frac{1}{u_n}\frac{d^2z}{ds^2}\hat{k} = \frac{1}{u_n}\frac{d^2\mathbf{r}}{ds^2} \tag{1.371}$$

$$u_n^2 = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 \tag{1.372}$$

The unit vectors $\hat{u}_t$, $\hat{u}_n$, and $\hat{u}_b$ make an orthogonal triad that is called the natural coordinate frame:

$$\hat{u}_t \times \hat{u}_n = \hat{u}_b \tag{1.373}$$

The *curvature* κ of a space curve is defined as the limit of the ratio of the angle $\triangle\theta$ between two tangents to the arc length $\triangle s$ of the curve between the tangents as the arc length approaches zero:

$$\kappa = \lim_{\triangle s \to 0} \frac{\triangle\theta}{\triangle s} \tag{1.374}$$

The directional cosines of the tangent line are dx/ds, dy/ds, dz/ds at point $P_1\,(x_1, y_1, z_1)$ and $dx/ds + (d^2x)/(ds^2)\triangle s$, $dy/ds + (d^2y)/(ds^2)\triangle s$, $dz/ds + (d^2z)/(ds^2)\triangle s$ at

$$P_2\,(x_2, y_2, z_2) = P_2\left(x_1 + \tfrac{dx}{ds}\triangle s,\, y_1 + \tfrac{dy}{ds}\triangle s,\, z_1 + \tfrac{dz}{ds}\triangle s\right)$$

Using the cross product of the unit vectors along the two tangent lines, we have

$$\sin^2\triangle\theta = \left[\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right)^2 + \left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right)^2 + \left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right)^2\right](\triangle s)^2 \tag{1.375}$$

Because of the constraint among the directional cosines and

$$\lim_{\triangle\theta\to 0}\frac{\sin\triangle\theta}{\triangle\theta}=1 \tag{1.376}$$

the coefficient of $(\triangle s)^2$ reduces to $(d^2x/ds^2)^2+(d^2y/ds^2)^2+(d^2z/ds^2)^2$ and we can calculate the curvature of the curve as

$$\kappa=\frac{d\theta}{ds}=\sqrt{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2} \tag{1.377}$$

Consider a circle with $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=0$. The curvature of the circle would be $\kappa=1/\rho$ because $ds=\rho\ d\varphi$. Equating the curvature of the curve with the curvature of the circle provides the radius of curvature of the curve:

$$\rho=\frac{1}{\kappa} \tag{1.378}$$

Using the radius of curvature, we may simplify the unit normal vector $\hat{u}_n$ to

$$\hat{u}_n=\rho\left(\frac{d^2x}{ds^2}\hat{\imath}+\frac{d^2y}{ds^2}\hat{\jmath}+\frac{d^2z}{ds^2}\hat{k}\right)=\rho\frac{d^2\mathbf{r}}{ds^2} \tag{1.379}$$

Because the unit vector $\hat{u}_t$ in (1.358) is tangent to the space curve in the direction of increasing curve length s, the velocity vector $\mathbf{v}$ must be tangent to the curve in the direction of increasing time t. Therefore, $\mathbf{v}$ is proportional to $\hat{u}_t$ where the proportionality factor is the speed $\dot{s}$ of P:

$$\mathbf{v}=\dot{s}\hat{u}_t=\dot{s}\left(\frac{dx}{ds}\hat{\imath}+\frac{dy}{ds}\hat{\jmath}+\frac{dz}{ds}\hat{k}\right)=\dot{s}\frac{d\mathbf{r}}{ds} \tag{1.380}$$

$$v=\dot{s} \tag{1.381}$$

The acceleration of P would be

$$\mathbf{a}=\ddot{s}\hat{u}_t+\dot{s}\frac{d}{dt}\hat{u}_t \tag{1.382}$$

However,

$$\frac{d}{dt}\hat{u}_t=\dot{s}\frac{d^2x}{ds^2}\hat{\imath}+\dot{s}\frac{d^2y}{ds^2}\hat{\jmath}+\dot{s}\frac{d^2z}{ds^2}\hat{k}=\frac{\dot{s}}{\rho}\hat{u}_n \tag{1.383}$$

which shows that

$$\mathbf{a}=\ddot{s}\hat{u}_t+\frac{\dot{s}^2}{\rho}\hat{u}_n \tag{1.384}$$

$$a=\sqrt{\ddot{s}^2+\frac{\dot{s}^4}{\rho^2}} \tag{1.385}$$

The natural coordinate frame $\hat{u}_t$, $\hat{u}_n$, and $\hat{u}_b$ may also be called the *Frenet frame*, *Frenet trihedron*, *repère mobile frame*, *moving frame*, or *path frame*. ■

Example 58 Osculating Plane to a Helix A point P is moving on a helix with equation

$$x = a\cos\varphi \qquad y = a\sin\varphi \qquad z = k\varphi \tag{1.386}$$

where a and k are constant and φ is an angular variable. The tangent line (1.357) to the helix at $\varphi = \pi/4$ is

$$-\frac{\sqrt{2}}{a}\left(x - \frac{1}{2}\sqrt{2}a\right) = \frac{\sqrt{2}}{a}\left(y - \frac{1}{2}\sqrt{2}a\right) = \frac{1}{k}\left(z - \frac{1}{4}\pi k\right) \tag{1.387}$$

Using

$$x_0 = \frac{\sqrt{2}}{2}a \qquad y_0 = \frac{\sqrt{2}}{2}a \qquad z_0 = k\frac{\pi}{4} \tag{1.388}$$

and

$$\frac{dx}{d\varphi} = -a\sin\varphi = -\frac{\sqrt{2}}{2}a \qquad \frac{dy}{d\varphi} = a\cos\varphi = \frac{\sqrt{2}}{2}a \tag{1.389}$$

$$\frac{dz}{d\varphi} = k \tag{1.390}$$

we can find the perpendicular plane (1.347) to the helix at $\varphi = \pi/4$:

$$-\sqrt{2}ax + \sqrt{2}ay + 2zk = \tfrac{1}{2}\pi k^2 \tag{1.391}$$

To find the osculating and rectifying planes, we need to calculate the second derivatives of the curve at $\varphi = \pi/4$,

$$\begin{gathered}\frac{d^2x}{d\varphi^2} = -a\cos\varphi = -\frac{\sqrt{2}}{2}a \qquad \frac{d^2y}{d\varphi^2} = -a\sin\varphi = -\frac{\sqrt{2}}{2}a \\ \frac{d^2z}{d\varphi^2} = 0\end{gathered} \tag{1.392}$$

substitute in Equation (1.369) for the osculating plane,

$$\sqrt{2}x - \sqrt{2}ky + 2az = \frac{1}{2}\pi ak \tag{1.393}$$

and substitute in Equation (1.392) for the rectifying plane,

$$\sqrt{2}x + \sqrt{2}y = 2a \tag{1.394}$$

Because of (1.392), the curvature of the helix at $\varphi = \pi/4$ is

$$\kappa = a \tag{1.395}$$

and therefore the curvature radius of the helix at that point is

$$\rho = \frac{1}{\kappa} = \frac{1}{a} \tag{1.396}$$

Having the equations of the three planes and the curvature radius ρ, we are able to identify the unit vectors $\hat{u}_t$, $\hat{u}_n$, and $\hat{u}_b$:

$$\hat{u}_t = \frac{1}{\sqrt{a^2+k^2}}\left(-\frac{\sqrt{2}}{2}a\hat{\imath} + \frac{\sqrt{2}}{2}a\hat{\jmath} + k\hat{k}\right) \tag{1.397}$$

$$\hat{u}_n = -\frac{\sqrt{2}}{2}\hat{\imath} - \frac{\sqrt{2}}{2}\hat{\jmath} \tag{1.398}$$

$$\hat{u}_b = \frac{1}{\sqrt{a^2+k^2}}\left(\frac{1}{2}\sqrt{2}k\hat{\imath} - \frac{1}{2}\sqrt{2}k\hat{\jmath} + a\hat{k}\right) \tag{1.399}$$

We can check and see that

$$\hat{u}_t \times \hat{u}_n = \hat{u}_b \tag{1.400}$$

A helix is a category of space curves with a constant curvature–torsion ratio:

$$\frac{\kappa}{\tau} = \text{const} \tag{1.401}$$

The circular helix is only a special case of the general helix curves.

Example 59 Uniform Motion on a Circle Consider a particle P that is moving on a circle with radius R around the origin of the coordinate frame at a constant speed v. The equation of the circle is

$$\mathbf{r}\cdot\mathbf{r} = r^2 \tag{1.402}$$

where r is the constant length of $\mathbf{r}$. Differentiating (1.402) with respect to time results in

$$\mathbf{r}\cdot\mathbf{v} = 0 \tag{1.403}$$

which shows that $\mathbf{r}$ and $\mathbf{v}$ are perpendicular when $\mathbf{r}$ has a constant length. If the speed of the particle is constant, then

$$\mathbf{v}\cdot\mathbf{v} = v^2 \tag{1.404}$$

which shows that

$$\mathbf{v}\cdot\mathbf{a} = 0 \tag{1.405}$$

Now differentiating (1.403) with respect to time results in

$$\mathbf{r}\cdot\mathbf{a} = -v^2 \tag{1.406}$$

It indicates that $\mathbf{r}$ and $\mathbf{a}$ are collinear and oppositely directed. So, the value of their product must be

$$\mathbf{r}\cdot\mathbf{a} = -ra \tag{1.407}$$

which determines the length of the acceleration vector a on a uniformly circular motion:

$$a = -\frac{v^2}{r} \tag{1.408}$$

Example 60 Curvature of a Plane Curve Let us consider a curve C in the (x, y)-plane as is shown in Figure 1.20, which is defined time parametrically as

$$x = x(t) \qquad y = y(t) \tag{1.409}$$

The curve increment ds is

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = dx^2 + dy^2 \tag{1.410}$$

which after dividing by dt would be

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 \tag{1.411}$$

Differentiating from the slop of the curve θ,

$$\tan\theta = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \tag{1.412}$$

we have

$$\dot{\theta}\left(1 + \tan^2\theta\right) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2} \tag{1.413}$$

and therefore

$$\dot{\theta} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2} \tag{1.414}$$

However, because of

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{ds}\frac{ds}{dt} = \dot{s}\frac{d\theta}{ds} = \frac{\dot{s}}{\rho} = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\rho} \tag{1.415}$$

we get

$$\kappa = \frac{1}{\rho} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}} \tag{1.416}$$

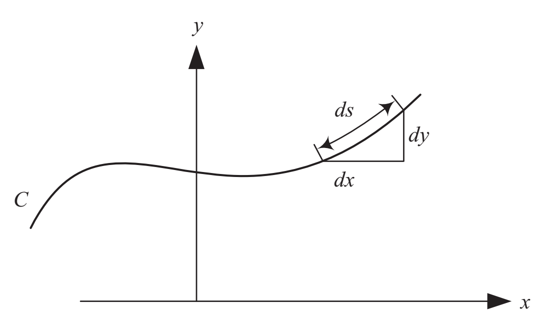

Figure 1.20 A curve C in the (x, y)-plane.

Whenever, instead of (1.409), we have the equation of the plane curve as

$$y = y(x) \tag{1.417}$$

then the curvature equation would simplify to

$$\kappa = \frac{1}{\rho} = \left| \frac{d^2y/dx^2}{\left(1 + (dy/dx)^2\right)^{3/2}} \right| \tag{1.418}$$

As an example, consider a plane curve given by the parametric equations

$$x = t \qquad y = 2t^2 \tag{1.419}$$

The curvature at $t = 3\,\text{s}$ is $2.2909 \times 10^{-3}\,\text{m}^{-1}$ because

$$\kappa = \frac{1}{\rho} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}} = \frac{4 - 0}{\left(1 + 16t^2\right)^{3/2}} = 2.2909 \times 10^{-3}\,\text{m}^{-1} \tag{1.420}$$

The same curve can be expressed by

$$y = 2x^2 \tag{1.421}$$

which has the same radius of curvature $\rho = 1/\kappa = 2.2909 \times 10^{-3}\,\text{m}^{-1} = 436.5\,\text{m}$ at $x = 3\,\text{m}$ because $dy/dx = 4x = 12$ and $d^2y/dx^2 = 4$:

$$\begin{aligned} \kappa = \frac{1}{\rho} &= \left| \frac{d^2y/dx^2}{\left[1 + (dy/dx)^2\right]^{3/2}} \right| = \left| \frac{4}{\left[1 + (12)^2\right]^{3/2}} \right| \\ &= 2.2909 \times 10^{-3}\,\text{m}^{-1} \end{aligned} \tag{1.422}$$

Example 61 Natural Coordinate Frame Is Orthogonal To show that the natural coordinate frame $\hat{u}_t$, $\hat{u}_n$, $\hat{u}_b$ in Equations (1.354)–(1.356) is orthogonal, we may differentiate the relation

$$\hat{u}_t \cdot \hat{u}_t = 1 \tag{1.423}$$

with respect to s and get

$$2\frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} = 0 \tag{1.424}$$

It indicates that $\hat{u}_t$ is orthogonal to $\hat{u}_n$. Equation (1.356) also shows that $\hat{u}_b$ is orthogonal to both $\hat{u}_t$ and $\hat{u}_n$.

Example 62 Vectorial Expression of Curvature Assume that the position vector of a moving point on a space curve is given by

$$\mathbf{r} = \mathbf{r}(s) \tag{1.425}$$

where s is the arc length on the curve measured from a fixed point on the curve. Then,

$$\mathbf{v} = \dot{s}\hat{u}_t \tag{1.426}$$

$$\hat{u}_t = \frac{d\mathbf{r}}{ds} \tag{1.427}$$

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{d}{ds}\hat{u}_t = \frac{1}{\rho}\hat{u}_n = \kappa\hat{u}_n \tag{1.428}$$

and therefore,

$$\kappa = \frac{1}{\rho} = \left|\frac{d^2\mathbf{r}}{ds^2}\right| \tag{1.429}$$

We may also employ the velocity and acceleration vectors of the moving point and determine the curvature of the curve. Because the outer product of $\mathbf{v}$ and $\mathbf{a}$ is

$$\mathbf{v} \times \mathbf{a} = \left(\dot{s}\hat{u}_t\right) \times \left(\ddot{s}\hat{u}_t + \frac{\dot{s}^2}{\rho}\hat{u}_n\right) = \mathbf{v} \times \mathbf{a}_n \tag{1.430}$$

$$|\mathbf{v} \times \mathbf{a}| = v a_n \tag{1.431}$$

we have

$$a_n = \frac{\dot{s}^2}{\rho} = \frac{v^2}{\rho} = \frac{|\mathbf{v} \times \mathbf{a}|}{v} \tag{1.432}$$

and therefore,

$$\kappa = \frac{1}{\rho} = \frac{|\mathbf{v} \times \mathbf{a}|}{v^3} = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \tag{1.433}$$

As an example, consider a moving point at

$$\mathbf{r} = \begin{bmatrix} t \\ 2t^2 \end{bmatrix} \tag{1.434}$$

Its velocity and acceleration are

$$\mathbf{v} = \begin{bmatrix} 1 \\ 4t \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \tag{1.435}$$

and therefore the curvature of the motion is

$$\kappa = \frac{1}{\rho} = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{4}{\left(\sqrt{16t^2 + 1}\right)^3} \tag{1.436}$$

The curvature at $t = 3\,\text{s}$ is $\kappa = 2.2909 \times 10^{-3}\,\text{m}^{-1}$.

Example 63 ★ Curvature Vector κ Using the definition of tangential unit vector $\hat{u}_t$,

$$\hat{u}_t = \frac{d\mathbf{r}}{ds} \tag{1.437}$$

and taking a curve length derivative we can define a curvature vector $\boldsymbol{\kappa}$ as

$$\boldsymbol{\kappa} = \frac{d\hat{u}_t}{ds} = \frac{d^2\mathbf{r}}{ds^2} = \kappa\hat{u}_n \tag{1.438}$$

that has a length κ and indicates the curvature center of the curve. So, the curvature vector $\boldsymbol{\kappa}$ points in the direction in which $\hat{u}_t$ is turning, orthogonal to $\hat{u}_t$. The length $\kappa = |\boldsymbol{\kappa}|$ gives the rate of turning. It can be found from

$$\kappa^2 = \frac{d^2\mathbf{r}}{ds^2} \cdot \frac{d^2\mathbf{r}}{ds^2} \tag{1.439}$$

Furthermore, because

$$\hat{u}_t = \frac{\mathbf{v}}{\dot{s}} \tag{1.440}$$

we may also define the curvature vector $\boldsymbol{\kappa}$ as

$$\boldsymbol{\kappa} = \frac{d}{ds}\frac{\mathbf{v}}{\dot{s}} = \frac{1}{\dot{s}}\frac{d}{dt}\frac{\mathbf{v}}{\dot{s}} = \frac{\mathbf{a}\dot{s} - \mathbf{v}\ddot{s}}{\dot{s}^3} \tag{1.441}$$

Example 64 ★ Frenet–Serret Formulas When the position vector of a moving point on a space curve is given as a function of the arc length s,

$$\mathbf{r} = \mathbf{r}(s) \tag{1.442}$$

we define the unit vectors $\hat{u}_t$, $\hat{u}_n$, and $\hat{u}_b$ and an orthogonal coordinate frame

$$\hat{u}_t \times \hat{u}_n = \hat{u}_b \tag{1.443}$$

that is carried by the point. Because s is the variable that indicates the point, it is useful to determine the derivatives of the unit vectors with respect to s.

Using Equation (1.383), we can find the s-derivative of the tangent unit vector $\hat{u}_t$:

$$\frac{d\hat{u}_t}{ds} = \frac{d\hat{u}_t}{dt}\frac{dt}{ds} = \frac{d\hat{u}_t}{dt}\frac{1}{\dot{s}} = \frac{1}{\rho}\hat{u}_n = \kappa\hat{u}_n \tag{1.444}$$

$$\left|\frac{d\hat{u}_t}{ds}\right| = \frac{1}{\rho} = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2} \tag{1.445}$$

To find $d\hat{u}_b/ds$, we may take a derivative from (1.443):

$$\frac{d\hat{u}_b}{ds} = \frac{d}{ds}\left(\hat{u}_t \times \hat{u}_n\right) = \frac{d\hat{u}_t}{ds} \times \hat{u}_n + \hat{u}_t \times \frac{d\hat{u}_n}{ds} = \hat{u}_t \times \frac{d\hat{u}_n}{ds} \tag{1.446}$$

Because $\hat{u}_b$ is a constant-length vector, $d\hat{u}_b/ds$ is perpendicular to $\hat{u}_b$. It must also be perpendicular to $\hat{u}_t$. So, $d\hat{u}_b/ds$ is parallel to $\hat{u}_n$:

$$\frac{d\hat{u}_b}{ds} = -\tau\hat{u}_n = -\frac{1}{\sigma}\hat{u}_n \tag{1.447}$$

The coefficient τ is called the *torsion of the curve*, while $\sigma = 1/\tau$ is called the *radius of torsion*. The torsion at a point of the curve indicates that the osculating plane rotates about the tangent to the curve as the point moves along the curve. The torsion is considered positive if the osculating plane rotates about $\hat{u}_t$ and negative if it rotates about $-\hat{u}_t$. A curve with $\kappa \neq 0$ is planar if and only if $\tau = 0$.

The derivative of the normal unit vector $d\hat{u}_n/ds$ may be calculated from

$$\begin{aligned}\frac{d\hat{u}_n}{ds} &= \frac{d}{ds}\left(\hat{u}_b \times \hat{u}_t\right) = \frac{d\hat{u}_b}{ds} \times \hat{u}_t + \hat{u}_b \times \frac{d\hat{u}_t}{ds} \\ &= -\frac{1}{\sigma}\left(\hat{u}_n \times \hat{u}_t\right) + \frac{1}{\rho}\left(\hat{u}_b \times \hat{u}_n\right) = \frac{1}{\sigma}\hat{u}_b - \frac{1}{\rho}\hat{u}_t\end{aligned} \tag{1.448}$$

Equations (1.444), (1.447), and (1.448) are called the Frenet–Serret formulas. The Frenet–Serret formulas may be summarized in a matrix form:

$$\begin{bmatrix} \dfrac{d\hat{u}_t}{ds} \\ \dfrac{d\hat{u}_n}{ds} \\ \dfrac{d\hat{u}_b}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_t \\ \hat{u}_n \\ \hat{u}_b \end{bmatrix} \tag{1.449}$$

It shows that the derivative of the natural coordinate unit vectors can be found by multiplying a skew-symmetric matrix and the coordinate unit vectors.

Having the Frenet–Serret formulas, we are able to calculate the kinematics of a moving point on the space curve:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\dot{s} = \dot{s}\hat{u}_t \tag{1.450}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{s}\hat{u}_t + \dot{s}\frac{d\hat{u}_t}{dt} = \ddot{s}\hat{u}_t + \dot{s}^2\frac{d\hat{u}_t}{ds} = \ddot{s}\hat{u}_t + \dot{s}^2\frac{1}{\rho}\hat{u}_n \tag{1.451}$$

$$\mathbf{j} = \frac{d\mathbf{a}}{dt} = \left(\dddot{s} - \frac{\dot{s}^3}{\rho^2}\right)\hat{u}_t + \frac{1}{\rho}\left(3\dot{s}\ddot{s} + \frac{\dot{s}^2}{\rho}\dot{\rho}\right)\hat{u}_n + \frac{\dot{s}^3}{\rho\sigma}\hat{u}_b \tag{1.452}$$

Frenet (1816–1900) and Serret (1819–1885) were two French mathematicians.

Example 65 Characteristics of a Space Curve Consider a space curve C with the parametric equation

$$\mathbf{r} = \mathbf{r}(t) \tag{1.453}$$

The natural coordinate frame and curve characteristics are

$$\hat{u}_t = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} \tag{1.454}$$

$$\hat{u}_b = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|} \tag{1.455}$$

$$\hat{u}_n = \hat{u}_b \times \hat{u}_t \tag{1.456}$$

$$\kappa = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^3} \tag{1.457}$$

$$\tau = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \dddot{\mathbf{r}}}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} \tag{1.458}$$

Employing these equations, the Frenet–Serret formulas (1.449) can be determined in time derivatives:

$$\frac{d\hat{u}_t}{dt} = \kappa \, |\dot{\mathbf{r}}| \, \hat{u}_n \tag{1.459}$$

$$\frac{d\hat{u}_n}{dt} = -\kappa \, |\dot{\mathbf{r}}| \, \hat{u}_t + \tau \, |\dot{\mathbf{r}}| \, \hat{u}_b \tag{1.460}$$

$$\frac{d\hat{u}_b}{dt} = -\tau \, |\dot{\mathbf{r}}| \, \hat{u}_b \tag{1.461}$$

Example 66 ★ Osculating Sphere The sphere that has a contact of third order with a space curve at a point $P(x, y, z)$ is called the osculating sphere of the curve at P. If the center of the sphere is denoted by $C\,(x_C, y_C, z_C)$, then the equation of the osculating sphere is

$$(x - x_C)^2 + (y - y_C)^2 + (z - z_C)^2 = R \tag{1.462}$$

where R is the radius of the sphere. Taking three derivatives from (1.462) provides a set of four equations to determine x_C, y_C, z_C and R. To set up the equations, we show the equation of the sphere as

$$(\mathbf{r}_C - \mathbf{r})^2 = R^2$$

where $\mathbf{r}_C - \mathbf{r}$ indicates the position of the center of the sphere from point P. Taking derivatives with respect to the arc length s provides

$$(\mathbf{r}_C - \mathbf{r}) \cdot \frac{d\mathbf{r}}{ds} = 0 \tag{1.463}$$

$$-1 + (\mathbf{r}_C - \mathbf{r}) \cdot \frac{d^2\mathbf{r}}{ds^2} = 0 \tag{1.464}$$

$$-\frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} + (\mathbf{r}_C - \mathbf{r}) \cdot \frac{d^3\mathbf{r}}{ds^3} = 0 \tag{1.465}$$

Employing the curvature τ and torsion κ, we can rewrite these equations:

$$(\mathbf{r}_C - \mathbf{r}) \cdot \hat{u}_t = 0 \tag{1.466}$$

$$-1 + (\mathbf{r}_C - \mathbf{r}) \cdot \kappa \hat{u}_n = 0 \tag{1.467}$$

$$(\mathbf{r}_C - \mathbf{r}) \cdot \left(\frac{d\kappa}{ds} \hat{u}_n + \kappa \left(\tau \hat{u}_b - \kappa \hat{u}_t \right) \right) = 0 \tag{1.468}$$

Expanding (1.468) yields

$$\frac{d\kappa}{ds} (\mathbf{r}_C - \mathbf{r}) \cdot \hat{u}_n + \kappa \tau (\mathbf{r}_C - \mathbf{r}) \cdot \hat{u}_b - \kappa^2 (\mathbf{r}_C - \mathbf{r}) \cdot \hat{u}_t = 0 \tag{1.469}$$

and using Equations (1.466) and (1.467), we find

$$\frac{1}{\kappa} \frac{d\kappa}{ds} + \kappa \tau (\mathbf{r}_C - \mathbf{r}) \cdot \hat{u}_b = 0 \tag{1.470}$$

Knowing that

$$\frac{d\rho}{ds} = \frac{d}{ds} \left(\frac{1}{\kappa} \right) = -\frac{1}{\kappa^2} \frac{d\kappa}{ds} \tag{1.471}$$

we can simplify Equation (1.470):

$$(\mathbf{r}_C - \mathbf{r}) \cdot \hat{u}_b = \sigma \frac{d\rho}{ds} \tag{1.472}$$

From Equations (1.466), (1.467), and (1.472), we have

$$(\mathbf{r}_C - \mathbf{r}) \cdot \hat{u}_t = 0 \tag{1.473}$$

$$(\mathbf{r}_C - \mathbf{r}) \cdot \hat{u}_n = \rho \tag{1.474}$$

$$(\mathbf{r}_C - \mathbf{r}) \cdot \hat{u}_b = \sigma \frac{d\rho}{ds} \tag{1.475}$$

that indicates $\mathbf{r}_C - \mathbf{r}$ lies in a perpendicular plane. The components of $\mathbf{r}_C - \mathbf{r}$ are ρ along $\hat{u}_n$ and $\sigma \, (d\rho/ds)$ along $\hat{u}_b$:

$$\mathbf{r}_C - \mathbf{r} = \rho \hat{u}_n + \sigma \frac{d\rho}{ds} \hat{u}_b \tag{1.476}$$

Therefore, the position vector of the center of the osculating sphere is at

$$\mathbf{r}_C = \mathbf{r} + \rho \hat{u}_n + \sigma \frac{d\rho}{ds} \hat{u}_b \tag{1.477}$$

and the radius of the osculating sphere is

$$R = |\mathbf{r}_C - \mathbf{r}| = \sqrt{\rho^2 + \sigma^2 \left(\frac{d\rho}{ds} \right)^2} \tag{1.478}$$

Example 67 ★ Taylor Series Expansion of a Space Curve Consider a point P that is moving on a space curve that is parametrically expressed as $\mathbf{r} = \mathbf{r}(s)$. If at $s = 0$ we have the position and velocity of P, it is possible to express the curve by a Taylor expansion:

$$\mathbf{r}(s) = \mathbf{r}(0) + \frac{d\mathbf{r}(0)}{ds}s + \frac{d^2\mathbf{r}(0)}{ds^2}\frac{s^2}{2!} + \frac{d^3\mathbf{r}(0)}{ds^3}\frac{s^3}{3!} + \cdots \tag{1.479}$$

Using the natural coordinate system, we have

$$\frac{d\mathbf{r}}{ds} = \hat{u}_t \tag{1.480}$$

$$\frac{d^2\mathbf{r}}{ds^2} = \kappa\,\hat{u}_n \tag{1.481}$$

$$\frac{d^3\mathbf{r}}{ds^3} = \frac{d}{ds}\left(\kappa\,\hat{u}_n\right) = \frac{d\kappa}{ds}\hat{u}_n + \kappa\left(-\kappa\hat{u}_t + \tau\hat{u}_b\right) \tag{1.482}$$

$$\begin{aligned}\frac{d^4\mathbf{r}}{ds^4} &= \frac{d^2\kappa}{ds^2}\hat{u}_n + \frac{d\kappa}{ds}\left(-\kappa\hat{u}_t + \tau\hat{u}_b\right) + \frac{d\kappa}{ds}\left(-\kappa\hat{u}_t + \tau\hat{u}_b\right) \\ &\quad + \kappa\left(-\frac{d\kappa}{ds}\hat{u}_t - \kappa^2\hat{u}_n + \frac{d\tau}{ds}\hat{u}_b - \tau^2\hat{u}_n\right) \\ &= -3\kappa\frac{d\kappa}{ds}\hat{u}_t + \left(\frac{d^2\kappa}{ds^2} - \kappa^3 - \kappa\tau^2\right)\hat{u}_n + \left(2\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right)\hat{u}_b\end{aligned} \tag{1.483}$$

and therefore,

$$\frac{d\mathbf{r}(0)}{ds} = \hat{u}_t(0) = \hat{u}_{t_0} \tag{1.484}$$

$$\frac{d^2\mathbf{r}(0)}{ds^2} = \kappa(0)\,\hat{u}_n(0) = \kappa_0\hat{u}_{n_0} \tag{1.485}$$

$$\frac{d^3\mathbf{r}(0)}{ds^3} = -\kappa_0^2\hat{u}_{t_0} + \frac{d\kappa_0}{ds}\hat{u}_{n_0} + \kappa_0\tau_0\hat{u}_{b_0} \tag{1.486}$$

$$\begin{aligned}\frac{d^4\mathbf{r}(0)}{ds^4} &= -3\kappa_0\frac{d\kappa_0}{ds}\hat{u}_{t_0} + \left(\frac{d^2\kappa_0}{ds^2} - \kappa_0^3 - \kappa_0\tau_0^2\right)\hat{u}_{n_0} \\ &\quad + \left(2\tau_0\frac{d\kappa_0}{ds} + \kappa_0\frac{d\tau_0}{ds}\right)\hat{u}_{b_0}\end{aligned} \tag{1.487}$$

Substituting these results in Equation (1.479) shows that

$$\begin{aligned}\mathbf{r} &= \mathbf{r}_0 + s\hat{u}_{t_0} + \frac{1}{2}\kappa_0 s^2\hat{u}_{n_0} + \frac{s^3}{6}\left(-\kappa_0^2\hat{u}_{t_0} + \frac{d\kappa_0}{ds}\hat{u}_{n_0} + \kappa_0\tau_0\hat{u}_{b_0}\right) \\ &\quad + \frac{s^4}{24}\left(-3\kappa_0\frac{d\kappa_0}{ds}\right)\hat{u}_{t_0} + \frac{s^4}{24}\left(\frac{d^2\kappa_0}{ds^2} - \kappa_0^3 - \kappa_0\tau_0^2\right)\hat{u}_{n_0} \\ &\quad + \frac{s^4}{24}\left(2\tau_0\frac{d\kappa_0}{ds} + \kappa_0\frac{d\tau_0}{ds}\right)\hat{u}_{b_0} + \cdots\end{aligned} \tag{1.488}$$

Let us rearrange the equation to determine the natural components of $\mathbf{r} - \mathbf{r}_0$:

$$\begin{aligned}\mathbf{r} - \mathbf{r}_0 &= \left(s - \frac{\kappa_0^2}{6} s^3 - \frac{\kappa_0}{8} \frac{d\kappa_0}{ds} s^4 + \cdots \right) \hat{u}_{t_0} \\ &\quad + \left[\frac{1}{2}\kappa_0 s^2 + \frac{1}{6}\frac{d\kappa_0}{ds} s^3 + \frac{1}{24}\left(\frac{d^2\kappa_0}{ds^2} - \kappa_0^3 - \kappa_0\tau_0^2 \right) s^4 + \cdots \right] \hat{u}_{n_0} \\ &\quad + \left[\frac{1}{6}\kappa_0\tau_0 s^3 + \frac{1}{24}\left(2\tau_0 \frac{d\kappa_0}{ds} + \kappa_0 \frac{d\tau_0}{ds} \right) s^4 + \cdots \right] \hat{u}_{b_0}\end{aligned} \tag{1.489}$$

It follows from these equations that in the neighborhood of a point at which $\kappa = 0$ the curve approximates a straight line. Furthermore, if $\tau = 0$ at a point, the curve remains on a plane. Accepting only the first term of each series, we may approximate a curve as

$$\mathbf{r}(s) - \mathbf{r}_0 \approx s\hat{u}_{t_0} + \tfrac{1}{2}\kappa_0 s^2 \hat{u}_{n_0} + \tfrac{1}{6}\kappa_0\tau_0 s^3 \hat{u}_{b_0} \tag{1.490}$$

Now assume that the position vector of the point P is expressed as a function of time $\mathbf{r} = \mathbf{r}(t)$. If at $t = t_0$ we have the position and velocity of P, it is possible to express the path of motion by a Taylor expansion:

$$\mathbf{r}(t) = \mathbf{r}_0 + (t - t_0)\,\dot{\mathbf{r}}_0 + \frac{(t - t_0)^2}{2!}\ddot{\mathbf{r}}_0 + \frac{(t - t_0)^3}{3!}\dddot{\mathbf{r}}_0 + \cdots \tag{1.491}$$

Using the natural coordinate system (1.454)–(1.461) and defining $\dot{s} = |\dot{\mathbf{r}}|$, we have

$$\dot{\mathbf{r}} = |\dot{\mathbf{r}}|\,\hat{u}_t = \dot{s}\hat{u}_t \tag{1.492}$$

$$\ddot{\mathbf{r}} = \ddot{s}\hat{u}_t + \kappa\dot{s}^2\hat{u}_n \tag{1.493}$$

$$\dddot{\mathbf{r}} = \left(\dddot{s} - \kappa^2\dot{s}^3 \right)\hat{u}_t + \kappa\left(3\dot{s}\ddot{s} + \dot{\kappa}\dot{s}^2 \right)\hat{u}_n + \kappa\tau\dot{s}^3\hat{u}_b \tag{1.494}$$

and therefore,

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + (t - t_0)\,\dot{s}\hat{u}_t + \frac{(t - t_0)^2}{2!}\left(\ddot{s}\hat{u}_t + \kappa\dot{s}^2\hat{u}_n \right) \\ &\quad + \frac{(t - t_0)^3}{3!}\left[\left(\dddot{s} - \kappa^2\dot{s}^3 \right)\hat{u}_t + \kappa\left(3\dot{s}\ddot{s} + \dot{\kappa}\dot{s}^2 \right)\hat{u}_n + \kappa\tau\dot{s}^3\hat{u}_b \right] + \cdots \\ &= \mathbf{r}_0 + \left((t - t_0)\,\dot{s} + \frac{(t - t_0)^2}{2!}\ddot{s} + \frac{(t - t_0)^3}{3!}\left(\dddot{s} - \kappa^2\dot{s}^3 \right) + \cdots \right)\hat{u}_t \\ &\quad + \left(\frac{(t - t_0)^2}{2!}\kappa\dot{s}^2 + \frac{(t - t_0)^3}{3!}\kappa\left(3\dot{s}\ddot{s} + \dot{\kappa}\dot{s}^2 \right) + \cdots \right)\hat{u}_n \\ &\quad + \left(\frac{(t - t_0)^3}{3!}\kappa\tau\dot{s}^3 + \cdots \right)\hat{u}_b\end{aligned} \tag{1.495}$$

Example 68 ★ Torsion of a Space Curve We may use (1.447) to determine the torsion of a curve analytically. Let us start with

$$\tau = -\hat{u}_n \cdot \frac{d\hat{u}_b}{ds} \tag{1.496}$$

and employ

$$\hat{u}_b = \hat{u}_t \times \hat{u}_n \qquad \hat{u}_t = \frac{d\mathbf{r}}{ds} \qquad \hat{u}_n = \rho \frac{d^2\mathbf{r}}{ds^2} \tag{1.497}$$

to get

$$\hat{u}_b = \rho \left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2} \right) \tag{1.498}$$

and hence

$$\tau = -\rho^2 \frac{d^2\mathbf{r}}{ds^2} \cdot \frac{d}{ds} \left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2} \right) = \rho^2 \frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3} \tag{1.499}$$

So, the scalar triple product of velocity, acceleration, and jerk $\left[\mathbf{v}, \mathbf{a}, \mathbf{j}\right]$ is

$$\left[\frac{d\mathbf{r}}{ds} \frac{d^2\mathbf{r}}{ds^2} \frac{d^3\mathbf{r}}{ds^3} \right] = \frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3} = \tau\kappa^2 \tag{1.500}$$

Example 69 ★ Darboux Vector By defining a vector $\mathbf{u}$ as

$$\mathbf{u} = \frac{1}{\rho}\hat{u}_b + \frac{1}{\sigma}\hat{u}_t \tag{1.501}$$

the Frenet–Serret formulas simplify to

$$\frac{d\hat{u}_t}{ds} = \mathbf{u} \times \hat{u}_t \qquad \frac{d\hat{u}_n}{ds} = \mathbf{u} \times \hat{u}_n \qquad \frac{d\hat{u}_b}{ds} = \mathbf{u} \times \hat{u}_b \tag{1.502}$$

The vector $\mathbf{u}$ is called the *Darboux vector*. Darboux (1842–1917) was a French mathematician.

Example 70 ★ Curvature as the Change of a Deformed Curve Curvature determines how the length of a curve changes as the curve is deformed. Consider an infinitesimal arc ds of a planar curve, as is shown in Figure 1.21. The arc length ds lies to second order on a circle of radius $\rho = 1/\kappa$. Let us push ds a distance dr in the direction of the curvature vector $\boldsymbol{\kappa}$. The arc length ds will change to $(1 - \kappa dr)\, ds$ that is on a new circle of radius $1/\kappa - dr = (1/\kappa)(1 - \kappa dr)$. In general, the displacement is not necessarily in direction $\boldsymbol{\kappa}$ and may be indicated by a vector $d\mathbf{r}$. In this case the change of the arc length is $1 - \boldsymbol{\kappa} \cdot d\mathbf{r}$ and hence, the rate of change of the curve length is $-\int \boldsymbol{\kappa} \cdot d\mathbf{v}\, ds$, where $\mathbf{v} = d\mathbf{r}/dt$.

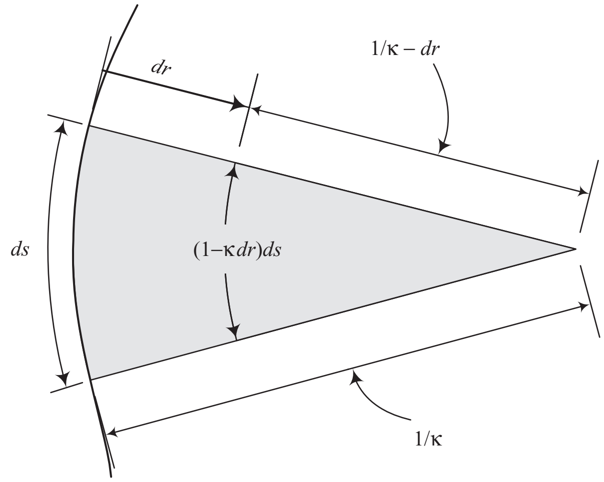

Figure 1.21 An infinitesimal arc *ds* of a planar curve.

Example 71 ★ Jerk in Natural Coordinate Frame $\hat{u}_t$, $\hat{u}_n$, $\hat{u}_b$ Employing Equation (1.353) and using the derivatives of the unit vectors of the natural coordinate frame,

$$\frac{d\hat{u}_t}{dt} = \frac{d\hat{u}_t}{dt} = \frac{\dot{s}}{\rho}\hat{u}_n \tag{1.503}$$

$$\frac{d\hat{u}_n}{dt} = \frac{\dot{s}}{\sigma}\hat{u}_b - \frac{\dot{s}}{\rho}\hat{u}_t \tag{1.504}$$

$$\frac{d\hat{u}_b}{dt} = -\frac{\dot{s}}{\sigma}\hat{u}_n \tag{1.505}$$

we can determine the jerk vector of a moving point in the natural coordinate frame:

$$\begin{aligned}
\mathbf{j} &= \frac{d}{dt}\mathbf{a} = \frac{d}{dt}\left(\ddot{s}\hat{u}_t + \frac{\dot{s}^2}{\rho}\hat{u}_n\right) \\
&= \dddot{s}\,\hat{u}_t + \ddot{s}\frac{d}{dt}\hat{u}_t + \frac{2\rho\dot{s}\ddot{s} - \dot{\rho}\dot{s}^2}{\rho^2}\hat{u}_n + \frac{\dot{s}^2}{\rho}\frac{d}{dt}\hat{u}_n \\
&= \dddot{s}\,\hat{u}_t + \ddot{s}\frac{\dot{s}}{\rho}\hat{u}_n + \frac{2\rho\dot{s}\ddot{s} - \dot{\rho}\dot{s}^2}{\rho^2}\hat{u}_n + \frac{\dot{s}^3}{\rho}\left(\frac{1}{\sigma}\hat{u}_b - \frac{1}{\rho}\hat{u}_t\right) \\
&= \left(\dddot{s} - \frac{\dot{s}^3}{\rho^2}\right)\hat{u}_t + \left(3\frac{\ddot{s}\dot{s}}{\rho} - \frac{\dot{\rho}\dot{s}^2}{\rho^2}\right)\hat{u}_n + \left(\frac{\dot{s}^3}{\rho\sigma}\right)\hat{u}_b
\end{aligned} \tag{1.506}$$

Example 72 ★ A Roller Coaster Figure 1.22 illustrates a roller coaster with the parametric equations

$$x = (a + b \sin\theta)\cos\theta$$
$$y = (a + b \sin\theta)\sin\theta \qquad (1.507)$$
$$z = b + b\cos\theta$$

for

$$a = 200\,\mathrm{m} \qquad b = 150\,\mathrm{m} \qquad (1.508)$$

Such a space curve is on the surface shown in Figure 1.23. The parametric equations of the surface are

$$x = (a + b \sin\theta)\cos\varphi$$
$$y = (a + b \sin\theta)\sin\varphi \qquad (1.509)$$
$$z = b + b\cos\theta$$

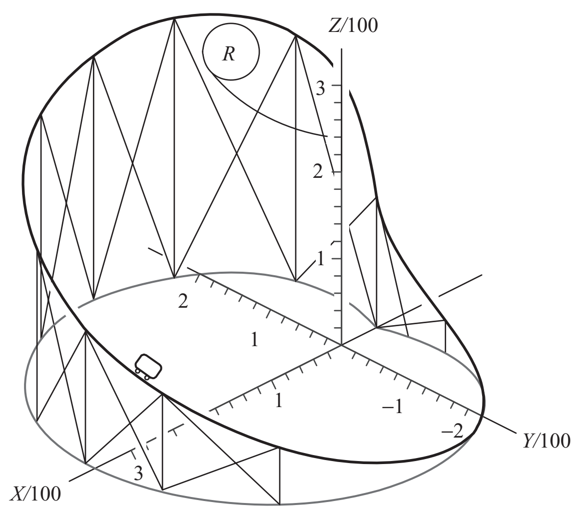

Figure 1.22 A roller coaster.

Let us assume that the car is a particle that moves on the roller coaster when the parameter θ is a function of time. The velocity and acceleration of the particle are

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = \begin{bmatrix} b\dot{\theta}\cos 2\theta - a\dot{\theta}\sin\theta \\ a\dot{\theta}\cos\theta + b\dot{\theta}\sin 2\theta \\ -b\dot{\theta}\sin\theta \end{bmatrix} \qquad (1.510)$$

$$\mathbf{a} = \frac{d}{dt}\mathbf{v} = \begin{bmatrix} (b\cos 2\theta - a\sin\theta)\,\ddot{\theta} - (a\cos\theta + 2b\sin 2\theta)\,\dot{\theta}^2 \\ (a\cos\theta + b\sin 2\theta)\,\ddot{\theta} + (2b\cos 2\theta - a\sin\theta)\,\dot{\theta}^2 \\ -b\ddot{\theta}\sin\theta - b\dot{\theta}^2\cos\theta \end{bmatrix} \qquad (1.511)$$

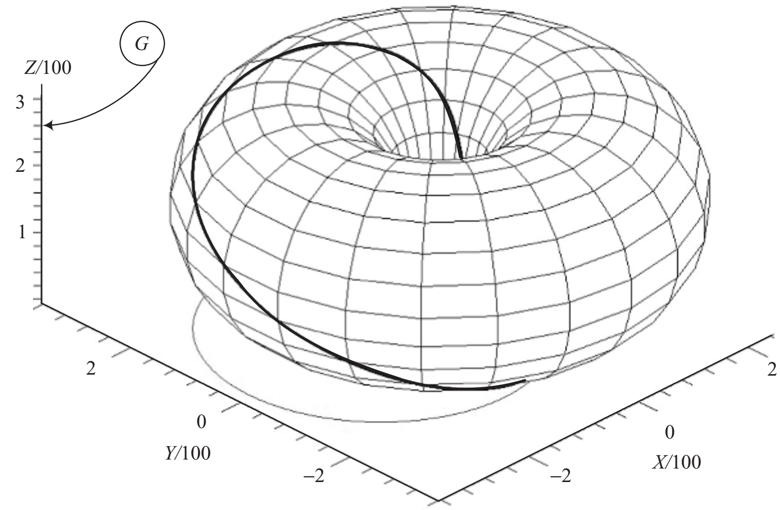

Figure 1.23 The path of the roller coaster is a space curve on the torus.

The equation of the tangent line (1.228) to the space curve is

$$\frac{x - x_0}{b\cos 2\theta - a\sin\theta} = \frac{y - y_0}{b\sin 2\theta + a\cos\theta} = \frac{z - z_0}{-b\sin\theta} \tag{1.512}$$

where

$$\begin{aligned} x_0 &= (a + b\sin\theta_0)\cos\theta_0 \\ y_0 &= (a + b\sin\theta_0)\sin\theta_0 \\ z_0 &= b + b\cos\theta_0 \end{aligned} \tag{1.513}$$

and

$$\begin{aligned} \frac{dx}{d\theta} &= b\cos 2\theta - a\sin\theta \\ \frac{dy}{d\theta} &= b\sin 2\theta + a\cos\theta \\ \frac{dz}{d\theta} &= -b\sin\theta \end{aligned} \tag{1.514}$$

As an example the tangent line at $\theta = \pi/4$ is

$$\frac{x - 216.42}{-141.42} = \frac{y - 216.42}{291.42} = \frac{z - 256.07}{-106.07} \tag{1.515}$$

because

$$\begin{aligned} x_0 &= \left(a + b\sin\frac{\pi}{4}\right)\cos\frac{\pi}{4} = 216.42 \text{ m} \\ y_0 &= \left(a + b\sin\frac{\pi}{4}\right)\sin\frac{\pi}{4} = 216.42 \text{ m} \\ z_0 &= b + b\cos\frac{\pi}{4} = 256.07 \text{ m} \end{aligned} \tag{1.516}$$

$$\begin{aligned}\frac{dx}{d\theta} &= b\cos 2\frac{\pi}{4} - a\sin\frac{\pi}{4} = -141.42\ \text{m/rad}\\ \frac{dy}{d\theta} &= b\sin 2\frac{\pi}{4} + a\cos\frac{\pi}{4} = 291.42\ \text{m/rad}\\ \frac{dz}{d\theta} &= -b\sin\frac{\pi}{4} = -106.07\ \text{m/rad}\end{aligned} \tag{1.517}$$

The arc length element ds of the space curve is

$$ds = \sqrt{\frac{d\mathbf{r}}{d\theta}\cdot\frac{d\mathbf{r}}{d\theta}}d\theta = \sqrt{\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial x}{\partial\theta}\right)^2}d\theta$$

$$= \frac{\sqrt{2}}{2}\sqrt{2a^2 + 3b^2 - b^2\cos 2\theta + 4ab\sin\theta}d\theta \tag{1.518}$$

The perpendicular plane (1.347) to the roller coaster curve is

$$(x - x_0)\frac{dx}{d\theta}\frac{d\theta}{ds} + (y - y_0)\frac{dy}{d\theta}\frac{d\theta}{ds} + (z - z_0)\frac{dz}{d\theta}\frac{d\theta}{ds} = 0 \tag{1.519}$$

$$(b\cos 2\theta - a\sin\theta)(x - x_0) + (b\sin 2\theta + a\cos\theta)(y - y_0) - b\sin\theta\,(z - z_0) = 0 \tag{1.520}$$

This perpendicular plane at $\theta = \pi/4$ is

$$-141.42x + 42y - 106.07z - 5302.7 = 0 \tag{1.521}$$

To find the osculating and rectifying planes, we also need to calculate the second derivatives of the curve:

$$\begin{aligned}\frac{d^2x}{d\theta^2} &= -a\cos\theta - 2b\sin 2\theta\\ \frac{d^2y}{d\theta^2} &= 2b\cos 2\theta - a\sin\theta\\ \frac{d^2z}{d\theta^2} &= -b\cos\theta\end{aligned} \tag{1.522}$$

The osculating plane (1.348) to the roller coaster curve can be found by the derivative with respect to the arc length ds:

$$\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right)(x - x_0) + \left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right)(y - y_0) + \left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right)(z - z_0) = 0 \tag{1.523}$$

The arc length is a function of θ, so we must transform (1.523) for the derivative with respect to θ. Consider d^2x/ds^2, which we may transform to a function of θ using (1.518):

$$\frac{d^2x}{ds^2} = \frac{d}{ds}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{dx}{d\theta}\frac{d\theta}{ds}\right) = \frac{d^2x}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2 + \frac{dx}{d\theta}\frac{d\theta}{ds}\frac{d}{d\theta}\left(\frac{d\theta}{ds}\right)$$
$$= \frac{4(-2b\sin 2\theta - a\cos\theta)}{6b^2 + 8ab\sin\theta + 4a^2 - 2b^2\cos 2\theta} - \frac{2(b\cos 2\theta - a\sin\theta)(8ab\cos\theta + 4b^2\sin 2\theta)}{\left(6b^2 + 8ab\sin\theta + 4a^2 - 2b^2\cos 2\theta\right)^2} \tag{1.524}$$

Following the same method, Equation (1.523) becomes

$$-\frac{b(a + 2b\sin\theta - 2b\sin\theta\cos^2\theta)}{a^2 + 2b^2 + 2ab\sin\theta - b^2\cos^2\theta}(x - x_0) + \frac{b^2\cos\theta(2\cos^2\theta - 3)}{a^2 + 2b^2 + 2ab\sin\theta - b^2\cos^2\theta}(y - y_0) + \frac{a^2 + 2b^2 + 3ab\sin\theta}{a^2 + 2b^2 + 2ab\sin\theta - b^2\cos^2\theta}(z - z_0) = 0 \tag{1.525}$$

This osculating plane at $\theta = \pi/4$ is

$$-0.395174x + 0.273892y + 1.27943z - 301.37061 = 0 \tag{1.526}$$

The rectifying plane (1.369) is

$$-3.45942x - 1.91823y - .65786z + 1332.29646 = 0 \tag{1.527}$$

Figure 1.24 shows the space curve and the three planes—perpendicular, osculating, and rectifying—at $\theta = \pi/4$.

The curvature κ of the space curve (1.507) from (1.377) and (1.518) is

$$\kappa = \frac{d\theta}{ds} = \frac{2}{\sqrt{4a^2 + 6b^2 - 2b^2\cos 2\theta + 8ab\sin\theta}} \tag{1.528}$$

and therefore the curvature radius of the helix at that point is

$$\rho = \frac{1}{\kappa} = \frac{1}{2}\sqrt{4a^2 + 6b^2 - 2b^2\cos 2\theta + 8ab\sin\theta} \tag{1.529}$$

The equations of the three planes and the curvature κ enable us to identify the unit vectors $\hat{u}_t$, $\hat{u}_n$, and $\hat{u}_b$. The tangent unit vector $\hat{u}_t$ is given as

$$\hat{u}_t = \frac{d\mathbf{r}}{ds} = \begin{bmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \\ \frac{dz}{ds} \end{bmatrix} = \begin{bmatrix} \frac{dx}{d\theta}\frac{d\theta}{ds} \\ \frac{dy}{d\theta}\frac{d\theta}{ds} \\ \frac{dz}{d\theta}\frac{d\theta}{ds} \end{bmatrix} = \kappa \begin{bmatrix} \frac{dx}{d\theta} \\ \frac{dy}{d\theta} \\ \frac{dz}{d\theta} \end{bmatrix} = \kappa \begin{bmatrix} b\cos 2\theta - a\sin\theta \\ b\sin 2\theta + a\cos\theta \\ -b\sin\theta \end{bmatrix} \tag{1.530}$$

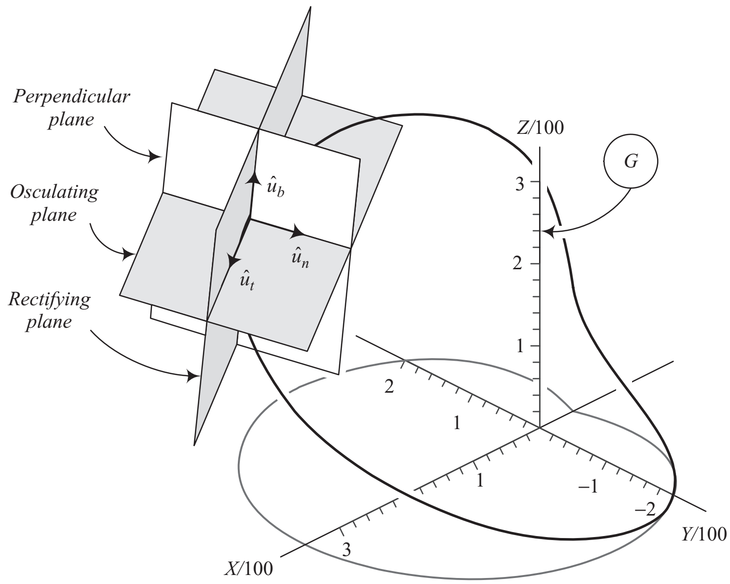

Figure 1.24 The space curve of a roller coaster and the three planes—perpendicular, osculating, and rectifying—at a specific point.

and the normal unit vector $\hat{u}_n$ as

$$\hat{u}_n = \rho \frac{d^2\mathbf{r}}{ds^2} = \rho \begin{bmatrix} \dfrac{d^2x}{ds^2} \\ \dfrac{d^2y}{ds^2} \\ \dfrac{d^2z}{ds^2} \end{bmatrix} = \begin{bmatrix} \kappa \dfrac{d^2x}{d\theta^2} + \dfrac{dx}{d\theta}\dfrac{d\kappa}{d\theta} \\ \kappa \dfrac{d^2y}{d\theta^2} + \dfrac{dy}{d\theta}\dfrac{d\kappa}{d\theta} \\ \kappa \dfrac{d^2z}{d\theta^2} + \dfrac{dz}{d\theta}\dfrac{d\kappa}{d\theta} \end{bmatrix} \tag{1.531}$$

where

$$\frac{d\kappa}{d\theta} = -\frac{b\,(a + b\sin\theta)\cos\theta}{\left(a^2 + 2b^2 - b^2\cos^2\theta + 2ab\sin\theta\right)^{3/2}} \tag{1.532}$$

and the other terms come from Equations (1.528), (1.522), and (1.514).

The bivector unit vector $\hat{u}_b$ from (1.366) and (1.525) is then

$$\hat{u}_b = \frac{\dfrac{d\mathbf{r}}{ds} \times \dfrac{d^2\mathbf{r}}{ds^2}}{\left|\dfrac{d\mathbf{r}}{ds} \times \dfrac{d^2\mathbf{r}}{ds^2}\right|} = \frac{1}{u_b} \begin{bmatrix} \dfrac{dy}{ds}\dfrac{d^2z}{ds^2} - \dfrac{dz}{ds}\dfrac{d^2y}{ds^2} \\ \dfrac{dz}{ds}\dfrac{d^2x}{ds^2} - \dfrac{dx}{ds}\dfrac{d^2z}{ds^2} \\ \dfrac{dx}{ds}\dfrac{d^2y}{ds^2} - \dfrac{dy}{ds}\dfrac{d^2x}{ds^2} \end{bmatrix}$$

$$= \frac{2}{\sqrt{Z}} \begin{bmatrix} -b(a + 2b\sin\theta - 2b\sin\theta\cos^2\theta) \\ b^2\cos\theta(2\cos^2\theta - 3) \\ a^2 + 2b^2 + 3ab\sin\theta \end{bmatrix} \tag{1.533}$$

$$Z = 4a^4 + 26b^4 + 38a^2b^2 + 4ab(6a^2 + 15b^2)\sin\theta - 6b^2(3a^2 + b^2)\cos 2\theta - 4ab^3 \sin 3\theta \tag{1.534}$$

Example 73 ★ Curvature Center of a Roller Coaster The position of the center of curvature of a space curve can be shown by a vector $\mathbf{r}_c$, where

$$\mathbf{r}_c = \rho \hat{u}_n \tag{1.535}$$

The radius of curvature and the normal unit vector of the roller coaster space curve (1.507) are give in Equations (1.529) and (1.531). Therefore, the position of the curvature center of the roller coaster is

$$\mathbf{r} + \mathbf{r}_c = \begin{bmatrix} (a + b\sin\theta)\cos\theta \\ (a + b\sin\theta)\sin\theta \\ b + b\cos\theta \end{bmatrix} + \begin{bmatrix} \dfrac{d^2x}{d\theta^2} + \rho\dfrac{dx}{d\theta}\dfrac{d\kappa}{d\theta} \\ \dfrac{d^2y}{d\theta^2} + \rho\dfrac{dy}{d\theta}\dfrac{d\kappa}{d\theta} \\ \dfrac{d^2z}{d\theta^2} + \rho\dfrac{dz}{d\theta}\dfrac{d\kappa}{d\theta} \end{bmatrix} \tag{1.536}$$

$$\rho = \frac{1}{2}\sqrt{4a^2 + 6b^2 - 2b^2\cos 2\theta + 8ab\sin\theta} \tag{1.537}$$

Figure 1.25 illustrates the path of motion and the path of curvature center. The initial positions at $\theta = 0$ are indicated by two small circles and the direction of motion by increasing θ is shown by two small arrows.

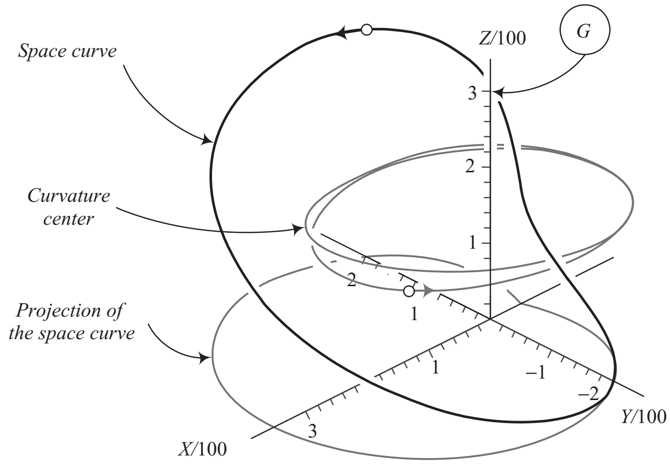

Figure 1.25 The path of motion of a roller coaster and the path of its curvature center.

1.6 FIELDS

A field is a domain of space in which there is a physical quantity associated with every point of the space. If the physical quantity is scalaric, the field is called a *scalar field*, and if the physical quantity is vectorial, the field is a *vector field*. Furthermore, a field is called *stationary* or *time invariant* if it is independent of time. A field that changes with time is a *nonstationary* or *time-variant* field.

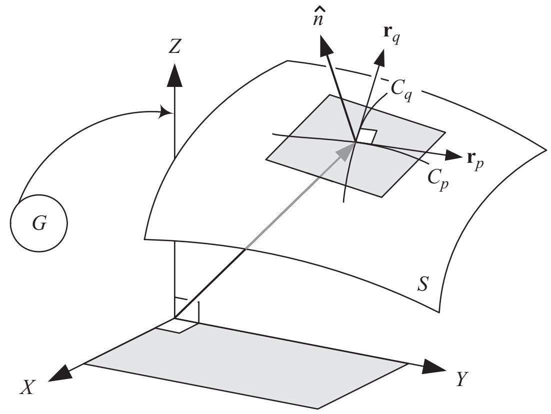

Figure 1.26 A surface ${}^G\mathbf{r} = {}^G\mathbf{r}(p, q)$ and partial derivatives $\mathbf{r}_p$ and $\mathbf{r}_q$.

1.6.1 Surface and Orthogonal Mesh

If the position vector ${}^G\mathbf{r}_P$ of a moving point P is such that each component is a function of two variables p and q,

$$ {}^G\mathbf{r} = {}^G\mathbf{r}(p, q) = x(p, q)\,\hat{\imath} + y(p, q)\,\hat{\jmath} + z(p, q)\,\hat{k} \tag{1.538}$$

then the end point of the vector indicates a surface S in G, as is shown in Figure 1.26. The surface ${}^G\mathbf{r} = {}^G\mathbf{r}(p, q)$ reduces to a curve on S if we fix one of the parameters q or p. The curves C_p and C_q on S at (p_0, q_0) are indicated by single-variable vectors ${}^G\mathbf{r}(p, q_0)$ and ${}^G\mathbf{r}(p_0, q)$, respectively.

At any specific point ${}^G\mathbf{r} = {}^G\mathbf{r}(p_0, q_0)$ there is a tangent plane to the surface that is indicated by a normal unit vector $\hat{n}$,

$$\hat{n} = \hat{n}(p_0, q_0) = \frac{\mathbf{r}_p \times \mathbf{r}_q}{\left|\mathbf{r}_p \times \mathbf{r}_q\right|} \tag{1.539}$$

where $\mathbf{r}_p$ and $\mathbf{r}_q$ are partial derivatives of ${}^G\mathbf{r}$:

$$\mathbf{r}_p = \frac{\partial\,\mathbf{r}(p, q_0)}{\partial p} \tag{1.540}$$

$$\mathbf{r}_q = \frac{\partial\,\mathbf{r}(p_0, q)}{\partial q} \tag{1.541}$$

Varying p and q provides a set of curves C_p and C_q that make a mesh of S. The mesh is called *orthogonal* if we have

$$\mathbf{r}_p \cdot \mathbf{r}_q = 0 \tag{1.542}$$

Proof: By fixing one of the variables, say $p = p_0$, we can make a single-variable vector function ${}^G\mathbf{r} = {}^G\mathbf{r}(p_0, q)$ to define a curve C_q lying on the surface S. Similarly, we may fix $q = q_0$ to define another single-variable vector function ${}^G\mathbf{r} = {}^G\mathbf{r}(p, q_0)$ and curve C_p. So, there are two curves C_p and C_q that pass through the point (p_0, q_0).

The vectors

$$\mathbf{r}_p = \frac{\partial\,\mathbf{r}\,(p, q_0)}{\partial p} = \frac{\partial\,x\,(p, q_0)}{\partial p}\hat{\imath} + \frac{\partial\,y\,(p, q_0)}{\partial p}\hat{\jmath} + \frac{\partial\,z\,(p, q_0)}{\partial p}\hat{k} \tag{1.543}$$

$$\mathbf{r}_q = \frac{\partial\,\mathbf{r}\,(p_0, q)}{\partial q} = \frac{\partial\,x\,(p_0, q)}{\partial q}\hat{\imath} + \frac{\partial\,y\,(p_0, q)}{\partial q}\hat{\jmath} + \frac{\partial\,z\,(p_0, q)}{\partial q}\hat{k} \tag{1.544}$$

that are tangent to the curves C_p and C_q are called the *partial derivatives* of ${}^{G}\mathbf{r}\,(p, q)$. The vectors $\mathbf{r}_p$ and $\mathbf{r}_q$ define a *tangent plane*. The tangent plane may be indicated by a unit *normal vector* $\hat{n}$:

$$\hat{n} = \hat{n}\,(p_0, q_0) = \frac{\mathbf{r}_p \times \mathbf{r}_q}{\left|\mathbf{r}_p \times \mathbf{r}_q\right|} \tag{1.545}$$

A surface for which a normal vector $\hat{n}$ can be constructed at any point is called *orientable*. An orientable surface has inner and outer sides. At each point $x\,(p_0, q_0)$, $y\,(p_0, q_0)$, $z\,(p_0, q_0)$ of an orientable surface S there exists a normal axis on which we can choose two directions $\hat{n}_0$, $-\hat{n}_0$. The positive normal vector $\hat{n}_0$ cannot be coincident with $-\hat{n}_0$ by a continuous displacement. The normal unit vector on the convex side is considered positive and the normal to the concave side negative.

If $\mathbf{r}_p \cdot \mathbf{r}_q = 0$ at any point on the surface S, the mesh that is formed by curves C_p and C_q is called an *orthogonal mesh*. The set of unit vectors of an orthogonal mesh,

$$\hat{u}_p = \frac{\mathbf{r}_p}{\left|\mathbf{r}_p\right|} \tag{1.546}$$

$$\hat{u}_q = \frac{\mathbf{r}_q}{\left|\mathbf{r}_q\right|} \tag{1.547}$$

$$\hat{n} = \hat{u}_p \times \hat{u}_q \tag{1.548}$$

defines an orthogonal coordinate system. These definitions are consistent with the definition of unit vectors in Equation (1.200).

★ We assume that the functions $x\,(p, q)$, $y\,(p, q)$, and $z\,(p, q)$ in the parametric expression of a surface in Equation (1.538) have continuous derivatives with respect to the variables q and p. For such a surface, we can define a Jacobian matrix $[J]$ using partial derivatives of the functions x, y, and z:

$$[J] = \begin{bmatrix} x_p & x_q \\ y_p & y_q \\ z_p & z_q \end{bmatrix} \tag{1.549}$$

The surface at a point $P\,(p_0, q_0)$ is called *regular* if and only if the rank of $[J]$ is not less than 2. A point P at which $[J]$ has rank 1 is called a *singular* point. At a regular point, we have

$$\mathbf{r}_p \times \mathbf{r}_q \neq 0 \tag{1.550}$$

Therefore, we can determine the tangent plane unit-normal vector $\hat{n}$ for every regular point. At a singular point, the rank of $[J]$ is 1 and we have

$$\mathbf{r}_p \times \mathbf{r}_q = 0 \tag{1.551}$$

which indicates $\mathbf{r}_p$ and $\mathbf{r}_q$ are parallel. There is not a unique tangent plane at a singular point.

A surface that has no singularity is called an *immersed surface*. ■

Example 74 Sphere and Orthogonal Mesh A sphere is defined as the position of all points (x, y, z) that have the same distance R from the center (x_0, y_0, z_0):

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2 \tag{1.552}$$

Consider a moving point P on a sphere with a center at the origin. The position vector of P is

$$\begin{aligned}\mathbf{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ &= (R\cos\theta\sin\varphi)\,\hat{\imath} + (R\sin\theta\sin\varphi)\,\hat{\jmath} + (R\cos\varphi)\,\hat{k}\end{aligned} \tag{1.553}$$

Eliminating θ and φ between x, y, and z generates the surface equation:

$$z = \pm\sqrt{R^2 - x^2 - y^2} \tag{1.554}$$

As a sample, when $(\theta, \varphi) = (\pi/3, \pi/4)$, point P is at $(x, y, z) = (0.35355R, 0.61237R, 0.70711R)$ and we may define two curves C_θ and C_φ as

$$C_\theta = \begin{cases} x = R\cos\dfrac{\pi}{3}\sin\varphi \\ y = R\sin\dfrac{\pi}{3}\sin\varphi \\ z = R\cos\varphi \end{cases} = \begin{cases} x = 0.5\,R\sin\varphi \\ y = 0.866\,03\,R\sin\varphi \\ z = R\cos\varphi \end{cases} \tag{1.555}$$

$$C_\varphi = \begin{cases} x = R\cos\theta\sin\dfrac{\pi}{4} \\ y = R\sin\theta\sin\dfrac{\pi}{4} \\ z = R\cos\dfrac{\pi}{4} \end{cases} = \begin{cases} x = 0.707\,11\,R\cos\theta \\ y = 0.707\,11\,R\sin\theta \\ z = 0.707\,11\,R \end{cases} \tag{1.556}$$

The tangent vectors to C_θ and C_φ at arbitrary θ and φ can be found by partial derivatives:

$$\mathbf{r}_\theta = \frac{\partial\,\mathbf{r}\,(\theta, \varphi)}{\partial\theta} = -R\sin\theta\sin\varphi\,\hat{\imath} + R\cos\theta\sin\varphi\,\hat{\jmath} \tag{1.557}$$

$$\mathbf{r}_\varphi = \frac{\partial\,\mathbf{r}\,(\theta, \varphi)}{\partial\varphi} = R\cos\theta\cos\varphi\,\hat{\imath} + R\sin\theta\cos\varphi\,\hat{\jmath} - R\sin\varphi\,\hat{k} \tag{1.558}$$

These tangent vectors at the point P reduce to

$$\mathbf{r}_\varphi = \frac{\partial\,\mathbf{r}\,(\pi/3, \varphi)}{\partial\varphi} = \begin{bmatrix} 0.5R\cos\varphi \\ 0.86603R\cos\varphi \\ -R\sin\varphi \end{bmatrix} \tag{1.559}$$

$$\mathbf{r}_\theta = \frac{\partial\,\mathbf{r}\,(\theta, \pi/4)}{\partial\theta} = \begin{bmatrix} -0.70711R\sin\theta \\ 0.70711R\cos\theta \\ 0 \end{bmatrix} \tag{1.560}$$

We can check the orthogonality of the curves C_θ and C_φ for different θ and φ by examining the inner product of $\mathbf{r}_\theta$ and $\mathbf{r}_\varphi$ from (1.557) and (1.558):

$$\mathbf{r}_\theta \cdot \mathbf{r}_\varphi = \begin{bmatrix} -R\sin\theta\sin\varphi \\ R\cos\theta\sin\varphi \\ 0 \end{bmatrix} \cdot \begin{bmatrix} R\cos\theta\cos\varphi \\ R\sin\theta\cos\varphi \\ -R\sin\varphi \end{bmatrix} = 0 \tag{1.561}$$

The tangent vectors $\mathbf{r}_\theta$ and $\mathbf{r}_\varphi$ define a tangent plane with a unit-normal vector $\hat{n}$:

$$\begin{aligned} \hat{n} = \hat{n}\left(\frac{\pi}{3}, \frac{\pi}{4}\right) &= \frac{\mathbf{r}_\theta \times \mathbf{r}_\varphi}{\left|\mathbf{r}_\theta \times \mathbf{r}_\varphi\right|} \\ &= \frac{1}{0.70711R^2} \begin{bmatrix} 0.25R^2 \\ 0.43301R^2 \\ 0.5R^2 \end{bmatrix} = \begin{bmatrix} 0.35355 \\ 0.61237 \\ 0.7071 \end{bmatrix} \end{aligned} \tag{1.562}$$

Therefore, we may establish an orthogonal coordinate system at $P(\theta, \varphi, r) = (\pi/3, \pi/4, R)$ with the following unit vectors:

$$\hat{u}_\theta = \frac{\mathbf{r}_\theta}{|\mathbf{r}_\theta|} = \frac{\partial\mathbf{r}/\partial\theta}{|\partial\mathbf{r}/\partial\theta|} = -0.86602\hat{\imath} + 0.5\hat{\jmath} \tag{1.563}$$

$$\hat{u}_\varphi = \frac{\mathbf{r}_\varphi}{\left|\mathbf{r}_\varphi\right|} = \frac{\partial\mathbf{r}/\partial\varphi}{|\partial\mathbf{r}/\partial\varphi|} = 0.35355\hat{\imath} + 0.61238\hat{\jmath} - 0.70711\hat{k} \tag{1.564}$$

$$\hat{n} = \frac{\mathbf{r}_\theta \times \mathbf{r}_\varphi}{\left|\mathbf{r}_\theta \times \mathbf{r}_\varphi\right|} = -0.35356\hat{\imath} - 0.61237\hat{\jmath} - 0.70711\hat{k} \tag{1.565}$$

Example 75 Surface-Analytic Expressions There are several methods to express a surface. Three of them are the most applied forms: parametric, *Monge*, and implicit.

The parametric expression of a surface is when the x-, y-, and z-components of a position vector are functions of two parameters:

$${}^G\mathbf{r} = {}^G\mathbf{r}(p, q) = x(p, q)\,\hat{\imath} + y(p, q)\,\hat{\jmath} + z(p, q)\,\hat{k} \tag{1.566}$$

The Monge expression of a surface is when we eliminate the parameters p and q from x, y, z and define z as a function of x and y:

$${}^G\mathbf{r}(x, y) = x\hat{\imath} + y\hat{\jmath} + z(x, y)\,\hat{k} \tag{1.567}$$

The implicit form of a surface is a nonlinear equation f of x, y, z:

$$f(x, y, z) = 0 \tag{1.568}$$

Example 76 Directional Cosines of Unit-Normal Vector $\hat{n}$ We are able to solve the first two equations of the parametric expression of a surface,

$$x = x(p, q) \qquad y = y(p, q) \qquad z = z(p, q) \tag{1.569}$$

for p and q, and define the surface by a function

$$z = z(x, y) = g(x, y) \tag{1.570}$$

and write the vector representation of the surface by the Monge expression

$${}^{G}\mathbf{r}(x, y) = x\hat{\imath} + y\hat{\jmath} + g(x, y)\hat{k} \tag{1.571}$$

The partial derivatives and the equation of the two curves C_x and C_y would be

$$\mathbf{r}_x = \frac{\partial \mathbf{r}}{\partial x} = \hat{\imath} + \frac{\partial g(x, y)}{\partial x}\hat{k} \tag{1.572}$$

$$\mathbf{r}_y = \frac{\partial \mathbf{r}}{\partial y} = \hat{\jmath} + \frac{\partial g(x, y)}{\partial y}\hat{k} \tag{1.573}$$

The cross product of $\mathbf{r}_x$ and $\mathbf{r}_y$ is

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g(x, y)}{\partial x}\hat{\imath} - \frac{\partial g(x, y)}{\partial y}\hat{\jmath} + \hat{k} \tag{1.574}$$

and hence the unit-normal vector $\hat{n}$ is

$$\hat{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\left|\mathbf{r}_x \times \mathbf{r}_y\right|} = \frac{-\dfrac{\partial z}{\partial x}\hat{\imath} - \dfrac{\partial z}{\partial y}\hat{\jmath} + \hat{k}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}} \tag{1.575}$$

The normal vector (1.575) can never be horizontal.

As an example, the position vector of a moving point on the northern hemisphere of the sphere,

$$z = +\sqrt{R^2 - x^2 - y^2} \tag{1.576}$$

is

$${}^{G}\mathbf{r}(x, y) = x\hat{\imath} + y\hat{\jmath} + \sqrt{R^2 - x^2 - y^2}\hat{k} \tag{1.577}$$

The partial derivatives $\mathbf{r}_x$ and $\mathbf{r}_y$ and the unit-normal vector $\hat{n}$ are

$$\mathbf{r}_x = \frac{\partial\, \mathbf{r}}{\partial x} = \hat{\imath} - \frac{x}{\sqrt{R^2 - x^2 - y^2}}\hat{k} \tag{1.578}$$

$$\mathbf{r}_y = \frac{\partial\, \mathbf{r}}{\partial y} = \hat{\jmath} - \frac{y}{\sqrt{R^2 - x^2 - y^2}}\hat{k} \tag{1.579}$$

$$\hat{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\left|\mathbf{r}_x \times \mathbf{r}_y\right|} = \frac{1}{R}\begin{bmatrix} x \\ y \\ \sqrt{R^2 - x^2 - y^2} \end{bmatrix} \tag{1.580}$$

It shows that the normal vector to a sphere is always in the direction of the position vector $\mathbf{r}$ and away from the center.

These vectors may be used to make an orthogonal coordinate system. At a point such as $(x, y, z) = (0.35355R, 0.61237R, 0.70711R)$, we have

$$\hat{u}_x = \frac{\mathbf{r}_x}{|\mathbf{r}_x|} = \frac{\hat{\imath} - 0.49999\hat{k}}{1.118} = 0.89445\hat{\imath} - 0.44722\hat{k}$$
$$\hat{u}_y = \frac{\mathbf{r}_y}{|\mathbf{r}_y|} = \frac{\hat{\jmath} - 0.86601\hat{k}}{1.3229} = 0.75592\hat{\jmath} - 0.65463\hat{k} \tag{1.581}$$
$$\hat{n} = \hat{u}_x \times \hat{u}_y = 0.33806\hat{\imath} + 0.58553\hat{\jmath} + 0.67613\hat{k}$$

Example 77 Equation of a Tangent Plane Consider a vector $\mathbf{n}$,

$$\mathbf{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} \tag{1.582}$$

that is perpendicular to a plane at a point (x_0, y_0, z_0). The analytic equation of the plane that includes the point (x_0, y_0, z_0) is indicated by position vector ${}^G\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ such that the vector ${}^G\mathbf{r} - {}^G\mathbf{r}_0$ is perpendicular to $\mathbf{n}$,

$$\left({}^G\mathbf{r} - {}^G\mathbf{r}_0\right) \cdot \mathbf{n} = 0 \tag{1.583}$$

which reduces to

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{1.584}$$

Because the normal vector to a surface $z = g(x, y)$ is

$$\mathbf{n} = -\frac{\partial g(x, y)}{\partial x}\hat{\imath} - \frac{\partial g(x, y)}{\partial y}\hat{\jmath} + \hat{k} \tag{1.585}$$

the equation of the tangent plane to the surface at a point (x_0, y_0, z_0) is

$$z - z_0 = \frac{\partial g(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial g(x_0, y_0)}{\partial y}(y - y_0) \tag{1.586}$$

As an example consider a surface $z = 10 - x^2 - y^2$ and a point P at $(x_0, y_0, z_0) = (1, 2, 5)$. The normal vector at P is

$$\mathbf{n} = 2\hat{\imath} + 4\hat{\jmath} + \hat{k} \tag{1.587}$$

and the tangent plane at P is

$$z - 5 = -2(x - 1) - 4(y - 2) \tag{1.588}$$

Example 78 Normal Vector to a Surface Let us eliminate the parameters p and q from the equations of a surface,

$$x = x(p, q) \qquad y = y(p, q) \qquad z = z(p, q) \tag{1.589}$$

and define the surface by a function

$$z = z(x, y) = g(x, y) \tag{1.590}$$

or alternatively by

$$f = f(x, y, z) \tag{1.591}$$

So, we theoretically have

$$f(x, y, z) = z - g(x, y) \tag{1.592}$$

The normal vector to surface (1.590) is

$$\mathbf{n} = -\frac{\partial z}{\partial x}\hat{\imath} - \frac{\partial z}{\partial y}\hat{\jmath} + \hat{k} \tag{1.593}$$

However, we may use expression (1.592) and substitute the partial derivatives

$$\frac{\partial f}{\partial x} = -\frac{\partial z}{\partial x} \qquad \frac{\partial f}{\partial y} = -\frac{\partial z}{\partial y} \qquad \frac{\partial f}{\partial z} = 1 \tag{1.594}$$

to define the normal vector to the surface (1.592) by

$$\mathbf{n} = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \tag{1.595}$$

Such an expression of a normal vector to a surface is denoted by $\mathbf{n} = \nabla f$ and is called the *gradient* of f:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \tag{1.596}$$

Example 79 ★ Curvature of a Surface Consider a point P on a surface z with a continuous second derivative, as is shown in Figure 1.27:

$$z = f(x, y) \tag{1.597}$$

To determine the curvature of the surface at P, we find the unit-normal vector $\hat{u}_n$ to the surface at P,

$$\hat{u}_n = \frac{\mathbf{n}}{|\mathbf{n}|} \tag{1.598}$$

$$\mathbf{n} = \left(-\frac{\partial g(x, y)}{\partial x}\hat{\imath} - \frac{\partial g(x, y)}{\partial y}\hat{\jmath} + \hat{k}\right) \tag{1.599}$$

and slice the surface by planes containing $\hat{u}_n$ to consider the curvature vector $\boldsymbol{\kappa}$ of the intersection curve. The curvature vector at P on any intersecting curve will be

$$\boldsymbol{\kappa} = \kappa \hat{u}_n \tag{1.600}$$

The value of κ will change by turning the plane around $\hat{u}_n$. The minimum and maximum values of κ are indicated by κ_1 and κ_2 and are called the *principal curvatures*, where κ_1 and κ_2 occur in orthogonal directions . They may be used to determine the curvature in any other directions.

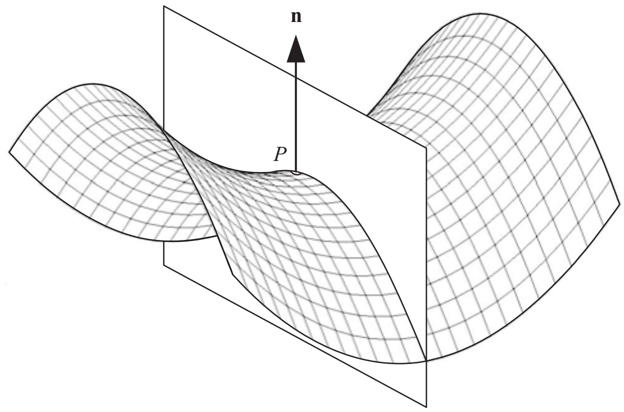

Figure 1.27 A point P on a surface $z = f(x, y)$ with a continuous second derivative.

If the unit vector tangent to the curve at P is shown by $\hat{u}_t$, the intersection curve is in the plane spanned by $\hat{u}_t$ and $\hat{u}_n$. The curvature of the curve κ is called the directional curvature at P in direction $\hat{u}_t$ and defined by

$$\kappa = \hat{u}_t^{\mathrm{T}} \; [D_2 z] \; \hat{u}_t = \hat{u}_t^{\mathrm{T}} \begin{bmatrix} \dfrac{\partial^2 z}{\partial x^2} & \dfrac{\partial z}{\partial x \partial y} \\ \dfrac{\partial z}{\partial y \partial x} & \dfrac{\partial^2 z}{\partial y^2} \end{bmatrix} \hat{u}_t \tag{1.601}$$

The matrix $[D_2 z]$ and vector $\hat{u}_t$ should be determined at point P.

The principal curvatures κ_1 and κ_2 are the eigenvalues of $[D_2 z]$ and their associated directions are called the principal directions of surface z at P. If the coordinate frame (x, y, z) is set up such that z is on $\hat{u}_n$ and x, y are in the principal directions, then the frame is called the principal coordinate frame. The second-derivative matrix $[D_2 z]$ in a principal coordinate frame would be

$$[D_2 z] = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \tag{1.602}$$

1.6.2 Scalar Field and Derivative

Consider a scalar function f of a vector variable $\mathbf{r}$,

$$f = f(\mathbf{r}) = f(x, y, z) \tag{1.603}$$

such that it provides a number f at a point $P(x, y, z)$. Having such a function is equivalent to associating a numeric value to every point of the space. The space that $f(x, y, z)$ is defined in is called a *scalar field*, and the function f is called the *scalar field function*. The field function is assumed to be smooth and differentiable. A smooth field has no singularity, jump, sink, or source.

Setting f equal to a specific value f_0 defines a surface

$$f(x, y, z) = f_0 \tag{1.604}$$

that is the loci of all points for which f takes the fixed value f_0. The surface $f(x, y, z) = f_0$ is called an *isosurface* and the associated field value is called the *isovalue* f_0.

The space derivative of f for an infinitesimal displacement $d\mathbf{r}$ is a vector:

$$\frac{df(\mathbf{r})}{d\mathbf{r}} = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = \nabla f \tag{1.605}$$

It can also be shown by

$$df = \nabla f \cdot d\mathbf{r} \tag{1.606}$$

where at any point ${}^G\mathbf{r} = {}^G\mathbf{r}(x, y, z)$ there exists a vector ∇f that indicates the value and direction of the maximum change in f for an infinitesimal change $d\mathbf{r}$ in position.

Figure 1.28 illustrates an isosurface f_0 and the vector ∇f at a point on the isosurface,

$$\nabla f = \nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \tag{1.607}$$

The vector ∇f is called the *gradient* of the scalar field . The gradient (1.607) can be expressed by a vectorial derivative operator ∇,

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \tag{1.608}$$

that operates on the scalar field f. The *gradient operator* ∇ is also called the *grad*, *del*, or *nabla* operator.

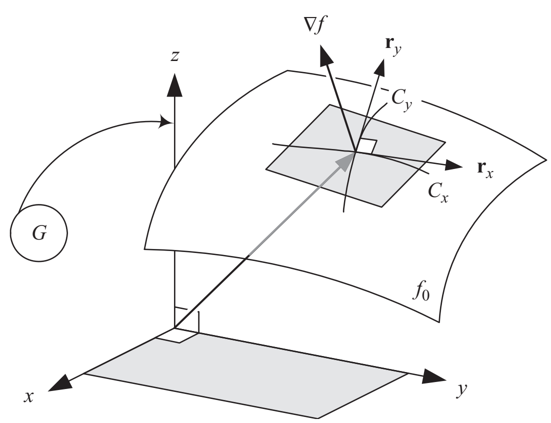

Figure 1.28 An isosurface f_0 and its gradient vector at a point on the isosurface.

Proof: By assigning various values to f, we obtain a family of isosurfaces of the scalar field $f = f(\mathbf{r}) = f(x, y, z)$ as is shown in Figure 1.29. These surfaces serve to geometrically visualize the field's characteristics.

An isosurface $f_0 = f(x, y, z)$ can be expressed by a position vector ${}^G\mathbf{r}$,

$$ {}^G\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1.609} $$

where its components x, y, z are constrained by the isosurface equation (1.604). Let us consider a point P at $\mathbf{r} = \mathbf{r}(x, y, z)$ on an isosurface $f(x, y, z) = f$. Any infinitesimal change

$$ d\mathbf{r} = dx\ \hat{\imath} + dy\ \hat{\jmath} + dz\ \hat{k} \tag{1.610} $$

in the position of P will move the point to a new isosurface with a field value $f + df$, where

$$ \begin{aligned} df &= f\,(x + dx, y + dy, z + dz) - f(x, y, z) \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(dx\ \hat{\imath} + dy\hat{\jmath} + dz\ \hat{k}\right) = \nabla f \cdot d\mathbf{r} \end{aligned} \tag{1.611} $$

So *df* can be interpreted as an inner product between two vectors ∇f and $d\mathbf{r}$. The first vector, denoted by

$$ \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \tag{1.612} $$

is a Cartesian expression of the *gradient* of the scalar function f, and the second vector $d\mathbf{r}$ is the displacement vector of the point. If the two nearby points lie on the same isosurface, then $df = 0$, $d\mathbf{r}$ would be a tangent vector to this isosurface, and

$$ df = \nabla f \cdot d\mathbf{r} = 0 \tag{1.613} $$

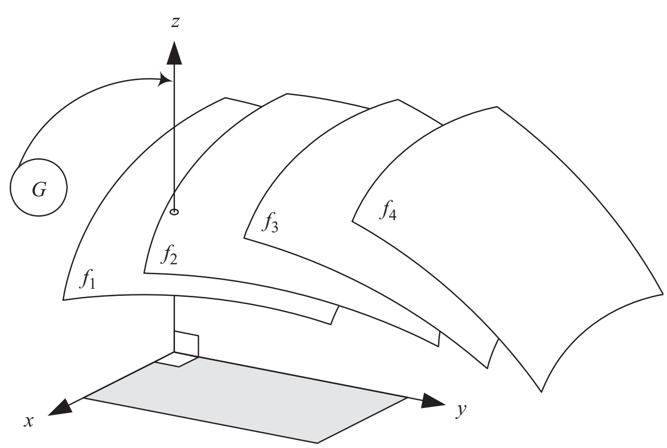

Figure 1.29 A family of isosurfaces of a scalar field $f = f\,(\mathbf{r})$.

Therefore, ∇f is perpendicular to $d\mathbf{r}$ and hence normal to the isosurface f. The gradient of a scalar field is a coordinate-independent property.

We examine a nonstationary field at the interested specific instant of time. ■

Example 80 Derivative of Scalar Function with Vector Variable If $\mathbf{c}$ is a constant vector and $f = \mathbf{c} \cdot \mathbf{r}$ is a scalar field, then

$$\text{grad } f = \text{grad } (\mathbf{c} \cdot \mathbf{r}) = \mathbf{c} \tag{1.614}$$

If $f = \mathbf{r}^2$, then

$$\text{grad } f = \text{grad } \mathbf{r}^2 = 2\mathbf{r} \tag{1.615}$$

If $f = |\mathbf{r}|$ and $g = \mathbf{r}^2$, then $f = g^{1/2}$, and therefore,

$$\text{grad } f = \frac{1}{2} g^{-1/2} \text{ grad } g = \frac{\mathbf{r}}{|\mathbf{r}|} \tag{1.616}$$

Example 81 Gradient of Scalar Field Consider a scalar field

$$f(x, y, z) = x + x^2 y + y^3 + y^2 x + z^2 = C \tag{1.617}$$

The gradient of the field is

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} = \begin{bmatrix} y^2 + 2xy + 1 \\ x^2 + 2xy + 3y^2 \\ 2z \end{bmatrix} \tag{1.618}$$

Now assume that the gradient (1.618) is given. To find the field function, we should integrate the components of the gradient:

$$\begin{aligned} f &= \int \left(y^2 + 2xy + 1\right) dx = x^2 y + xy^2 + x + g_1(y, z) \\ &= \int \left(x^2 + 2xy + 3y^2\right) dy = x^2 y + xy^2 + y^3 + g_2(x, z) \\ &= \int (2z)\, dz = z^2 + g_3(x, y) \end{aligned} \tag{1.619}$$

Comparison shows that

$$f(x, y, z) = x + x^2 y + y^3 + y^2 x + z^2 = C \tag{1.620}$$

Example 82 Examples of Scalar and Vector Fields A field is another useful man-made concept to describe physical quantities. We call a function $f = f(x, y, z)$ a scalar field function if it assigns a numeric value to any point $P(x, y, z)$ of space. We call a function $\mathbf{f} = \mathbf{f}(x, y, z)$ a vector field function if it assigns a vector to any point $P(x, y, z)$ of space.

Temperature, density, and humidity are a few examples of scalar fields, and electric, magnetic, and velocity are a few examples of vector fields.

Example 83 Time Derivative of Scalar Field Consider a time-varying scalar field of a vector variable

$$f = f(\mathbf{r}(t)) \tag{1.621}$$

The time derivative of f is

$$\frac{df}{dt} = \frac{df}{d\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = \nabla f \cdot \mathbf{v} \tag{1.622}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is called the velocity of the position vector $\mathbf{r}$:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\hat{\imath} + \frac{dy}{dt}\,\hat{\jmath} + \frac{dz}{dt}\,\hat{k} \tag{1.623}$$

Following the derivative rule of a scalar field function (1.611), we may confirm that the time derivative of a scalar field is

$$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \nabla f \cdot \mathbf{v} \tag{1.624}$$

Example 84 Alternative Definition of Gradient Consider the scalar field function

$$f = f(\mathbf{r}) = f(x, y, z) \tag{1.625}$$

When the position vector moves from a point at $\mathbf{r} = \mathbf{r}(x, y, z)$ to a close point at $\mathbf{r} + d\mathbf{r}$, the field function changes from $f(\mathbf{r}) = f$ to $f(\mathbf{r} + d\mathbf{r}) = f + df$:

$$\begin{aligned} f(\mathbf{r} + d\mathbf{r}) &= f(x + dx, y + dy, z + dz) \\ &\approx f(x, y, z) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \cdots \end{aligned} \tag{1.626}$$

Therefore, a change in the field due to an infinitesimal change in position is given as

$$\begin{aligned} df &= f(\mathbf{r} + d\mathbf{r}) - f(\mathbf{r}) \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \nabla f \cdot d\mathbf{r} \end{aligned} \tag{1.627}$$

where df is called the *total derivative* of f.

Example 85 Directional Derivative An isosurface $f(x, y, z) = f$ can be expressed by the position vector ${}^G\mathbf{r}$

$${}^G\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1.628}$$

where its coordinates (x, y, z) are constrained by the isosurface equation (1.604). So, ${}^{G}\mathbf{r}$ is a two-variable vector function where its end point indicates a surface in G. To show this, let us consider a point P at $\mathbf{r} = \mathbf{r}(x, y, z)$ on an isosurface $f(x, y, z) = f$. Any change $d\mathbf{r}$ in the position of P will move the point to a new isosurface with a field value $f + df$:

$$\begin{aligned} df &= f(x + dx, y + dy, z + dz) - f(x, y, z) \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \end{aligned} \tag{1.629}$$

Let us move P on a space curve $\mathbf{r} = \mathbf{r}(q)$,

$$\mathbf{r} = x(q)\hat{\imath} + y(q)\hat{\jmath} + z(q)\hat{k} \tag{1.630}$$

The unit vector tangent to the curve at P is

$$\hat{u}_q = \frac{\partial \mathbf{r}/\partial q}{|\partial \mathbf{r}/\partial q|} = \frac{(\partial x/\partial q)\hat{\imath} + (\partial y/\partial q)\hat{\jmath} + (\partial z/\partial q)\hat{k}}{\sqrt{(dx/dq)^2 + (dy/dq)^2 + (dz/dq)^2}} \tag{1.631}$$

For an infinitesimal motion on the curve, we have

$$\frac{df}{dq} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q} \tag{1.632}$$

which can be interpreted as a dot product between two vectors:

$$\frac{df}{dq} = \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial x}{\partial q}\hat{\imath} + \frac{\partial y}{\partial q}\hat{\jmath} + \frac{\partial z}{\partial q}\hat{k}\right) \tag{1.633}$$

$$= \nabla f \cdot \frac{d\mathbf{r}}{dq} = \nabla f \cdot \hat{u}_q \left|\frac{d\mathbf{r}}{dq}\right| \tag{1.634}$$

The first vector, denoted by

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

is the gradient of the scalar function f expressed in terms of Cartesian coordinates, and the second vector,

$$\frac{d\mathbf{r}}{dq} = \frac{\partial x}{\partial q}\hat{\imath} + \frac{\partial y}{\partial q}\hat{\jmath} + \frac{\partial z}{\partial q}\hat{k} \tag{1.635}$$

is a tangent vector to the space curve (1.630) in the direction of increasing q. The dot product $\nabla f \cdot \hat{u}_q$ calculates the projection of ∇f on the tangent line to the space curve at P. To maximize this product, the angle between ∇f and $\hat{u}_q$ must be zero. It happens when ∇f and $d\mathbf{r}/dq$ are parallel:

$$\frac{\partial f/\partial x}{\partial x/\partial q} = \frac{\partial f/\partial y}{\partial y/\partial q} = \frac{\partial f/\partial z}{\partial z/\partial q} \tag{1.636}$$

A space curve (1.630) with condition (1.636) is perpendicular to the surface (1.628) and is called the *normal* or *flow curve*. Flow curves are perpendicular to isosurfaces of a scalar field f and show the lines of maximum change in field f.

The gradient of the scalar field indicates the direction to move for maximum change in the field, and its magnitude indicates the change in the field for a unit-length move. The product $\nabla f \cdot \hat{u}_q$, which determines the change in the field for a unit-length move in direction $\hat{u}_q$, is called the *directional derivative*.

Example 86 Direction of Maximum Rate of Increase Consider the scalar field

$$\varphi = 10 + xyz \tag{1.637}$$

A point $P(0.5, 0.4, z)$ on an isosurface will have the following z-component:

$$z = \frac{\varphi - 10}{xy} = \frac{\varphi - 10}{0.5 \times 0.4} = 5\varphi - 50 \tag{1.638}$$

The gradient vector $\nabla\varphi$ at point $P(0.5, 0.4, -50)$ on the isosurface $\varphi = 0$ is

$$\nabla\varphi = \begin{bmatrix} \dfrac{\partial\varphi}{\partial x} \\ \dfrac{\partial\varphi}{\partial y} \\ \dfrac{\partial\varphi}{\partial z} \end{bmatrix} = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} = \begin{bmatrix} -20 \\ -25 \\ 0.2 \end{bmatrix} \tag{1.639}$$

Example 87 Directional Derivative of a Field at a Point Consider the scalar field

$$f = xy^2 + yz^4 \tag{1.640}$$

Its rate of change at point $P(2,1,1)$ in the direction $\mathbf{r} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ is found by the inner product of its gradient at P,

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} y^2 \\ z^4 + 2xy \\ 4yz^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} \tag{1.641}$$

and $\hat{u}_{\mathbf{r}} = \mathbf{r}/\left|\mathbf{r}\right|$,

$$df = \nabla f \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{1}{\sqrt{6}} \\ \dfrac{2}{\sqrt{6}} \\ \dfrac{1}{\sqrt{6}} \end{bmatrix} = 6.1237 \tag{1.642}$$

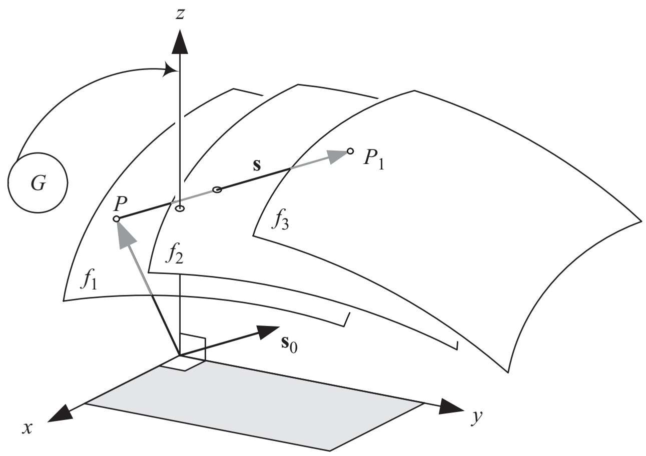

Figure 1.30 Directional derivative of a scalar field $f(\mathbf{r})$ in direction $\mathbf{s}_0$ at a point P is defined by $df/d\mathbf{s}$.

Example 88 ★ Isosurfaces Have No Common Point Consider a field $f = f(\mathbf{r})$ that is defined over a domain Ω of space. The isosurfaces corresponding to different $f = c$ fill the entire of Ω, and no two surfaces $f(\mathbf{r}) = c_1$ and $f(\mathbf{r}) = c_2$, $c_1 \neq c_2$ have common points. The isosurfaces, also called level surfaces, enable us to qualitatively judge the rate of change of the scalar field $f(\mathbf{r})$ in a give direction.

Consider a point P at $\mathbf{r}_P$ in a scalar field $f(\mathbf{r})$ and a fixed direction $\mathbf{s}_0$ as are shown in Figure 1.30. We draw a straight line $\mathbf{s}$ through P parallel to $\mathbf{s}_0$ and pick a point P_1 to define the directional derivative of $f(\mathbf{r})$:

$$\frac{df}{d\mathbf{s}} = \lim_{P_1 \to P} \frac{f(\mathbf{r}_P) - f\left(\mathbf{r}_{P_1}\right)}{PP_1} \tag{1.643}$$

Such a limit, if it exists, is called the directional derivative of the scalar field $f(\mathbf{r})$ in direction $\mathbf{s}_0$ at point P. Using Equation (1.606), we may show that

$$\frac{df}{d\mathbf{s}} = \nabla f \cdot \hat{u}_{\mathbf{s}} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma \tag{1.644}$$

where α, β, and γ are the directional cosines of $\mathbf{s}$.

1.6.3 Vector Field and Derivative

Consider a vector function $\mathbf{f}$ of a vector variable $\mathbf{r}$,

$$\mathbf{f} = \mathbf{f}(\mathbf{r}) = \mathbf{f}(x, y, z) = f_x(\mathbf{r})\,\hat{\imath} + f_y(\mathbf{r})\,\hat{\jmath} + f_z(\mathbf{r})\,\hat{k} \tag{1.645}$$

so it provides a vector $\mathbf{f}$ at a point $P(x, y, z)$. Having such a function is equivalent to associating a vector to every point of the space. The space in which there exist an $\mathbf{f}(x, y, z)$ is called a *vector field*, and the function $\mathbf{f}$ is called the *vector field function*.

The space derivative of $\mathbf{f}(\mathbf{r})$ is a quaternion product of ∇ and $\mathbf{f}$,

$$\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} = \nabla\mathbf{f}(\mathbf{r}) = \nabla \times \mathbf{f} - \nabla \cdot \mathbf{f} = \operatorname{curl}\mathbf{f} - \operatorname{div}\mathbf{f} \tag{1.646}$$

where

$$\begin{aligned}\nabla \times \mathbf{f} = \operatorname{curl}\mathbf{f} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} \\ &= \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{k}\end{aligned} \tag{1.647}$$

and

$$\begin{aligned}\nabla \cdot \mathbf{f} = \operatorname{div}\mathbf{f} &= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}\right) \\ &= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\end{aligned} \tag{1.648}$$

The first term, $\nabla \times \mathbf{f}$, is a vector and is called the *curl* of the vector field. The second term, $\nabla \cdot \mathbf{f}$, is a scalar and is called the *divergence* of the vector field. The curl of $\mathbf{f}$ indicates the change in direction and the divergence of $\mathbf{f}$ indicates the change in the magnitude of $\mathbf{f}$.

Proof: If $\mathbf{f}(\mathbf{r})$ is the function of a vector field, then each component of $\mathbf{f}$ is a scalar function of a vector variable,

$$\mathbf{f}(\mathbf{r}) = f_x(\mathbf{r})\,\hat{\imath} + f_y(\mathbf{r})\,\hat{\jmath} + f_z(\mathbf{r})\,\hat{k} \tag{1.649}$$

So, the differential of the vector field function $\mathbf{f}(\mathbf{r})$ with respect to a change in the position $\mathbf{r}$ is equal to the gradient of each component of $\mathbf{f}(\mathbf{r})$:

$$\begin{aligned}\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} &= \frac{d}{d\mathbf{r}}\left(f_x(\mathbf{r})\,\hat{\imath} + f_y(\mathbf{r})\,\hat{\jmath} + f_z(\mathbf{r})\,\hat{k}\right) = \nabla f_x\hat{\imath} + \nabla f_y\hat{\jmath} + \nabla f_z\hat{k} \\ &= \left(\frac{\partial f_x}{\partial x}\hat{\imath} + \frac{\partial f_x}{\partial y}\hat{\jmath} + \frac{\partial f_x}{\partial z}\hat{k}\right)\hat{\imath} + \left(\frac{\partial f_y}{\partial x}\hat{\imath} + \frac{\partial f_y}{\partial y}\hat{\jmath} + \frac{\partial f_y}{\partial z}\hat{k}\right)\hat{\jmath} \\ &\quad + \left(\frac{\partial f_z}{\partial x}\hat{\imath} + \frac{\partial f_z}{\partial y}\hat{\jmath} + \frac{\partial f_z}{\partial z}\hat{k}\right)\hat{k}\end{aligned} \tag{1.650}$$

Knowing that

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1 \tag{1.651}$$

$$\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath} = \hat{k} \qquad \hat{\jmath}\hat{k} = -\hat{k}\hat{\jmath} = \hat{\imath} \qquad \hat{k}\hat{\imath} = -\hat{\imath}\hat{k} = \hat{\jmath} \tag{1.652}$$

we can simplify Equation (1.650) to

$$\begin{aligned}\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} &= \left(-\frac{\partial f_x}{\partial x} - \frac{\partial f_x}{\partial y}\hat{k} + \frac{\partial f_x}{\partial z}\hat{\jmath}\right) + \left(\frac{\partial f_y}{\partial x}\hat{k} - \frac{\partial f_y}{\partial y} - \frac{\partial f_y}{\partial z}\hat{\imath}\right) \\ &\quad + \left(-\frac{\partial f_z}{\partial x}\hat{\jmath} + \frac{\partial f_z}{\partial y}\hat{\imath} - \frac{\partial f_z}{\partial z}\right) \\ &= \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{k} \\ &\quad - \frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} - \frac{\partial f_z}{\partial z}\end{aligned} \tag{1.653}$$

which is equal to

$$\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} = \nabla\mathbf{f}(\mathbf{r}) = \nabla \times \mathbf{f} - \nabla \cdot \mathbf{f} \tag{1.654}$$

The divergence of the gradient of a scalar field f is a fundamental partial differential equation in potential theory called the **Laplacian** of f. The Laplacian of f is shown by $\nabla^2 f$ and is equal to:

$$\begin{aligned}\nabla^2 f &= \operatorname{div}\operatorname{grad} f = \nabla \cdot \nabla f \\ &= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned} \tag{1.655}$$

■

Example 89 Derivative of a Vector Function with a Vector Variable If $\mathbf{c}$ is a constant vector and $\mathbf{f} = \mathbf{c} \times \mathbf{r}$ is a vector field, then

$$\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} = \operatorname{grad}\mathbf{f} = \operatorname{grad}(\mathbf{c} \times \mathbf{r}) = 3\mathbf{c} \tag{1.656}$$

because $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and

$$\begin{aligned}\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} &= \nabla \times \mathbf{f} - \nabla \cdot \mathbf{f} = \nabla \times (\mathbf{c} \times \mathbf{r}) - \nabla \cdot (\mathbf{c} \times \mathbf{r}) \\ &= (\nabla \cdot \mathbf{r})\,\mathbf{c} - (\nabla \cdot \mathbf{c})\,\mathbf{r} - (\nabla \times \mathbf{c}) \cdot \mathbf{r} \\ &= \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)\mathbf{c} = 3\mathbf{c}\end{aligned} \tag{1.657}$$

However, if $\mathbf{f} = c\mathbf{r}$, where c is a constant scalar, then its space derivative is a scalar,

$$\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} = \operatorname{grad}\mathbf{f} = \operatorname{grad} c\mathbf{r} = 3c \tag{1.658}$$

because

$$\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} = \nabla \times \mathbf{f} - \nabla \cdot \mathbf{f} = \nabla \times c\mathbf{r} - \nabla \cdot c\mathbf{r}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} - c\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3c \tag{1.659}$$

Example 90 Matrix Form of Vector Field Derivative We may arrange the derivative of a vector field $\nabla\mathbf{f}(\mathbf{r})$,

$$\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} = \nabla\mathbf{f}(\mathbf{r}) = \nabla f_x \hat{\imath} + \nabla f_y \hat{\jmath} + \nabla f_z \hat{k} \tag{1.660}$$

in matrix form:

$$\frac{d\mathbf{f}(\mathbf{r})}{d\mathbf{r}} = \left[\frac{\partial f_i}{\partial x_j}\hat{\imath}_j\hat{\imath}_i\right] = \begin{bmatrix} \frac{\partial f_x}{\partial x}\hat{\imath}\hat{\imath} & \frac{\partial f_y}{\partial x}\hat{\imath}\hat{\jmath} & \frac{\partial f_z}{\partial x}\hat{\imath}\hat{k} \\ \frac{\partial f_x}{\partial y}\hat{\jmath}\hat{\imath} & \frac{\partial f_y}{\partial y}\hat{\jmath}\hat{\jmath} & \frac{\partial f_z}{\partial y}\hat{\jmath}\hat{k} \\ \frac{\partial f_x}{\partial z}\hat{k}\hat{\imath} & \frac{\partial f_y}{\partial z}\hat{k}\hat{\jmath} & \frac{\partial f_z}{\partial z}\hat{k}\hat{k} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\partial f_x}{\partial x} & \frac{\partial f_y}{\partial x}\hat{k} & -\frac{\partial f_z}{\partial x}\hat{\jmath} \\ -\frac{\partial f_x}{\partial y}\hat{k} & -\frac{\partial f_y}{\partial y} & \frac{\partial f_z}{\partial y}\hat{\imath} \\ \frac{\partial f_x}{\partial z}\hat{\jmath} & -\frac{\partial f_y}{\partial z}\hat{\imath} & -\frac{\partial f_z}{\partial z} \end{bmatrix} \tag{1.661}$$

The trace of the matrix indicates the divergence of $\mathbf{f}$:

$$\operatorname{tr}\left[\frac{\partial f_i}{\partial x_j}\hat{\imath}_j\hat{\imath}_i\right] = \nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \tag{1.662}$$

Example 91 Symmetric and Skew-Symmetric Derivative Matrix Recalling that every matrix $[A]$ can be decomposed into a symmetric plus a skew-symmetric matrix,

$$[A] = \tfrac{1}{2}\left[[A] + [A]^{\mathrm{T}}\right] + \tfrac{1}{2}\left[[A] - [A]^{\mathrm{T}}\right] \tag{1.663}$$

we may determine the symmetric and skew-symmetric matrices of the derivative matrix:

$$\left[\frac{\partial f_i}{\partial x_j}\hat{\imath}_j\hat{\imath}_i\right]^{\mathrm{T}} = \begin{bmatrix} -\frac{\partial f_x}{\partial x} & -\frac{\partial f_x}{\partial y}\hat{k} & \frac{\partial f_x}{\partial z}\hat{\jmath} \\ \frac{\partial f_y}{\partial x}\hat{k} & -\frac{\partial f_y}{\partial y} & -\frac{\partial f_y}{\partial z}\hat{\imath} \\ -\frac{\partial f_z}{\partial x}\hat{\jmath} & \frac{\partial f_z}{\partial y}\hat{\imath} & -\frac{\partial f_z}{\partial z} \end{bmatrix} \tag{1.664}$$

$$\left[\frac{\partial f_i}{\partial x_j}\hat{\imath}_j\hat{\imath}_i\right] + \left[\frac{\partial f_i}{\partial x_j}\hat{\imath}_j\hat{\imath}_i\right]^{\mathrm{T}} = \begin{bmatrix} -2\frac{\partial f_x}{\partial x} & \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{k} & \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\hat{\jmath} \\ \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{k} & -2\frac{\partial f_y}{\partial y} & \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{\imath} \\ \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\hat{\jmath} & \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{\imath} & -2\frac{\partial f_z}{\partial z} \end{bmatrix} \tag{1.665}$$

$$\left[\frac{\partial f_i}{\partial x_j}\hat{\imath}_j\hat{\imath}_i\right] - \left[\frac{\partial f_i}{\partial x_j}\hat{\imath}_j\hat{\imath}_i\right]^{\mathrm{T}} = \begin{bmatrix} 0 & \left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right)\hat{k} & \left(-\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right)\hat{\jmath} \\ \left(-\frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x}\right)\hat{k} & 0 & \left(\frac{\partial f_z}{\partial y} + \frac{\partial f_y}{\partial z}\right)\hat{\imath} \\ \left(\frac{\partial f_x}{\partial z} + \frac{\partial f_z}{\partial x}\right)\hat{\jmath} & \left(-\frac{\partial f_y}{\partial z} - \frac{\partial f_z}{\partial y}\right)\hat{\imath} & 0 \end{bmatrix} \tag{1.666}$$

The skew-symmetric matrix is an equivalent form for $-\nabla \times \mathbf{f}$:

$$\nabla \times \mathbf{f} = \left[\frac{\partial f_i}{\partial x_j}\hat{\imath}_j\hat{\imath}_i\right]^{\mathrm{T}} - \left[\frac{\partial f_i}{\partial x_j}\hat{\imath}_j\hat{\imath}_i\right] \tag{1.667}$$

Example 92 **div r = 3** ***and*** **grad** $f(r) \cdot \mathbf{r} = r\ \partial f/\partial r$ Direct calculation shows that if

$$\mathbf{f} = \mathbf{r} \tag{1.668}$$

then

$$\begin{aligned} \operatorname{div}\mathbf{r} = \nabla \cdot \mathbf{r} &= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\right) \\ &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \end{aligned} \tag{1.669}$$

To calculate grad $f(r) \cdot \mathbf{r}$ we use

$$r = \sqrt{x^2 + y^2 + z^2} \tag{1.670}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r} \tag{1.671}$$

and show that

$$\begin{aligned}
\operatorname{grad} f(r) \cdot \mathbf{r} = \nabla f(r) \cdot \mathbf{r} &= \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\right) \\
&= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + y\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + z\frac{\partial f}{\partial r}\frac{\partial r}{\partial z} \\
&= \frac{x^2}{r}\frac{\partial f}{\partial r} + \frac{y^2}{r}\frac{\partial f}{\partial r} + \frac{z^2}{r}\frac{\partial f}{\partial r} = r\frac{\partial f}{\partial r}
\end{aligned} \tag{1.672}$$

As an application, consider a vector function field $\mathbf{f}$ that generates a vector $f(r)\mathbf{r}$ at every point of space,

$$\mathbf{f} = f(r)\mathbf{r} \tag{1.673}$$

Divergence of $\mathbf{f}$ would then be

$$\begin{aligned}
\operatorname{div}\mathbf{f} &= \operatorname{div}(f(r)\mathbf{r}) = \nabla \cdot f(r)\mathbf{r} \\
&= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[xf(r)\,\hat{\imath} + yf(r)\,\hat{\jmath} + zf(r)\,\hat{k}\right] \\
&= \frac{\partial}{\partial x}xf(r) + \frac{\partial}{\partial y}yf(r) + \frac{\partial}{\partial z}zf(r) \\
&= f(r)\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) + x\frac{\partial f(r)}{\partial x} + y\frac{\partial f(r)}{\partial y} + z\frac{\partial f(r)}{\partial z} \\
&= f(r)\nabla \cdot \mathbf{r} + \nabla f \cdot \mathbf{r} = 3f(r) + r\frac{\partial f(r)}{\partial r}
\end{aligned} \tag{1.674}$$

Example 93 Second Derivative of a Scalar Field Function The first space derivative of a scalar field function $f = f(\mathbf{r})$ is the gradient of f:

$$\frac{df}{d\mathbf{r}} = \nabla f \tag{1.675}$$

The second space derivative of $f = f(\mathbf{r})$ is

$$\begin{aligned}
\frac{d^2 f}{d\mathbf{r}^2} &= \frac{d}{d\mathbf{r}}\left(\frac{df}{d\mathbf{r}}\right) = \nabla(\nabla f) = \nabla \times \nabla f - \nabla \cdot \nabla f = -\nabla^2 f \\
&= -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)
\end{aligned} \tag{1.676}$$

Example 94 Trajectory of a Vector Field A space curve $\mathbf{r}(s)$ whose tangent at every point has the same direction as a vector field $\mathbf{v}(\mathbf{r})$ is called a *trajectory* of the field.

The trajectories of the vector field $\mathbf{f} = \nabla\varphi$ are the orthogonal curves to the iso-surfaces $\varphi = const$ at every point of space. Therefore, the trajectories are the lines of most rapid change of the function $\varphi = \varphi(t)$.

Consider a stationary velocity field $\mathbf{v}(\mathbf{r})$ of a moving fluid:

$$\mathbf{v} = \mathbf{v}(\mathbf{r}) = \mathbf{v}(x, y, z) \tag{1.677}$$

The trajectory of a velocity vector field is called the *streamline* and shows the path of motion of fluid particles. So, the trajectory of a fluid particle is a space curve $\mathbf{r} = \mathbf{r}(s)$ such that

$$d\mathbf{r} \times \mathbf{v}(\mathbf{r}) = 0 \tag{1.678}$$

or equivalently

$$\frac{dx}{v_x(x, y, z)} = \frac{dy}{v_y(x, y, z)} = \frac{dz}{v_z(x, y, z)} \tag{1.679}$$

Equation (1.678) is the vectorial differential equation of the trajectories of the vector field $\mathbf{v}(\mathbf{r})$. Integration of the differential equation provides the family of trajectories of the field. If the vector field is nonstationary, $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$, the streamlines change with time and do not necessarily coincide with the actual path of particles at a specific time. So, Equation (1.678) will become

$$d\mathbf{r} \times \mathbf{v}(\mathbf{r}, t) = 0 \tag{1.680}$$

or

$$\frac{dx}{v_x(x, y, z, t)} = \frac{dy}{v_y(x, y, z, t)} = \frac{dz}{v_z(x, y, z, t)} \tag{1.681}$$

If $\mathbf{v}(\mathbf{r}) = 0$ at a point P, Equation (1.678) would be indeterminate. Such a point is called a *singular* point.

Example 95 Time Derivative of Vector Field Consider a time-varying vector field of a vector variable:

$$\mathbf{f} = \mathbf{f}(\mathbf{r}(t)) \tag{1.682}$$

The time derivative of $\mathbf{f}$ is

$$\begin{aligned} \frac{d\mathbf{f}}{dt} &= \frac{df_x}{dt}\hat{\imath} + \frac{df_y}{dt}\hat{\jmath} + \frac{df_z}{dt}\hat{k} \\ &= (\nabla f_x \cdot \mathbf{v})\,\hat{\imath} + \left(\nabla f_y \cdot \mathbf{v}\right)\hat{\jmath} + (\nabla f_z \cdot \mathbf{v})\,\hat{k} \end{aligned} \tag{1.683}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the velocity of position vector $\mathbf{r}$.

Example 96 ★ Laplacian of $\varphi = 1/|\mathbf{r}|$ Consider a scalar field φ that is proportional to the distance from a fixed point. If we set up a Cartesian coordinate frame at the point, then

$$\varphi = \frac{k}{|\mathbf{r}|} \tag{1.684}$$

This is an acceptable model for gravitational and electrostatic fields. The Laplacian of such a field is zero,

$$\nabla^2 \varphi = \nabla^2 \frac{k}{|\mathbf{r}|} = 0 \tag{1.685}$$

because

$$\operatorname{grad} \frac{k}{|\mathbf{r}|} = -\frac{k}{|\mathbf{r}|^2} \operatorname{grad} |\mathbf{r}| = -\frac{k\mathbf{r}}{|\mathbf{r}|^3} \tag{1.686}$$

and therefore,

$$\begin{aligned} \operatorname{div} \operatorname{grad} \frac{k}{|\mathbf{r}|} &= \nabla \cdot \frac{-k\mathbf{r}}{|\mathbf{r}|^3} = -\frac{k}{|\mathbf{r}|^3} \nabla \cdot \mathbf{r} - k\mathbf{r} \cdot \nabla \frac{1}{|\mathbf{r}|^3} \\ &= -3\frac{k}{|\mathbf{r}|^3} + 3\frac{k\mathbf{r}}{|\mathbf{r}|^4} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} \end{aligned} \tag{1.687}$$

Example 97 ★ Tensor Fields Recalling that tensor is a general name for any type of physical quantity, such that a tensor of rank 1 is a scalar, rank 2 is a vector, and rank 3 is a 3 × 3 matrix, we can define a tensor field as a mathematical rule to assign a unique value of a tensor to each point of a certain domain of space. Traditionally tensor is used to indicate a tensor of rank 2 only.

Stress σ and strain ϵ are examples of the fundamental tensors in solid mechanics. A stress field is defined by

$$\left[\sigma_{ij}(\mathbf{r})\right] = \begin{bmatrix} \sigma_x(\mathbf{r}) & \tau_{xy}(\mathbf{r}) & \tau_{xz}(\mathbf{r}) \\ \tau_{yx}(\mathbf{r}) & \sigma_y(\mathbf{r}) & \tau_{yz}(\mathbf{r}) \\ \tau_{zx}(\mathbf{r}) & \tau_{zy}(\mathbf{r}) & \sigma_z(\mathbf{r}) \end{bmatrix} \tag{1.688}$$

A tensor field may be nonstationary if it is a function of space and time. So, for a nonstationary stress field $\sigma_{ij}(\mathbf{r}, t)$, we may define a stress tensor at a specific instant of time.

Example 98 Gradient of a Scalar Field Makes a Vector Field Consider a scalar field $f = f(\mathbf{r})$. The gradient of f assigns a vector ∇f at any position $\mathbf{r}$, and hence, $\mathbf{f} = \nabla f$ defines a vector field in the same definition domain of $f(\mathbf{r})$.

Example 99 Index Notation and Vector Analysis We may show a function $f = f(x, y, z)$ by $f = f(x_1, x_2, x_3)$ or in general by $f = f(q_1, q_2, q_3)$ to make it proper for index notation. If we show the partial derivative of a scalar field function $f = f(q_1, q_2, q_3)$ with respect to q_i by a comma,

$$\frac{\partial f}{\partial q_i} = f_{,i} \tag{1.689}$$

then it is possible to write the vector analysis operations by index notation:

1. Gradient of a scalar field $f = f(q_1, q_2, q_3)$:

$$\nabla f = \operatorname{grad} f = \sum_{i=1}^{3} f_{,i} \hat{u}_i \tag{1.690}$$

2. Laplacian of a scalar function $f = f(q_1, q_2, q_3)$:

$$\nabla^2 f = \sum_{i=1}^{3} f_{,ii} \tag{1.691}$$

3. Divergence of a vector field $\mathbf{r} = \mathbf{r}(q_1, q_2, q_3)$:

$$\nabla \cdot \mathbf{r} = \operatorname{div} \mathbf{r} = \sum_{i=1}^{3} r_{i,i} \tag{1.692}$$

4. Curl of a vector field $\mathbf{r} = \mathbf{r}(q_1, q_2, q_3)$:

$$\nabla \times \mathbf{r} = \operatorname{curl} \mathbf{r} = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \hat{u}_i r_{j,k} \tag{1.693}$$

Example 100 ★ Nabla Identities If **x** and **y** are two vector functions of **r** and φ is a scalar function of **r**, then we can verify the following identities:

$$\nabla (\mathbf{x} \cdot \mathbf{y}) = (\nabla \mathbf{x}) \cdot \mathbf{y} + \mathbf{x} \cdot (\nabla \mathbf{y}) \tag{1.694}$$

$$\nabla (\mathbf{x} \times \mathbf{y}) = (\nabla \mathbf{x}) \times \mathbf{y} + \mathbf{x} \times (\nabla \mathbf{y}) \tag{1.695}$$

$$\nabla \cdot \varphi \mathbf{x} = \nabla \varphi \cdot \mathbf{x} + \varphi (\nabla \cdot \mathbf{x}) \tag{1.696}$$

$$\nabla \times \varphi \mathbf{x} = \nabla \varphi \times \mathbf{x} + \varphi \nabla \times \mathbf{x} \tag{1.697}$$

$$\nabla \times \nabla \varphi = 0 \tag{1.698}$$

$$\nabla \cdot (\mathbf{x} \times \mathbf{y}) = (\nabla \times \mathbf{x}) \cdot \mathbf{y} + \mathbf{x} \cdot (\nabla \times \mathbf{y}) \tag{1.699}$$

$$\nabla \cdot (\nabla \times \mathbf{x}) = 0 \tag{1.700}$$

$$\nabla \times (\mathbf{x} \times \mathbf{y}) = (\mathbf{y} \cdot \nabla) \mathbf{x} - (\mathbf{x} \cdot \nabla) \mathbf{y} + \mathbf{x} (\nabla \cdot \mathbf{y}) - \mathbf{y} (\nabla \cdot \mathbf{x}) \tag{1.701}$$

$$\mathbf{x} \times (\nabla \times \mathbf{y}) = \nabla \mathbf{y} \cdot \mathbf{x} - \mathbf{x} \cdot \nabla \mathbf{y} \tag{1.702}$$

$$\nabla \times (\nabla \times \mathbf{x}) = \nabla (\nabla \cdot \mathbf{x}) - \nabla^2 \mathbf{x} \tag{1.703}$$

KEY SYMBOLS

$\mathbf{0}$	zero vector
$a, \ddot{x}, \mathbf{a}, \dot{\mathbf{v}}$	acceleration
a_{ijk}	inner product constant of $\mathbf{x}_i$
$\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{p}, \mathbf{q}$	vectors, constant vectors
$[\mathbf{abc}]$	scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
A, B	points
A, B, C	axes of triad, constant parameters
A, B, C, D	axes of tetrad, coefficient of a plane equation
$\hat{b}_1, \hat{b}_2, \hat{b}_3$	nonorthogonal unit vectors

$\mathbf{b} = \dot{\mathbf{z}}$	bong
b_{ijk}	inner product of $\mathbf{x}_i$, $\dot{\mathbf{x}}_j$
$B(oxyz)$, B_1, B_2	body coordinate frames
c	constant coefficient
c_i	weight factors of vector addition
c_{ijk}	inner product of $\dot{\mathbf{x}}_i$, $\dot{\mathbf{x}}_j$
$\mathbf{c} = \dot{\mathbf{s}}$	crackle
C	space curve
C_p, C_q	space curves on S for constant q and p at (p_0, q_0)
df	total derivative of f
$d\mathbf{r}$	infinitesimal displacement
ds	arc length element
${}^G\mathbf{d}_o$	position vector of B in G
$[D_2 z]$	second-derivative matrix
$f = f(\mathbf{r})$	scalar field function
f, g, h	functions of x, y, z of q_1, q_2, q_3
f_0	isovalue
$f(x, y, z)$	equation of a surface
$f(x, y, z) = f_0$	isosurface of the scalar field $f(\mathbf{r})$ for f_0
$\mathbf{f} = \mathbf{f}(\mathbf{r})$	vector field function
G	gravitational constant
$G(OXYZ)$	global coordinate frame
G_i	kinematic constants of three bodies
$\hat{\imath}, \hat{\jmath}, \hat{k}$	unit vectors of a Cartesian coordinate frame
$\hat{I}, \hat{J}, \hat{K}$	unit vectors of a global Cartesian system G
$\mathbf{j}, \dot{\mathbf{a}}, \ddot{\mathbf{v}}, \dddot{\mathbf{r}}$	jerk
$[J]$	Jacobian matrix
k	scalar coefficient
l	a line
n	number of dimensions of an nD space, controlled digit for vector interpolation
$\mathbf{n}$	perpendicular vector to a surface $z = g(x, y)$
$\hat{n}$	perpendicular unit vector
O	origin of a triad, origin of a coordinate frame
$OABC$	a triad with axes A, B, C
$(Ouvw)$	an orthogonal coordinate frame
$(Oq_1q_2q_3)$	an orthogonal coordinate system
P	point, particle
q, p	parameters, variables
$\mathbf{q} = \dot{\mathbf{b}}$	jeeq
$r = \lvert\mathbf{r}\rvert$	length of $\mathbf{r}$
$\mathbf{r}$	position vector
$\mathbf{r}_c$	position vector of curvature center of a space curve
${}_B\mathbf{r}_A$	position vector of A relative to B
$\mathbf{r}_p, \mathbf{r}_q$	partial derivatives of ${}^G\mathbf{r}$
$\mathbf{r}_\parallel, \mathbf{r}_\perp$	parallel and perpendicular components of $\mathbf{r}$ on l
R	radius
s	arc length parameter
$\mathbf{s} = d\mathbf{j}/dt$	snap, jounce
S	surface
t	time

$T = [\tau_1, \tau_2]$	the set in which a vector function is defined
u, v, w	components of a vector $\mathbf{r}$ in $(Ouvw)$
$\mathbf{u}$	Darboux vector
$\mathbf{u} = \dot{\mathbf{q}}$	sooz
u_1, u_2, u_3	components of $\hat{u}_r$
$\hat{u}^{\mathrm{T}}$	transpose of $\hat{u}$
$\hat{u}_l$	unit vector on a line l
$\hat{u}_r$	a unit vector on $\mathbf{r}$
$\hat{u}_1, \hat{u}_2, \hat{u}_3$	unit vectors along the axes q_1, q_2, q_3
$\hat{u}_r, \hat{u}_\theta, \hat{u}_\varphi$	unit vectors of a spherical coordinate system
$\hat{u}_t, \hat{u}_n, \hat{u}_b$	unit vectors of natural coordinate frame
$\hat{u}_u, \hat{u}_v, \hat{u}_w$	unit vectors of $(Ouvw)$
$\hat{u}_\parallel, \hat{u}_\perp$	parallel and perpendicular unit vectors of l
v	speed
$v, \dot{x}, \mathbf{v}$	velocity
$\mathbf{v}(\mathbf{r})$	velocity field
$\mathbf{v}(\mathbf{r}) = 0$	singular points equation
v	vector space
x, y, z	axes of an orthogonal Cartesian coordinate frame
x_0, y_0, z_0	coordinates of an interested point P
$\mathbf{x}, \mathbf{y}$	vector functions
$\mathbf{x}_i$	relative position vectors of three bodies
X, Y, Z	global coordinate axes
$\mathbf{X}_i$	global position vectors of three bodies
$\mathbf{z} = \ddot{\mathbf{p}}$	larz
Z	short notation symbol

Greek

α	angle between two vectors, angle between $\mathbf{r}$ and l
α, β, γ	directional cosines of a line
$\alpha_1, \alpha_2, \alpha_3$	directional cosines of $\mathbf{r}$ and $\hat{u}_r$
δ_{ij}	Kronecker delta
ϵ	strain
$\epsilon_i = 1/\lvert\mathbf{x}_i\rvert^3$	relative position constant of three bodies
ϵ_{ijk}	Levi-Civita symbol
θ	angle, angular coordinate, angular parameter
κ	curvature
$\boldsymbol{\kappa} = \kappa \hat{u}_n$	curvature vector
ρ	curvature radius
σ	stress tensor, normal stress
$\left[\sigma_{ij}(\mathbf{r})\right]$	stress field
τ	curvature torsion, shear stress
$\varphi = \varphi(\mathbf{r})$	scalar field function
ω	angular speed

Symbol

$\cdot$	inner product of two vectors
D	dimension
$\times$	outer product of two vectors
∇	gradient operator
$\nabla \mathbf{f}(\mathbf{r})$	gradient of $\mathbf{f}$

$\nabla \times \mathbf{f} = \text{curl}\,\mathbf{f}$	curl of $\mathbf{f}$
$\nabla \cdot \mathbf{f} = \text{div}\,\mathbf{f}$	divergence of $\mathbf{f}$
$\nabla^2 f$	Laplacian of f
$\nabla f = \text{grad}\, f$	gradient of f
$\mathbf{P} = \dot{\mathbf{c}}$	pop
$\triangle$	difference symbol
$\mathbb{R}$	set of real numbers
$\parallel$	parallel
$\perp$	perpendicular

EXERCISES

1. **Position Vector Characteristics** Three position vectors $\mathbf{r}_1 = OA$, $\mathbf{r}_2 = DB$, and $\mathbf{r}_3 = EB$ are illustrated in Figure 1.31.

   (a) Determine the length of OA, DB, and EB.
   (b) Determine the directional cosines of OA, DB, EB, and AO, BD, and BE.
   (c) Determine the angle between OA and DB.
   (d) Determine a vector to be perpendicular to both OA and DB.
   (e) Determine the surface area of the box by using the vectors OA, DB, and EB.
   (f) Determine the volume of the box by using the vectors OA, DB, and EB.
   (g) Determine the equation of the perpendicular plane to OA, DB, and EB.
   (h) ★ Determine the area of the triangle that is made up by the intersection of the planes in (g) if the plane of OA includes point O, the plane of DB includes point D, and the plane of EB includes point E.

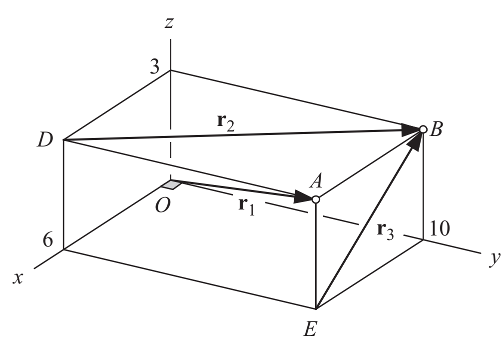

Figure 1.31 Three position vectors OA, DB, and EB.

2. ★ **Independent Orthogonal Coordinate Frames in Euclidean Spaces** In 3D Euclidean space, we need a triad to locate a point. There are two independent and nonsuperposable triads. How many different nonsuperposable Cartesian coordinate systems can be imagined in 4D Euclidean space? How many Cartesian coordinate systems do we have in an nD Euclidean space?

3. **Vector Algebra** Using

$$\mathbf{a} = 2\hat{\imath} - \hat{k} \qquad \mathbf{b} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k} \qquad \mathbf{c} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$$

determine

(a) $(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b})$ (b) $\mathbf{b} \cdot \mathbf{c} \times \mathbf{a}$

(c) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ (d) $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$

(e) A unit vector perpendicular to **b** and **c**

4. **Bisector** Assume that in $\triangle OAB$ of Figure 1.32 we have $\angle AOC = \angle BOC$. Show that the vector **c** divides the side AB such that

$$\frac{AC}{CB} = \frac{|\mathbf{a}|}{|\mathbf{b}|}$$

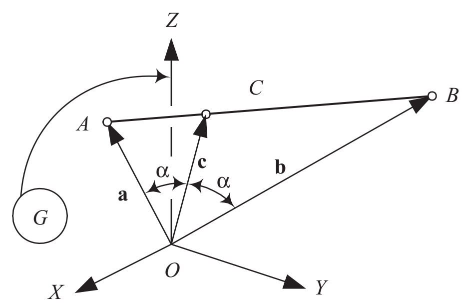

Figure 1.32 The bisector **c** divides the side AB such that $AC/CB = |\mathbf{a}| / |\mathbf{b}|$.

5. **Vector Interpolation** Determine a vector $\mathbf{r} = \mathbf{r}(q)$, $0 \le q \le 1$, to interpolate between two position vectors with the tip points A and B:

(a) $A\,(1,0,0)\,,\;\; B\,(0,1,0)$ (b) $A\,(1,1,0)\,,\;\; B\,(-1,1,0)$

(c) ★$A\,(1,0,0)\,,\;\; B\,(-1,1,0)$

6. **Vectorial Equation** Solve for **x**:

$$a\mathbf{x} + \mathbf{x} \times \mathbf{b} = \mathbf{c}$$

7. **Loci of Tip Point of a Vector** Find the locus of points (x, y, z) such that a vector from point $(2, -1, 4)$ to point (x, y, z) will always be perpendicular to the vector from $(2, -1, 4)$ to $(3, 3, 2)$.

8. **Rotating Triangle** The triangle in Figure 1.33 remains equilateral while point A is moving on an ellipse with a center at O. Assume a corner of the triangle is fixed at O.

(a) What is the path of point B?

(b) What is the area of the triangle?

(c) ★ If the side OA is turning with a constant angular velocity ω, then what is the area of the triangle as a function of time t?

(d) ★ If point A is moving with a constant speed v, then what is the area of the triangle as a function of time t?

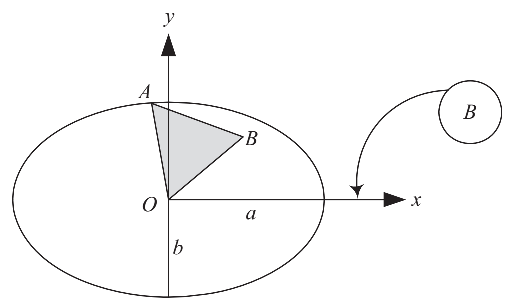

Figure 1.33 A rotating triangle.

9. **Components of an Unknown Vector** Consider a given vector **a**:

$$\mathbf{a} = \frac{b}{2}\sqrt{x^2 + y^2}\left(\hat{\imath} - \hat{\jmath}\right)$$

Solve the following equations for the components of the vector $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$:

$$\mathbf{r} \times \mathbf{a} = \frac{b}{2}\sqrt{x^2 + y^2}\sinh cy\,\hat{k} \qquad \mathbf{r} \cdot \mathbf{a} = 0$$

10. **Cosine Law** Consider a triangle ABC where its sides are expressed by vectors as

$$\overrightarrow{AB} = \mathbf{c} \qquad \overrightarrow{AC} = \mathbf{b} \qquad \overrightarrow{CB} = \mathbf{a} \qquad \mathbf{c} = \mathbf{a} + \mathbf{b}$$

Use vector algebra and prove the cosine law,

$$c^2 = a^2 + b^2 - 2ab\cos\alpha$$

where

$$\alpha = \angle ACB$$

11. **Trigonometric Equation** Use two planar vectors **a** and **b** which respectively make angles α and β with the x-axis and prove the following trigonometric equation:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

12. **Spherical Trigonometric Equations** Use vectors to prove the following spherical trigonometric equations in a spherical triangle $\triangle ABC$ with sides a, b, c and angle α, β, γ:

$$\cos a = \cos b\cos c + \sin b\sin c\cos\alpha$$

$$\cos b = \cos c\cos a + \sin c\sin a\cos\beta$$

$$\cos c = \cos a\cos b + \sin a\sin b\cos\gamma$$

13. **Three Colinear Points** Consider three points A, B, and C at **a**, **b**, and **c**. If the points are colinear, then

$$\frac{c_x - a_x}{b_x - a_x} = \frac{c_y - a_y}{b_y - a_y} = \frac{c_z - a_z}{b_z - a_z}$$

Show that this condition can be expressed as

$$(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a}) = 0$$

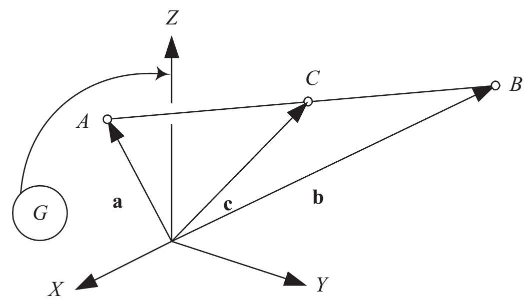

Figure 1.34 Dividing a line in a given ratio.

14. Dividing a Line in a Given Ratio The points A and B are at positions **a** and **b** as shown in Figure 1.34.

(a) Find the position vector **c** of point C that divides the line AB in the ratio of x/y:

$$\frac{AC}{CB} = \frac{x}{y}$$

(b) Show that the equation of a line is

$$\mathbf{r} - \mathbf{a} = k\,(\mathbf{b} - \mathbf{a})$$

(c) Show that the equation of a plane going through A and B and parallel to a vector **u** is

$$[(\mathbf{r} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})] \cdot \mathbf{u}$$

(d) Find the equation of a line going through a point A and parallel to a given vector **u**.

15. Volume of a Parallelepiped Consider three points A, B, C and determine the volume of the parallelepiped made by the vectors OA, OB, OC.

(a) $A\,(1, 0, 0)$, $B\,(0, 1, 0)$, $C\,(0, 0, 1)$

(b) $A\,(1, 0, 0)$, $B\,(0, 1, 0)$, C is the center of the parallelepiped in part (a)

(c) ★ $A\,(1, 0, 0)$, $B\,(0, 1, 0)$, C is at a point that makes the volume of the parallelepiped equal to 2. Determine and discuss the possible loci of C.

16. Moving on x-Axis The displacement of a particle moving along the x-axis is given by

$$x = 0.01t^4 - t^3 + 4.5t^2 - 10 \qquad t \geq 0$$

(a) Determine t_1 at which x becomes positive.

(b) For how long does x remain positive after $t = t_1$?

(c) How long does it take for x to become positive for the second time?

(d) When and where does the particle reach its maximum acceleration?

(e) Derive an equation to calculate its acceleration when its speed is given.

17. Moving on a Cycloid A particle is moving on a planar curve with the following parametric expression:

$$x = r\,(\omega t - \sin \omega t) \qquad y = r\,(1 - \cos \omega t)$$

(a) Determine the speed of the particle at time t.

(b) Show that the magnitude of acceleration of the particle is constant.

(c) Determine the tangential and normal accelerations of the particle.

(d) Using $ds = v\,dt$, determine the length of the path that the particle travels up to time t.

(e) Check if the magnitude of acceleration of the particle is constant for the following path:

$$x = a\,(\omega t - \sin \omega t) \qquad y = b\,(1 - \cos \omega t)$$

18. Areal Velocity Point A in Figure 1.35 is moving on the following circle such that its position vector **r** sweeps out with a constant areal velocity h:

$$x^2 - 2Rx + y^2 = 0$$

Determine the velocity and acceleration of the point.

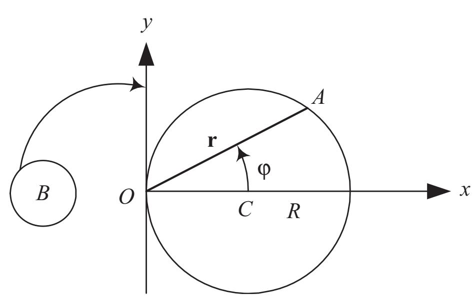

Figure 1.35 A moving point on a circle with constant areal velocity.

19. Velocity *v* as a Function of Position *x* Determine the acceleration of a particle that is moving according to the following equations:

(a) $v^2 = 2\,(x \sin x + \cos x)$

(b) $v^2 = 2\,(x \sinh x + \cosh x)$

(c) $v^2 = 4x - x^2$

20. **Relative Frequency** Consider a body B that is moving along the x-axis with a constant velocity u and every T seconds emits small particles which move with a constant velocity c along the x-axis. If f denotes the frequency and λ the distance between two successively emitted particles, then we have

$$f = \frac{1}{T} = \frac{c-u}{\lambda}$$

Now suppose that an observer moves along the x-axis with velocity v. Let us show the number of particles per second that the observer meets by the relative frequency f' and the time between meeting the two successive particles by the relative period T', where

$$f' = \frac{c-v}{\lambda}$$

Show that

$$f' \approx f\left(1 - \frac{v-u}{c}\right)$$

21. ★ **A Velocity–Acceleration–Jerk Equation** Show that if the path of motion of a moving particle,

$$\mathbf{r} = \mathbf{r}(t)$$

is such that the scalar triple product of its velocity–acceleration–jerk is zero,

$$\mathbf{v}\cdot(\mathbf{a}\times\mathbf{j}) = 0 \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} \qquad \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} \qquad \mathbf{j} = \frac{d^3\mathbf{r}}{dt^3}$$

then $\mathbf{r}(t)$ is a planar curve.

22. **Velocity of End Point of a Stick** Point A of the stick in Figure 1.36 has a constant velocity $\mathbf{v}_A = v\hat{\imath}$ on the x-axis. What is the velocity of point B?

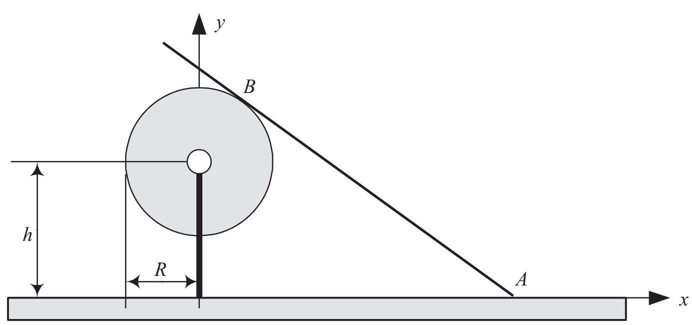

Figure 1.36 A sliding stick.

23. ★ **Disadvantages of a Nonorthogonal Triad** Why do we use an orthogonal triad to define a Cartesian space? Can we define a 3D space with nonorthogonal triads?

24. ★ **Usefulness of an Orthogonal Triad** Orthogonality is the common property of all useful coordinate systems, such as Cartesian, cylindrical, spherical, parabolic, and ellipsoidal coordinate systems. Why do we only define and use orthogonal coordinate systems? Do you think the ability to define a vector based on the inner product and unit vectors of the coordinate system, such as

$$\mathbf{r} = (\mathbf{r}\cdot\hat{\imath})\hat{\imath} + (\mathbf{r}\cdot\hat{\jmath})\hat{\jmath} + (\mathbf{r}\cdot\hat{k})\hat{k}$$

is the main reason for defining the orthogonal coordinate systems?

25. ★ **Three Coplanar Vectors** Show that if $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = 0$, then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar.

26. **A Derivative Identity** If $\mathbf{a} = \mathbf{a}(t)$ and $\mathbf{b}$ is a constant vector, show that

$$\frac{d}{dt}\left[\mathbf{a} \cdot (\dot{\mathbf{a}} \times \mathbf{b})\right] = \mathbf{a} \cdot (\ddot{\mathbf{a}} \times \mathbf{b})$$

27. **Lagrange and Jacobi Identities**

(a) Show that for any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ the Lagrange identity is correct:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

(b) Show that for any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ the following identities are correct:

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{abd}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{d}$$

$$[\mathbf{abc}]\,\mathbf{d} = [\mathbf{dbc}]\,\mathbf{a} + [\mathbf{dca}]\,\mathbf{b} + [\mathbf{dab}]\,\mathbf{c}$$

(c) Show that for any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ the Jacobi identity is correct:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = 0$$

28. ★ **Flight and Local Time** Figure 1.37 illustrates Earth of radius R with its local coordinate frame E that is turning about the Z-axis of a global coordinate frame G with a constant angular velocity ω. Consider an airplane that is flying at a height h above the spherical Earth. The local time of the airplane is the time of the associated point on Earth right below the airplane. So, the local time of the airplane is determined by its global coordinates. The speed of the airplane $\mathbf{v}$ can be indicated by an angle α with respect to the local constant latitude circle.

(a) An airplane is flying from Tokyo, Japan (35°41′6″N/139°45′5″E), to Tehran, Iran (35°40′19″N/51°25′27″E). What would be the velocity of the plane to have a constant local time. For simplicity assume that both cities are at 35°41′ N.

(b) An airplane is flying from Tehran, Iran (35°40′19″N/51°25′27″E), to Oklahoma City, Oklahoma (35°28′3″N/97°30′58″W). What would be the velocity of the plane to have a constant local time. For simplicity assume that both cities are at 35°40′N.

(c) An airplane is flying from Tehran, Iran (35°40′19″N/51°25′27″E), to Toronto, Canada (43°40′0″N/79°25′0″W). What would be the velocity of the plane to have a constant local time.

(d) An airplane flies from Toronto, Canada (43°40′0″N/79°25′0″W), to Tehran, Iran (35°40′19″N/51°25′27″E). What would be the local time at Tehran if the plane flies with a constant average velocity of part (c) and begins its flight at 1 AM.

(e) An airplane flies from Melbourne, Australia (37°49′0″S/144°58′0″E), to Dubai by the Persian Gulf (25°15′8″N/55°16′48″E). What would be the velocity of the airplane to have a constant local time.

(f) An airplane flies from Melbourne, Australia (37°49′0″S/144°58′0″E), to Dubai by the Persian Gulf (25°15′8″N/55°16′48″E) and returns to Melbourne with no stop. What would be the local time at Melbourne when the airplane is back. Assume the velocity of the airplane on the way to Dubai is such that its local time remains constant and the airplane keeps the same velocity profile on the way back.

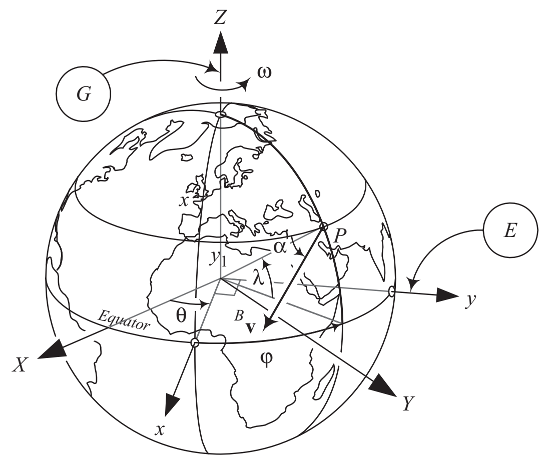

Figure 1.37 Flight and local time on Earth.

29. ★ **Vector Function and Vector Variable** A vector function is defined as a dependent vectorial variable that relates to a scalar independent variable:

$$\mathbf{r} = \mathbf{r}(t)$$

Describe the meaning and define an example for a vector function of a vector variable

$$\mathbf{a} = \mathbf{a}(\mathbf{b})$$

and a scalar function of a vector variable

$$f = f(\mathbf{b})$$

30. ★ **Index Notation** Expand the mass moment of n particles $m_1, m_2, \ldots, m_n$ about a line $\hat{u}$,

$$I_{\hat{u}} = \sum_{i=1}^{n} m_i \left(\mathbf{r}_i \times \hat{u}\right)^2$$

and express $I_{\hat{u}}$ by an index equation.

31. ★ **Frame Dependent and Frame Independent** A vector function of scalar variables is a frame-dependent quantity. Is a vector function of vector variables frame dependent? What about a scalar function of vector variables?

32. ★ **Coordinate Frame and Vector Function** Explain the meaning of ${}^B\mathbf{v}_P({}^G\mathbf{r}_P)$ if $\mathbf{r}$ is a position vector, $\mathbf{v}$ is a velocity vector, and $\mathbf{v}(\mathbf{r})$ means $\mathbf{v}$ is a function of $\mathbf{r}$.

33. **A Vector Product Identity** Show that for any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in a Cartesian coordinate frame, we have

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

34. **Expansion of a Vector with Respect to Two Vectors** Consider two linearly independent vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Show that every vector $\mathbf{r}_3$ coplanar with $\mathbf{r}_1$ and $\mathbf{r}_2$ has a unique expansion

$$\mathbf{r}_3 = -c_1\mathbf{r}_1 - c_2\mathbf{r}_2$$

35. ★ **Natural Coordinate System of a Parametric Path** Assume the path s can be expressed by a one-parameter function $s = s(\alpha)$, where α is the parameter. Show that

$$\textbf{(a)}\quad \mathbf{r}''s' = s''s'\hat{u}_t + \frac{(s')^3}{\rho}\hat{u}_n$$

$$\textbf{(b)}\quad \hat{u}_n = \frac{\rho}{(s')^3}\left(\mathbf{r}''s' - s''\mathbf{r}'\right)$$

$$\textbf{(c)}\quad \frac{1}{\rho} = \frac{1}{(s')^3}\sqrt{(\mathbf{r}''\cdot\mathbf{r}'')(s')^2 - (\mathbf{r}'\cdot\mathbf{r}'')^2}$$

36. ★ **Natural Coordinate System of a Planar Path** Show that if a planar path is given by a set of equations of a parameter α which is not necessarily the path length

$$x = x(\alpha) \qquad y = y(\alpha)$$

then the natural tangential unit vector and derivatives of the path are

$$\hat{u}_t = \frac{1}{\sqrt{x'^2 + y'^2}}\begin{bmatrix} x' \\ y' \end{bmatrix} \qquad \frac{d\hat{u}_t}{d\alpha} = \frac{x'y'' - x''y'}{x'^2 + y'^2}$$

$$\frac{ds}{d\alpha} = \sqrt{x'^2 + y'^2} \qquad \frac{1}{R} = \frac{\left|x'y'' - x''y'\right|}{\left(x'^2 + y'^2\right)^{3/2}}$$

Use the equations and show that the radius of curvature of the parabola $y = x^2/(4a)$ is

$$R = \frac{4a^2}{\left(4a^2 + x^2\right)^{3/2}}$$

37. ★ **Natural Coordinate System and Important Planes** Consider the space curve

$$x = (10 + 2\sin\theta)\cos\theta \qquad y = (10 + 2\sin\theta)\sin\theta \qquad z = 2 + 2\cos\theta$$

(a) Find the equations of osculating, perpendicular, and rectifying planes and determine them at $\theta = 45^\circ$.

(b) Find the radius and coordinates of the center of curvature of the curve.

38. **Moving on a Given Curve** A particle is moving on a curve $y = f(x)$ such that the x-component of the velocity of the particle remains constant. Determine the acceleration and jerk of the particle.

(a) $y = x^2$

(b) $y = x^3$

(c) $y = e^x$

(d) Determine the angle between velocity vectors of curves (a) and (b) at their intersection.

(e) Determine the exponent n of $y = x^n$ such that the angle between velocity vectors of this curve and curve (a) at their intersection is 45 deg.

39. ★ **A Wounding Cable** Figure 1.38 illustrates a turning cone and wounding cable that supports a hanging box. If the cone is turning with angular velocity ω, determine:

(a) Velocity, acceleration, and jerk of the box

(b) The angular velocity ω such that the velocity of the box remains constant

(c) The angular velocity ω such that the acceleration of the box remains constant

(d) The angular velocity ω such that the jerk of the box remains constant

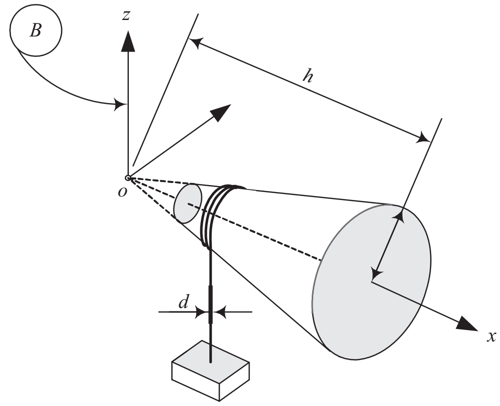

Figure 1.38 A wounding cable on a cone.

40. ★ **Natural Coordinate Unit Vectors** A particle is moving on the following curves. Determine the tangential $\hat{u}_t$, normal $\hat{u}_n$, and binormal $\hat{u}_b$ unit vectors.

(**a**) $x = \sin\alpha \qquad y = 8x^2$
(**b**) $x = \cos\alpha \qquad y = -x^2$
(**c**) $x = \sin^2 t \qquad y = -\sqrt{x}$

41. ★ **Cylindrical Coordinate System and a Helix** A particle is moving on a helix of radius R and pitch a at a constant speed, where

$$z = \frac{a\theta}{2\pi}$$

(a) Express the position, velocity, and acceleration of the particle in the cylindrical coordinate system.
(b) Determine the unit vectors in the cylindrical coordinate system.
(c) Determine the radius of curvature.

42. ★ **Torsion and Curvature of a Helix** A point P is moving with arc length parameter s on a space curve,

$$\mathbf{r}(s) = 10\cos\frac{\sqrt{2}s}{20}\hat{\imath} + 10\sin\frac{\sqrt{2}s}{20}\hat{\jmath} + 10\sqrt{2}s\hat{k} = \begin{bmatrix} 10\cos\frac{\sqrt{2}s}{20} \\ 10\sin\frac{\sqrt{2}s}{20} \\ 10\sqrt{2}s \end{bmatrix}$$

Determine the curvature κ and torsion τ.

43. **Arc Length Element** Find the square of the element of arc length ds in cylindrical and spherical coordinate systems.

44. **Plane through Three Points** Show that the equation of a plane that includes the three points

$$P_1\,(0,1,2) \qquad P_2\,(-3,2,1) \qquad P_3\,(1,0,-1)$$

is

$$4x + 10y - 2z = 6$$

45. Vectorial Operation of Scalar Fields Consider a scalar field $\phi(x, y, z)$,

$$\phi(x, y, z) = \tfrac{1}{2}ax^2 + \tfrac{1}{2}by^2 + c^z$$

(a) Determine $\operatorname{grad}\phi(x, y, z) = \nabla\phi(x, y, z)$.
(b) Determine $\operatorname{curl}\operatorname{grad}\phi(x, y, z) = \nabla \times \nabla\phi(x, y, z)$.
(c) Show that $\nabla \times \nabla\phi = 0$ regardless of the form of ϕ.
(d) Show that $\nabla \cdot (\nabla \times \mathbf{a}) = 0$ regardless of the form of $\mathbf{a}$.
(e) Show that $\nabla \cdot (\phi\mathbf{a}) = \nabla\phi \cdot \mathbf{a} + \phi\,(\nabla \cdot \mathbf{a})$.